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# More about Embeddings of Almost Homogeneous Heisenberg Groups

**J. Hoheisel and M. Stroppel**

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**Abstract.** Heisenberg groups are simply connected nilpotent Lie groups of class 2. A group is called almost homogeneous if its automorphism group acts with at most 3 orbits. Several open problems about the existence of embeddings between almost homogeneous Heisenberg groups have been posed in a previous paper by the second author. Most of these problems are solved.
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## 1. Introduction.

An important class of nilpotent groups is formed by the so-called *Heisenberg groups*, which are defined as follows: Let $V$ and $Z$ be vector spaces of finite dimension over $\mathbb{R}$, and let $\beta := \langle\cdot,\cdot\rangle\colon V \times V \to Z$ be a symplectic bilinear map. Then $[(v,x),(w,y)] := (0,\langle v,w\rangle)$ gives a Lie bracket on the vector space $V \times Z$; the Lie algebra thus defined will be denoted by $\mathfrak{gh}(V,Z,\beta)$. The corresponding simply connected group is the topological space $V \times Z$, endowed with the multiplication

$$(v,x)\cdot(w,y) := \left(v+w, x+y+\tfrac{1}{2}\langle v,w\rangle\right) .$$

We denote this group by $\mathrm{GH}(V,Z,\beta)$.

A (topological) group $G$ is called *almost homogeneous* if the group $\mathrm{Aut}(G)$ of (topological) automorphisms acts with at most 3 orbits on $G$. The discrete case has been investigated in [3]. For a locally compact connected group $G$, the assumption that $\mathrm{Aut}(G)$ acts with less than $2^{\aleph_0}$ orbits already implies that $G$ is a simply connected, nilpotent Lie group, cf. [7] 5.1. Thus the locally compact connected almost homogeneous groups are exactly those groups that have been determined in [5]. Some of the results in the present paper are contained in the first author's thesis [2]. Another part of that thesis was incorporated in [8] 3.8.

**Theorem 1.1.** *Let $H$ be a locally compact connected group.*

**a.** *If $\mathrm{Aut}(H)$ acts with 2 orbits then $H$ is isomorphic to $\mathbb{R}^n$, for some natural number $n$.*

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**b.** *If* $\mathrm{Aut}(H)$ *acts with* 3 *orbits then* $H$ *is a Heisenberg group, and the pair of dimensions* $\big(\dim(H/H'), \dim H'\big)$ *belongs to the set*

$$\big\{(2n,1) \mid n \geq 1\big\} \ \cup \ \big\{(4n,2) \mid n \geq 1\big\} \ \cup \ \big\{(4n,3) \mid n \geq 1\big\} \\ \cup \ \big\{ (3,3), (6,6), (7,7), (8,5), (8,6), (8,7)\big\}.$$

*Moreover, the pair* $(\dim(H/H'), \dim H')$ *determines* $H$*, up to isomorphism.*

The pattern of possible dimensions already suggests that there are three infinite series, and 6 isolated examples. This is indeed the case, cf. [5]:

**Remark 1.2.** **a.** Every Heisenberg group $\mathrm{GH}(V,Z,\beta)$ with $\dim Z = 1$ is obtained from a symplectic form $\beta : V \times V \to Z \cong \mathbb{R}$. Such a group is almost homogeneous if, and only if, the form $\beta$ is non-degenerate. In this case, (the isomorphism type of) $\mathrm{GH}(V,Z,\beta)$ is denoted by $\mathrm{H}^n_{\mathbb{R}}$, where $n = \dim_{\mathbb{R}} V$.

**b.** A Heisenberg group $\mathrm{GH}(V,Z,\beta)$ with $\dim Z = 2$ is almost homogeneous if, and only if, the spaces $V$ and $Z$ can be made vector spaces over $\mathbb{C}$ in such a way that $\beta$ is complex bilinear, and non-degenerate. In this case, (the isomorphism type of) $\mathrm{GH}(V,Z,\beta)$ is denoted by $\mathrm{H}^{2n}_{\mathbb{C}}$, where $n = \dim_{\mathbb{C}} V$.

**c.** A Heisenberg group $\mathrm{GH}(V,Z,\beta)$ with $\dim V = 4n$ and $\dim Z = 3$ is almost homogeneous if, and only if, the space $V$ can be made a vector space over Hamilton's quaternions $\mathbb{H}$ in such a way that $\beta$ is the pure part of a positive definite hermitian form from $\mathbb{H}^n \times \mathbb{H}^n \cong V \times V$ to $Z \cong \mathrm{Pu}(\mathbb{H}) := \big\{h \in \mathbb{H} \,\big|\, \bar{h} = -h\big\}$. In this case, (the isomorphism type of) $\mathrm{GH}(V,Z,\beta)$ is denoted by $\mathrm{H}^{4n}_{\mathbb{H}}$, where $n = \dim_{\mathbb{H}} V$.

The almost homogeneous Heisenberg groups with $(v,z) := \big(\dim(H/H'), \dim H'\big)$ in $\big\{ (3,3), (6,6), (7,7), (8,5), (8,6), (8,7)\big\}$ will be denoted by $\mathrm{H}^v_z$; the corresponding Lie algebra by $\mathfrak{h}^v_z$. The Lie algebras $\mathfrak{h}^8_5$ and $\mathfrak{h}^8_6$ will be introduced by certain module projections in 2.7 below. Even more explicit descriptions may be found in 3.1 and 3.2.

**Problems 1.3.** There exist quite obvious embeddings between almost homogeneous Heisenberg groups: $\mathrm{H}^{2n}_{\mathbb{R}} \hookrightarrow \mathrm{H}^{4n}_{\mathbb{C}}$, $\mathrm{H}^{2n}_{\mathbb{R}} \hookrightarrow \mathrm{H}^{4n}_{\mathbb{H}}$ and $\mathrm{H}^3_3 \hookrightarrow \mathrm{H}^6_6 \hookrightarrow \mathrm{H}^7_7$; see [6]. It appeared natural to pose the general embeddability problem. Apart from a few cases, this problem was solved in [6]; only the following problems remained open:

**(a)** Is there an embedding of $\mathrm{H}^4_{\mathbb{R}}$ into $\mathrm{H}^8_6$ ?

**(b)** Is there an embedding of $\mathrm{H}^4_{\mathbb{C}}$ into $\mathrm{H}^8_5$ ?

**(c)** Is there an embedding of $\mathrm{H}^4_{\mathbb{H}}$ into $\mathrm{H}^8_5$, or into $\mathrm{H}^8_6$ ?

**(d)** Is there an embedding of $\mathrm{H}^6_{\mathbb{R}}$ into $\mathrm{H}^8_5$ ?

**(e)** For which pairs $(k,n)$ is there an embedding of $\mathrm{H}^{4k}_{\mathbb{C}}$ into $\mathrm{H}^{8n}_{\mathbb{H}}$ ?

In the present paper we solve Problems (a) – (d), but Problem (e) remains open.

We are dealing with simply connected nilpotent Lie groups here, where the exponential map is a homeomorphism (even a diffeomorphism, see [9] Thm. 3.6.2). Therefore, embeddings (where we have a homeomorphism onto the image) correspond bijectively to embeddings of Lie algebras (i.e., injective Lie algebra homomorphisms), and embeddings are nothing else than injective continuous group homomorphisms. This remark is made only to indicate the deeper reason why it suffices to consider Lie algebra homomorphisms in the sequel. Of course, non-existence of an embedding for the algebras implies non-existence of an embedding for the groups.

**Lemma 1.4.** *Let $G = \mathrm{GH}(S,Y,\gamma)$ and $H = \mathrm{GH}(V,Z,\beta)$ be almost homogeneous Heisenberg groups. Assume that $\varphi\colon G \to H$ is an injective continuous homomorphism. Then the following hold.*

**a.** *The commutator groups satisfy $(\{0\}\times Y)^\varphi = G'^\varphi = H' \cap G^\varphi = (\{0\}\times Z)\cap G^\varphi$.*

**b.** *There are injective linear maps $\varphi_1\colon S \to V$ and $\varphi_2\colon Y \to Z$ such that the following diagram commutes:*

$$\begin{tikzcd} S\times S \arrow[r, "\gamma"] \arrow[d, "{(\varphi_1,\varphi_1)}"'] & Y \arrow[d, "\varphi_2"] \\ V\times V \arrow[r, "\beta"'] & Z \end{tikzcd}$$

**c.** *Every pair $(\varphi_1,\varphi_2)$ as in assertion* b *yields an embedding*

$$(\varphi_1,\varphi_2) : \mathrm{GH}(S,Y,\gamma) \to \mathrm{GH}(V,Z,\beta) : (s,y) \mapsto (s^{\varphi_1}, y^{\varphi_2})\,.$$

*However, if $\varphi_1$ and $\varphi_2$ are obtained from an embedding $\varphi$ as in assertion* b*, it may happen that $(\varphi_1,\varphi_2)$ and $\varphi$ are different.*

**d.** *For every $g \in G$, we have $\dim \mathrm{C}_G(g) - \dim G' \le \dim \mathrm{C}_H(g^\varphi) - \dim H'$.*

**Proof.** The first two statements were proved in [6] 3.2. Every linear map $\tau : S \to Y$ defines an automorphism $\tilde\tau : (s,y) \mapsto (s, y+s^\tau)$ of $\mathrm{GH}(S,Y,\gamma)$, and $\varphi$ and $\tilde\tau\varphi$ yield the same pair $(\varphi_1,\varphi_2)$. The rest of assertion c is verified by a simple computation. The last assertion follows from the observation that $\varphi$ induces an injective linear map from $\mathrm{C}_G(g)/G'$ to $\mathrm{C}_H(g^\varphi)/H'$. ∎

## 2. Notation.

As our treatment involves some explicit computations, we have to fix descriptions for rather well known objects.

Complex numbers will be considered as $2\times 2$ matrices over $\mathbb{R}$; then complex conjugation is obtained by transposition:

$$\mathbb{C} := \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \middle|\ a,b \in \mathbb{R} \right\}, \quad \overline{\begin{pmatrix} a & b \\ -b & a \end{pmatrix}} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

In the Lie algebra $\mathfrak{gl}_{2n}\mathbb{R}$ of all $2n \times 2n$ matrices over $\mathbb{R}$, we have the subalgebra

$$\mathfrak{gl}_n\mathbb{C} := \left\{ A = (a_{j,k})_{j,k=1}^{2n} \in \mathfrak{gl}_{2n}\mathbb{R} \ \middle|\ \forall j,k \in \{1,\dots,n\} : \begin{array}{rcl} a_{2j,2k} &=& a_{2j-1,2k-1} \\ a_{2j-1,2k} &=& -a_{2j,2k-1} \end{array} \right\}$$

of all complex matrices, considered as matrices with special block structure.

The transpose of a real matrix $A$ is denoted by $A'$. For the sake of distinction, the transpose of a complex matrix $C = (c_{j,k})_{j,k=1}^n \in \mathfrak{gl}_n\mathbb{C}$ will be written $C^\intercal := (c_{k,j})_{j,k=1}^n$, and $\overline{C}$ is obtained by conjugation of each entry:

$$\overline{(c_{j,k})_{j,k=1}^n} = (\overline{c_{j,k}})_{j,k=1}^n.$$

Note that $\overline{C}^\intercal$ is just the transpose of the *real* matrix. The complex trace of $C$ is $\mathrm{tr}_\mathbb{C}(c_{j,k})_{j,k=1}^n = \sum_{j=1}^n c_{j,j}$.

Hamilton's quaternions form a subalgebra of $\mathfrak{gl}_2\mathbb{C}$, namely

$$\mathbb{H} := \left\{ \begin{pmatrix} x & y \\ -\overline{y} & \overline{x} \end{pmatrix} \middle| \; x, y \in \mathbb{C} \right\}.$$

We will use the standard involution $h \mapsto \tilde{h} := h' = \bar{h}^\intercal$ on $\mathbb{H}$. This involution fixes the real scalar multiples of the identity, while the complementary eigenspace is

$$\mathrm{Pu}\,(\mathbb{H}) := \left\{ h \in \mathbb{H} \,\middle|\, \tilde{h} = -h \right\} = \left\{ \begin{pmatrix} ri & b \\ -\bar{b} & -ri \end{pmatrix} \middle| \; r \in \mathbb{R}, b \in \mathbb{C} \right\}.$$

Again, elements of $\mathfrak{gl}_n\mathbb{H}$ will be interpreted as special elements of $\mathfrak{gl}_{2n}\mathbb{C}$ or of $\mathfrak{gl}_{4n}\mathbb{R}$, respectively.

The real $n \times n$ identity matrix will be denoted by $\mathbf{1}_n$; note that for $n = 2h$ even, this is the complex $h \times h$ identity matrix $\mathbf{1}_h^\mathbb{C}$, and the quaternion $q \times q$ identity matrix $\mathbf{1}_q^\mathbb{H}$ for $n = 4q$.

**Definition 2.1.** Let $\mathfrak{o}_{2n}\mathbb{R}$ be the Lie algebra of all skew symmetric $2n \times 2n$ matrices over $\mathbb{R}$. We consider the subalgebras

$$\begin{aligned} \mathfrak{u}_n\mathbb{C} := \mathfrak{o}_{2n}\mathbb{R} \cap \mathfrak{gl}_n\mathbb{C} = \quad & \left\{ (c_{j,k})_{j,k=1}^n \in \mathfrak{gl}_n\mathbb{C} \,\middle|\, \forall j,k : c_{j,k} = -\overline{c_{k,j}} \right\} \\ \text{and} \qquad \mathfrak{su}_n\mathbb{C} := \quad & \left\{ M \in \mathfrak{u}_n\mathbb{C} \,\middle|\, \mathrm{tr}_\mathbb{C} M = 0 \right\}. \end{aligned}$$

For $n = 4$, we will use a convenient description of the latter subalgebra by complex block matrices:

$$\mathfrak{su}_4\mathbb{C} = \left\{ \begin{pmatrix} A & B \\ -\overline{B}^\intercal & C \end{pmatrix} \middle| \; \begin{array}{c} A, C \in \mathfrak{u}_2\mathbb{C}, B \in \mathfrak{gl}_2\mathbb{C}, \\ \mathrm{tr}_\mathbb{C} A = -\mathrm{tr}_\mathbb{C} C \end{array} \right\}$$

In the sequel, we will consider representations of (Lie algebras of) compact connected Lie groups. By Weyl's trick, these representations are completely reducible, cf. [1] I.6.2, Thm 2, p. 52.

The adjoint representation of $\mathfrak{o}_8\mathbb{R}$ restricts to an $\mathbb{R}$-linear representation of $\mathfrak{su}_4\mathbb{C}$ on $\mathfrak{o}_8\mathbb{R}$. We will decompose $\mathfrak{o}_8\mathbb{R}$ as a direct sum of irreducible $\mathfrak{su}_4\mathbb{C}$-modules. To this end, we introduce some more notation. We put

$$S := \begin{pmatrix} \sigma & & & \\ & \sigma & & \\ & & \sigma & \\ & & & \sigma \end{pmatrix}, \quad \text{where } \sigma = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}.$$

Note that, for each $c \in \mathbb{C}$, we have $\sigma c = \overline{c}\sigma$. We will also use

$$i := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathbb{C} \quad \text{and} \quad J := \begin{pmatrix} 0 & \mathbf{1}_2 \\ -\mathbf{1}_2 & 0 \end{pmatrix} \in \mathbb{H}.$$

**Lemma 2.2.** *For $A \in \mathfrak{u}_4\mathbb{C}$ and*

$$MS \in U := (\mathfrak{o}_4\mathbb{C})S = \left\{ \left( \begin{array}{cc} tJ & X \\ -X^{\intercal} & uJ \end{array} \right) S \,\middle|\, t,u \in \mathbb{C}, X \in \mathfrak{gl}_2\mathbb{C} \right\},$$

*we have* $[A, MS] = (AM - (AM)^{\intercal})\, S$.

**Proof.** Using $\sigma c = \bar{c}\sigma$ and the defining properties for $A \in \mathfrak{u}_4\mathbb{C}$ and $M \in US^{-1}$, we compute $MSA = M^{\intercal}S\bar{A}^{\intercal} = M^{\intercal}A^{\intercal}S = (AM)^{\intercal}S$, and the assertion follows. ∎

Comparing dimensions and using Lemma 2.2, one sees that $U$ is a $\mathfrak{u}_4\mathbb{C}$-invariant complement of $\mathfrak{u}_4\mathbb{C}$ in $\mathfrak{o}_8\mathbb{R}$. However, the $\mathfrak{u}_4\mathbb{C}$-module $U$ is not irreducible. We consider the $\mathbb{R}$-subspaces

$$W_0 := \left\{ \left( \begin{array}{cc} tJ & X \\ -X^{\intercal} & -\bar{t}J \end{array} \right) S \,\middle|\, t \in \mathbb{C}, X \in \mathbb{H} \right\}$$

$$\text{and} \quad W := \left\{ \left( \begin{array}{cc} tJ & Y \\ -Y^{\intercal} & \bar{t}J \end{array} \right) S \,\middle|\, t \in \mathbb{C}, Y \in \mathbb{H}\left( \begin{array}{cc} \mathbf{1}_2 & 0 \\ 0 & -\mathbf{1}_2 \end{array} \right) \right\}.$$

**Proposition 2.3.** *The $\mathfrak{su}_4\mathbb{C}$-module $\mathfrak{o}_8\mathbb{R}$ splits as a direct sum of $\mathfrak{su}_4\mathbb{C}$-modules $\mathfrak{o}_8\mathbb{R} = \mathfrak{u}_4\mathbb{C} \oplus U$, and $U = W_0 \oplus W$. The $\mathfrak{su}_4\mathbb{C}$-modules $W_0$ and $W$ are irreducible, and $\mathfrak{u}_4\mathbb{C}$ splits as sum of the irreducible submodules $\mathfrak{su}_4\mathbb{C}$ and $\mathfrak{z}(\mathfrak{u}_4\mathbb{C})$.*

**Proof.** The $\mathfrak{su}_4\mathbb{C}$-module $\mathfrak{su}_4\mathbb{C}$ is irreducible because $\mathfrak{su}_4\mathbb{C}$ is a simple Lie algebra (submodules are just ideals here). Note that $W = [I, W_0]$ is the image of $W_0$ under the adjoint action of the element $I = i\mathbf{1}_4^{\mathbb{C}}$ of the center $\mathfrak{z}(\mathfrak{u}_4\mathbb{C})$ of $\mathfrak{u}_4\mathbb{C}$. Therefore, it suffices to check that $W_0$ is an irreducible $\mathfrak{su}_4\mathbb{C}$-submodule of $\mathfrak{o}_8\mathbb{R}$. A straightforward calculation shows $\{0\} \neq [\mathfrak{su}_4\mathbb{C}, W_0] \subseteq W_0$. Every nontrivial $\mathfrak{su}_4\mathbb{C}$-module has dimension at least $6 = \dim W_0$, cf. [4] p. 624. Therefore, the module $W_0$ is irreducible. ∎

**Remark 2.4.** The $\mathfrak{su}_4\mathbb{C}$-modules $W_0$ and $W$ provide explicit models for the representation that gives rise to the exceptional isomorphism $\mathfrak{su}_4\mathbb{C} \cong \mathfrak{o}_6\mathbb{R}$ (which is a restriction of "the" obvious isomorphism between simple complex Lie algebras of type $A_3$ and $D_3$). ∎

Our next aim is to obtain a decomposition of $\mathfrak{o}_8\mathbb{R}$ as a $\mathfrak{u}_2\mathbb{H}$-module. The Lie algebra $\mathfrak{u}_2\mathbb{H}$ is obtained as intersection $\mathfrak{u}_2\mathbb{H} := \mathfrak{o}_8\mathbb{R} \cap \mathfrak{gl}_2\mathbb{H}$, where

$$\mathfrak{gl}_2\mathbb{H} := \left\{ \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) \,\middle|\, A,B,C,D \in \mathbb{H} \right\}.$$

Note that $\mathfrak{u}_2\mathbb{H}$ is contained in $\mathfrak{su}_4\mathbb{C}$; in fact, one has $\mathfrak{u}_2\mathbb{H} = \mathfrak{su}_4\mathbb{C} \cap \mathfrak{gl}_2\mathbb{H}$.

We define the vector space

$$Z := \left\{ \left( \begin{array}{cc} ri\mathbf{1}_2^{\mathbb{C}} & Hi \\ \widetilde{H}^{\intercal}i & -ri\mathbf{1}_2^{\mathbb{C}} \end{array} \right) \,\middle|\, r \in \mathbb{R}, H \in \mathbb{H} \right\}.$$

**Proposition 2.5.** *The $\mathfrak{u}_2\mathbb{H}$-module $\mathfrak{o}_8\mathbb{R}$ splits as a direct sum of submodules $\mathfrak{o}_8\mathbb{R} = \mathfrak{u}_2\mathbb{H} \oplus Z \oplus \mathfrak{z}(\mathfrak{u}_4\mathbb{C}) \oplus U$, and $\mathfrak{su}_4\mathbb{C} = \mathfrak{u}_2\mathbb{H} \oplus Z$. The $\mathfrak{u}_2\mathbb{H}$-modules $\mathfrak{u}_2\mathbb{H}$ and $Z$ are both irreducible.*

**Proof.** Since $\mathfrak{u}_2\mathbb{H}$ is contained in $\mathfrak{su}_4\mathbb{C}$, we have $[\mathfrak{u}_2\mathbb{H}, \mathfrak{su}_4\mathbb{C}] \subseteq \mathfrak{su}_4\mathbb{C}$ and $[\mathfrak{u}_2\mathbb{H}, \mathfrak{z}(\mathfrak{u}_4\mathbb{C})] = \{0\}$ as well as $[\mathfrak{u}_2\mathbb{H}, U] \subseteq [\mathfrak{su}_4\mathbb{C}, U] = U$. A straightforward calculation shows $\mathfrak{su}_4\mathbb{C} = \mathfrak{u}_2\mathbb{H} \oplus Z$, and $\{0\} \neq [\mathfrak{u}_2\mathbb{H}, Z] \subseteq Z$. The module $\mathfrak{u}_2\mathbb{H}$ is irreducible because $\mathfrak{u}_2\mathbb{H}$ is a simple Lie algebra. Every nontrivial $\mathfrak{u}_2\mathbb{H}$-module has dimension at least $5 = \dim Z$, cf. [4] p. 624. Thus the assertion follows. ■

**Remark 2.6.** The $\mathfrak{u}_2\mathbb{H}$-module $Z$ provides a concrete model for the representation that gives rise to the exceptional isomorphism $\mathfrak{u}_2\mathbb{H} \cong \mathfrak{o}_5\mathbb{R}$ (which is a restriction of "the" obvious isomorphism between Lie algebras of type $C_2$ and $B_2$).

The $\mathfrak{u}_2\mathbb{H}$-module $U$ splits as the sum of two one-dimensional and two five-dimensional simple submodules. This follows from the observation that $\mathfrak{u}_2\mathbb{H}$ is embedded in $\mathfrak{su}_4\mathbb{C}$ like $\mathfrak{o}_5\mathbb{R}$ in $\mathfrak{o}_6\mathbb{R}$; the $\mathfrak{u}_2\mathbb{H}$-modules $W_0$ and $W$ split accordingly. We will not use that information, however: for our purposes, it suffices to identify the irreducible $\mathfrak{u}_2\mathbb{H}$-submodule $Z$ together with the complementary submodule $\mathfrak{u}_2\mathbb{H} \oplus \mathfrak{z}(\mathfrak{u}_4\mathbb{C}) \oplus U$. ■

**Definition 2.7.** The module decompositions obtained in 2.3 and in 2.5 are used to introduce the almost homogeneous Heisenberg algebras $\mathfrak{h}_5^8$ and $\mathfrak{h}_6^8$. In both cases, we describe a skew symmetric bilinear map $\beta_\ell$ from $\mathbb{R}^8 \times \mathbb{R}^8$ to some vector space $C_\ell$ by giving the corresponding linear map $\bar{\beta}_\ell : \mathbb{R}^8 \wedge \mathbb{R}^8 = \mathfrak{o}_8\mathbb{R} \to C_\ell$. Then the Lie algebra $\mathfrak{h}_\ell^8 := \mathfrak{gh}(\mathbb{R}^8, C_\ell, \beta_\ell)$ is obtained as $\mathbb{R}^8 \times C_\ell$, with commutator $[(x,c),(y,d)] := (0, (x,y)^{\beta_\ell})$.

For $\mathfrak{h}_5^8$, we put $C_5 := Z$ and let $\pi_5 : \mathfrak{o}_8\mathbb{R} \to Z$ be the projection modulo the $\mathfrak{u}_2\mathbb{H}$-submodule $\mathfrak{u}_2\mathbb{H} + \mathfrak{z}(\mathfrak{u}_4\mathbb{C}) + U$. Then $\pi_5$ and $\bar{\beta}_5 := 4\pi_5$ are homomorphisms of $\mathrm{U}_2\mathbb{H}$-modules.

For $\mathfrak{h}_6^8$, we put $C_6 := W$ and let $\pi_6 : \mathfrak{o}_8\mathbb{R} \to W$ be the projection modulo the $\mathfrak{su}_4\mathbb{C}$-submodule $\mathfrak{u}_4\mathbb{C} + W_0$. Then $\pi_6$ and $\bar{\beta}_6 := 4\pi_6$ are homomorphisms of $\mathrm{SU}_4\mathbb{H}$-modules.

Using the module decompositions obtained above, it is now easy (if tedious) to determine the values $(v_j, v_k)^{\beta_\ell} \in C_\ell$ for the standard basis $v_1, \ldots, v_8$ of $\mathbb{R}^8$: one has to express $v_j \wedge v_k = v_j \otimes v_k - v_k \otimes v_j = v_j' v_k - v_k' v_j$ as a sum of $(v_j, v_k)^{\pi_\ell} \in C_\ell$ and $R_{j,k}^\ell \in \ker \pi_\ell$.

In the sequel, we are going to use this in order to determine the subalgebras of $\mathfrak{h}_5^8$ and $\mathfrak{h}_6^8$ generated by vector subspaces of $\mathbb{R}^8$.

## 3. Explicit Computations.

We are now going to give explicit descriptions of the symplectic maps $\beta_5$ and $\beta_6$ used to define the Lie algebras $\mathfrak{h}_5^8$ and $\mathfrak{h}_6^8$. We describe bilinear maps from $\mathbb{R}^n \times \mathbb{R}^n$ to some vector space $M$ by matrices with vector entries, as follows:

For any $n \times n$ matrix $R = (r_{j,k})_{j,k=1}^n$ with entries from $M$ and vectors $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, we write $xRy' := \sum_{j,k=1}^n x_j y_k r_{j,k} \in M$.

**Lemma 3.1.** *We define the following elements of* $\mathfrak{gl}_4\mathbb{C}$*:*

$$z_1 := \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}, \quad z_2 := \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \quad z_3 := \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$
$$z_4 := \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \quad z_5 := \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

*Then* $\{z_1, \dots, z_5\}$ *is a basis of* $Z \cong \mathbb{R}^5$*. With respect to this basis, the symplectic map* $\beta_5$ *is described by* $(x,y)^{\beta_5} = xRy'$*, where*

$$R = \begin{pmatrix} 0 & z_1 & 0 & 0 & -z_3 & z_2 & -z_5 & z_4 \\ -z_1 & 0 & 0 & 0 & -z_2 & -z_3 & -z_4 & -z_5 \\ 0 & 0 & 0 & z_1 & -z_5 & -z_4 & z_3 & z_2 \\ 0 & 0 & -z_1 & 0 & z_4 & -z_5 & -z_2 & z_3 \\ z_3 & z_2 & z_5 & -z_4 & 0 & -z_1 & 0 & 0 \\ -z_2 & z_3 & z_4 & z_5 & z_1 & 0 & 0 & 0 \\ z_5 & z_4 & -z_3 & z_2 & 0 & 0 & 0 & -z_1 \\ -z_4 & z_5 & -z_2 & -z_3 & 0 & 0 & z_1 & 0 \end{pmatrix}.$$

**Proof.** Elements of $\ker \pi_5 = \mathfrak{u}_2\mathbb{H} \oplus \mathfrak{z}(\mathfrak{u}_4\mathbb{C}) \oplus U$ and $C_5 = Z$ have the form

$$\begin{pmatrix} ai & b & c & d \\ * & -ai & -\bar{d} & \bar{c} \\ * & * & ei & f \\ * & * & * & -ei \end{pmatrix} + \begin{pmatrix} gi & 0 & 0 & 0 \\ * & gi & 0 & 0 \\ * & * & gi & 0 \\ * & * & * & gi \end{pmatrix} + \begin{pmatrix} 0 & t\sigma & m\sigma & n\sigma \\ * & 0 & p\sigma & q\sigma \\ * & * & 0 & u\sigma \\ * & * & * & 0 \end{pmatrix} \text{ and}$$

$$\begin{pmatrix} ri & 0 & xi & yi \\ * & ri & -\bar{y}i & \bar{x}i \\ * & 0 & -ri & 0 \\ * & * & * & -ri \end{pmatrix}, \text{ respectively, with uniquely determined entries}$$

$$a, e, g, r \in \mathbb{R} \quad \text{and} \quad b, c, d, f, t, u, m, n, p, q, x, y \in \mathbb{C}.$$

The entries marked $*$ are determined by those given explicitly because we consider elements of $\mathfrak{o}_8\mathbb{R}$. Using the fact that $\mathbf{1}_1^{\mathbb{C}} = \mathbf{1}_2$ and $\sigma$ are linearly independent in the complex vector space $\mathbb{R}^{2\times 2}$, together with observations like $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) = \frac{1}{2}\left(\mathbf{1}_1^{\mathbb{C}} + \sigma\right)$ and $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) = \frac{1}{2}(i - i\sigma)$, one finds the image of $v_j \wedge v_k$ under $\bar{\beta}_5 = 4\pi_5$ that forms the $(j,k)$-entry in $R$. Computational details are left to the interested reader. ∎

**Lemma 3.2.** *We define the following elements of* $\mathfrak{gl}_4\mathbb{C}$*:*

$$w_1 := \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} S, \qquad w_2 := \begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} S,$$

$$w_3 := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} S, \qquad w_4 := \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} S,$$

$$w_5 := \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} S, \qquad w_6 := \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} S.$$

*Then* $\{w_1, \dots, w_5\}$ *is a basis of* $W \cong \mathbb{R}^6$*. With respect to this basis, the symplectic map* $\beta_6$ *is described by* $(x,y)^{\beta_6} = xTy'$*, where*

$$T = \begin{pmatrix} 0 & 0 & w_1 & -w_2 & w_3 & -w_4 & -w_5 & w_6 \\ 0 & 0 & -w_2 & -w_1 & -w_4 & -w_3 & w_6 & w_5 \\ -w_1 & w_2 & 0 & 0 & -w_5 & -w_6 & -w_3 & -w_4 \\ w_2 & w_1 & 0 & 0 & -w_6 & w_5 & -w_4 & w_3 \\ -w_3 & w_4 & w_5 & w_6 & 0 & 0 & w_1 & w_2 \\ w_4 & w_3 & w_6 & -w_5 & 0 & 0 & w_2 & -w_1 \\ w_5 & -w_6 & w_3 & w_4 & -w_1 & -w_2 & 0 & 0 \\ -w_6 & -w_5 & w_4 & -w_3 & -w_2 & w_1 & 0 & 0 \end{pmatrix}.$$

**Proof.** We proceed as in the proof of 3.1, using 2.1 and 2.3: Elements of $\ker \pi_6 = \mathfrak{u}_4\mathbb{C} \oplus W_0$ and $C_6 = W$ have the form

$$\begin{pmatrix} ai & b & c & d \\ * & ei & f & g \\ * & * & hi & j \\ * & * & * & ki \end{pmatrix} + \begin{pmatrix} 0 & t\sigma & x\sigma & y\sigma \\ * & 0 & -\bar{y}\sigma & \bar{x}\sigma \\ * & * & 0 & -\bar{t}\sigma \\ * & * & * & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & z\sigma & u\sigma & -v\sigma \\ * & 0 & -\bar{v}\sigma & -\bar{u}\sigma \\ * & * & 0 & \bar{z}\sigma \\ * & * & * & 0 \end{pmatrix},$$

respectively, where the entries $a, e, h, k \in \mathbb{R}$ and $b, c, d, f, g, j, t, x, y, z, u, v \in \mathbb{C}$ are uniquely determined. Again, the entries marked $*$ are determined by those given explicitly because we consider elements of $\mathfrak{o}_8\mathbb{R}$. ■

**Remark 3.3.** In order to solve Problem 1.3(c), we describe $\mathrm{H}^4_{\mathbb{H}}$ more explicitly, using the hermitian form $\langle x|y\rangle = x\tilde{y}$ and the corresponding symplectic map $\gamma^1_{\mathbb{H}} : \mathbb{H}^1 \times \mathbb{H}^1 \to \mathrm{Pu}(\mathbb{H}) : (x,y) \mapsto \mathrm{Pu}(\langle x|y\rangle) = \frac{1}{2}(x\tilde{y} - y\tilde{x})$. With respect to the basis

$$p_1 := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad p_2 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad p_3 := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

for $\mathrm{Pu}\,(\mathbb{H})$, and the basis $p_0 := \mathbf{1}_4$, $p_1$, $p_2$, $p_3$ for $\mathbb{H}^1$, the symplectic map $\gamma^1_{\mathbb{H}}$ is described by the matrix

$$H := \begin{pmatrix} 0 & -p_1 & -p_2 & -p_3 \\ p_1 & 0 & -p_3 & p_2 \\ p_2 & p_3 & 0 & -p_1 \\ p_3 & -p_2 & p_1 & 0 \end{pmatrix} \quad \text{via} \quad \left(\sum_{j=0}^{3} x_j p_j, \sum_{j=0}^{3} y_j p_j\right) \mapsto xHy'\,. \qquad \blacksquare$$

**Remark 3.4.** After the identification $(a+ib, c+id) \leftrightarrow (a,b,c,d)$, a Heisenberg group of type $\mathrm{H}^4_{\mathbb{C}}$ is obtained as $\mathrm{GH}(\mathbb{R}^4, \mathbb{C}, \beta^2_{\mathbb{C}})$, where $\beta^4_{\mathbb{C}} : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{C}$ is given by

$$(x,y) \mapsto x \begin{pmatrix} 0 & 0 & 1 & i \\ 0 & 0 & i & -1 \\ -1 & -i & 0 & 0 \\ -i & 1 & 0 & 0 \end{pmatrix} y'\,. \qquad \blacksquare$$

## 4. Embeddings into $\mathrm{H}^8_5$.

Throughout this section, let $v_1, \dots, v_8$ be the standard basis for $\mathbb{R}^8$.

We want to determine the isomorphism types of subgroups $H \le \mathrm{H}^8_\ell$ with $\dim H/H' = d$. To this end, we have to consider vector subspaces $V$ of dimension $d$ in $\mathbb{R}^8$, and determine the image of $V \times V$ under $\beta_\ell$. Using subgroups of $\mathrm{Aut}(\mathrm{H}^8_\ell)$, we can reduce this problem considerably.

The following reduction helped to find candidates for embeddings.

**Lemma 4.1.** *If searching for the isomorphism types of subgroups $H < \mathrm{H}^8_5$ with $\dim H/H' = 4$, it suffices to consider the subspaces of $\mathbb{R}^8$ generated by independent sets of the following two types:*

$$\mathcal{B}_1 : \quad \{v_1, v_2, v_3, v_4\}, \qquad \text{or} \qquad \mathcal{B}_2 : \quad \{v_1, d_2, d_3, d_4\},$$

*where $d_2, d_3 \in \langle v_2, v_3, v_4, v_6, v_7, v_8\rangle_{\mathbb{R}}$ and $d_4 \in v_5 + \langle v_2, v_3, v_4\rangle_{\mathbb{R}}$.*

**Proof.** We consider a 4-dimensional subspace $V$ of $\mathbb{R}^8$. By the very construction of $\mathrm{H}^8_5$ (via $\mathfrak{u}_2\mathbb{H}$-submodules of $\mathfrak{o}_8\mathbb{R}$), the group $\mathrm{U}_2\mathbb{H} = \exp(\mathfrak{u}_2\mathbb{H})$ is a subgroup of $\mathrm{Aut}(\mathrm{H}^8_5)$. Since this group acts transitively on $\mathbb{R}^8 \setminus \{0\}$, we may assume that $v_1$ is contained in $V$. If $V = \langle v_1\rangle_{\mathbb{H}} = \langle v_1, v_2, v_3, v_4\rangle_{\mathbb{R}}$, there is nothing left to do. Therefore, we may assume that $V$ contains an element $x \in \langle v_2, v_3, v_4, v_5, v_6, v_7, v_8\rangle_{\mathbb{R}} \setminus \langle v_2, v_3, v_4\rangle_{\mathbb{R}}$.

The stabilizer of $v_1$ in the group $\mathrm{U}_2\mathbb{H}$ is

$$(\mathrm{U}_2\mathbb{H})_{v_1} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} \,\middle|\, d \in \mathbb{H}, d\tilde{d} = 1 \right\}.$$

Using an element $\lambda$ of this stabilizer, we may map $x$ to some real scalar multiple of an element $d_4 \in v_5 + \langle v_2, v_3, v_4\rangle_{\mathbb{R}}$. Extending $\{v_1, d_4\}$ to a suitable basis for the image of $V$ under $\lambda$, we establish the claim. $\blacksquare$

The image of $\langle v_1\rangle_{\mathbb{H}}{}^2 = \langle v_1, v_2, v_3, v_4\rangle_{\mathbb{R}}{}^2$ under $\beta_5$ obviously is $\langle z_1\rangle_{\mathbb{R}}$; cf. 3.1. Thus $\langle v_1\rangle_{\mathbb{H}} \times \{0\}$ does not generate a subgroup isomorphic to $\mathrm{H}^4_{\mathbb{C}}$ or $\mathrm{H}^4_{\mathbb{H}}$. Closer inspection reveals that the subgroup generated by $\langle v_1\rangle_{\mathbb{H}} \times \{0\}$ is isomorphic to $\mathrm{H}^4_{\mathbb{R}}$. This is the embedding of $\mathrm{H}^4_{\mathbb{R}}$ into $\mathrm{H}^8_5$ found in [6] 3.9.

**Theorem 4.2.** *There is an embedding of* $\mathrm{H}^4_{\mathbb{C}}$ *into* $\mathrm{H}^8_5$.

*In fact, for* $V := \langle v_1, v_3, v_5, v_7\rangle_{\mathbb{R}} \le \mathbb{R}^8$ *the subset* $V \times \{0\}$ *generates a subgroup of* $\mathrm{H}^8_5$ *that is isomorphic to* $\mathrm{H}^4_{\mathbb{C}}$.

**Proof.** Using 3.1 one easily computes that the image of $V \times V$ generates the subspace $U := \langle z_3, z_5\rangle_{\mathbb{R}}$ in $W$. In order to see that $\mathrm{H}^4_{\mathbb{C}}$ is isomorphic to $\mathrm{GH}(V, U, \beta_5|_{V\times V})$, we define maps $\varphi_1 : \mathbb{H} \to V$ and $\varphi_2 : \mathrm{Pu}(\mathbb{H}) \to U$ by linear extension of $(1,0)^{\varphi_1} := v_1$, $(i,0)^{\varphi_1} := -v_3$, $(0,1)^{\varphi_1} := v_7$, $(0,i)^{\varphi_1} := v_5$, $1^{\varphi_2} := -z_5$, and $i^{\varphi_2} := -z_3$. A simple calculation shows that $\varphi_1 \times \varphi_2$ is an isomorphism from $\mathrm{H}^4_{\mathbb{C}}$ onto $\mathrm{GH}(V, U, \beta_5|_{V\times V})$. (See 3.1 and 3.4 for the structure constants.) ■

**Theorem 4.3.** *There is an embedding of* $\mathrm{H}^4_{\mathbb{H}}$ *into* $\mathrm{H}^8_5$.

*In fact, for* $V := \langle v_1, v_2, v_5, v_6\rangle_{\mathbb{R}} \le \mathbb{R}^8$ *the subset* $V \times \{0\}$ *of* $\mathbb{R}^8 \times W$ *generates a subgroup of* $\mathrm{H}^8_5$ *that is isomorphic to* $\mathrm{H}^4_{\mathbb{H}}$.

**Proof.** The image of $V \times V$ under $\beta_5$ generates the 3-dimensional subspace $U := \langle z_1, z_2, z_3\rangle_{\mathbb{R}}$. An isomorphism $(\varphi_1, \varphi_2)$ from $\mathrm{H}^4_{\mathbb{H}}$ onto $\mathrm{GH}(V, U, \beta_5|_{V\times V})$ is obtained by linear extension of $p_0^{\varphi_1} := v_1$, $p_1^{\varphi_1} := v_2$, $p_2^{\varphi_1} := v_6$, $p_3^{\varphi_1} := v_5$, and $p_1^{\varphi_2} := -z_1$, $p_2^{\varphi_2} := -z_2$, $p_3^{\varphi_2} := z_3$, cf. 3.1 and 3.3. ■

We conclude this section with a negative result.

**Theorem 4.4.** *There is no continuous injective homomorphism from the group* $\mathrm{H}^6_{\mathbb{R}}$ *into the group* $\mathrm{H}^8_5$.

**Proof.** First, we determine the centralizer $\mathrm{C}_{\mathrm{H}^8_5}((v_1, 0))$. From 3.1 we read off that the image of the set $\{(v_1, v_j) \mid j \in \{2,5,6,7,8\}\}$ spans $Z$, and that $\{v_1, v_3, v_4\}$ is contained in the centralizer. Therefore, one has $\dim \mathrm{C}_{\mathrm{H}^8_5}(v_1) = 8$ and $\mathrm{C}_{\mathrm{H}^8_5}(v_1) = \langle v_1, v_3, v_4\rangle_{\mathbb{R}} + Z$. Since $\mathrm{H}^8_5$ is almost homogeneous, this implies $\dim \mathrm{C}_{\mathrm{H}^8_5}(g) - \dim(\mathrm{H}^8_5)' = 3$ for each $g \in \mathrm{H}^8_5 \setminus (\mathrm{H}^8_5)'$.

According to 1.4, the existence of an injective continuous homomorphism from $\mathrm{H}^6_{\mathbb{R}}$ into $\mathrm{H}^8_5$ would require $\dim \mathrm{C}_{\mathrm{H}^6_{\mathbb{R}}}(x) - \dim(\mathrm{H}^6_{\mathbb{R}})' \le 3$ for each $x \in \mathrm{H}^6_{\mathbb{R}} \setminus (\mathrm{H}^6_{\mathbb{R}})'$. However, it is easy to see that every element in $\mathrm{H}^6_{\mathbb{R}}$ has a centralizer of dimension at least 6. Thus $\dim \mathrm{C}_{\mathrm{H}^6_{\mathbb{R}}}(x) - \dim(\mathrm{H}^6_{\mathbb{R}})' \ge 5$, and no injection is possible. ■

## 5. Embeddings into $\mathrm{H}^8_6$.

Again, let $v_1, \ldots, v_8$ be the standard basis for $\mathbb{R}^8$. We start with positive results.

**Theorem 5.1.** *There is an embedding of* $\mathrm{H}^4_{\mathbb{H}}$ *into* $\mathrm{H}^8_6$.

*In fact, for* $V := \langle v_1, v_3, v_5, v_7\rangle_{\mathbb{R}} \le \mathbb{R}^8$ *the subset* $V \times \{0\}$ *of* $\mathbb{R}^8 \times W$ *generates a subgroup of* $\mathrm{H}^8_6$ *that is isomorphic to* $\mathrm{H}^4_{\mathbb{H}}$.

**Proof.** Using 3.2 we compute that the image of $V \times V$ under $\beta_6$ generates the 3-dimensional subspace $U := \langle w_1, w_3, w_5\rangle_{\mathbb{R}}$ of $W$. An isomorphism from $\mathrm{H}^4_{\mathbb{H}}$ onto $\mathrm{GH}(V, U, \beta_6|_{V\times V})$ is given by the maps $\varphi_1 : \mathbb{R}^4 \to V$ and $\varphi_2 : \mathbb{R}^3 \to U$ obtained by linear extension of $p_0^{\varphi_1} = v_1$, $p_1^{\varphi_1} = v_3$, $p_2^{\varphi_1} = v_5$, $p_3^{\varphi_1} = v_7$, $p_1^{\varphi_2} = -w_1$, $p_2^{\varphi_2} = -w_3$, and $p_3^{\varphi_2} = w_5$. ∎

Embeddings of $\mathrm{H}^4_{\mathbb{C}}$ into $\mathrm{H}^6_6$ and of $\mathrm{H}^6_6$ into $\mathrm{H}^8_6$ were given in [6] 3.8, 3.11. We take the opportunity to exhibit an explicit embedding of $\mathrm{H}^4_{\mathbb{C}}$ into $\mathrm{H}^8_6$.

**Theorem 5.2.** *Let $V := \langle v_1, v_2, v_3, v_4\rangle_{\mathbb{R}} \le \mathbb{R}^8$. Then $V \times \{0\}$ generates a subgroup of $\mathrm{H}^8_6$ that is isomorphic to $\mathrm{H}^4_{\mathbb{C}}$.*

**Proof.** The image of $V \times V$ under $\beta_6$ generates the subspace $U := \langle w_1, w_2\rangle_{\mathbb{R}}$ in $W$. An isomorphism $(\varphi_1, \varphi_2)$ from $\mathrm{H}^4_{\mathbb{C}}$ onto $\mathrm{GH}(V, U, \beta_6|_{V\times V})$ is obtained by linear extension of $(1,0)^{\varphi_1} := v_1$, $(i,0)^{\varphi_1} := v_2$, $(0,1)^{\varphi_1} := v_3$, $(0,i)^{\varphi_1} := v_4$, $1^{\varphi_2} := w_1$, and $i^{\varphi_2} := -w_2$. (See 3.2 and 3.4 for the structure constants.) ∎

We conclude this section by another negative result.

**Theorem 5.3.** *There is no continuous injective homomorphism from the group $\mathrm{H}^4_{\mathbb{R}}$ into the group $\mathrm{H}^8_6$.*

**Proof.** The 6-dimensional space $W$ is generated by the images

$$\begin{array}{rclrclrcl} (v_1,v_3)^{\beta_6} &=& w_1, & (v_1,v_4)^{\beta_6} &=& -w_2, & (v_1,v_5)^{\beta_6} &=& w_3, \\ (v_1,v_6)^{\beta_6} &=& -w_4, & (v_1,v_7)^{\beta_6} &=& -w_5, & (v_1,v_8)^{\beta_6} &=& w_6, \end{array}$$

cf. 3.2. Thus $\dim \mathrm{C}_{\mathrm{H}^8_6}((v_1,0)) - \dim(\mathrm{H}^8_6)' = 2$, and the same value is obtained for any $g \in \mathrm{H}^8_6 \setminus (\mathrm{H}^8_6)'$ because the group $\mathrm{H}^8_6$ is almost homogeneous.

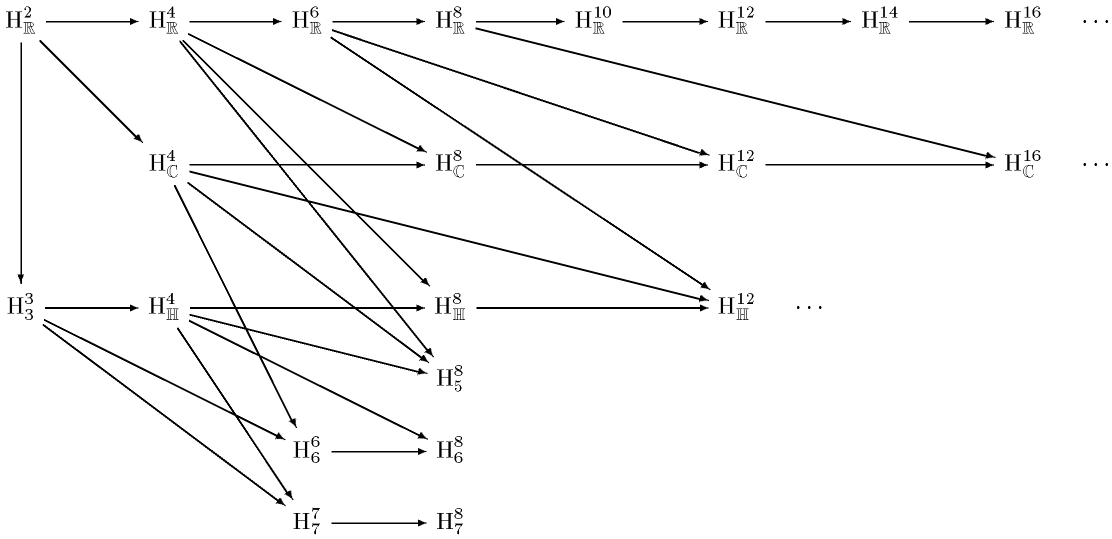
For each element $x \in \mathrm{H}^4_{\mathbb{R}}$, one easily sees that $\dim \mathrm{C}_{\mathrm{H}^4_{\mathbb{R}}}(x) - \dim(\mathrm{H}^4_{\mathbb{R}})'$ is at least 3. Thus a continuous injection is impossible by 1.4. ∎

## 6. Conclusion.

The following diagram is a modification of the diagram in [6], taking into account the information obtained in the present paper. The diagram attempts to visualize all embeddings between almost homogeneous Heisenberg groups of dimension at most 18. In particular, this range comprises all exceptional ones.

For the sake of readability, triangles have been suppressed in the diagram: an embedding may be designated by a *path* of arrows.

Absence of (paths of) arrows indicates that no embedding exists.




There remains the following question, stated as Problem (e) in [6]:

**Problem 6.1.** For which pairs $(k, n)$ of positive integers is there an embedding of $\mathrm{H}^{4k}_{\mathbb{C}}$ into $\mathrm{H}^{4n}_{\mathbb{H}}$ ?

(Partial answers have been given in [6] 3.4, 3.5, 3.6.)

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M. Stroppel and J. Hoheisel
Institut für Geometrie und Topologie
Universität Stuttgart
70550 Stuttgart
Germany
stroppel@mathematik.uni-stuttgart.de



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