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On the Cartan Subalgebras of Lie Algebras over Small Fields

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Abstract. In this note we study Cartan subalgebras of Lie algebras defined over finite fields. We prove that a possible Lie algebra of minimal dimension without Cartan subalgebras is semisimple. Subsequently, we study Cartan subalgebras of $\mathfrak{gl}(n, F)$.
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1. Introduction

Let L be a finite-dimensional Lie algebra over a field F. A subalgebra H of L is called a *Cartan subalgebra* of L if it is nilpotent and self-normalizing, i.e. $N_L(H) = H$.

They play a fundamental role in the theory of Lie algebra of characteristic 0 and they represent also an important tool for studying modular Lie algebras.

We recall that, for any $x \in L$, the Fitting null component relative to $\operatorname{ad} x$ is defined by

$$L_0(\operatorname{ad} x) = \{y \in L \mid y(\operatorname{ad} x)^n = 0 \text{ for certain } n \in \mathbf{N}\}.$$

An element $h \in L$ is said to be *regular* if, for any $x \in L$:

$$\dim_F L_0(\operatorname{ad} h) \leq \dim_F L_0(\operatorname{ad} x)$$

In 1967, Barnes [1] showed that if $\dim_F L < |F|$ then a subalgebra H is a Cartan subalgebra of minimal dimension if and only if there exists a regular element x in L such that $H = L_0(\operatorname{ad} x)$.

On the contrary, when the base field has not more than $\dim_F L$ elements, the existence of Cartan subalgebras of finite-dimensional Lie algebras is still an open problem.

However, in some cases the hypothesis on the cardinality of F can be dropped. For example, one can demonstrate that solvable Lie algebras always

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have Cartan subalgebras (when $\operatorname{char} F = 0$, they are conjugate under the action of inner automorphisms, too: see [4] or [5]); restricted Lie algebras have Cartan subalgebras and they coincide with the centralizers of the maximal tori (see [5]).

In this paper, we analyze some properties of a possible minimal counterexample of Lie algebra which does not have a Cartan subalgebra. Indeed, we reduce the general problem of the existence of the Cartan subalgebras of Lie algebras to the case of semisimple non restricted Lie algebras over a "small" field with respect to the dimension.

In the second part, we determine the Cartan subalgebras of $\mathfrak{gl}(n, F)$ when the field F has at least $n^2 + 1$ elements generalizing a result already obtained by Jacobson for algebraically closed fields.

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2. Properties of a possible minimal counterexample

For the proof of the next result we refer to [5].

Lemma 2.1. *Let $\mathfrak{L}$, $\overline{\mathfrak{L}}$ be two Lie algebras on a field F and let $f : \mathfrak{L} \to \overline{\mathfrak{L}}$ be a homomorphism. Then if H is a Cartan subalgebra of $\mathfrak{L}$, $f(H)$ is a Cartan subalgebra of $f(\mathfrak{L})$. Also, if $\overline{H}$ is a Cartan subalgebra of $f(\mathfrak{L})$ and H is a Cartan subalgebra of $f^{-1}(\overline{H})$, then H is a Cartan subalgebra of $\mathfrak{L}$.*

Using this result, we are able to prove:

Proposition 2.2. *Let $\mathfrak{L}$ be a minimal example of a finite dimensional Lie algebra over a field F without Cartan subalgebras. Then*

1. *F is a finite field and $|F| \leq \dim_F \mathfrak{L}$;*
2. *$\mathfrak{L}$ is not a restricted Lie algebra;*
3. *$\mathfrak{L}$ is semisimple*

Proof. The assertions 1. and 2. are consequence of the remarks already stated in the introduction.

In order to prove 3., suppose by contradiction that $\mathfrak{L}$ is not semisimple. Then the radical $Rad\,\mathfrak{L}$ is not $\{0\}$ and, because $Rad\,\mathfrak{L}$ is solvable, there exists a positive integer k such that $Rad^{(k)}\,\mathfrak{L} \neq \{0\}$ and $Rad^{(k+1)}\,\mathfrak{L} = \{0\}$.

Now, set $J = Rad^{(k)}\,\mathfrak{L}$; then it results $J \neq \{0\}$ and $[J, J] = \{0\}$, in other words J is a nonzero abelian ideal of $\mathfrak{L}$.

Consequently, considered the quotient Lie algebra $\mathfrak{L}/J$, we have that $0 \leq \dim_F \mathfrak{L}/J < \dim_F \mathfrak{L}$. By minimality of the dimension of $\mathfrak{L}$, it follows that there exists a Cartan subalgebra $\overline{H}$ of $\mathfrak{L}/J$ and then $\mathfrak{L}$ has a subalgebra H such that $\overline{H} = H/J$.

Since H/J is nilpotent and J is abelian, in particular H/J and J are solvable, also H is solvable, therefore by [4] (or [5]) this assures the existence of a Cartan subalgebra $\mathfrak{H}$ of H.

Consider now the canonical mapping

$$\chi : \mathfrak{L} \to \mathfrak{L}/J.$$

We conclude that χ is a homomorphism between the Lie algebras $\mathfrak{L}$ and $\mathfrak{L}/J$, $\overline{H}$ is a Cartan subalgebra of $\mathfrak{L}/J = \chi(\mathfrak{L})$ and $\mathfrak{H}$ is a Cartan subalgebra of $H = \chi^{-1}(\overline{H})$.

Hence, in view of Lemma [5] $\mathfrak{H}$ is a Cartan subalgebra of $\mathfrak{L}$,contradicting the original assumption on $\mathfrak{L}$. ■

3. Cartan subalgebras of $\mathfrak{gl}(n, F)$

In [3], Jacobson proved that if F is an algebraically closed field, $A \in \mathfrak{gl}(n, F)$ is a regular element if and only if its characteristic polynomial $\det(\lambda I - A)$ has n distinct roots. Moreover the Cartan subalgebra determined by A coincides with the centralizer $C_{\mathfrak{gl}(n,F)}(A)$. We will apply the Barnes' theorem to prove the following generalization:

Theorem 3.1. *Let $n \in \mathbf{N}$ and F a field with $|F| > n^2$. Then a subalgebra H of $\mathfrak{gl}(n, F)$ is a Cartan subalgebra if and only if there exists $X \in H$ having n characteristic roots in some extension of F such that $H = C_{\mathfrak{gl}(n,F)}(X)$.*

Proof. Denote by V the n-dimensional vector space over F.

Since the property of an element to be regular or not is unchanged on extending the base field, we can assume, without loss of generality, that the characteristic roots of any element are always in F.

Let $A \in \mathfrak{gl}(n, F)$, let $\alpha_1, \alpha_2, \dots, \alpha_r \in F$ be the distinct characteristic roots of A and let $V_{\alpha_1}, V_{\alpha_2}, \dots V_{\alpha_r}$ be the corresponding characteristic spaces.

Since the Lie algebra $\mathfrak{L} = FA$ is nilpotent and the subspaces V_{α_i} are invariant under $\mathfrak{L}$ we have that $\alpha_1, \alpha_2, \dots, \alpha_r$ are the weights for $\mathfrak{L}$ and

$$V = V_{\alpha_1} \oplus V_{\alpha_2} \oplus \dots \oplus V_{\alpha_r} \tag{1}$$

is the decomposition of V into the weight spaces relative to $\mathfrak{L}$.

Moreover, for every $1 \le i \le r$, we recall that the dual module $V^*_{-\alpha_i}$ of V_{α_i} is a weight module for $\mathfrak{L}$ with the weight $-\alpha_i$, hence (1) induces a decomposition of the dual space V^* given by

$$V^* = V^*_{-\alpha_1} \oplus V^*_{-\alpha_2} \dots \oplus V^*_{-\alpha_r}.$$

Now, if R is the representation of $\mathfrak{L}$ relative to the module $V \otimes_F V^*$ and $\tilde{A} = AR$, following the same argument used in [1] we see that the Fitting null component relative to $\tilde{A}$ in $V \otimes_F V^*$ is

$$\bigoplus_{i=1}^{r} V_{\alpha_i} \otimes_F V^*_{-\alpha_i}.$$

For $1 \le i \le r$ set $n_i = \dim_F V_{\alpha_i}$. Then

$$\sum_{i=1}^{r} \dim_F \left(V_{\alpha_i} \otimes_F V^*_{-\alpha_i}\right) = \sum_{i=1}^{r} \left(\dim_F V_{\alpha_i} \cdot \dim_F V^*_{-\alpha_i}\right) = \sum_{i=1}^{r} n_i^2$$

and by (1) we have also $n = \dim_F V = \sum_{i=1}^r n_i$.

Consider now the system of diophantine equations

$$\begin{cases} \sum_{i=1}^r n_i^2 = m \\ \\ \sum_{i=1}^r n_i = n \end{cases} \tag{2}$$

For any positive integers $x_1, x_2, \ldots, x_r$ we have

$$(x_1 + x_2 + \ldots + x_r)^2 \leq x_1^2 + x_2^2 + \ldots + x_r^2,$$

thus m is minimal if and only if $r = n$ and $n_i = 1$ for all i.

By assumption $|F| > n^2$, in particular the field F (and obviously any extension) has at least n elements. It is well known that the module $V \otimes_F V^*$ is isomorphic in a natural way to $\mathfrak{gl}(n,F)$ (as a module over $\mathfrak{L}$, the corresponding representation is the adjoint mapping). Thus the dimensionality of $\mathfrak{gl}(n,F)_0(\operatorname{ad} A)$ is minimal (i.e. A is regular) if and only if A has n distinct eigenvalues.

Now, since $|F| > n^2 = \dim_F \mathfrak{gl}(n,F)$ and the Cartan subalgebras of $\mathfrak{gl}(n,F)$ are equidimensional (see Theorem 3.2, below), in view of Barnes' theorem the Cartan subalgebras of $\mathfrak{gl}(n,F)$ are exactly the Fitting null component relative to the adjoint mapping of the regular elements of $\mathfrak{gl}(n,F)$. So a subalgebra H of $\mathfrak{gl}(n,F)$ is a Cartan subalgebra if and only there exists $X \in \mathfrak{gl}(n,F)$ whose characteristic polynomial has n distinct roots in its splitting field such that $H = L_0(\operatorname{ad} X)$.

Then X is diagonalizable, so V admits a basis $\{f_i\}_{i=1}^n$ consisting of eigenvectors of X. The centralizer $C_{\mathfrak{gl}(n,F)}(X)$ contains all elements of $\mathfrak{gl}(n,F)$ whose matrix is diagonal relative to the basis $\{f_i\}_{i=1}^n$, so $\dim_F C_{\mathfrak{gl}(n,F)}(X) \geq n$. But $C_{\mathfrak{gl}(n,F)}(X) \subseteq L_0(\operatorname{ad} X) = H$ and $\dim_F L_0(\operatorname{ad} X) = \sum_{i=1}^n n_i = n$, therefore $H = C_{\mathfrak{gl}(n,F)}(X)$. ■

Generally, the result fails without having any hypothesis on the cardinality of the field. If, for example, $n > 2$, F is a field with less than n elements and H is the subalgebra consisting of the diagonal elements of $\mathfrak{gl}(n,F)$, then H is a Cartan subalgebra of $\mathfrak{gl}(n,F)$. On the other hand, the eigenvalues of each $X \in H$ are exactly the elements of the principal diagonal of its associated matrix, which are elements of F and thus X cannot have n distinct elements.

Nevertheless, a part of Theorem 3.1 can be generalized: indeed, the Cartan subalgebras of $\mathfrak{gl}(n,F)$, F an arbitrary field, are all n-dimensional.

For the definition of the concept of a torus we refer to [5].

Theorem 3.2. *Let F be an arbitrary field and let H be a subalgebra of $\mathfrak{gl}(n,F)$. Then H is a Cartan subalgebra if and only if H is a maximal torus of $\mathfrak{gl}(n,F)$. Moreover,* $\dim_F H = n$.

Proof. In characteristic 0 the result is trivial as all Cartan subalgebras are conjugate over the algebraic closure of F.

Suppose thus F is of characteristic $p > 0$.

We remark that $\mathfrak{gl}(n,F)$ is obviously restricted. By [5], Theorem 4.5.17, we have that H is a Cartan subalgebra of $\mathfrak{gl}(n,F)$ if and only if $H = C_{\mathfrak{gl}(n,F)}(T)$, where T is a maximal torus.

Let K be the algebraic closure of F. Then $T \otimes_F K$ is a maximal torus of $\mathfrak{gl}(n,F) \otimes_F K$. And $T \otimes_F K$ is conjugate under the action of $GL(n,K)$ to the maximal torus D of $\mathfrak{gl}(n,F) \otimes_F K$ consisting of diagonal matrices. So $H \otimes_F K$ is conjugate to $C_{\mathfrak{gl}(n,F)\otimes_F K}(D) = D$.

It follows that $\dim H = n$, and $H = T$. ■

For $n = 2$ a complete determination of the Cartan subalgebras of $\mathfrak{gl}(2,F)$ can be obtained without restriction on the field by some elementary considerations. This not only yields the previous result, but establishes also a one-to-one corrispondence between the Cartan subalgebras and certain elements of $\mathfrak{gl}(2,F)$.

In the following, we identify an element x of $\mathfrak{gl}(2,F)$ with its associated matrix, $\{E_{ij}\}$ is the canonical basis of $\mathfrak{gl}(2,F)$ and L_x denotes the Lie subalgebra generated by x and the identity matrix I.

First we show some preliminary lemmas.

Lemma 3.3. *Let F a field,* $\operatorname{char} F \neq 2$, *and* $x = \left(\begin{smallmatrix} 0 & a_{12} \\ a_{21} & 0 \end{smallmatrix}\right)$ *where* $a_{12}, a_{21} \in F \setminus \{0\}$. *Then L_x is a Cartan subalgebra of $\mathfrak{gl}(2,F)$ and $L_x = C_{\mathfrak{gl}(2,F)}(x)$.*

Proof. Since the structure constants of L_x are 0, L_x is abelian and in particular nilpotent,hence it remains only to show that $N_{\mathfrak{gl}(2,F)}(L_x) \subseteq L_x$.

If $y = \left(\begin{smallmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{smallmatrix}\right) \in N_{\mathfrak{gl}(2,F)}(L_x)$, we have that

$$[x,y] = \begin{pmatrix} a_{12}b_{21} - a_{21}b_{12} & a_{12}(b_{22} - b_{11}) \\ a_{21}(b_{11} - b_{22}) & a_{21}b_{12} - a_{12}b_{21} \end{pmatrix} \tag{3}$$

and by the assumption on y there exist $h, k \in F$ such that $[x,y] = kI + hx$.

But $tr([x,y]) = 0$, so $2k = 0$ and then $k = 0$ (as $\operatorname{char} F \neq 2$) and from (3) it follows $b_{21} = \frac{a_{21}}{a_{12}} b_{12}$ and $b_{11} = b_{22}$.

For $\hat{h} = \frac{b_{12}}{a_{12}}$ and $\hat{k} = b_{11}$ we have $\hat{k}I + \hat{h}x = y$ so $y \in L_x$.

Therefore L_x is a Cartan subalgebra of $\mathfrak{gl}(2,F)$ and it is easy to show that L_x coincides with $C_{\mathfrak{gl}(2,F)}(x)$. ■

Remark 3.4. In Lemma 3.3, if we have $a_{21} = 0$ and $a_{12} \neq 0$, then $[x, E_{11}] = -x$ with $E_{11} \notin L_x$, thus L_x is not a self-normalizing subalgebra.

Analogously if $a_{12} = 0$ and $a_{21} \neq 0$.

If $a_{12} = a_{21} = 0$, $L_x = FI$ and in Theorem 3.2 we have seen that L_x cannot be a Cartan subalgebra.

Therefore the condition $a_{12}, a_{21} \neq 0$ in the Lemma is necessary.

If $\operatorname{char} F = 2$ and x is as in Lemma 3.3, then $[x, E_{11}] = x$ with $E_{11} \notin L_x$, hence also the assumption $\operatorname{char} F \neq 2$ is essential.

Lemma 3.5. *Let F be an arbitrary field and let* $x = \left(\begin{smallmatrix} 1 & a_{12} \\ a_{21} & 0 \end{smallmatrix}\right)$ *with* a_{12}, a_{21} $\in F$ *such that* $1 + 4a_{12}a_{21} \neq 0$. *Then L_x is a Cartan subalgebra of $\mathfrak{gl}(2,F)$ and* $L_x = C_{\mathfrak{gl}(2,F)}(x)$.

Proof. As in Lemma 3.3, it suffices to prove that, for every $y = \left(\begin{smallmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{smallmatrix}\right) \in \mathfrak{gl}(2,F)$, if $[x,y] \in L_x$ then $y \in L_x$. We have

$$[x,y] = \begin{pmatrix} a_{12}b_{21} - a_{21}b_{12} & b_{12} + a_{12}(b_{22} - b_{11}) \\ a_{21}(b_{11} - b_{22}) - b_{21} & a_{21}b_{12} - a_{12}b_{21} \end{pmatrix} \tag{4}$$

and $[x,y] = kI + hx$, for some $h, k \in F$.

From $tr([x,y]) = 0$ it follows $h = -2k$ and combining with (4) we obtain the pair of relations

$$\begin{cases} 2a_{12}(a_{12}b_{21} - a_{21}b_{12}) = b_{12} + a_{12}(b_{22} - b_{11}) \\ \\ 2a_{21}(a_{12}b_{21} - a_{21}b_{12}) = a_{21}(b_{11} - b_{22}) - b_{21}. \end{cases} \tag{5}$$

We distinguish a few cases.

In the first case, $a_{12} \neq 0, a_{21} \neq 0$. Then by (5) we have

$$b_{12}(1 + 4a_{12}a_{21}) = \frac{a_{12}}{a_{21}} b_{21}(1 + 4a_{12}a_{21})$$

and, since $1 + 4a_{12}a_{21} \neq 0$, this forces $b_{12} = \frac{a_{12}}{a_{21}} b_{21}$ and $b_{11} = b_{22} + \frac{b_{12}}{a_{12}}$.

For $\hat{k} = b_{22}$ and $\hat{h} = \frac{b_{12}}{a_{12}}$, we have $y = \hat{k}I + \hat{h}x \in L_x$.

In the second case, $a_{12} = 0, a_{21} \neq 0$. Then (5) give immediately $b_{12} = 0$ and $b_{11} = b_{22} + \frac{b_{21}}{a_{21}}$, so for $\hat{k} = b_{22}$ and $\hat{h} = \frac{b_{21}}{a_{21}}$ we have $y = \hat{k}I + \hat{h}x \in L_x$.

In the third case, $a_{12} \neq 0, a_{21} = 0$. Analogous to the previous case .

Fourth case, $a_{12} = a_{21} = 0$. In this case, L_x is the set of all diagonal matrices which is a Cartan subalgebra of $\mathfrak{gl}(2,F)$.

In conclusion, $N_{\mathfrak{gl}(2,F)}(L_x) = L_x$ and, clearly, $L_x = C_{\mathfrak{gl}(2,F)}(x)$. ∎

Remark 3.6. The requirement $1+4a_{12}a_{21} \neq 0$ is always satisfied if char $F = 2$. When char $F \neq 2$, this condition cannot be omitted: in fact, for $a_{21} = -\frac{1}{4a_{12}}$, and $y = \left(\begin{smallmatrix} 0 & 4a_{12}^2 \\ 1 & 0 \end{smallmatrix}\right)$, then $y \notin L_x$, while $[x,y] = a_{12}(-2I + 4x) \in L_x$.

Remark 3.7. Lemmas 3.3, 3.5 can also be proved using Theorem 3.2. For example, if char $F \neq 2$, then $1 + 4a_{12}a_{21} \neq 0$ is equivalent to x being semisimple.

Proposition 3.8. *Let F be an arbitrary field. Then the Cartan subalgebras of $\mathfrak{gl}(2,F)$ are precisely the centralizers $C_{\mathfrak{gl}(2,F)}(x)$, where*

if char $F \neq 2$*:*

$$x = \begin{cases} \begin{pmatrix} 0 & a_{12} \\ a_{21} & 0 \end{pmatrix} & \textit{with } a_{12}, a_{21} \in F \setminus \{0\} \qquad \textit{or} \\ \begin{pmatrix} 1 & \overline{a}_{12} \\ \overline{a}_{21} & 0 \end{pmatrix} & \textit{with } \overline{a}_{12}, \overline{a}_{21} \in F \textit{ and } 1 + 4\overline{a}_{12}\overline{a}_{21} \neq 0 \end{cases}$$

if char $F = 2$*:*

$$x = \begin{pmatrix} 1 & c_{12} \\ c_{21} & 0 \end{pmatrix} \qquad \textit{with } c_{12}, c_{21} \in F.$$

Proof. By lemmas 3.3 and 3.5, the centralizers defined in the statement are Cartan subalgebras.

Conversely, let H be a Cartan subalgebra of $\mathfrak{gl}(2,F)$. In view of Theorem 3.2, H is two-dimensional and $I \in H$, hence there exists $\overline{x} = \left(\begin{smallmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{smallmatrix}\right) \in H$ such that $\{I, \overline{x}\}$ is a basis for H.

If $x_{11} \neq x_{22}$, setting

$$x = \begin{pmatrix} 1 & \frac{x_{12}}{x_{11}-x_{22}} \\ \frac{x_{21}}{x_{11}-x_{22}} & 0 \end{pmatrix}$$

we have $\overline{x} = x_{22}I + (x_{11} - x_{22})x$, thus $H \subseteq L_x$,which together with $\dim_F H = \dim_F L_x = 2$ implies $H = L_x$.

If $x_{11} = x_{22}$, considering

$$x = \begin{pmatrix} 0 & x_{12} \\ x_{21} & 0 \end{pmatrix}$$

we have $\overline{x} = x_{11}I + x$ and as above $H = L_x$. ■

We observe that if x_1 and x_2 are as in the statement of Proposition 3.8 and they are not proportional, then the respective subalgebras determined by them are evidently distinct.

Therefore there is a one-to-one correspondence between the Cartan subalgebras of $\mathfrak{gl}(2,F)$ and the non proportional elements x.

In particular

Corollary 3.9. *Let F be a finite field, $|F| = p^n$. Then the number of Cartan subalgebras of $\mathfrak{gl}(2,F)$ is p^{2n}.*

Proof. We compute the number of the elements of $\mathfrak{gl}(2,F)$ whose centralizers determine the Cartan subalgebras.

If $\operatorname{char} F = 2$, the number of the pairs $(a_{12}, a_{21}) \in F^2$ is p^{2n}, so thanks to Proposition 3.8 the assertion follows.

If $\operatorname{char} F \neq 2$, the number of non proportional pairs (a_{12}, a_{21}) of nonzero elements of F is $p^n - 1$. Moreover, the number of solutions (in F) for the equation $1 + 4a_{12}a_{21} = 0$ is $p^n - 1$. So the number of the pairs (a_{12}, a_{21}) such that $1 + 4a_{12}a_{21} \neq 0$ is $p^{2n} - (p^n - 1)$. Hence the number of the Cartan subalgebras is again p^{2n}. ■

References

[1] Barnes, R. E., *On Cartan Subalgebras of Lie Algebras*, Math. Z. **101** (1967), 350–355.

[2] Gastineau-Hills, H. M., "Homomorphisms and Formations of Soluble Lie Algebras," M. Sc. Thesis, University of Sydney, 1967.

[3] Jacobson, N., "Lie Algebras," Wiley Interscience, New York London, 1962.

[4] Seligman, G. B., "Modular Lie algebras," Springer-Verlag , New York etc., 1967.

[5] Winter, D. J., "Abstract Lie Algebras," M. I. T. Press, Cambridge, Mass., 1972.

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