Journal of Lie Theory
Volume **13** (2003) 481–510

Strange Phenomena Related to Ordering Problems in Quantizations

Hideki Omori[§], Yoshiaki Maeda[†], Naoya Miyazaki[◇], and Akira Yoshioka[*]

Communicated by P. Olver

Abstract. We introduce an object which in a sense extends the notion of a covering space. Such an object is required to understand the "group" generated by the exponential functions of quadratic forms in the Weyl algebra, and gives in some sense a complexification of the metaplectic group.

1. Introduction

The quantum picture is basically set up by the Weyl algebra. It is derived from the differential calculus via correspondence principle: Let u be the operator $x\cdot$ of multiplication by the coordinate function x on $\mathbb{R}$ acting on the space of all C^∞ functions on $\mathbb{R}$, and let v be the differential operator $i\hbar\partial_x$. u and v generate an algebra $W_\hbar$, called the *Weyl algebra*. Thus, the Weyl algebra is an associative algebra generated over $\mathbb{C}$ by u, v with the fundamental relation $[u,v]=-i\hbar$.

However, the correspondence principle, $u \leftrightarrow x\cdot$, $v \leftrightarrow i\hbar\partial_x$, raises many mathematical questions. We meet immediately the ordering problem (see §1). That occurs mainly in Schrödinger quantization procedure which assigns a differential operator defined on a configuration space to every classical observable.

Avoiding configuration spaces, the Heisenberg procedure for quantum mechanics is a formalism built from von Neumann algebras or C^* algebras (cf. [Co]). In this formalism, the ordering problem comes down to expressing an element of

[§] Partially supported by Grant-in-Aid for Scientific Research (#14540092), Ministry of Education, Culture, Sports, Science and Technology, Japan.
[†] Partially supported by Grant-in-Aid for Scientific Research (#12440022), Ministry of Education, Culture, Sports, Science and Technology, Japan.
[◇] Partially supported by Grant-in-Aid for Scientific Research (#13740049), Ministry of Education, Culture, Sports, Science and Technology, Japan.
[*] Partially supported by Grant-in-Aid for Scientific Research (#13640088), Ministry of Education, Culture, Sports, Science and Technology, Japan.

ISSN 0949–5932 / $2.50

the algebra in the unique way. Whenever the expression is fixed, it is possible to put a topology on the algebra and to take the topological completion (cf. §1). However, in this formalism it is difficult to relate the quantum world to the classical world, as it is very arduous to work out in the theory of selfadjoint operators.

In this paper, we first take several topological completions of the Weyl algebra. Here, we are not restricted to work within C^*-algebras or operator algebras, as we only treat $\hbar$ as a deformation parameter (a positive real parameter), following the notion of deformation quantization initiated by [BFFLS]. Noting that many laws in physics are expressed as evolution equations, we will consider the evolution equation $\frac{d}{dt}f_t = i\hbar p_*(u,v)*f_t$ given by a polynomial $p_*(u,v)$ by using the product in $W_\hbar$. Thus, solving the evolution equation, we have to know the individual phenomenon.

In the theory of formal deformation quantization where $\hbar$ is treated as a formal parameter, there is no problem in solving evolution equations. In the theory where $\hbar$ is treated as a positive real parameter, the existence of solutions of evolution equations is not so obvious. However, it is easy to see the uniqueness of a real analytic solution if it exists. To obtain solutions, we have to construct some topological completion of the Weyl algebra, in which one can define the exponential function $e_*^{itp_*(u,v)}$.

Along the configuration space approach of Schrödinger, Hörmander introduced the notion of pseudo-differential operators (ψDO) and of Fourier integral operators on any manifold [9] to treat $e_*^{itp_*(u,v)}$ whenever $p(u,v)$ is in the symbol class of order one with respect to v. Furthermore, Hörmander [10] proposed a Weyl calculus on $\mathbb{R}^{2n}$ using extended notion of ψDO's where u and v have the same weight. In these theories, the essential self-adjointness of $p_*(u,v)$ is crucial, because the evolution equations for $e_*^{itp_*(u,v)}$ are treated as partial differential equations.

On the other hand, there is another classical way of treating such evolution equations. This is indeed the method of Lie theory, which treats such evolution equations within the system of ordinary differential equations. In order to use this method, we restrict our attention to the linear hull over $\mathbb{C}$ of $*$-exponential functions of polynomials of degree ≤ 2.

However, within these restricted objects, we have encountered pathological phenomena: A typical phenomenon is that the region where the product is defined depends on the ordering of expressions (see Lemma 10). In spite of this, one can obtain product formulas by collecting all possible ordering expressions. Moreover, it happens that an element has *two different inverses.* Since this brakes the associativity (see §1), we cannot treat such a system as an associative algebra.

Motivated by such pathological phenomena, we investigate more precisely the types of difficulties which occur with such objects. To extend products, we have to treat intertwiners between different ordering expressions. It happens, however, that intertwiners are defined only 2-to-2 mappings on the space of exponential functions of quadratic forms, because of the ambiguity of taking a square root $\sqrt{\ }$ in the calculation (see § 6). Thus, ambiguity can not be eliminated by taking an appropriate double covering spaces (see § 4).

Thinking about the serious meaning of such a pathological phenomenon, we are forced to consider the notion of manifolds which do not form point sets. We propose in this paper the idea of two-valued elements.

Besides such strange phenomena, we have another motivation for treating $\hbar$ as a genuine parameter. The deformation quantization of [BFFLS] made us free from operator theory. In particular, if we treat the deformation parameter $\hbar$ as a formal parameter and consider everything in the category of formal power series of $\hbar$ (formal deformation), then the quantization problem goes through very smoothly. Kontsevich [K] showed every Poisson algebra on a manifold is formally deformation quantizable. However it is apparent that formal deformation quantization plays only a probe for the quantum world with exact physical significance.

After Kontsevich's result, a next generation of deformation theory is developing, called the exact deformation theory. We have to make an effort to revise the deformation theory more close to the theories where C^* algebras or von Neumann algebras are explicitly used (cf. Connes [Co]). Actually, Rieffel [R] has proposed a notion of such a deformation theory, called strictly deformation quantization, and has pointed out many serious problems.

In this paper we point out several serious difficulties are still involved in the theory of classical ordering.

This paper is organized as follows:

In §2, we give several basic facts, several different orderings, and product formulas. We also explain also several pathological phenomena, and how such phenomena appear naturally in exact deformation quantization theory. However, no problem occurs for exponential functions of linear functions of generators, (see Theorem 3, and the equation (30)).

Thus, in §3, we restrict our attention to the space of exponential functions of quadratic forms. Infinitesimal actions of quadratic forms is computed in Weyl ordering and normal ordering, and these define involutive distributions on the space of exponential functions. We easily obtain maximal integral submanifolds.

In §4, we give the explicit formula for $*$-exponential functions in Weyl ordering and in normal ordering. Via these explicit expressions, we find an "element" ε_{00}, called the *polar element*, having such a strange property that one must call this is a "two valued" element, although such a notion has never appeared in ordinary mathematics.

In spite of this, ε_{00} is very useful in computation. We give in §5 several product formulas, and show that $*$-exponential functions of quadratic forms generate a group-like object, which looks like a non-trivial double cover of $SL_{\mathbb{C}}(2)$. Nevertheless, technicality is involved in a standard classical Lie theory. To understand why such a strange element appears, we define in §6 intertwiners between several ordering expressions. We see that our strange phenomena are caused by the ambiguity of $\frac{1}{\sqrt{}}$ of intertwiners. Because of this ambiguity, intertwiners are defined only as "2-to-2 diffeomorphisms" on the set of exponential functions of quadratic forms.

Hence in §7 we describe the corresponding glued object. Similar phenomena occurs in the magnetic monopole theory, and was treated mathematically by Brylinski (see the last chapter of [**?**]) using the theory of gerbes of Giraud. However, we prefer to use the notion of two-valued element, because it is very simple and intuitive. To clarify these, we propose the notion of blurred $\mathbb{C}_*$-bundles.

Our conclusion in this paper is that $*$*-exponential functions of quadratic* *forms generate a group-like object which is not a point set, but this object can be understood as a non-trivial double cover of $SL_{\mathbb{C}}(2)$. It contains the non trivial double covers of $SL_{\mathbb{R}}(2)$ and $SU(1,1)$ as real forms. Hence this object may be understood as a complexification of metaplectic group $Mp(2,\mathbb{R})$ [GiS]. It is known that there is no complexification of these groups as genuine Lie groups.*

2. The Weyl algebra and extensions

We consider the Weyl algebra $W_\hbar$ generated by u,v over $\mathbb{C}$ with the fundamental relation $[u,v]\,(=u{*}v-v{*}u)=-i\hbar$ where $\hbar$ is a positive constant. The pair (u,v) of generators is called a *canonical conjugate pair.*

2.1. Orderings and product formulas.

To express elements of the Weyl algebra $W_\hbar$, we introduce several orderings. Namely, we choose the typical orderings in $W_\hbar$; normal ordering, anti-normal ordering, and Weyl ordering, respectively. For the normal ordering (resp. the anti-normal ordering), we write elements in the form $\sum a_{m,n}u^m{*}v^n$ (resp. $\sum a_{m,n}v^m{*}u^n$) by arranging u to the left (resp. right) hand side in each term. In the Weyl ordering elements are written in the form $\sum a_{m,n}u^m\odot v^n$ defined by using the symmetric product $\cdot$ given by $u{\cdot}v=\frac{1}{2}(u{*}v+v{*}u)$. (See [17] §1.2, but we have no need to know about the symmetric product, since the product formulas are given concretely.)

Using such orderings, one can identify the Weyl algebra $W_\hbar$ with the space $\mathbb{C}[u,v]$ of all polynomials on $\mathbb{C}^2$ with coordinates u,v. Thus, the Weyl algebra $W_\hbar$ can be viewed as a noncommutative associative product structure defined on the space $\mathbb{C}[u,v]$ by fixing an ordering of $W_\hbar$. According to the normal, anti-normal, Weyl orderings of $W_\hbar$, we have noncommutative products on $\mathbb{C}[u,v]$, and denoted by $*_N, *_{\tilde{N}}, *_M$, respectively.

Product formulas. Let $f(u,v), g(u,v)\in\mathbb{C}[u,v]$. We denote the ordinary commutative product of functions by $\circ, \bullet, \cdot$ solely to distinguish the orderings of $W_\hbar$.

- *The normal ordering*: the product $*$ of the Weyl algebra is given by the ΨDO-*product formula* as follows: (Note this coincides with the product formula of ΨDO's,)

$$f(u,v)*_N g(u,v)=f\exp\{i\hbar(\overleftarrow{\partial_v}\circ\overrightarrow{\partial_u})\}g. \tag{1}$$

- *The anti-normal ordering*: the product $*$ of the Weyl algebra is given by the $\overline{\Psi}$DO-*product formula* as follows:

$$f(u,v)*_{\bar{N}} g(u,v)=f\exp\{-i\hbar(\overleftarrow{\partial_u}\bullet\overrightarrow{\partial_v})\}g. \tag{2}$$

- *The Weyl ordering*: the product $*$ of the Weyl algebra is given by the *Moyal product formula* as follows:

$$f(u,v)*_M g(u,v)=f\exp\frac{i\hbar}{2}\{\overleftarrow{\partial_v}\stackrel{\cdot}{\wedge}\overrightarrow{\partial_u}\}g \tag{3}$$

where $\overleftarrow{\partial_v} \dot{\wedge} \overrightarrow{\partial_u} = \overleftarrow{\partial_v} \cdot \overrightarrow{\partial_u} - \overleftarrow{\partial_u} \cdot \overrightarrow{\partial_v}$, and

$$f(\overleftarrow{\partial_v} \cdot \overrightarrow{\partial_u} - \overleftarrow{\partial_u} \cdot \overrightarrow{\partial_v})g = \partial_v f \cdot \partial_u g - \partial_u f \cdot \partial_v g.$$

Every product formula yields $u*v - v*u = -i\hbar$, and recovers the Weyl algebra $W_\hbar$.

Within the Weyl algebra $W_\hbar$, $\frac{1}{i\hbar}\mathrm{ad}(u)$ and $-\frac{1}{i\hbar}\mathrm{ad}(v)$ are mutually commutating pair of derivations. These derivations also reproduce commutative products $\circ, \bullet, \cdot$ from the $*$-product by reversing formulas (cf. [17]). Such inverse expressions ensure that there is *no other relation within $W_\hbar$ produced by the ordering.*

For elements $p_*(u,v)$, $q_*(u,v) \in W_\hbar$, we have various expressions according to the ordering. The product is given as follows:

$$p_*(u,v) * q_*(u,v) = f_.(u,v) *_M g_.(u,v) = f_\circ(u,v) *_N g_\circ(u,v) = f_\bullet(u,v) *_{\bar N} g_\bullet(u,v).$$

If no confusion is suspected, we omit the suffix $M, N, \bar N$ in the $*$-product.

Let $Hol(\mathbb{C}^2)$ be the space of all entire functions on $\mathbb{C}^2$ with the compact-open topology. $Hol(\mathbb{C}^2)$ is a complete topological linear space in the compact open topology. Every product formula (1), (2), (3) has the following properties:

Proposition 1. *(1) $f*g$ is defined if one of f,g is a polynomial.*
(2) For every polynomial $p=p(u,v)$, the left- (resp. right-) multiplication $p*$ (resp. $*p$) is a continuous linear mapping of $Hol(\mathbb{C}^2)$ into itself in the compact-open topology.

We call such a system $(Hol(\mathbb{C}^2), \mathbb{C}[u,v], *)$ a $(\mathbb{C}[u,v];*)$*-bimodule.*

By the polynomial approximation theorem, the associativity

$$f*(g*h) = (f*g)*h$$

holds if two of f,g,h are polynomials. We call this *2-p-associativity.*

2.2. Canonical conjugate pairs.

For every $A \in SL_{\mathbb{C}}(2)$, we have a change of generators

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = A \begin{pmatrix} u \\ v \end{pmatrix}.$$

It is obvious that $[u',v']_* = -i\hbar$, and hence u',v' may be viewed as generators. The replacement (pull-back) A^* of u,v by u',v' gives an algebra isomorphism of $W_\hbar$. Thus, we may consider the ordering problem by using u',v' instead of u,v.

The following is the most useful property of Moyal product formula (3):

Proposition 2. *For every $A \in SL_{\mathbb{C}}(2)$ and $\binom{\alpha}{\beta} \in \mathbb{C}^2$, let Φ^* be the replacement (pull-back) of u,v into u',v' by the combination of the linear transformation by the matrix A and the parallel displacement $\binom{\alpha}{\beta}$:*

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = A \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \qquad A \in SL_{\mathbb{C}}(2), \quad (\alpha,\beta) \in \mathbb{C}^2.$$

Then, Φ^* is an isomorphism both on $(\mathbb{C}[u,v], \cdot)$ and $(\mathbb{C}[u,v], *)$.

We remark that the normal and the anti-normal orderings do not have such a property. It is easily seen that

$$(au+bv)_*^m = (au+bv)_{\cdot}^m, \quad \text{but} \quad (au+bv)_*^m \neq (au+bv)_\circ^m \quad \text{for} \quad ab \neq 0.$$

For the proof of Proposition 2, we have only to remark the following identity:

$$\overleftarrow{\partial_v} \dot{\wedge} \overrightarrow{\partial_u} = \overleftarrow{\partial_{v'}} \dot{\wedge} \overrightarrow{\partial_{u'}}.$$

2.3. Evolution equations.

In the $(\mathbb{C}[u,v],*)$-bimodule $(Hol(\mathbb{C}),\mathbb{C}[u,v],*)$, we consider the evolution equation

$$\frac{d}{dt}f_t = p_*(u,v) * f_t, \quad f_0 = f(u,v) \tag{4}$$

for every polynomial $p_*(u,v)$. If $p(u,v) = u^2 + (\frac{i}{\hbar}v)^2$, the equation corresponds to that of standard harmonic oscillator. For a complex parameter t, the evolution equation (4) may not be necessarily solved for arbitrary initial function. However a real analytic solution for (4) in t is unique if it exists. The solution, if it exists for the initial function $f_0 = 1$, will be denoted by $e_*^{tp_*(u,v)}$. If the real analytic solution of (4) exists, then we denote it by $e_*^{tp(u,v)} * f(u,v)$.

If the infinite series $\sum \frac{t^k}{k!} p(u,v)_*^k$ converges, then it must be the solution of (4). Since $\sum \frac{t^k}{k!}(\alpha u + \beta v)_*^k$ converges, we use the $*$-exponential function $e_*^{t(\alpha u+\beta v)}$ to define the intertwiners between different orderings, (see §6).

2.4. Extensions of product formulas.

Starting from $(\mathbb{C}[u,v];*)$, we extend the $*$-product to a wider class of functions. For every positive real number p, we set

$$\mathcal{E}_p(\mathbb{C}^2) = \{f \in Hol(\mathbb{C}^2) \mid \|f\|_{p,s} = \sup |f|\, e^{-s|\xi|^p} < \infty,\ \forall s > 0\} \tag{5}$$

where $|\xi| = (|u|^2 + |v|^2)^{1/2}$. The family of seminorms $\{\|\cdot\|_{p,s}\}_{s>0}$ induces a topology on $\mathcal{E}_p(\mathbb{C}^2)$ and $(\mathcal{E}_p(\mathbb{C}^2), \cdot)$ is an associative commutative Fréchet algebra, where the dott $\cdot$ is the ordinary product for functions in $\mathcal{E}_p(\mathbb{C}^2)$. The product $\cdot$ may be replaced by $\circ$ or $\bullet$ to indicate the ordering. It is easily seen that for $0 < p < p'$, there is a continuous embedding

$$\mathcal{E}_p(\mathbb{C}^2) \subset \mathcal{E}_{p'}(\mathbb{C}^2) \tag{6}$$

as commutative Fréchet algebras (cf. [GS]), and that $\mathcal{E}_p(\mathbb{C}^2)$ is $SL_{\mathbb{C}}(2)$-invariant.

It is obvious that every polynomial is contained in $\mathcal{E}_p(\mathbb{C}^2)$ and $\mathbb{C}[u,v]$ is dense in $\mathcal{E}_p(\mathbb{C}^2)$ for any $p > 0$ in the Fréchet topology defined by the family of seminorms $\{\|\ \|_{p,s}\}_{s>0}$.

We note that every exponential function $e_{\cdot}^{\alpha u+\beta v}$ is contained in $\mathcal{E}_p(\mathbb{C}^2)$ for any $p > 1$, but not in $\mathcal{E}_1(\mathbb{C}^2)$, and such functions as $e_{\cdot}^{au^2+bv^2+2cuv}$ are contained in $\mathcal{E}_p(\mathbb{C}^2)$ for any $p > 2$, but not in $\mathcal{E}_2(\mathbb{C}^2)$. Functions such as $\sum \frac{1}{(k!)^{\frac{1}{p}}} u^k$ is contained in $\mathcal{E}_q(\mathbb{C}^2)$ for any $q > p$, but not in $\mathcal{E}_p(\mathbb{C}^2)$.

The following theorem is the main result of [18]: [1]

[1] In [18], the proof is given for the Weyl ordering, but the same proof works for other orderings.

Theorem 3. *The product formulas* (1), (2), (3) *extend to give the following:*
(i) For $0 < p \leq 2$*, the space* $(\mathcal{E}_p(\mathbb{C}^2), *)$ *forms a complete topological associative algebra.*
(ii) For $p > 2$*, every product formula gives continuous bi-linear mappings of*

$$\mathcal{E}_p(\mathbb{C}^2) \times \mathcal{E}_{p'}(\mathbb{C}^2) \to \mathcal{E}_p(\mathbb{C}^2), \quad \mathcal{E}_{p'}(\mathbb{C}^2) \times \mathcal{E}_p(\mathbb{C}^2) \to \mathcal{E}_p(\mathbb{C}^2), \tag{7}$$

for every p' *such that* $\frac{1}{p} + \frac{1}{p'} \geq 1$.

Let $\mathcal{E}_{2+}(\mathbb{C}^2) = \bigcap_{p>2} \mathcal{E}_p(\mathbb{C}^2)$. Thus, $\mathcal{E}_{2+}(\mathbb{C}^2)$ is a Fréchet space for the natural intersection topology. Note that $e^{\frac{1}{\hbar}(au^2+bv^2+2cuv)}$ is continuous in $\mathcal{E}_{2+}(\mathbb{C}^2)$ in $(a,b,c) \in \mathbb{C}^3$.

By Theorem 3, it is easy to treat $(\mathcal{E}_p(\mathbb{C}^2), *)$ for $0 < p \leq 2$. We now focus on the space $\mathcal{E}_{2+}(\mathbb{C}^2)$. As we mention in §2.5, the extended space $\mathcal{E}_{2+}(\mathbb{C}^2)$ has several strange phenomena.

2.5. Vacuums, half-inverses and the break down the associativity.

A direct calculation using the Moyal product formula (3) shows that the coordinate function v has a right inverse $v^{\circ} = \frac{1}{v}(1 - e^{\frac{2i}{\hbar}uv})$ and a left inverse $v^{\bullet} = \frac{1}{v}(1 - e^{-\frac{2i}{\hbar}uv})$ respectively in $\mathcal{E}_{2+}(\mathbb{C}^2)$, i.e,

$$v * v^{\circ} = 1 = v^{\bullet} * v, \quad v^{\circ} * v = 1 - 2e^{\frac{2i}{\hbar}uv}, \quad v * v^{\bullet} = 1 - 2e^{-\frac{2i}{\hbar}uv},$$

where uv means $u \cdot v$ in precise. The $\cdot$-sign is occasionally omitted in the Weyl ordering.

If associativity holds in $\mathcal{E}_{2+}(\mathbb{C}^2)$, then v° should coincide with $v^{\bullet}$. Hence $\frac{1}{v}\sin\frac{2}{\hbar}uv = 0$, a contradiction (cf. [18]). Thus, we lose associativity in $\mathcal{E}_{2+}(\mathbb{C}^2)$. This is one of the typical phenomena showing the lack of the associativity, namely that *coordinate functions have different left- and right-inverses.*

By the Moyal product formula (3), we also have

$$v * e^{\frac{2i}{\hbar}uv} = 0 = e^{\frac{2i}{\hbar}uv} * u, \quad u * e^{-\frac{2i}{\hbar}uv} = 0 = e^{-\frac{2i}{\hbar}uv} * v.$$

We set by $\varpi_{00} = 2e^{\frac{2i}{\hbar}uv}$, $\bar{\varpi}_{00} = 2e^{-\frac{2i}{\hbar}uv}$ to be a *vacuum* and a *bar-vacuum*, respectively. Using the Moyal product formula and the 2-p-associativity, we easily have

$$(uv - \tfrac{i\hbar}{2}) * e^{\frac{2i}{\hbar}uv} = u * v * e^{\frac{2i}{\hbar}uv} = 0. \tag{8}$$

In Lemma 4 in §4, we show that $e_*^{\frac{it}{\hbar}uv} = \frac{1}{\cosh\frac{t}{2}} e^{\frac{i}{\hbar}(\tanh\frac{t}{2})2uv}$ in the Weyl ordering. Note that $\int_{-\infty}^{\infty} \frac{1}{\cosh\frac{t}{2}} e^{\frac{i}{\hbar}(\tanh\frac{t}{2})2uv} dt < \infty$ in the space $\mathcal{E}_{2+}(\mathbb{C}^2)$. Setting

$$(uv)_{+i0}^{-1} = -i\hbar \int_0^{\infty} e_*^{\frac{it}{\hbar}uv} dt, \quad (uv)_{-i0}^{-1} = i\hbar \int_{-\infty}^{0} e_*^{\frac{it}{\hbar}uv} dt,$$

we see that uv has *two different* inverses, since the difference is given as

$$(uv)_{+i0}^{-1} - (uv)_{-i0}^{-1} = -i\hbar \int_{-\infty}^{\infty} e_*^{\frac{it}{\hbar}uv} dt. \tag{9}$$

The r.h.s. of (9) has the following expression by the Hansen-Bessel formula:

$$\int_{-\infty}^{\infty} e_*^{\frac{it}{\hbar}uv}dt = \int_{-\infty}^{\infty} \frac{1}{\cosh\frac{t}{2}} e^{\frac{i}{\hbar}(\tanh\frac{t}{2})2uv}dt = \frac{\pi}{2}J_0(\frac{2}{\hbar}uv),$$

where J_0 is the Bessel function. This is obviously non zero, causing another breakdown of associativity. Thus, it is impossible to treat both $(uv)^{-1}_{+i0}$ and $(uv)^{-1}_{-i0}$ in the same associative algebra.

Since the r.h.s. of (9) can be viewed as the $*$-Fourier transform of the constant function 1, it may be regarded as the $*$-*delta function* $-i\delta_*(uv)$ (cf. [18]). This can actually be expressed as the difference of two holomorphic functions and has several nice relations to Sato's hyperfunctions are observed [16],[18], (see also [12]).

Hence, the $*$-delta function $\delta_*(uv)$ is expressed as an entire function in terms of the Weyl ordering. We are very interested in such phenomena, since these may be useful in nano-technology.

3. Quadratic forms

These strange phenomena as in §2 are deeply related to $*$-exponential functions, such as $e_*^{\frac{t}{\hbar}u\cdot v}$, defined by the evolution equation (4) of quadratic forms.

It is easy to see that the set of all quadratic forms in $W_\hbar$ is closed under the commutator bracket $[a,b]=a{*}b-b{*}a$. Set $X=\frac{1}{\hbar\sqrt{8}}u^2, Y=\frac{1}{\hbar\sqrt{8}}v^2, H=\frac{i}{2\hbar}uv$, where $uv=u{*}sv+\frac{i\hbar}{2}$. Then, they form a basis of the Lie algebra $\mathfrak{sl}_{\mathbb{C}}(2)$: We see

$$[H,X]=-X, \quad [H,Y]=Y, \quad [X,Y]=-H,$$

and $\{X,Y,H\}$ generate an associative algebra in the space $\mathbb{C}[u,v]$, which is an enveloping algebra of $\mathfrak{sl}_{\mathbb{C}}(2)$. Setting $\mathrm{ad}(W)V=[W,V]$, we see

$$\mathrm{ad}(\frac{i}{2\hbar}(au^2+bv^2+2cuv))\begin{bmatrix}u\\v\end{bmatrix}=\begin{bmatrix}-c & -b\\ a & c\end{bmatrix}\begin{bmatrix}u\\v\end{bmatrix}. \tag{10}$$

Thus, $\mathrm{ad}(\frac{i}{2\hbar}(au^2+bv^2+2cuv))$ generates the complex Lie group $SL_{\mathbb{C}}(2)$, which will be useful to fix the product formula involving $*$-exponential functions of quadratic forms (see (30)). In a $(\mathbb{C}[u,v];*)$-bimodule $(Hol(\mathbb{C}^2),\mathbb{C}[u,v],*)$ with an ordering expression as in §2, we consider the evolution equation (4) for every quadratic form $q(u,v)$ with the initial function f.

However, following standard method in Lie theory, we change a partial differential equation to a system of ordinary differential equations.

3.1. Singular distributions in the Weyl ordering.

In the following, we identify $(a,b,c;s)\in\mathbb{C}^3\times\mathbb{C}_*$ with

$$se^{\frac{1}{\hbar}(au^2+bv^2+2cuv)}\in\mathcal{E}_{2+}(\mathbb{C}^2), \text{ i.e. } (a,b,c;s)\Longleftrightarrow se^{\frac{1}{\hbar}(au^2+bv^2+2cuv)},$$

if no confusion is possible. s and $\frac{1}{\hbar}(au^2+bv^2+2cuv)$ are called the *amplitude* and the *phase* respectively. The function $e^{\frac{1}{\hbar}(au^2+bv^2+2cuv)}$ is called the *phase part*.

For every point $(a,b,c;s)$ in $\mathbb{C}^4$, we consider a curve $s(t)e^{\frac{1}{\hbar}(a(t)u^2+b(t)v^2+2c(t)uv)}$ starting at $se^{\frac{1}{\hbar}(au^2+bv^2+2cuv)}$. The tangent vector of this curve at $t=0$ is given as

$$\big(\frac{1}{\hbar}(a'u^2+b'v^2+2c'uv)s+s'\big)e^{\frac{1}{\hbar}(au^2+bv^2+2cuv)}.$$

We now compute the derivative of the $*$-product $e^{\frac{t}{\hbar}(a'u^2+b'v^2+2c'uv)}*se^{\frac{1}{\hbar}(au^2+bv^2+2cuv)}$ at $t=0$. Using the Moyal product formula, we have

$$\begin{aligned}
&\frac{d}{dt}\Big|_{t=0} e^{\frac{t}{\hbar}(a'u^2+b'v^2+2c'uv)} * se^{\frac{1}{\hbar}(au^2+bv^2+2cuv)}\\
&\quad=\frac{1}{\hbar}(a'u^2+b'v^2+2c'uv)*se^{\frac{1}{\hbar}(au^2+bv^2+2cuv)}\\
&\quad=\frac{1}{\hbar}(a'u^2+b'v^2+2c'uv)se^{\frac{1}{\hbar}(au^2+bv^2+2cuv)}\\
&\quad+\frac{2i}{\hbar}\{(b'v+c'u)(au+cv)-(a'u+c'v)(bv+cu)\}se^{\frac{1}{\hbar}(au^2+bv^2+2cuv)}\\
&\quad-\frac{1}{2\hbar}\{b'(\hbar a+2(au+cv)^2)-2c'(\hbar c+2(au+cv)(bv+cu))\\
&\qquad\qquad+a'(\hbar b+2(bv+cu)^2)\}se^{\frac{1}{\hbar}(au^2+bv^2+2cuv)}.
\end{aligned}\tag{11}$$

Then, (11) is written as

$$\begin{aligned}
&\frac{d}{dt}\Big|_{t=0} e^{\frac{t}{\hbar}(a'u^2+b'v^2+2c'uv)} * se^{\frac{1}{\hbar}(au^2+bv^2+2cuv)}\\
&\quad=\frac{1}{\hbar}(a',b',c')M(a,b,c;s)\begin{bmatrix}u^2\\ v^2\\ 2uv\\ \hbar\end{bmatrix}se^{\frac{1}{\hbar}(au^2+bv^2+2cuv)},
\end{aligned}\tag{12}$$

where

$$M(a,b,c;s)=\begin{bmatrix}-(c+i)^2, & -b^2, & -b(c+i); & -\frac{b}{2}\\ -a^2, & -(c-i)^2, & -a(c-i); & -\frac{a}{2}\\ 2a(c+i), & 2b(c-i), & 1+ab+c^2; & c\end{bmatrix}.\tag{13}$$

We denote by $M(a,b,c)$ the submatrix of the first three columns of $M(a,b,c;s)$. Note that

$$\det M(a,b,c)=(c^2-ab+1)^3.\tag{14}$$

It is seen that every radial direction is an eigenvector of $M(a,b,c)$:

$$(a,b,c)M(\tau a,\tau b,\tau c)=(1+(c^2-ab)\tau^2)(a,b,c).\tag{15}$$

If $c^2-ab+1=0$, then we can write

$$au^2+bv^2+2cuv=2i(\alpha u+\beta v)(\gamma u+\delta v),\quad \alpha\delta-\beta\gamma=1.$$

Clearly, $[\alpha u+\beta v,\gamma u+\delta v]=-i\hbar$. For $u'=\alpha u+\beta v$, $v'=\gamma u+\delta v$, (u',v') is a canonical conjugate pair. Applying (3) to (u',v'), we easily see that

$$(\gamma u+\delta v)*e^{\frac{2i}{\hbar}(\alpha u+\beta v)(\gamma u+\delta v)}=0,\quad \text{for}\quad \alpha\delta-\beta\gamma=1.\tag{16}$$

It follows by 2-p-associativity that

$$\begin{aligned}&(\gamma u+\delta v)^2_* * e^{\frac{1}{\hbar}(au^2+bv^2+2cuv)}=0,\\&(\alpha u+\beta v)*(\gamma u+\delta v)*e^{\frac{1}{\hbar}(au^2+bv^2+2cuv)}=0.\end{aligned}\tag{17}$$

The second identity of (17) yields $(a,b,c)M(a,b,c)=0$, if $c^2-ab+1=0$, which corresponds to (15), and the first identity of (17) yields

$$(\gamma^2,\delta^2,\gamma\delta)M(a,b,c)=0,\qquad c^2-ab+1=0.$$

Hence $M(a,b,c)$ has rank 1 at the point $c^2-ab+1=0$, but the rank of $M(a,b,c;s)$ is 2 there. Setting $u'=\alpha u+\beta v$, $v'=\gamma u+\delta v$, we call $2e^{\frac{2i}{\hbar}u'v'}$ the *vacuum* w.r.t. (u',v'). Thus, it makes sense to call the bar-vacuum $2e^{-\frac{2i}{\hbar}u'v'}$ the *vacuum* w.r.t. $(-v',u')$.

We consider a holomorphic singular distribution $\mathcal{D}_\mu$ on $\mathbb{C}^3\times\mathbb{C}_*$ given by

$$\mathcal{D}_\mu(a,b,c;s)=\{(a',b',c')M(a,b,c;s)|\ (a',b',c')\in\mathbb{C}^3\}.$$

Let $\pi:\mathbb{C}^3\times\mathbb{C}_*\to\mathbb{C}^3$ be the natural projection. Set

$$V_\mu=\{(a,b,c);c^2-ab+1=0\}\quad\text{(phase part of vacuums)}.\tag{18}$$

Then, $2e^{\frac{1}{\hbar}(au^2+bv^2+2cuv)}$, $(a,b,c)\in V_\mu$ is a vacuum. Though $\mathcal{D}_\mu$ is singular on the submanifold $V_\mu\times\mathbb{C}_*$, it gives an ordinary involutive distribution on $(\mathbb{C}^3-V_\mu)\times\mathbb{C}_*$. Hence, there is the 3-dimensional maximal integral holomorphic submanifold M^3 of $\mathcal{D}_\mu$ through the origin $(0,0,0;1)$. Since

$$M(a,b,c)^{-1}=\frac{1}{(1+c^2-ab)^2}\begin{bmatrix}-(c-i)^2, & -b^2, & -b(c-i)\\ -a^2, & -(c+i)^2, & -a(c+i)\\ 2a(c-i), & 2b(c+i), & c^2+ab+1\end{bmatrix},$$

the distribution $\mathcal{D}_\mu$ on $(\mathbb{C}^3-V_\mu)\times\mathbb{C}_*$ is given by

$$\begin{bmatrix}1, & 0, & 0; & \frac12\partial_a\log(1+c^2-ab)\\ 0, & 1, & 0; & \frac12\partial_b\log(1+c^2-ab)\\ 0, & 0, & 1; & \frac12\partial_c\log(1+c^2-ab)\end{bmatrix}.$$

Hence M^3 is given by

$$(a,b,c;\sqrt{1+c^2-ab})\Longleftrightarrow\sqrt{1+c^2-ab}\ e^{\frac{1}{\hbar}(au^2+bv^2+2cuv)},\quad(a,b,c)\in\mathbb{C}^3-V_\mu.\tag{19}$$

Since $\sqrt{\ }$ is two-valued function, M^3 is in fact a non-trivial double cover of $\mathbb{C}^3-V_\mu$, (see also Proposition 5 below).

3.2. Singular distributions in the normal ordering.

Since $uv=u\circ v+\frac{i\hbar}{2}$, we have $au^2+2cuv+bv^2=au^2+2cu\circ v+bv^2+\hbar ci$. In this subsection, we compute $e_*^{\frac{t}{\hbar}(au^2+bv^2+2cuv)}=e^{cit}e_*^{\frac{t}{\hbar}(au^2+bv^2+2cu\circ v)}$ by the ΨDO-product formula (1). Setting

$$e_*^{\frac{t}{\hbar}(au^2+bv^2+2cuv)}=s(t)e_\circ^{\frac{1}{\hbar}(a(t)u^2+b(t)v^2+2c(t)u\circ v)},\tag{20}$$

and computing as in §3.1, we have:

$$\begin{aligned}
&\frac{d}{dt}\Big|_{t=0} e^{\frac{t}{\hbar}(a'u^2+b'v^2+2c'uv)} * se^{\frac{1}{\hbar}(au^2+bv^2+2cuv)}\\
&=\frac{1}{\hbar}(a'u^2+b'v^2+2c'(u{\circ}v+\frac{i\hbar}{2})) * se_\circ^{\frac{1}{\hbar}(au^2+bv^2+2cu\circ v)}\\
&=\{\frac{1}{\hbar}(a'u^2+b'v^2+2c'(u{\circ}v+\frac{i\hbar}{2}))+\frac{i}{\hbar}(2b'v+2c'u){\circ}(2au+2cv)\\
&\quad+\frac{-1}{\hbar}\frac{1}{2}(2b')((2au+2cv)^2+2a\hbar)\}{\circ}se_\circ^{\frac{1}{\hbar}(au^2+bv^2+2cu\circ v)}.
\end{aligned} \tag{21}$$

The r.h.s. of (21) equals

$$\frac{1}{\hbar}(a',b',c')N(a,b,c;s)\begin{bmatrix} u^2\\ v^2\\ 2u{\circ}v\\ \hbar\end{bmatrix}{\circ}se_\circ^{\frac{1}{\hbar}(au^2+bv^2+2cu\circ v)}, \tag{22}$$

where

$$N(a,b,c;s)=\begin{bmatrix} 1, & 0, & 0; & 0\\ -4a^2, & (1+2ci)^2, & 2ai(1+2ci); & -2a\\ 4ai, & 0, & 1+2ci; & i\end{bmatrix}. \tag{23}$$

We denote by $N(a,b,c)$ the submatrix of the first three columns of $N(a,b,c;s)$. The determinant of $N(a,b,c)$ is $(1+2ci)^3$, which is zero at $e_\circ^{\frac{1}{\hbar}(au^2+bv^2+iu\circ v)}$. This is in fact a phase part of a vacuum computed in the normal ordering w.r.t. a certain canonical conjugate pair, (see Proposition 22 below). Let $\mathcal{D}_\nu$ be the the singular distribution given by $N(a,b,c;s)$. Let

$$V_\nu=\{(a,b,c);1+2ci=0\} \quad \text{(phase part of vacuums)}. \tag{24}$$

Since

$$N(a,b,c)^{-1}=\frac{1}{(1+2ci)^2}\begin{bmatrix} (1+2ci)^2, & 0, & 0,\\ -4a^2, & 1, & -2ai\\ -4ai(1+2ci), & 0, & 1+2ci\end{bmatrix},$$

$\mathcal{D}_\nu$ is an ordinary involutive distribution on $(\mathbb{C}^3-V_\nu)\times\mathbb{C}_*$ given by

$$\mathcal{D}_\nu(a,b,c)=\{(a',b',c';\frac{c'i}{1+2ci});(a',b',c')\in\mathbb{C}^3\}.$$

The maximal integral holomorphic submanifold N^3 of $\mathcal{D}_\nu$ through the origin $(0,0,0;1)$ is given by

$$(a,b,c;\sqrt{1+2ci})\Longleftrightarrow\sqrt{1+2ci}\ e_\circ^{\frac{1}{\hbar}(au^2+bv^2+2cu\circ v)}. \tag{25}$$

Since $\sqrt{\ \ }$ is a two-valued function, N^3 is the non-trivial double cover of $\mathbb{C}^3-V_\nu$.

4. $*$-exponential functions and vacuums

We now consider the evolution equation (4) for an arbitrary quadratic form as an integral curve of the distributions mentioned in §3. To define the $*$-exponential function $e_*^{t(au^2+bv^2+2cuv)}$, we set $e_*^{t(au^2+bv^2+2cuv)} = F(t,u,v)$, and consider the evolution equation

$$\frac{\partial}{\partial t}F(t,u,v) = (au^2+bv^2+2cuv)*F(t,u,v), \quad F(0,u,v) = 1. \tag{26}$$

First, we compute the r.h.s. of (26) by the Moyal product formula (3). Keeping in mind that a real analytic solution of (26) in t is unique if it exists, we assume that $F(t,u,v)$ has the form $s(t)e^{a(t)u^2+b(t)v^2+2c(t)uv}$. Then, we solve the system of ordinary differential equations:

$$\begin{aligned}&(a'(t),b'(t),c'(t);s'(t)/s(t)) = (a,b,c)M(a(t),b(t),c(t);s(t)),\\ &(a(0),\, b(0), c(0);s(0)) = (0,0,0;1).\end{aligned} \tag{27}$$

Lemma 4. (Cf. [1], [13]) *The solution of* (26) *is given by*

$$f_t(x) = \frac{1}{\cosh(\hbar\sqrt{ab-c^2}t)}\exp\frac{x}{\hbar\sqrt{ab-c^2}}\Big\{\tanh(\hbar\sqrt{ab-c^2}\ t)\Big\},$$

where $x = au^2+bv^2+2cuv$.

Lemma 4 holds even in the case $ab-c^2=0$ as we may set

$$\frac{1}{\hbar\sqrt{ab-c^2}}\tanh(\hbar\sqrt{ab-c^2}\ t) = t,$$

by a Taylor expansion. By Lemma 4, we have

$$\begin{aligned}e_*^{\frac{t}{\hbar}(au^2+bv^2+2cuv)} &= \frac{1}{\cosh(\sqrt{ab-c^2}\ t)}e^{\frac{1}{\hbar}(au^2+bv^2+2cuv)\left(\frac{1}{\sqrt{ab-c^2}}\tanh(\sqrt{ab-c^2}\ t)\right)}\\ &= \frac{1}{\cos(\sqrt{c^2-ab}\ t)}e^{\frac{1}{\hbar}(au^2+bv^2+2cuv)\left(\frac{1}{\sqrt{c^2-ab}}\tan(\sqrt{c^2-ab}\ t)\right)}.\end{aligned} \tag{28}$$

The ambiguity of $\pm\sqrt{ab-c^2}$ does not affect the result.

By (28), we have in particular, if $c^2\neq ab$, then $e_*^{\frac{\pi}{\hbar\sqrt{c^2-ab}}(au^2+bv^2+2cuv)} = -1$, although $e_*^{\frac{\pi}{2\hbar\sqrt{c^2-ab}}(au^2+bv^2+2cuv)}$ diverges in the Weyl ordering. Let Π_μ be the subset of $\mathbb{C}^3$ where $e_*^{\frac{1}{\hbar}(au^2+bv^2+2cuv)}$ is singular in the Weyl ordering:

$$\Pi_\mu = \{(a,b,c)\in\mathbb{C}^3; \sqrt{c^2-ab} = \pi(\mathbb{Z}+\frac{1}{2})\}.$$

The $*$-exponential mapping $\exp_*$ is a holomorphic mapping of $\mathbb{C}^3-\Pi_\mu$ into M^3. Using (19) and Lemma 4, we have

Proposition 5. *M^3 is a non-trivial double cover of $\mathbb{C}^3 - V_\mu$, and*

$$M^3 = \{\pm\sqrt{c^2-ab+1}\; e^{\frac{1}{\hbar}(au^2+bv^2+2cuv)};\; c^2-ab+1 \neq 0\}.$$

$\{e_*^{\frac{1}{\hbar}(au^2+bv^2+2cuv)}\}$ covers the open dense subset

$$M^3 - \{-e^{\frac{1}{\hbar}(au^2+bv^2+2cuv)};\; c^2-ab=0, (a,b,c)\neq(0,0,0)\}$$

of M^3.

Proof. Suppose $Q \in M^3$. Set $\pi Q = (a,b,c)$. Then $c^2-ab+1 \neq 0$. Since the exceptional values of $\tan z$ are $\pm i$, there exists θ such that $\tan\theta = \sqrt{c^2-ab}$. By (28), we have

$$\sqrt{c^2-ab+1}e^{\frac{1}{\hbar}(au^2+bv^2+2cuv)} = e_*^{\frac{\theta}{\hbar\sqrt{c^2-ab}}(au^2+bv^2+2cuv)}.$$

Recall that $\lim_{\theta\to 0}\frac{\tan\theta}{\theta} = 1$. For $c^2-ab=0$, $\frac{1}{\sqrt{c^2-ab}}\theta$ is taken to be 1.

Remark that $e_*^{t(au^2+bv^2+2cuv)} \in M^3$, whenever this is defined. The differing periodicities of cosine and tangent imply that if $c^2-ab\neq 0$, then

$$\pi^{-1}\pi\{e_*^{\frac{1}{\hbar}(au^2+bv^2+2cuv)}; (a,b,c)\notin\Pi_\mu\} = \{\pm e_*^{\frac{1}{\hbar}(au^2+bv^2+2cuv)}; (a,b,c)\notin\Pi_\mu\}. \quad (29)$$

However, we have to take $\sqrt{1}=1$ in the case $c^2-ab=0$ to get the initial value 1 at $t=0$. Thus we cannot get $-e^{\frac{1}{\hbar}(au^2+bv^2+2cuv)}$ by the exponential function, if $(a,b,c)\neq(0,0,0)$. This proves the last assertion.

Note that $e_*^{\frac{2\pi}{\hbar}uv} = -1 \in M^3$ implies that the integral submanifold through $(0,0,0;-1)$ is in M^3. These arguments together with (19) give the first and second assertions. $\square$

In the following we denote by M_*^3 the set of elements of M^3 expressed in the form of $*$-exponential functions:

$$M_*^3 = \{\pm e_*^{\frac{1}{\hbar}(au^2+bv^2+c(u*v+v*u))};\; \text{its Weyl ordering} \in M^3\}.$$

Similarly, we denote for each canonical conjugate pair (u,v),

$$N_*^3 = \{\pm e_*^{\frac{1}{\hbar}(au^2+bv^2+c(u*v+v*u))};\; \text{its normal ordering} \in N^3\}.$$

By the uniqueness of analytic solutions of the evolution equation (4), the exponential law $e_*^{isx} * e_*^{itx} = e_*^{i(s+t)x}$ for a quadratic function in x holds whenever both sides are defined. Using this, we have

Lemma 6. *For $s,\sigma\in\mathbb{C}$ such that $1+s\sigma(ab-c^2)\neq 0$, we have*

$$e^{\frac{s}{\hbar}(au^2+bv^2+2cuv)} * e^{\frac{\sigma}{\hbar}(au^2+bv^2+2cuv)} = \frac{1}{1+s\sigma(ab-c^2)}e^{\frac{s+\sigma}{\hbar(1+s\sigma(ab-c^2))}(au^2+bv^2+2cuv)}.$$

In particular, we have an idempotent element

$$2e^{\frac{1}{\hbar\sqrt{ab-c^2}}(au^2+bv^2+2cuv)} * 2e^{\frac{1}{\hbar\sqrt{ab-c^2}}(au^2+bv^2+2cuv)} = 2e^{\frac{1}{\hbar\sqrt{ab-c^2}}(au^2+bv^2+2cuv)}.$$

Recall that $2e^{\frac{1}{\hbar\sqrt{ab-c^2}}(au^2+bv^2+2cuv)}$ is a vacuum as defined in §2.

Corollary 7. *Vacuums are obtained as the limit point of $*$-exponential functions; i.e.*

$$2e^{\frac{1}{\hbar\sqrt{ab-c^2}}(au^2+bv^2+2cuv)} = \lim_{t\to\infty} e^t e_*^{\frac{t}{\hbar\sqrt{ab-c^2}}(au^2+bv^2+2cuv)}$$

is a vacuum for every (a,b,c) such that $c^2-ab\neq 0$.

This shows that vacuums may be regarded as certain equilibrium states (cf. [3]).

Remarks. Let $\mathrm{Ad}(g)h = g*h*g^{-1}$. Using (10) and uniqueness of solutions, we see that

$$\mathrm{Ad}(\pm e_*^{\frac{it}{2\hbar}(au^2+bv^2+2cuv)}) \begin{bmatrix} u \\ v \end{bmatrix} = \Big(\exp t \begin{bmatrix} -c & -b \\ a & c \end{bmatrix}\Big) \begin{bmatrix} u \\ v \end{bmatrix}. \tag{30}$$

Remark that $\mathrm{Ad}(\pm e_*^{\frac{it}{2\hbar}(au^2+bv^2+2cuv)})$ has no singularity in t, and that the sign of $\pm e_*^{\frac{it}{2\hbar}(au^2+bv^2+2cuv)}$ makes no difference. Hence, the "group" generated by the $*$-exponential functions of quadratic forms looks like a "double covering group" of $SL_{\mathbb{C}}(2)$, which is known to be simply connected.

Moreover, (30) is useful to make the product formula involving elements f, g of $\mathcal{E}_p(\mathbb{C}^2)$, $p<2$. We compute as follows:

$$(f*e_*^{p(u,v)})*(g*e_*^{q(u,v)}) = (f*(e^{\mathrm{ad}(p(u,v))}g))*(e_*^{p(u,v)}*e_*^{q(u,v)}).$$

This is well defined whenever $e_*^{p(u,v)}*e_*^{q(u,v)}$ is well defined. Hence, we have only to care about the product formula $e_*^{p(u,v)}*e_*^{q(u,v)}$.

The following lemma is useful to compute these transcendental products. It is proved by showing that both quantities satisfy the same partial differential equation with the same initial condition, but intuitively this is given by the trivial identity $v*(u*v)^m = (v*u)^m*v$, which explains the name of the next lemma:

Lemma 8. (Bumping lemma)

$$v*e_*^{itu*v} = e_*^{itv*u}*v, \quad e_*^{itu*v}*u = u*e_*^{itv*u}.$$

4.1. $*$-exponential functions by the normal ordering.

Although $e_*^{\pm\frac{\pi}{\hbar}uv}$ diverge in the Weyl ordering, we prove in this subsection that such elements make sense in the normal ordering. We now consider the evolution equation (26) in the normal ordering. Assuming that

$$e_*^{\frac{t}{\hbar}(au^2+bv^2+2cu*v)} = \psi(t)e_\circ^{\phi_1(t)u^2+\phi_2(t)v^2+2\phi_3(t)u\circ v},$$

we solve the system of ordinary differential equations:

$$\begin{cases} \phi_1'(t) = \dfrac{1}{\hbar}a + 4ic\phi_1(t) - 4\hbar b\phi_1(t)^2 \\ \phi_2'(t) = \dfrac{1}{\hbar}b + 4ib\phi_3(t) - 4\hbar b\phi_3(t)^2 \\ \phi_3'(t) = \dfrac{1}{\hbar}c + 2ic\phi_3(t) + 2ib\phi_1(t) - 4\hbar b\phi_1(t)\phi_3(t) \\ \psi'(t) = -2\hbar b\phi_1(t)\psi(t) \end{cases} \tag{31}$$

with the initial condition $\phi_i(0)=0$ and $\psi(0)=1$.

Proposition 9. *There exists a unique analytic solution of* (31) *given by the following form:*

$$\begin{cases} \phi_1(t) = \dfrac{a}{2\hbar}\dfrac{\sin(2\sqrt{D}\,t)}{\sqrt{D}\cos(2\sqrt{D}\,t) - ic\sin(2\sqrt{D}\,t)}, \\ \phi_2(t) = \dfrac{b}{2\hbar}\dfrac{\sin(2\sqrt{D}\,t)}{\sqrt{D}\cos(2\hbar\sqrt{D}\,t) - ic\sin(2\sqrt{D}\,t)}, \\ \phi_3(t) = \dfrac{i}{2\hbar}\Big(1 - \dfrac{\sqrt{D}}{\sqrt{D}\cos(2\sqrt{D}\,t) - ic\sin(2\sqrt{D}\,t)}\Big), \\ \psi(t) = e^{-cit}\Big(\dfrac{\sqrt{D}}{\sqrt{D}\cos(2\sqrt{D}\,t) - ic\sin(2\sqrt{D}\,t)}\Big)^{1/2} \end{cases} \tag{32}$$

where $D = c^2 - ab$. (For the case $D = 0$, we set $\frac{1}{\sqrt{D}}\sin(2\sqrt{D}\,t) = 2t$ via Taylor expansion.) The sign ambiguity of $\sqrt{D}$ does not affect the result, but the $\pm$ ambiguity of $(\ )^{1/2}$ remains in the expression of $\psi(t)$.

Note that taking the *complex conjugate* of (32), we obtain the formula of $*$-exponential function in the anti-normal ordering. By this observation, we have the following:

Lemma 10. *In every ordering, the $*$-exponential function $e_*^{\frac{1}{\hbar}au^2+bv^2+2cuv}$ has singularities in $(a,b,c) \in \mathbb{C}^3$. However, there is no common singularity of the normal ordering and of the Weyl ordering.*

Noting that $2uv = u{*}v + v{*}u = 2u{*}v + i\hbar = 2u{\circ}v + i\hbar$, we can use Lemma 9 to obtain the formula of $e_*^{\frac{t}{\hbar}(au^2+bv^2+c(u*v+v*u))}$. Remark that $e_*^{\frac{t}{\hbar}(au^2+bv^2+2cuv)}$ is a curve contained in N^3, that is, $\sqrt{1+2i\hbar\phi_3(t)} = e^{cit}\psi(t)$ must hold by (25). This can be checked by direct calculation. For the special case $ab = 0$, we have

$$e_*^{\frac{t}{\hbar}(au^2+bv^2+2cu\circ v)} = e_\circ^{\frac{1}{4ci\hbar}(e^{4cit}-1)(au^2+bv^2)+\frac{1}{2i\hbar}(e^{2cit}-1)2u\circ v}, \quad ab = 0, \tag{33}$$

because by setting $\sqrt{c^2} = c$, (33) gives the real analytic solution of (31) with initial data 1. Remark (33) has no singularity in $t \in \mathbb{C}$. Using (33), we have

$$e_*^{\frac{t}{\hbar}(au^2+bv^2+c(u*v+v*u))} = e^{cit}e_\circ^{\frac{1}{4ci\hbar}(e^{4cit}-1)(au^2+bv^2)+\frac{1}{2i\hbar}(e^{2cit}-1)2u\circ v}, \quad ab = 0. \tag{34}$$

By recalling that $u{*}v + v{*}u = 2u{*}v + i\hbar$ again, Proposition 9 gives a very strange formula of $e_*^{\frac{\pi}{2\hbar}(au^2+bv^2+c(u*v+v*u))}$ for $c^2 - ab = 1$:

Lemma 11. *In the normal ordering w.r.t. (u,v), the $*$-exponential function $e_*^{\frac{\pi}{2\hbar}(au^2+bv^2+c(u*v+v*u))}$ for $c^2 - ab = 1$ is given identically as $\sqrt{-1}e_\circ^{\frac{2i}{\hbar}u\circ v}$.*

4.2. Polar element.

Here a new question arises whether the sign ambiguity of $\sqrt{-1}$ of Lemma 11 can be eliminated for all $a,b,c \in \mathbb{C}$. Our conclusion in this subsection is that the ambiguity can *not* be eliminated.

Note first that $e_*^{\frac{\pi}{2\hbar}(au^2+bv^2+c(u*v+v*u))}$ diverges in the Weyl ordering for $c^2-ab=1$. By Lemma 11, $e_*^{\frac{\pi}{2\hbar}(au^2+bv^2+c(u*v+v*u))}$ is independent of a,b,c whenever $c^2-ab=1$. Thus, it must be viewed as a single element. We call it the *polar element* and denote by ε_{00}. We have in particular that

$$\varepsilon_{00}*\varepsilon_{00}=-1,\quad \mathrm{Ad}(\varepsilon_{00})=-I, \tag{35}$$

but ε_{00} has several strange features.

4.3. It looks like a contradiction.

It is clear that $\pi(\varepsilon_{00})=(0,0,i)$. Moreover $\varepsilon_{00}*\varepsilon_{00}=-1$ by the exponential law. But this does not imply that $\varepsilon_{00}=\sqrt{-1}$, because the following holds by the bumping Lemma 8:

Proposition 12. $u*\varepsilon_{00}+\varepsilon_{00}*u=0$, $v*\varepsilon_{00}+\varepsilon_{00}*v=0$. *In particular,* ε_{00} *commutes with every* $\frac{t}{\hbar}(au^2+bv^2+2cuv)$, *and hence with* $e_*^{\frac{t}{\hbar}(au^2+bv^2+2cuv)}$.

Moreover, since $(a,b,c)=(0,0,1)$ and $(0,0,-1)$ are arcwise connected in the set $c^2-ab=1$, Lemma 11 gives

$$e_*^{\frac{\pi}{2\hbar}(u*v+v*u)}=\sqrt{-1}e_\circ^{\frac{2i}{\hbar}u\circ v}=e_*^{-\frac{\pi}{2\hbar}(u*v+v*u)}.$$

Considering the exponential law of the $*$-exponential function $e_*^{\frac{t}{2\hbar}(u*v+v*u)}$ for $t\in\mathbb{C}-\{\text{singular set}\}$, we must set by (34)

$$e_*^{\frac{\pi}{2\hbar}(u*v+v*u)}=ie_\circ^{\frac{2i}{\hbar}u\circ v},\quad e_*^{-\frac{\pi}{2\hbar}(u*v+v*u)}=-ie_\circ^{\frac{2i}{\hbar}u\circ v}.$$

If one wants to fix the sign ambiguity, the exponential law gives

$$-1=e_*^{\frac{\pi}{2\hbar}(u*v+v*u)}*e_*^{\frac{\pi}{2\hbar}(u*v+v*u)}=e_*^{\frac{\pi}{2\hbar}(u*v+v*u)}*e_*^{-\frac{\pi}{2\hbar}(u*v+v*u)}=1.$$

Remark that $(-v,u)$ is a canonical conjugate pair and the set of all canonical conjugate pairs is arcwise connected. Thus, it seems rigorous to set $\varepsilon_{00}=ie_\circ^{\frac{2i}{\hbar}u\circ v}=ie_\diamond^{\frac{2i}{\hbar}(-v)\diamond u}$, but this causes the same trouble. Therefore, we must conclude that the sign ambiguity in Proposition 9 cannot be eliminated. One has to set $\varepsilon_{00}=\sqrt{-1}e_\circ^{\frac{2i}{\hbar}u\circ v}$ with the sign ambiguity. It is better to understood ε_{00} as a "two-valued" element. But since such a notion does not exist in the set theory, it seems to be impossible to define ε_{00} as a point in a point set. Because of this anomalous character of ε_{00} we had spent a great deal of time to check our calculation, and come to a conclusion that *the polar element* ε_{00} *should be understood as a "two-valued" element.* Remark that if one considers m-tensor powers of our system, we have an element $\prod_{i=1}^m\varepsilon_{00}^{(i)}$ which should be treated as a 2^m-valued element. In §5, we claim this anomalous element is still useful in the calculation of $*$-products.

Olver [Ol] gives some examples of local Lie groups which do not form groups, because the associativity breaks down globally. Though these are examples within point sets, the set up in [Ol] seems similar and helpful to understand this ambiguity. We have to study such phenomena more closely, in order to understand the difficulties we must overcome to treat exact deformation quantizations, (cf. also [R]).

To understand ε_{00} rigorously within set theory, we give several formulations in §7.

5. Product formulas, restriction to real forms

We now study the "group" generated by $e_*^{aH+bX+cY}$, using the Weyl ordering and the polar element ε_{00}. We will see that the $*$-product

$$e_*^{aH+bX+cY} * e_*^{a'H+b'X+c'Y}$$

is defined in general only up to the sign ambiguity of $\sqrt{\cdot}$, and that the ambiguity can only be eliminated locally.

5.1. Product formula with $\pm$ ambiguity and singularity.

We first want to establish the product formula with $\pm$ ambiguity. If we use the Weyl ordering, Proposition 2 implies that the general product formula for quadratic exponential functions can be obtained from the two cases as follows:

$$e^{tu^2} * e^{au^2+bv^2+2cuv}, \quad e^{\tau uv} * e^{au^2+bv^2+2cuv}.$$

Solving (27) with the general initial condition

$$(a(0), b(0), c(0); s(0)) = (a, b, c; 1), \tag{36}$$

we see that the first case can be written as

$$e_*^{\frac{t}{\hbar}u^2} * e^{\frac{1}{\hbar}(au^2+bv^2+2cuv)} = \frac{1}{\sqrt{1+bt}} e^{\frac{1}{\hbar(1+bt)}\{(a+(ab-c^2-2ci+1)t)u^2+bv^2+2(c-ibt)uv\}} \tag{37}$$

where the ambiguity in $\pm\sqrt{1+bt}$ cannot be eliminated for all t, b. Note also that the discriminant of $(a+(ab-c^2-2ci+1)t)u^2+bv^2+2(c-ibt)uv$ is $(c^2-ab+1)(1+bt)-(1+bt)^2$. Thus, $e^{\frac{t}{\hbar}u^2} * \sqrt{c^2-ab+1}\, e^{\frac{1}{\hbar}(au^2+bv^2+2cuv)}$ is contained in M^3 as the discriminant of the phase function is $+1$.)

Remarking $e_*^{\frac{t}{\hbar}u^2} = e^{\frac{t}{\hbar}u^2}$, we have the following:

Lemma 13. *For $Q \in M^3$ such that $\pi(Q) = e^{\frac{1}{\hbar}(au^2+bv^2+2cuv)}$, Q is written in the form $\sqrt{c^2-ab+1}\, e^{\frac{1}{\hbar}(au^2+bv^2+2cuv)}$, and the product $e^{\frac{t}{\hbar}u^2} * Q$ is an element of M^3, if $bt \neq -1$.*

As in (37), we have in the Weyl ordering that

$$e_*^{\frac{t}{\hbar}v^2} * e^{\frac{1}{\hbar}(au^2+bv^2+2cuv)} = \frac{1}{\sqrt{1+at}}\, e^{\frac{1}{\hbar(1+at)}\{au^2+(b+(ab-c^2+2ci+1)t)v^2+2(c+iat)uv\}}, \tag{38}$$

and hence we have the result similar to Lemma 13.

Note that $e_*^{\frac{t}{\hbar}2uv} = \sqrt{1+s^2}\, e^{\frac{s}{\hbar}2uv}$, where $s = \tan t$. Solving (27) with the initial condition (36), we have in the Weyl ordering that

$$e^{\frac{s}{\hbar}2uv} * e^{\frac{1}{\hbar}(au^2+bv^2+2cuv)} =$$

$$\frac{1}{\sqrt{1-2cs+(c^2-ab)s^2}} e^{\frac{1}{\hbar(1-2cs+(c^2-ab)s^2)}(a(1+is)^2u^2+b(1-is)^2v^2+(c-(c^2-ab-1)s-cs^2)2uv)} \tag{39}$$

where the ambiguity in $\pm\sqrt{1-2cs+(c^2-ab)s^2}$ cannot be eliminated.

Note the following identity for the computation of the discriminant of the phase function of (39):

$$\begin{aligned}\big(1-2cs+(c^2-ab)s^2\big)^2+\big(c-(c^2-ab-1)s-cs^2\big)^2-ab(1+is)^2(1-is)^2\\ =(c^2-ab+1)(1+s^2)\big((c^2-ab)s^2-2cs+1\big).\end{aligned} \tag{40}$$

Using (39) and (40) for the computation of 1+discriminant, we see:

Lemma 14. *If $Q_1, Q_2 \in M^3$ are such that*

$$\pi(Q_1)=e^{\frac{s}{\hbar}2uv}, \quad \pi(Q_2)=e^{\frac{1}{\hbar}(au^2+bv^2+2cuv)},$$

then

$$Q_1=\pm\sqrt{1+s^2}\,e^{\frac{s}{\hbar}2uv}, \qquad Q_2=\pm\sqrt{c^2-ab+1}\,e^{\frac{1}{\hbar}(au^2+bv^2+2cuv)}$$

with $1+s^2\neq 0$, $c^2-ab+1\neq 0$. Furthermore, if $1-2cs+(c^2-ab)s^2\neq 0$, then the $*$-product Q_1*Q_2 is defined as an element of M^3.

Though every product formula in the Weyl ordering has some singularity, this does not mean that the $*$-product cannot be defined at such points. Recall that every quadratic form $Q(u,v)$ is written in the form $(\alpha u+\beta v)^2$ if $ab-c^2=0$, or the form $\lambda(\alpha u+\beta v)(\gamma u+\delta v)$ with $\alpha\delta-\beta\gamma=1$ otherwise. Solving the evolution equation by using the ΨDO-product formula (1) via the standard procedure of Lie theory, we have the following product formula in the normal ordering:

$$\begin{aligned} e_{\circ}^{\frac{t}{\hbar}u^2}*e_{\circ}^{\frac{1}{\hbar}(au^2+bv^2+2cuv)}&=e_{\circ}^{\frac{1}{\hbar}((a+t)u^2+bv^2+2cuv)},\\ e_{\circ}^{\frac{t}{\hbar}uv}*e_{\circ}^{\frac{1}{\hbar}(au^2+bv^2+2cuv)}&=e_{\circ}^{\frac{1}{\hbar}(a(1+it)^2u^2+bv^2+(c+(ci+\frac{1}{2})t)2uv)}.\end{aligned} \tag{41}$$

This shows that every product can be computed without ambiguity by a canonical conjugate pair whose choice depends on the elements to be calculated. However, since we have to use various canonical conjugate pairs to write elements in the above standard form, this does not imply that $*$-products can be defined without ambiguity.

Here, we close this subsection with a remark on associativity. Although all of our formulas above are written in the form $e^{\frac{1}{\hbar}(au^2+bv^2+2cuv)}$, we can replace (a,b,c) by $(\hbar a,\hbar b,\hbar c)$. After this replacement, all formulas (except the formula involving elements like $e^{\frac{2i}{\hbar}uv}$, where one cannot eliminate $\hbar$ in the expression) are changed into formulas which are real analytic in $\hbar$ and these are meaningful at $\hbar=0$.

If $\hbar$ is viewed as a formal parameter and all formulas are considered as formal power series in $\hbar$, we see that associativity holds, and the sign ambiguities disappear in the product formula. Using a Taylor expansion in $\hbar$, we have:

Proposition 15. *The associativity*

$$\begin{aligned}&\big(e^{au^2+bv^2+2cuv}*e^{a'u^2+b'v^2+2c'uv}\big)*e^{a''u^2+b''v^2+2c''uv}\\ &\qquad=e^{au^2+bv^2+2cuv}*\big(e^{a'u^2+b'v^2+2c'uv}*e^{a''u^2+b''v^2+2c''uv}\big)\end{aligned}$$

holds if both sides are defined.

5.2. Product formulas involving the polar element.

Even though ε_{00} is viewed as a two-valued element, we can derive product formulas. We begin with the following lemma, which shows that the mapping $a \to \varepsilon_{00} * a$ is better to be understood as a 2-to-2-mapping:

Lemma 16. *If $D = c^2 - ab \neq 0$, then*

$$\varepsilon_{00} * e^{\frac{1}{\hbar}(au^2+bv^2+2cuv)} = \frac{1}{\sqrt{c^2-ab}} e^{-\frac{1}{\hbar(c^2-ab)}(au^2+bv^2+2cuv)}.$$

Proof. Set $s = \tan t$. Then we have $e_*^{\pm\frac{t}{\hbar}2uv} = \sqrt{1+s^2}\, e^{\frac{s}{\hbar}2uv}$. We remark that $s \to \pm\infty$ as $t \to \pm\frac{\pi}{2}$. Multiplying $\sqrt{1+s^2}$ to (39) and taking the limit $s^2 \to \infty$ yields Lemma. $\square$

Since the ambiguity in $\sqrt{c^2-ab}$ depends on the choice of paths for $s^2 \to \infty$, the equality of Lemma 16 is better to be understand as

$$\varepsilon_{00} * \pm e^{\frac{1}{\hbar}(au^2+bv^2+2cuv)} = \frac{\pm 1}{\sqrt{c^2-ab}} e^{-\frac{1}{\hbar(c^2-ab)}(au^2+bv^2+2cuv)}.$$

Remark that ε_{00} commutes with every $e^{\frac{1}{\hbar}(au^2+bv^2+2cuv)}$ and ${\varepsilon_{00}}_*^2 = -1$. Thus, $\varepsilon_{00}*$ gives a 2-to-2-diffeomorphism of $M^3 - M_0^3$ onto itself, where

$$M_0^3 = \{e_*^{\frac{1}{\hbar}(au^2+bv^2+2cuv)}; c^2 - ab = 0\}.$$

For a point P of M_0^3, the computation of $\varepsilon_{00} * P$ is reduced to the case $P = e^{\frac{1}{\hbar}au^2}$. Since we see for $t \neq \pm\frac{\pi}{2}$ that

$$e_*^{\frac{t}{\hbar}2uv} * e^{\frac{1}{\hbar}au^2} = \sqrt{1+s^2}\, e^{\frac{1}{\hbar}(a(1+is)^2u^2+2suv)}, \quad \tan t = s, \tag{42}$$

this is written in the form of a $*$-exponential function and is therefore a member of M^3. As $t \to \pm\frac{\pi}{2}$, then the r.h.s of (42) diverges. Hence, we see that $e_*^{\pm\frac{\pi}{\hbar}uv} * e^{\frac{1}{\hbar}au^2}$ cannot be a member of M^3. Thus, we see that $M^3 \cap \varepsilon_{00} * M_0^3 = \emptyset$.

In the normal ordering w.r.t. (u, v), we see by (41)

$$\varepsilon_{00} * e_\circ^{\frac{1}{\hbar}au^2} = \sqrt{-1} e_\circ^{\frac{1}{\hbar}au^2}, \quad \varepsilon_{00} * e_\circ^{\frac{1}{\hbar}(au^2+bv^2+2cuv)} = \sqrt{-1} e_\circ^{\frac{1}{\hbar}(au^2+bv^2+(i-c)2uv)}.$$

Since $\varepsilon_{00}^2 = -1$, we see that $N_*^3 = \varepsilon_{00} * N_*^3$ for every canonical conjugate pair. In other words, we see that

$$N_*^3 \subset M_*^3 \cup (\varepsilon_{00} * M_*^3) \quad \text{for every canonical conjugate pair.} \tag{43}$$

Note that $e_*^{\frac{1}{\hbar}t(\alpha u+\beta v)^2} \in N_*^3$ in the normal ordering w.r.t. a certain canonical conjugate pair $u' = \alpha u + \beta v, v' = \gamma u + \delta v$, and $\varepsilon_{00} * N_*^3 = N_*^3$. Therefore, every element of $\varepsilon_{00} * M_0^3$ is contained in N_*^3 w.r.t. a certain canonical conjugate pair. Combining these arguments with (43), we see that

$$M_*^3 \cup (\bigcup_{(u,v)} N_*^3) = M_*^3 \cup (\varepsilon_{00} * M_*^3) \tag{44}$$

where the union is taken over all canonical conjugate pairs which are pairwise linearly related.

By (44), we have to consider the "set" obtained by "gluing" M^3_* and $\varepsilon_{00}{*}M^3_*$ by the "2-to-2-diffeomorphism" $\varepsilon_{00}*$ given by Lemma 16.

5.3. General product formulas.

By the argument in § 5.1, we have two cases that $Q_1 * Q_2$ are not defined in the Weyl ordering:

$$e^{\frac{t}{\hbar}u^2} * \sqrt{c^2-ab+1}\, e^{\frac{1}{\hbar}(au^2+bv^2+2cuv)}, \quad \text{for} \quad 1+bt=0,$$

and

$$\sqrt{1+s^2}\, e^{\frac{s}{\hbar}2uv} * \sqrt{c^2-ab+1}\, e^{\frac{1}{\hbar}(au^2+bv^2+2cuv)}, \text{ for } 1-2cs+(c^2-ab)s^2=0.$$

By Lemma16 and (37) we obtain the following:

Corollary 17. *If $1+bt=0$, then*

$$e^{\frac{t}{\hbar}u^2} * (\varepsilon_{00} * \sqrt{c^2-ab+1}\, e^{\frac{1}{\hbar}(au^2+bv^2+2cuv)}) = e^{\frac{1}{\hbar(c^2-ab+1)}((a+(ab+(ci-1)^2)t)u^2+bv^2+2(c-ibt)uv)}$$

and the r.h.s. can be written in the form $e^{\frac{1}{\hbar}(\alpha u+\beta v)^2} \in M^3_0$.

If $1-2cs+(c^2-ab)s^2=0$, then by (40) again, we have

$$(\varepsilon_{00} * \sqrt{1+s^2}\, e^{\frac{s}{\hbar}2uv}) * \sqrt{c^2-ab+1}\, e^{\frac{1}{\hbar}(au^2+bv^2+2cuv)}$$
$$= e^{\frac{1}{\hbar(c^2-ab+1)(1+s^2)}(a(1+is)^2u^2+b(1-is)^2v^2+(c-(c^2-ab-1)s-cs^2)2uv)}.$$

The r.h.s. can be written in the form $e^{\frac{1}{\hbar}(\alpha u+\beta v)^2}$, since the discriminant of the r.h.s. vanishes.

As a consequence, we have the following:

Theorem 18. *$M^3_* \cup (\varepsilon_{00} * M^3_*)$ is closed under the $*$-product. Moreover, the set $\{e_*^{aH+bX+cY}; a,b,c \in \mathbb{C}\}$ generates $M^3_* \cup (\varepsilon_{00} * M^3_*)$.*

For the proof, it is enough to remark the following: For $Q_1, Q_2 \in M^3$, if $Q_1 * Q_2$ is not defined in the Weyl ordering, then $Q_1 * (\varepsilon_{00} * Q_2)$ or $(Q_1 * \varepsilon_{00}) * Q_2$ is defined in the Weyl ordering as an element of M^3 and

$$Q_1 * (\varepsilon_{00} * Q_2) = (Q_1 * \varepsilon_{00}) * Q_2$$

holds, whenever both sides are defined. If $Q_1 * Q_2$, $Q_1 * (\varepsilon_{00} * Q_2)$, and $(\varepsilon_{00} * Q_1) * Q_2$ are defined, then we have $Q_1 * (\varepsilon_{00} * Q_2) = (\varepsilon_{00} * Q_1) * Q_2 = \varepsilon_{00} * (Q_1 * Q_2)$. Moreover, $(\varepsilon_{00} * Q_1) * (\varepsilon_{00} * Q_2)$ is defined as $-Q_1 * Q_2$.

$M^3_* \cup (\varepsilon_{00} * M^3_*)$ is "locally" isomorphic to $SL_{\mathbb{C}}(2)$. As is noted at (30), $M^3_* \cup \varepsilon_{00} * M^3_*$ may be viewed as a non-trivial double cover of $SL_{\mathbb{C}}(2)$, although it cannot be treated as a point set.

Looking at the product formulas (37), (38), (39) and Lemma 16 more carefully, we see the following:

Theorem 19. *If the coefficients a, b, c are restricted in the real number $\mathbb{R}$ and if we consider $(ia, ib, ic) \in \mathbb{C}^3$, or $(a, b, ic) \in \mathbb{C}^3$, then all product formulas are closed in these subspaces respectively. That is, $\{e_*^{\frac{i}{\hbar}(au^2+bv^2+2cuv)}\}$ and $\{e_*^{\frac{1}{\hbar}(au^2+bv^2+2ciuv)}\}$ generate Lie subgroup-like objects under the $*$-product whose Lie algebras are $sl_{\mathbb{R}}(2), su(1,1)$ respectively.*

Since the first homotopy groups π_1 of $SL_{\mathbb{R}}(2), SU(1,1)$ are both $\mathbb{Z}$, we see that the sign ambiguity can be treated as genuine double coverings in such subgroup-like objects. Hence, we see that $M_*^3 \cup (\varepsilon_{00} * M_*^3)$ contains the double covering group of $SL_{\mathbb{R}}(2) = Sp(2;\mathbb{R})$, which can be regarded as the metaplectic group $Mp(2;\mathbb{R})$. Thus, $M_*^3 \cup (\varepsilon_{00} * M_*^3)$ may be viewed as the complexification of $Mp(2;\mathbb{R})$. It is obvious that there is no such Lie group in the standard group theory. Similarly, $M_*^3 \cup (\varepsilon_{00} * M_*^3)$ contains the double covering group of $SU(1,1)$.

6. Intertwiners and their extensions

Recall that ε_{00} was defined by using different orderings. To understand the anomalous element ε_{00}, and anomalous phenomena related to $*$-exponential functions of quadratic forms, we introduce the notion of intertwiners.

We mentioned in §4 that ε_{00} is viewed as a two-valued element w.r.t. a canonical conjugate pair (u, v). If we take another canonical conjugate pair $u' = au + bv, v' = cu + dv$ with $ad - bc = 1$, then $e_*^{\frac{\pi}{2\hbar}(u'*v'+v'*u')}$ must equal $\sqrt{-1}e_\diamond^{\frac{2i}{\hbar}u'\diamond v'}$ where $\diamond$ indicates that this element is expressed in the normal ordering w.r.t. (u', v'). We view ε_{00} as the collection of expressions in the various normal orderings. Thus, it is not clear whether one may identify $\sqrt{-1}e_\circ^{\frac{2i}{\hbar}u\circ v}$ with $\sqrt{-1}e_\diamond^{\frac{2i}{\hbar}u'\diamond v'}$. (See Proposition 23 for our conclusion.)

To consider this problem, we consider intertwiners between varoius ordering expressions. In particular, we consider the intertwiner between normal ordering and the Weyl ordering w.r.t (u, v).

6.1. Intertwiners or coordinate transformations.

The construction of intertwiners is based on the following expressions of the $*$-exponential function $e_*^{\alpha u+\beta v}$ obtained as follows by solving the evolution equation (4):

$$\begin{aligned}
e_{*_M}^{\alpha u+\beta v} &= e_{\cdot}^{\alpha u+\beta v}, && \text{(in Weyl ordering)}\\
e_{*_N}^{\alpha u+\beta v} &= e^{\frac{i\hbar}{2}\alpha\beta}e_\circ^{\alpha u+\beta v}, && \text{(in normal ordering)}\\
e_{*_{\bar N}}^{\alpha u+\beta v} &= e^{-\frac{i\hbar}{2}\alpha\beta}e_\bullet^{\alpha u+\beta v}, && \text{(in antinormal ordering).}
\end{aligned}$$

We define the intertwiner from the Weyl ordering to the normal ordering w.r.t. (u, v) by the densely defined linear operator

$$I_\cdot^\circ = e^{\frac{i\hbar}{2}\partial_u\partial_v}. \tag{45}$$

In particular, we note that $I_\cdot^\circ(f *_M g) = (I_\cdot^\circ f) *_N (I_\cdot^\circ g)$, if both sides are defined, where $*_M$, $*_N$ denote the Moyal-product and ΨDO-product respectively. This is because that both sides are the expressions of $f * g$ in different orderings.

Suppose (u,v) and (u',v') are related by

$$u' = au + bv, \quad v' = cu + dv, \quad ad - bc = 1, \quad u = du' - bv', \quad v = -cu' + av'.$$

The $*$-exponential function is given by $e_*^{\alpha' u' + \beta' v'} = e^{\frac{i\hbar}{2}\alpha'\beta'} e_\diamond^{\alpha' u' + \beta' v'}$ w.r.t. a canonical conjugate pair (u',v'), where $\diamond$ indicates the normal ordering expression w.r.t. (u',v'). Thus, we must identify $e^{\frac{i\hbar}{2}\alpha\beta} e_\circ^{\alpha u + \beta v}$ with $e^{\frac{i\hbar}{2}\alpha'\beta'} e_\diamond^{\alpha' u' + \beta' v'}$. Hence, we have to define the intertwiner $I_\circ^\diamond$ from the normal ordering w.r.t. (u,v) to that w.r.t. (u',v') as follows:

$$I_\circ^\diamond f = e^{\frac{i\hbar}{2}\partial_{u'}\partial_{v'} - \frac{i\hbar}{2}\partial_u\partial_v} f. \tag{46}$$

To make precise, we must consider the exponential of the operator

$$\partial_{u'}\partial_{v'} - \partial_u\partial_v = -bd\partial_u^2 + (ad + bc - 1)\partial_u\partial_v - ac\partial_v^2.$$

If

$$g_\diamond(u,v) = e^{\frac{i\hbar}{2}(-bd\partial_u^2 + (ad+bc-1)\partial_u\partial_v - ac\partial_v^2)} f_\circ(u,v),$$

then $g_\diamond(u,v)$ is the normal ordering of $f_\circ(u,v)$ w.r.t. (u',v'). By the decomposition

$$e^{-bd\partial_u^2 + (ad+bc-1)\partial_u\partial_v - ac\partial_v^2} = e^{-bd\partial_u^2} e^{(ad+bc-1)\partial_u\partial_v} e^{-ac\partial_v^2},$$

we may treat these intertwiners separately. We remark that these intertwiners are well defined if $\hbar$ is a formal parameter. For a real parameter $\hbar$, intertwiners are first defined on the space $\mathbb{C}[u,v]$, and then extended as follows:

Theorem 20. *For every $0 < p \le 2$, $I_\circ^\diamond$ extends to a continuous linear isomorphism of $\mathcal{E}_p(\mathbb{C}^2)$ onto itself. $I_\circ^\diamond$ is also an algebra isomorphism of $(\mathcal{E}_p(\mathbb{C}^2)_\circ; *)$ onto $(\mathcal{E}_p(\mathbb{C}^2)_\diamond; *)$.*

Proof. We first remark that by [18], Proposition 6.1, the system of seminorms (5) can be replaced by the following system of seminorms: Set $\tau = \frac{1}{p}$, and for $f = \sum a_{m,n} u^m v^n$ we define

$$\|f\|_{\tau,s} = \sum_{m,n} |a_{m,n}| (m+n)^{\tau(m+n)} s^{\tau(m+n)}, \quad s > 0.$$

This system defines the same Fréchet space as $\mathcal{E}_p(\mathbb{C}^2)$ for every $p > 0$.

We show $e^{\alpha\partial_u\partial_v}$, $e^{\beta\partial_u^2}$ and $e^{\gamma\partial_v^2}$ extend to continuous linear isomorphisms of $\mathcal{E}_p(\mathbb{C}^2)$ onto itself for every $0 < p \le 2$. For every $f = \sum a_{m,n} u^m v^n$, we see that

$$e^{\alpha\partial_u\partial_v} f = \sum_{m,n,k} \frac{\alpha^k}{k!} \frac{(m+k)!}{k!} \frac{(n+k)!}{k!} a_{m+k,n+k} u^m v^n.$$

Hence,

$$\|e^{\alpha\partial_u\partial_v} f\|_{\tau,s} = \sum_{m,n} \sum_{0 \le k \le m \wedge n} \frac{\alpha^k}{k!} \frac{m!}{(m-k)!} \frac{n!}{(n-k)!} |a_{m,n}| (m+n-2k)^{\tau(m+n-2k)} s^{\tau(m+n-2k)}$$

$$< \sum_{m,n} \sum_{0 \le k \le m \wedge n} \frac{\alpha^k}{k!} (mn)^{(1-\tau)k} s^{\tau(m+n)} |a_{m,n}| (m+n)^{\tau(m+n)}.$$

If $\frac{1}{2} \leq \tau$, then we have $(mn)^{(1-\tau)} \leq K(m+n)$. Thus, we have

$$\|e^{\alpha\partial_u\partial_v} f\|_{\tau,s} \leq |a_{m,n}|(m+n)^{\tau(m+n)}(e^{K/\tau}\alpha s)^{\tau(m+n)}.$$

Since $e^{-\alpha\partial_u\partial_v}$ is the inverse of $e^{\alpha\partial_u\partial_v}$ we have that for every $p \leq 2$, $e^{\alpha\partial_u\partial_v}$ is a linear isomorphism of $\mathcal{E}_p(\mathbb{C}^2)$ onto itself. By the similar proof, we obtain the same result for $e^{\beta\partial_u^2}$ and $e^{\gamma\partial_v^2}$.

We now choose α, β, γ appropriately so that $e^{\alpha\partial_u\partial_v}e^{\beta\partial_u^2}e^{\gamma\partial_v^2}$ defines an intertwiner $I_\circ^\diamond$. Since it is clear that $I_\circ^\diamond$ is an algebra isomorphism of $\mathbb{C}[u,v]$ onto itself, the continuity of $I_\circ^\diamond$ gives the second half of the theorem. □

It is remarkable that the composition of intertwiners gives another intertwiner, symbolically denoted as $I_\diamond^{\circ'} I_\circ^\diamond(f) = I_\circ^{\circ'}(f)$ for $f \in \mathcal{E}_2(\mathbb{C}^2)$. This holds even for $f = e^{au^2+bv^2+2cuv}$ if both sides are defined. This is because both sides are real analytic w.r.t. $\hbar$ and the formula holds on the formal level w.r.t. $\hbar$.

6.2. Strange characters of extended intertwiners.

However, these intertwiners do not extend to the space $\mathcal{E}_{2+}(\mathbb{C}^2)$. In such spaces, intertwiners are defined only for exponential functions of quadratic forms $f = e^{au^2+bv^2+2cuv}$. It is not clear to what extent intertwiners can be defined.

Normal orderings have less symmetries than Weyl ordering. Thus, it seems to be natural to make the following definition:

Definition 21. Let A and B be elements of $\mathcal{E}_p$, defined by normal ordering expressions w.r.t. some canonical conjugate pairs. We denote by $A_\circ$, $B_\diamond$ normal ordering expressions of A, B w.r.t. $(u, v; \circ)$, $(u', v'; \diamond)$ respectively. We define *the notion of equal $A = B$, if and only if $I_\circ^\diamond(A_\circ) = B_\diamond$* through the intertwiner between normal ordering expressions w.r.t. these canonical conjugate pairs.

However, as it will be seen below, *it occurs that intertwiners are defined only as 2-to-2 mappings* because of the ambiguity of the sign of $\sqrt{\ }$. This shows that Definition 21 is not strictly defined. We have to understand intertwiners as 2-to-2 mappings.

Consider the differential equation

$$\frac{\partial}{\partial t} f = i\hbar\partial_u\partial_v f, \tag{47}$$

A real analytic solution in t is unique, if it exists. The solution with the initial function e^{au+bv} is given by $e^{i\hbar abt}e^{au+bv}$.

To obtain the solution with initial function $e^{\frac{1}{\hbar}(\alpha u^2+\beta v^2+2\gamma uv)}$, we set:

$$f = s(t)e^{\frac{1}{\hbar}(\phi_1(t)u^2+\phi_2(t)v^2+\phi_3(t)2uv)}.$$

Then, (47) is rewritten as a system of ordinary differential equations:

$$\begin{aligned}(\phi_1'(t),&\phi_2'(t), \phi_3'(t); s'(t)) \\ &= \big(4i\phi_1(t)\phi_3(t),\, 4i\phi_2(t)\phi_3(t),\, 2i\big(\phi_1(t)\phi_2(t)+\phi_3(t)^2\big);\, 2is(t)\phi_3(t)\big).\end{aligned} \tag{48}$$

We see that

$$\phi_1(t) = \alpha e^{4i\int_0^t \phi_3(\tau)d\tau}, \quad \phi_2(t) = \beta e^{4i\int_0^t \phi_3(\tau)d\tau}$$

Setting $x(t)=\int_0^t \phi_3(\tau)d\tau$, we have

$$x''(t)=2i\alpha\beta e^{8ix(t)}+2ix'(t)^2, \quad x(0)=0,\ x'(0)=\gamma.$$

We regard x as an independent variable and set $\phi_3(t)=p(x(t))$. Then since $\phi_3=x'$, we have $x''(t)=p\frac{dp}{dx}$. It follows that

$$\frac{1}{2}\frac{dp^2}{dx}-2ip^2=2i\alpha\beta e^{8ix}, \quad p(0)=\gamma,$$

and we have $p^2(x)=(\gamma^2-\alpha\beta)e^{4ix}+\alpha\beta e^{8ix}$. Thus, we obtain

$$e^{\hbar ti\partial_u\partial_v}e^{\frac{1}{\hbar}(\alpha u^2+\beta v^2+2\gamma uv)} = \frac{1}{\sqrt{1-4i\gamma t-4(\gamma^2-\alpha\beta)t^2}}\, e^{\frac{1}{1-4i\gamma t-4(\gamma^2-\alpha\beta)t^2}\frac{1}{\hbar}(\alpha u^2+\beta v^2+(\gamma-2i(\gamma^2-\alpha\beta)t)2uv)}, \tag{49}$$

where the ambiguity of the sign of $\sqrt{1-4i\gamma t+4(\alpha\beta-\gamma^2)t^2}$ will be discussed in §6.3.

Set $t=\frac{1}{2}$. Then, we have the intertwiner $I_{\cdot}^{\circ}$ from the Weyl ordering to the normal ordering:

$$(a',b',c';s')=I_{\cdot}^{\circ}(a,b,c;s)=\frac{1}{1-2ic-D}(a,b,c-iD;s\sqrt{1-2ic-D}), \tag{50}$$

where $D=c^2-ab$.

Proposition 22. $I_{\cdot}^{\circ}(e_{\cdot}^{\frac{1}{\hbar}(au^2+bv^2+2cu\cdot v)})$ *is singular if and only if* $(1-ci)^2+ab=0$.

It is easy to see that $D'=(c')^2-a'b'=\frac{D}{1-2ic-D}$, and the inverse mapping $I_{\circ}^{\cdot}=(I_{\cdot}^{\circ})^{-1}$ is given by setting $t=-\frac{1}{2}$. Indeed, we have

$$I_{\circ}^{\cdot}(a',b',c';s)=\frac{1}{1+2ic'-D'}(a',b',c'+iD';s'\sqrt{1+2ic'-D'}).$$

To confirm the result, we check that applying the intertwiner $I_{\cdot}^{\circ}$ given by (50) through (49) to the l.h.s. of (28) gives the normal ordering expression given in Proposition 9.

6.3. The case $e^{-i\hbar\partial_u^2}$.

Remark first that the intertwiner from the normal ordering w.r.t. (u,v) to that w.r.t. $(u',v')=(u+v,v)$ is given by $e^{-\frac{i\hbar}{2}\partial_u^2}$. Consider now the operator $\frac{d}{d\tau}f=i\hbar\partial_u^2 f$, and set

$$f=s(\tau)e^{\frac{1}{\hbar}(\phi_1(\tau)u^2+\phi_2(\tau)v^2+2\phi_3(\tau)uv)}. \tag{51}$$

Thus, we have

$$(\phi_1'(\tau),\phi_2'(\tau),\phi_3'(\tau);s'(\tau))=(4i\phi_1(\tau)^2,4i\phi_3(\tau)^2,4i\phi_1(\tau)\phi_3(\tau);2i\phi_1(\tau)s(\tau)).$$

The solution (51) with the initial data $(\phi_1(0),\phi_2(0),\phi_3(0);s(0))=(a,b,c;s)$ is obtained as follows:

$$(\phi_1(t),\phi_2(t),\phi_3(t);s(t))=(\frac{a}{1-4iat},\frac{b+4iDt}{1-4iat},\frac{c}{1-4iat};\frac{s}{\sqrt{1-4iat}}) \tag{52}$$

where $D=c^2-ab$. Setting $t=-\frac{1}{2}$, we obtain the intertwiner

$$(a'',b'',c'';s'')=I_{\circ}^{\diamond}(a,b,c,s)=\frac{1}{1+2ia}(a,\ b{-}2iD,c;\ s\sqrt{1{+}2ia}). \tag{53}$$

It is easy to see $D''=(c'')^2-a''b''=\frac{D}{1+2ia}$. The inverse relation for (53) is given by setting $t=\frac{1}{2}$:

$$(a,b,c;s)=(\frac{a''}{1{-}2ia''},\frac{b''{+}2iD''}{1{-}2ia''},\frac{c''}{1{-}2ia''};\frac{s''}{\sqrt{1{-}2ia''}}).$$

By a similar calculation, we can compute $e^{ti\hbar\partial_v^2}$: Consider the operator $\frac{d}{d\tau}f=i\hbar\partial_v^2 f$, and set f as in (51). The solution with the initial data $(\phi_1(0),\phi_2(0),\phi_3(0);s(0))=(a,b,c;s)$ is given by:

$$(\phi_1(t),\phi_2(t),\phi_3(t);s(t))=\left(\frac{a+4iDt}{1-4ibt},\frac{b}{1-4ibt},\frac{c}{1-4ibt};\frac{s}{\sqrt{1-4ibt}}\right). \tag{54}$$

We obtain the intertwiner from the normal ordering w.r.t. (u,v) to that w.r.t. $(u,u+v)$ is given by

$$I_{\circ}^{\bullet}(a,b,c;s)=\frac{1}{1-2ib}(a+2iD,b,c;s\sqrt{1-2ib}).$$

We now combine these results. The general intertwiner is obtained by composing (50), (52), (54). For instance, the intertwiner $I_{\cdot}^{\diamond}$ from the Weyl ordering to the normal ordering w.r.t. $\frac{1}{\sqrt{2}}(u-v,u+v)$ is given by

$$I_{\cdot}^{\diamond}(a,b,c;s)=\frac{1}{1{+}i(b{-}a){-}D}(a{-}iD,b{+}iD,c;\sqrt{1{+}i(b{-}a){-}D}).$$

It is remarkable that intertwiners between exponential functions of quadratic forms contain always a sign ambiguity in the amplitude.

The following shows that the polar element is defined globally only as a two-valued element:

Proposition 23. *For a canonical conjugate pair $u'=au+bv, v'=cu+dv$ with $ad-bc=1$, we have*

$$e^{\frac{i\hbar}{2}(-bd\partial_u^2+(ad+bc-1)\partial_u\partial_v-ac\partial_v^2)}\sqrt{-1}e_{\circ}^{\frac{2i}{\hbar}u\circ v}=\sqrt{-1}e_{\diamond}^{\frac{2i}{\hbar}u'\diamond v'}$$

where $\diamond$ denotes the normal ordering w.r.t. (u',v').

7. Gluing via intertwiners

In this section we first want to glue $\mathbb{C}^3\times\mathbb{C}_*$ and $\mathbb{C}^3\times\mathbb{C}_*$ together by the intertwiner $I_{\cdot}^{\circ}$, where $I_{\cdot}^{\circ}$ is the intertwiner from the Weyl ordering to the normal ordering w.r.t. (u,v). Let $I_{\circ}^{\cdot}$ be the inverse of $I_{\cdot}^{\circ}$. Let $P_{\cdot}^{\circ}$, $P_{\circ}^{\cdot}$ be the phase parts of the intertwiners $I_{\cdot}^{\circ}$, $I_{\circ}^{\cdot}$;

$$P_{\cdot}^{\circ}(a,b,c)=\frac{1}{1{-}2ic{-}D}(a,b,c-iD),\quad P_{\circ}^{\cdot}(a',b',c')=\frac{1}{1{+}2ic'{-}D'}(a',b',c'+iD').$$

Recall that
$$I^\circ_\cdot(a,b,c;s)=\Big(P^\circ_\cdot(a,b,c);\frac{s}{\sqrt{1-2ic-D}}\Big).$$
By Proposition 22 it is not hard to see that $P^\circ_\cdot(V_\mu-\{c=-i\})\subset V_\nu$ and $P^\cdot_\circ(V_\nu-\{4a'b'=-1\})\subset V_\mu$. Then, the space of vacuums is preserved by the intertwiner.

To understand the gluing, we define $\Sigma=\{(x,y,z)\in\mathbb{C}^3\mid z^2-xy=0\}$. Make a copy $\mathbb{C}^3-\Sigma'$ of $\mathbb{C}^3-\Sigma$, and consider a holomorphic diffeomorphism
$$T_0:\mathbb{C}^3-\Sigma\to\mathbb{C}^3-\Sigma',\quad T_0(x,y,z)=-\frac{1}{z^2-xy}(x,y,z),$$
Gluing two copies of $\mathbb{C}^3$ by T_0, we obtain a complex 3-dimensional manifold B^3.

On the other hand, consider
$$\Delta:\mathbb{C}^3\to\mathbb{C},\quad \Delta(x,y,z)=z^2-xy. \tag{55}$$
Since $\Delta T_0(x,y,z)=\frac{1}{\Delta(x,y,z)}$, Δ naturally extends to the mapping of B^3 onto the riemann sphere $S^2=\mathbb{C}\cup\{\infty\}$. We denote this mapping also by $\Delta:B^3\to S^2$.

We now consider the functions given by the intertwiners:
$$f_\mu(a,b,c)=1-2ic-D,\quad f_\nu(a',b',c')=1+2ic'-D', \tag{56}$$
where $D=c^2-ab$, and $D'=(c')^2-a'b'$. Take new coordinate functions as follows:
$$(x,y,z)=(a,b,-(c+i)),\quad (x',y',z')=(a',b',c'-i).$$
We see easily that $\Delta(x,y,z)=-f_\mu(a,b,c)$, $\Delta(x',y',z')=-f_\nu(a',b',c')$. Hence, the manifold B^3 glued by T_0 is obtained also via the gluing diffeomorphism $T_{\mu\nu}:\mathbb{C}^3-\{f_\mu=0\}\to\mathbb{C}^3-\{f_\nu=0\}$:
$$(a',b',c')=T_{\mu\nu}(a,b,c)=\frac{1}{f_\mu(a,b,c)}(a,b,c-iD). \tag{57}$$
Indeed (57) is equivalent to
$$\big(a',b',c'-i\big)=-\frac{1}{(c+i)^2-ab}\big(a,b,-(c+i)\big).$$
Considering the path replacing i by $si; s\in[0,1]$ in the above equality, we see the mapping $f_\mu\cup f_\nu:B^3\to S^2$ is homotopic to Δ. Therefore, we must consider the gluing of $\mathbb{C}^3\times\mathbb{C}_*$ and $\mathbb{C}^3\times\mathbb{C}_*$ by $\tilde T_{\mu\nu}$, where
$$\tilde T_{\mu\nu}(a,b,c;s)=\big(\frac{1}{f_\mu}a,\frac{1}{f_\mu}b,\frac{1}{f_\mu}(c-iD);\frac{1}{\sqrt{f_\mu}}s\big). \tag{58}$$

As we are looking for the group-like object generated by exponential functions of quadratic forms, we want to glue $\mathbb{C}^3-V_\mu$ and $\mathbb{C}^3-V_\nu$ by $P^\circ_\cdot(a,b,c)$. We denote the glued manifold by $\tilde B^3=B^3-\{\text{vacuums}\}$, where $\tilde B^3$ is simply connected. Through the adjoint mapping $\mathrm{Ad}(g)$, $\tilde B^3$ is diffeomorphic to $SL_{\mathbb{C}}(2)$. Hence, we must glue $(\mathbb{C}^3-V_\mu)\times\mathbb{C}_*$ and $(\mathbb{C}^3-V_\nu)\times\mathbb{C}_*$ by $\tilde T_{\mu\nu}$ given in (58). However, the glued object is impossible to realize as a manifold, as we would obtain a nontrivial double over of $SL_{\mathbb{C}}(2)$.

Thus, we need a little broad notion, perhaps similar to *an object of a gerbe* of Giraud, or more precisely a flat unitary Diximier-Douady sheaf of groupoids (cf. [Br]). Since these mathematical lingo does not fit directly to our context, we prefer to use other terminology. This will be given in the next subsection.

7.1. Blurred $\mathbb{C}_*$-bundles.

We introduce a notion of *blurred* $\mathbb{C}_*$*-bundles* on S^2 as follows: for a simple open covering $\mathcal{U}=\{U_\alpha\}_{\alpha\in\Lambda}$ of S^2, we give a system of holomorphic transition functions $t_{\alpha\beta}: U_\alpha\cap U_\beta\to\mathbb{C}_*$ such that $t_{\alpha\alpha}=1$, $t_{\alpha\beta}=t_{\beta\alpha}^{-1}$, but $t_{\alpha\beta}t_{\beta\gamma}t_{\gamma\alpha}\in\{e^{\frac{2\pi ik}{m}}; k\in\mathbb{Z}\}$ on $U_\alpha\cap U_\beta\cap U_\gamma\neq\emptyset$. $t_{\alpha\beta}$ is viewed as a gluing diffeomorphism

$$T_{\alpha\beta}: U_\beta\times\mathbb{C}_*\to U_\alpha\times\mathbb{C}_*,\quad T_{\alpha\beta}(p,z)=(p,t_{\alpha\beta}(p)z).$$

Set $t_{\alpha\beta}(p)=e^{2\pi i\lambda_{\alpha\beta}(p)}$, $\lambda_{\alpha\beta}(p)\in\mathbb{C}$, where $\lambda_{\alpha\beta}=-\lambda_{\beta\alpha}$, $\lambda_{\alpha\alpha}=0$.

For $U_\alpha\cap U_\beta\cap U_\gamma\neq\emptyset$, we set $\ell_{\alpha\beta\gamma}=(\delta\lambda)_{\alpha\beta\gamma}$ so that $t_{\alpha\beta}t_{\beta\gamma}t_{\gamma\alpha}=e^{2\pi i\ell_{\alpha\beta\gamma}}$. Then, $\ell=\{\ell_{\alpha\beta\gamma}\}$ is a Čech 2-cocycle over $\frac{1}{m}\mathbb{Z}$. Two such systems $\{\mathcal{U},\{t_{\alpha\beta}\}\}$, $\{\mathcal{U},\{\tilde{t}_{\alpha\beta}\}\}$ are said to be *equivalent*, if $\{\ell_{\alpha\beta\gamma}\}$ and $\{\tilde{\ell}_{\alpha\beta\gamma}\}$ defines the same cohomology class in $H^2(S^2,\frac{1}{m}\mathbb{Z})$. We call this equivalence class a *blurred* $\mathbb{C}_*$*-bundle* over S^2 and denote this by $\mathcal{M}_{S^2}^{(\frac{1}{m})}$. This notion seems to be a simple example of *gerbe* of Giraud (cf. [Br] §4 and §7).

Proposition 24. *If* $\{\mathcal{U},\{t_{\alpha\beta}\}\}$, $\{\mathcal{U},\{\tilde{t}_{\alpha\beta}\}\}$ *are equivalent, then* $\{t_{\alpha\beta}^m\}$ *and* $\{\tilde{t}_{\alpha\beta}^m\}$ *define the same* $\mathbb{C}_*$*-bundle.*

Proof. Suppose there is a 1-cochain $\{\xi_{\alpha\beta}\}\subset\frac{1}{m}\mathbb{Z}$ such that

$$\tilde{\ell}_{\alpha\beta\gamma}-\ell_{\alpha\beta\gamma}=\xi_{\alpha\beta}+\xi_{\beta\gamma}+\xi_{\gamma\alpha}.$$

Setting $\lambda_{\alpha\beta}-\tilde{\lambda}_{\alpha\beta}-\xi_{\alpha\beta}=M_{\alpha\beta}$ so that $\tilde{t}_{\alpha\beta}^{-1}t_{\alpha\beta}e^{-2\pi i\xi_{\alpha\beta}}=e^{2\pi iM_{\alpha\beta}}$, we see that $\{M_{\alpha\beta}\}$ is a Čech 1-cocycle over holomorphic functions $\mathcal{O}$. Note that $H^1(S^2,\mathcal{O})=\{0\}$. Thus, we set $M_{\alpha\beta}=\eta_\alpha-\eta_\beta$, where η_α,η_β are holomorphic functions on U_α, U_β respectively. The replacing $\tilde{t}_{\alpha\beta}$ by $e^{2\pi i\eta_\alpha}\tilde{t}_{\alpha\beta}e^{-2\pi i\eta_\beta}$ is a gauge transformation. Since

$$\lambda_{\alpha\beta}-\xi_{\alpha\beta}=\eta_\alpha+\tilde{\lambda}_{\alpha\beta}-\eta_\beta,$$

and $e^{2\pi m\xi_{\alpha\beta}}=1$, $\{\tilde{t}_{\alpha\beta}^m\}$ and $\{t_{\alpha\beta}^m\}$ define the same $\mathbb{C}_*$-bundle over S^2. □

Thus, if $H^2(N;\frac{1}{m}\mathbb{Z})=\{0\}$, then the restriction $\mathcal{M}_N^{(\frac{1}{m})}$ on a subset N of S^2 gives a genuine $\mathbb{C}_*$-bundle over N. We see that restrictions $\mathcal{M}_{S^2}^{(\frac{1}{m})}\big|_{\mathbb{C}}$ and $\mathcal{M}_{S^2}^{(\frac{1}{m})}\big|_{\mathbb{C}'}$, where $\mathbb{C}'=S^2-\{0\}$, are trivial $\mathbb{C}_*$-bundles. We denote also by $\mathcal{M}_{S^2}^{(1)}$ the $\mathbb{C}_*$-bundle defined by using $\{t_{\alpha\beta}^m\}$ as transition functions. The projections $\pi:\mathcal{M}_{S^2}^{(\frac{1}{m})}\to S^2$, $\tilde{\pi}:\mathcal{M}_{S^2}^{(1)}\to S^2$ are well defined. $\mathcal{M}_{S^2}^{(\frac{1}{m})}$ is naturally viewed as a m-covering space of $\mathcal{M}_{S^2}^{(1)}$. We call $\mathcal{M}_{S^2}^{(\frac{1}{m})}$ the *blurred* m*-covering space* of $\mathcal{M}_{S^2}^{(1)}$.

Now, we define the blurred $\mathbb{C}_*$-bundle $\mathcal{M}_{S^2}^{(\frac{1}{2})}$ such that $\mathcal{M}_{S^2}^{(1)}$ is the tautological $\mathbb{C}_*$-bundle of $\mathbb{C}^2-\{0\}$, and consider the desired glued object as the *pull back* of $\mathcal{M}_{S^2}^{(\frac{1}{2})}$. Consider the pull-back $\Delta^*\mathcal{M}_{S^2}^{(\frac{1}{2})}$ of $\mathcal{M}_{S^2}^{(\frac{1}{2})}$ via the map $\Delta:\tilde{B}^3\to S^2$ given by (55). We remark that blurred bundles over B^3 are always considered as pull-back bundles. That is, we use only coverings of B^3 obtained by the pull back ${\Delta'}^{-1}\mathcal{U}$ by Δ' which is homotopically equivalent with Δ. Hence, this will be denoted by $\Delta^*\mathcal{M}_{S^2}^{(\frac{1}{2})}$.

7.2. Involutive distributions.

Let $\Delta^*\mathcal{M}_{S^2}^{(\frac{1}{2})}$ be a blurred $\mathbb{C}_*$-bundle over a manifold B^3. Though this forms neither a manifold nor has an underlying point set, several notions defined on manifolds extend under the condition that they are invariant under the 2-to-2 local coordinate transformations.

The projection $\Delta^*\mathcal{M}_{S^2}^{(\frac{1}{2})} \to B^3$ is well defined. The notion of distributions is also well defined on $\Delta^*\mathcal{M}_{S^2}^{(\frac{1}{2})}$, and a notion of involutive distribution can be given. An involutive distribution is understood as a horizontal distribution of a flat connection on an object of a gerbe.

If an involutive distribution is restricted to an open subset N where the restriction $\Delta^*\mathcal{M}_{S^2}^{(\frac{1}{m})}\big|_N$ is a genuine $\mathbb{C}_*$-bundle, then we can take an integral submanifold as a point set. By viewing S^2 as the riemann sphere $\mathbb{C}\cup\mathbb{C}'$ glued by $z \leftrightarrow z' = \frac{1}{z}$, the restricted bundles $\Delta^*\mathcal{M}_{S^2}^{(\frac{1}{m})}\big|_{\Delta^{-1}\mathbb{C}}$, $\Delta^*\mathcal{M}_{S^2}^{(\frac{1}{m})}\big|_{\Delta^{-1}\mathbb{C}'}$, form genuine $\mathbb{C}^*$-bundles respectively.

The distributions $\mathcal{D}_\mu$ and $\mathcal{D}_\nu$ are glued together by $\tilde{T}_{\mu\nu}$, and give a distribution on $\Delta^*\mathcal{M}_{S^2}^{(\frac{1}{2})}$. If one understands this distribution as a horizontal distribution of a connection, then the curvature of this connection is identically 0. In spite of this, integral submanifolds M^3 and N^3 given in §3.1, §3.2 cannot be glued together as a manifold. How the union $M^3_*\cup N^3_*$ is considered ? Apparently we have no way to explain such an object directly in set theoretical terms. The only possible way is to give the alternative collection of usage, or axioms in total generality. The notion of gerbes is the one of this direction.

¿From this point of view, we prefer the following explanation, because this is simple and intuitive: $M^3_*\cup N^3_*$ is the maximal "blurred integral submanifold" of $\mathcal{D}_\mu\cup\mathcal{D}_\nu$, glued together by a 2-to-2 local diffeomorphism. This looks like a non-trivial double cover of $SL_{\mathbb{C}}(2)$.

7.3. $*$-exponential mapping.

In §4, we showed in that the $*$-exponential mapping $\exp_*$ is a holomorphic mapping of $\mathbb{C}^3-\Pi_\mu$ into $M^3\subset(\mathbb{C}^3-V_\mu)\times\mathbb{C}_*$. Let Π_ν be the subset where $e_*^{\frac{1}{\hbar}(au^2+bv^2+2cuv)}$ is singular in the normal ordering w.r.t. (u,v). Then, $\exp_*$: $\mathbb{C}^3-\Pi_\nu\to(\mathbb{C}^3-V_\nu)\times\mathbb{C}_*$ is a holomorphic mapping, and $\exp_*(\mathbb{C}^3-\Pi_\nu)\subset N^3_*$. Since $\Pi_\mu\cap\Pi_\nu=\emptyset$, the $*$-exponential mapping is defined from $\mathbb{C}^3$ into the "space" $\Delta^*\mathcal{M}_{S^2}^{(\frac{1}{2})}$.

8. Concluding remarks

We have seen that $\mathcal{D}_\mu\cup\mathcal{D}_\nu$ can be viewed as a horizontal distribution of a connection defined on $\Delta^*\mathcal{M}_{S^2}^{(\frac{1}{2})}$. Since this is involutive, the curvature of this connection vanishes identically on $\tilde{B}^3$. However, it is natural for physicists to say that the curvature tensor is supported only on equilibrium states (cf. Cor. 7).

The maximal integral submanifold $M^3_*\cup(\varepsilon_{00}*M^3_*)$ contains the double covering group of $SL_{\mathbb{R}}(2)=Sp(2;\mathbb{R})$, which may be regarded as the metaplectic group $Mp(2;\mathbb{R})$. Thus, $M^3_* \cup (\varepsilon_{00} * M^3_*)$ may be viewed as a complexification of $Mp(2;\mathbb{R})$. It is obvious that there is no such Lie group in the standard group theory. Moreover, $M^3_* \cup (\varepsilon_{00} * M^3_*)$ contains the double covering group of $SU(1,1)$ as a real form different from $Mp(2;\mathbb{R})$.

As a matter of course, $M^3 \cup (\varepsilon_{00} * M^3)$ cannot be recognized as a genuine object of mathematics based on the point set theory, since it is *not* a point set. In spite of this, we want to claim that such objects should have an appropriate position in rigorous mathematics after relaxing the definition of manifolds.

Strange elements such as ε_{00} are not recognized as an element in set theory. However, for physicists such elements are easily acceptable, because they are computable. Physicists may have already used such elements heuristically in the calculus of Feynmann diagrams. We might think that the connection between mathematics and physics is *not* so straight forward in set theoretical mathematics.

References

[1] Bayen, F., *Star exponentials of the elements of the homogeneous symplectic Lie algebra*, Lett. Math. Phys. **6** (1982), 491–497.

[2] Bayen, F., M. Flato, C. Fronsdal, A. Lichnerowicz, and D. Sternheimer, *Deformation theory and quantization I*, Ann. Phs. **111** (1978), 61–110.

[3] Basart, H., and A. Lichnerowicz, *Conformal symplectic geometry, deformations, rigidity and geometrical KMS conditions*, Lett. Math. Phys. **10** (1985), 167–177.

[4] Brylinski, J.L., "Loop Spaces, Characteristic Classes and Geometric Quantization," Progress in Mathematics, Birkhäuser, Basel 1992.

[5] Connes, A., "Non commutative Geometry," Academic Press, 1994.

[6] Etingof, P., and O. Schiffman, "Lectures on Quantum Groups," International Press, 1998.

[7] Gel'fand, I. M., and G. E. Shilov, "Generalized Functions, 2," Academic Press, 1968.

[8] Guillemin, V., and S. Sternbeg, "Symplectic Techniques in Physics," Cambridge Univ. Press, 1984.

[9] Hörmander, L., *Fourier integral operators I*, Acta Math. **127** (1971), 79–183.

[10] —, *Weyl calculus of pseudo-differential operators*, Comm. Pure Appl. Math. **32** (1979), 359–443.

[11] Kontsevich, M., *Deformation quantization of Poisson manifolds, I*, Preprint qalg/9709040.

[12] Morimoto, M., "An introduction to Sato's Hyperfunctions," Amer. Math. Soc. Trans. Mono. **129**, 1993.

[13] Moreno, C., and A. P. da Silva, *Star products, spectral analysis and hyperfunctions*, Math. Phys. Studies. **22** (2000), 211–214.

[14] Olver, P. J., *Non-associative local Lie groups*, J. Lie Theory **6** (1996), 23–59.

[15] Omori, H., "Infinite Dimensional Lie Groups," Amer. Math. Soc. Trans. Mono. **158**, 1997.

[16] —, *One must break symmetry in order to keep associativity*, Banach Center Publ. **55** (2002), 153–163.

[17] Omori, H., Y. Maeda, and A. Yoshioka, *Weyl manifolds and deformation quantization*, Adv. Math. **85** (1991), 224–255.

[18] Omori, H., Y.Maeda, N. Miyazaki, and A.Yoshioka, *Deformation quantization of Fréchet-Poisson algebras –Convergence of the Moyal product–*, Math. Phys. Studies. **22** (2000), 233–246.

[19] —, *Deformation quantizations of the Poisson algebra of Laurent polynomials*, Lett. Math. Phys. **46** (1998), 171–180.

[20] —, *Deformation quantizations of Fréchet-Poisson algebras associated with Heisenberg Lie algebra*, Contemp. Math. **288** (2001), 391–395.

[21] —, *Singular systems of exponential functions*, Math. Phys. Studies **23** (2001), 169–187.

[22] —, *Star exponential functions for quadratic forms and polar elements*, Contemp. Math. (2002), to appear.

[23] Rieffel, M., "Deformation quantization for actions of $\mathbb{R}^n$," Memoirs Amer. Math. Soc. **106**, 1993.

H.Omori
Department of Mathematics, Tokyo University of Science,
Noda, Chiba, 278-8510, Japan.
omori@ma.noda.tus.ac.jp

Y.Maeda
Department of Mathematics, Faculty of Science and Technology,
Keio University, Hiyoshi,
Yokohama, 223-8522, Japan.
maeda@math.keio.ac.jp

N.Miyazaki
Department of Mathematics, Faculty of Economics,
Keio University, Hiyoshi, Yokohama, 223-8521, Japan.
miyazaki@math.hc.keio.ac.jp

A.Yoshioka
Department of Mathematics, Tokyo University of Science, Kagurazaka,
Tokyo, 102-8601, Japan.
yoshioka@rs.kagu.sut.ac.jp

Received June 19, 2002
and in final form October 21, 2002