Journal of Lie Theory
Volume **13** (2003) 359–382

Groups of Strings

Wojciech Wojtyński

Communicated by K.-H. Neeb

Abstract. Certain non-commutative analogues of topological vector spaces and their relation to Lie groups and Lie algebras are investigated.
2000 Mathematics Subject Classification: Primary 22E65; Secondary 17B65

1. Introduction

One of the fundamental facts in the theory of finite dimensional Lie groups is the theorem stating that the group multiplication in such groups is described (locally, in exponential coordinates) by the Baker-Campbell-Hausdorff series (B-C-H series for short).

Since the B-C-H series depends on linear combinations of iterated Lie brackets of two Lie algebra variables, the theorem provides an explicit realization of the main principle of Lie group theory - the one-to-one correspondence of structural elements in a Lie group and in the associated Lie algebra.

This way of relating Lie groups with Lie algebras seems to be especially suitable in infinite dimensions. However the attempts in this direction have not been successful so far due to the following serious obstacles:

First, there is the problem of convergence of the B-C-H series. For Banach-Lie algebras (and in particular for finite dimensional ones) the problem was solved by E. V. Dynkin ([3]) (cf. Section 2 for exact formulation). He proved that in this case the B-C-H series converges absolutely for variables in some universal neighbourhood of zero in the product. Consequently, the B-C-H formula defines a local Banach-Lie group attached to a Banach-Lie algebra. The method depends on the Banach structure and it cannot be applied for topological Lie algebras (in fact it may then happen that in each neighbourhood of zero there are pairs of elements for which the series diverges).

Another problem is that the exponential map may not be locally surjective for 'Lie groups' of infinite dimension, e.g. for the group of all C^∞ diffeomorphisms of a compact manifold (cf. [7], [8]). This also excludes application of

ISSN 0949–5932 / \$2.50

classical scheme.

The aim of the present paper is to indicate a new method of overcoming these obstacles. The procedure relies on the concept of 'groups of strings', the main subject of this note.

Before we pass to the details we shall briefly sketch the idea.

To circumvent the first of the above-mentioned obstacles let us observe that the convergence problem of the B-C-H series disappears if a Lie algebra K has a suitable graded structure. Namely suppose that $K = \prod_{j=1}^{\infty} M_j$ is the product of linear spaces M_j, $j = 1, 2, \ldots$, and that the Lie bracket in K satisfies the conditions:

$$[M_j, M_k] \subset M_{j+k} \tag{1.1}$$

(algebras of this form will be called **N**-gradation Lie algebras* below).

The B-C-H series with variables in an **N**-gradation Lie algebra has the property that each coordinate of the terms of this series is zero for large indices and so the series converges in the product topology for each choice of topologies on the coordinates.

This is due to the graded structure of the series. Namely the n-th coefficient of the B-C-H series is a linear combination of $(n-1)$-fold Lie brackets and thus (1.1) implies that the coordinates of this coefficient with indices less than n vanish. Moreover the sum of the series depends continuously on the arguments in each product topology. It is well known (cf [2] Chapter II sec.6.5 Proposition 4) that then the globally convergent B-C-H series defines on K a global (topological) group structure. The resulting group will be denoted by $\exp(K)$.

In view of this we propose the following procedure enabling us to relate the group multiplication in a topological group G from an appropriate class S with the multiplication in $\exp(K)$ for a suitably chosen **N**-gradation Lie algebra K.

(I) The class S is composed of topological groups having 'rich' families of continuous one-parameter subgroups. (We shall make this precise later on.)

(II) With a topological group G of class S we canonically associate $S(G)$, a topological group with an action of the real numbers satisfying some natural conditions (in what follows groups of this type are called 'groups of strings'), in such a way that

$$G = S(G)/\Gamma \tag{1.2}$$

where Γ is a suitable closed normal subgroup of $S(G)$. Let us observe that for G connected the existence of exponential coordinates implies

* *Saying '**N**-gradation' instead of the more expected '**N**-graded' we want to stress that the Lie algebra in question has an **N**-gradation but it is a Lie algebra and not a graded Lie algebra, and moreover the gradation is a special one - e.g. it is complete with respect to the convergence of B-C-H series.*

(1.2), and that (1.2) may be thought of as a weak (but global) analogue of such coordinates.

(III) With an arbitrary topological group of strings P one may canonically associate an $\mathbf{N}$-gradation Lie algebra $L(P)$.

(IV) If P satisfies the condition that the intersection of the closed descending central series of P is trivial then P may be homomorphically injected in $\exp(L(P))$.

The above condition called the 'analyticity condition' is an algebraic counterpart of analyticity in the sense of analytic function theory observed (or absent) for differential geometry-based models of 'infinite dimensional Lie groups'.

As a result of I,II,III and IV, for a group G of class S, its group multiplication may be (globally but only in the analytic case) described as the quotient of the B-C-H multiplication restricted to a subgroup of $\exp\big(L(S(G))\big)$. In particular this applies to Banach-Lie groups. Moreover the $\mathbf{N}$-gradation Lie algebra $L(S(G))$ then admits a simple description in terms of the Lie algebra g of G. This provides a new interpretation of the fundamental theorem mentioned at the beginning of the paper.

It remains to specify what the class S is.

In our opinion the condition of having a manifold structure is too restrictive when trying to build a general Lie group theory. Instead we propose to consider topological groups which have 'rich' families of continuous one-parameter subgroups. For the purpose of this paper appropriate is the class S composed of those connected topological groups for which the set of all elements which are in the image of some one-parameter subgroup generates the whole group algebraically. (For a discussion see Section 3 and Section 7.)

The idea of giving priority to one-parameter subgroups of topological groups (rather then to a differentiable structure) has already appeared in some papers e.g. [1], [9], [11].

The paper is organized as follows: In Section 2 we fix notation, briefly discuss the B-C-H series and introduce the main objects: groups of strings and $\mathbf{N}$-gradation Lie algebras.

In Section 3 we discuss four types of examples of groups of strings. Especially important are two of them: the groups of strings attached to topological groups and the groups of strings attached to Lie algebras.

Section 4 deals with topological groups of strings. The notion of analyticity for such groups is introduced.

In Section 5 we introduce the concept of the $\mathbf{N}$-gradation Lie algebra $L(P)$ attached to a topological group of strings P (Theorem 5.3) and we describe the integration procedure for such Lie algebras (Proposition 5.7).

The resulting Lie functor is studied in Section 6. It culminates in Theorem 6.9 which provides a representation of an arbitrary topological group of strings P as an algebraic extension :

$$0 \longrightarrow \bar{P}_\infty \stackrel{j}{\longrightarrow} P \stackrel{\pi}{\longrightarrow} S_\Delta \longrightarrow 0, \tag{1.3}$$

where S_Δ is the maximal subgroup of strings in $\exp(L(P))$ for the attached $\mathbf{N}$-gradation Lie algebra $L(P)$, and $\bar{P}_\infty$ is the intersection of the closures of the central descending series subgroups of P.

In Section 7 we indicate further applications of groups of strings.

2. Notation and basic definitions

Throughout the paper we shall denote the set of real numbers by $\mathbf{R}$, the set of integers by $\mathbf{Z}$ and the set of positive integers by $\mathbf{N}$.

For elements $f_1, f_2, \ldots$ of a group we shall use the abbreviations $\{f_1, f_2\}$ for $f_1 \cdot f_2 \cdot f_1^{-1} \cdot f_2^{-1}$ and $\{f_1, \ldots, f_k\}$ for $\{f_1, \{f_2, \ldots \{f_{k-1}, f_k\} \ldots\}\}$.

Similarly, for a Lie algebra with Lie bracket $[\cdot, \cdot]$ we denote by $[a_1, \ldots, a_k]$ the iterated bracket $[a_1, [a_2, \ldots, [a_{k-1}, a_k] \ldots]$.

Definition 2.1. An $\mathbf{R}$- group is a group P which is additionally equipped with multiplication by real numbers

$$\mathbf{R} \times P \ni (s, f) \mapsto s * f \in P$$

which is related to the group multiplication in P by the following conditions valid for $s, t \in \mathbf{R}$ and $f, f_1, f_2 \in P$:

$$\begin{array}{lll} & (a) & s * (t * f) = (st) * f \\ (2.1) & (b) & 1 * f = f, \quad 0 * f = e \\ & (c) & s * (f_1 f_2) = (s * f_1)(s * f_2) \end{array}$$

where e is the unit of P.

The $\mathbf{R}$-groups form a category with $\mathbf{R}$-homomorphisms, $\mathbf{R}$-subgroups, etc. having the obvious meaning.

The additional condition, which is valid in linear spaces:

$$(s + t) * \phi = (s * \phi)(t * \phi) \quad \text{for all } s, t \in \mathbf{R}, \tag{2.2}$$

does not hold true for all $\phi \in P$ in a general $\mathbf{R}$-group P. The elements $\phi \in P$ for which (2.2) holds are called *exponential* and the set of all such elements in P will be denoted by $E(P)$.

Definition 2.2. A group of strings is an $\mathbf{R}$- group P in which $E(P)$ generates P.

The following proposition, which is partially well known, will be used frequently in what follows.

Proposition 2.3. *Let P be a group of strings. The following conditions are equivalent:*

(a) *P is a linear space.*
(b) *$E(P) = P$.*
(c) *P is commutative.*
(d) *$(-1) * f = f^{-1}$ for all $f \in P$.*

For the proof see [14] Proposition 2.3.

The Baker-Campbell-Hausdorff series is a real power series in non-commutative formal variables x, y which is obtained as the composition $\Theta = W \circ Z$ where

$$W(z) = \log(1+z) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{z^n}{n}$$

and

$$Z(x,y) = e^x e^y - 1 = \sum_{j+k\geq 1} \frac{x^j}{j!} \frac{y^k}{k!}.$$

Gathering together terms of a given order we obtain

$$\Theta(x,y) = \sum_{m=1}^{\infty} \Theta_m(x,y) \tag{2.3}$$

where $\Theta_m(x,y)$ is the (finite) sum of all homogeneous terms of order m.

One of the cornerstones of Lie theory is the observation (cf. [2],[13]) that each Θ_m is a *Lie polynomial*, i.e. it may be expressed as a finite linear combination of $(m-1)$-fold commutators of x and y. In particular $\Theta_1(x,y) = x + y$ and $\Theta_2(x,y) = \frac{1}{2}(xy - yx)$.

If K is a Lie algebra then substituting for the formal variables x and y in (2.3) arbitrary elements of K and replacing the commutators by Lie brackets in K we obtain the evaluated series with the terms $\Theta_m(x,y)$ in K.

It is known ([2], [3]) that for an arbitrary Banach-Lie algebra K which is normed in such a way that $\| [a,b] \| \leq \| a \| \cdot \| b \|$ we have $\sum_{m=1}^{\infty} \| \Theta_m(x,y) \| \leq \infty$ for $(x,y) \in Q$ where $Q = \{(x,y) \in K \times K : \| x \| + \| y \| \leq \ln 2\}$. In particular this implies that the function $Q \ni (x,y) \to \Theta(x,y) \in K$ is jointly continuous and defines a local Lie group structure on Q.

Let $K = \prod_{j=1}^{\infty} M_j$ be an $\mathbf{N}$-gradation Lie algebra. Then K is said to be a *topological* $\mathbf{N}$*-gradation Lie algebra* if K is a topological Lie algebra, each M_j is a topological vector space and the topology of K is the product topology. As observed in Introduction, for each pair of elements of a topological $\mathbf{N}$-gradation Lie algebra the B-C-H series converges, providing K with the structure of a topological group $\exp(K)$.

Proposition 2.4. (cf. [12] Proposition 7 and Proposition 8 (a)).

(a) *The group structure in* $\exp(K)$ *may be completed to an* $\mathbf{R}$*-group structure by the* $\mathbf{R}$*-multiplication defined for* $s \in \mathbf{R}$ *and* $f = (a_1, a_2, \ldots) \in \exp(K)$ *by*

$$(s * f)(t) = (s^1 a_1, s^2 a_2, \ldots, s^n a_n, \ldots). \tag{2.4}$$

(b) *The set* $E(\exp(K))$ *consists of all elements for which* $a_i = 0$ *for* $i \geq 2$.

Proof. The proof of (a) is straightforward and we omit it.

(b): By the form of the B-C-H series for each $f \in \exp(K)$ the function

$$\mathbf{R} \ni t \to t \cdot f \in \exp(K) \tag{2.5}$$

(where '$t{\cdot}f$' denotes the product of t and f in the linear space K) is a continuous one parameter subgroup of $\exp(K)$. Moreover each continuous one-parameter subgroup is of this form.

In fact, given a continuous one-parameter subgroup $\mathbf{R} \ni t \to \phi(t) \in \exp(\mathrm{K})$, let $x = \phi(1)$. Then for $\psi(t) = t \cdot x$ we get $\phi(1) = \psi(1)$ and from the uniqueness of square root operation in $\exp(K)$ one gets that $\phi(t) = \psi(t)$ for all t having finite binary expansions. By continuity of ϕ and ψ this extends to arbitrary t.

Now note that by (2.2) each 'multiplicative line' $\mathbf{R} \ni s \to s * f \in P$ passing through an exponential element f is simultaneously a one-parameter subgroup. By the above characterization of one-parameter subgroups in $\exp(K)$ this amounts to the condition

$$s * f = (s^1 a_1, s^2 a_2, \ldots, s^n a_n, \ldots) = s \cdot f = (s^1 a_1, s^1 a_2, \ldots, s^1 a_n, \ldots)$$

valid for each $s \in \mathbf{R}$. This implies (b). ■

Let K be an $\mathbf{N}$-gradation Lie algebra. Then K contains a subalgebra K_0 consisting of all elements in K which have a finite number of non-zero coordinates only. Clearly K_0 is dense in K in the product of discrete topologies on each M_j. K_0 will be referred to as the *direct sum subalgebra.*

Let K, L be (topological) $\mathbf{N}$-gradation Lie algebras. A Lie algebra homomorphism $\psi : K \to L$ is said to be an *$\mathbf{N}$-gradation homomorphism*, (*a continuous $\mathbf{N}$-gradation homomorphism*) if ψ acts (continuously) coordinatewise. It is clear that each $\mathbf{N}$-gradation homomorphism restricts to a homomorphism $\psi_0 : K_0 \to L_0$ of the corresponding direct sum subalgebras, and conversely each coordinatewise acting (continuous) homomorphism $\psi_0 : K_0 \to L_0$ extends to a continuous $\mathbf{N}$-gradation homomorphism $\psi : K \to L$. Also observe that ψ being a continuous $\mathbf{N}$-gradation homomorphism is equivalent to the following condition:

$$\psi([a,b]) = [\psi(a), \psi(b)] \text{ for each } a \in M_j, b \in M_k. \tag{2.6}$$

3. Examples

In this section we indicate a few ways in which groups of strings may be introduced. We take it for granted that the arising structures indeed satisfy the postulates (2.1) (a)-(c), and (2.2), leaving the standard verifications to the reader.

A. Groups of strings over a topological group

Examples of this type are linked with families Ω of continuous one-parameter subgroups of a topological group G.

First we specify the groups $S(G)$ appearing in (1.2), simultaneously explaining the formula. Let G be a connected topological group and $\Lambda(G)$ be the family of all continuous one-parameter subgroups of G. Let $\exp : \Lambda(G) \ni \phi \to \phi(1) \in G$ be the evaluation map. In order to describe G in terms of $\Lambda(G)$, it is reasonable to consider groups for which

$$\exp(\Lambda(G)) \text{ generates } G. \tag{3.1}$$

(Note that the set $\exp(\Lambda(G))$ consists of all elements of G which are in the image of some one-parameter subgroup.)

To put (3.1) in a more suitable algebraic form, consider the family $C(\mathbf{R}, G)$ of all continuous G-valued functions on the real line $\mathbf{R}$. Then $C(\mathbf{R}, G)$ with pointwise multiplication is a group and $\Lambda(G) \subset C(\mathbf{R}, G)$.

Definition 3.1. Let $S(G)$ be the subgroup of $C(\mathbf{R}, G)$ generated by $\Lambda(G)$. The elements of $S(G)$ will be called *strings* and $S(G)$ will be called *the group of strings over* G.

We claim that $S(G)$ with $\mathbf{R}$-multiplication defined by $(s * f)(t) = f(st)$ for $s, t \in \mathbf{R}$ and for $f \in S(G)$ is an $\mathbf{R}$-group. Moreover the set of exponential elements of $S(G)$ coincides with $\Lambda(G)$. Thus $S(G)$ is a group of strings in the sense of Definition 2.2. In particular, if G is abelian, then Proposition 2.3 implies that $S(G)$ is a linear space and $S(G) = \Lambda(G)$.

Observe that exp is a homomorphism of $S(G)$ into G and note that (3.1) is equivalent to the statement:

$$\exp : S(G) \to G \text{ is surjective.} \tag{3.2}$$

This is equivalent to the condition (1.2). Since this condition is purely algebraic, if we wish to stay within the class of topological groups, we may consider $S(G)$ with a suitable $\mathbf{R}$-group topology for which exp is continuous. An example is the compact-open topology. Other suitable topologies may be introduced depending on particular properties of G. We leave a systematic discussion of this subject to a subsequent paper.

The construction of $S(G)$ may be generalized as follows: For $\Omega \subset \Lambda(G)$ closed with respect to the *-multiplication we consider the subgroup $S_\Omega(G)$ of $C(\mathbf{R}, G)$ generated by Ω. Clearly $S_\Omega(G)$ is a group of strings and a subgroup of $S(G)$.

B. The group of strings over a Lie algebra.

For a real Lie algebra g let $g^{\mathbf{N}}$ be the family of all formal power series $f(s) = \sum_{n=1}^{\infty} a_n s^n$ with coefficients in g. Let us equip $g^{\mathbf{N}}$ with the Cauchy-Lie bracket, i.e. for $f(s) = \sum_{n=1}^{\infty} a_n s^n$ and $g(s) = \sum_{n=1}^{\infty} b_n s^n$,

$$[f, g](s) = \sum_{n=1}^{\infty} c_n s^n \quad \text{where} \quad c_n = \sum_{k+j=n} [a_j, b_k]$$

Then $g^{\mathbf{N}}$ is an example of an $\mathbf{N}$-gradation Lie algebra, i.e. $g^{\mathbf{N}} = \prod_{j=1}^{\infty} M_j$ where M_j for $j = 1, 2, 3, ..$ is the linear space of all series with only the j-th coefficient nonvanishing. Clearly the Cauchy-Lie bracket in $g^{\mathbf{N}}$ satisfies then the condition (1.1).

Let $\exp(g^{\mathbf{N}})$ be the associated B-C-H group. According to Proposition 2.4 (a) this group structure may be complemented by the $\mathbf{R}$-multiplication defined for $t \in \mathbf{R}$ and $f \in \exp(g^{\mathbf{N}})$ by

$$(t * f)(s) = \sum_{n=1}^{\infty} t^n a_n s^n. \tag{3.3}$$

Moreover $E(\exp(g^{\mathbf{N}}))$ consists of all the series with the first coefficient $x \in g$ and the remaining ones zero. We denote such a series by $x \otimes s$.

Definition 3.2. The subgroup of $\exp(g^{\mathbf{N}})$ generated by all the elements $x \otimes s$ with $x \in g$ will be called *the group of strings over* g and denoted by $S(g)$.

The following proposition together with its proof can be found in [13] Proposition 3.3

Proposition 3.3. *Let G be a Banach-Lie group with Lie algebra g. The groups of strings $S(G)$ and $S(g)$ are algebraically isomorphic.*

Comment: Both $S(G)$ and $S(g)$ may be considered in the various $\mathbf{R}$-group topologies where the compact-open topology seems to be natural for $S(G)$ whereas the product topology is easy to observe for $S(g)$. Neither the isomorphism I introduced in the proof of the above proposition nor its inverse are continuous with respect to those topologies. We intend to give more complete discussion of this in the forthcoming paper [15].

C. The groups of strings associated with an N-gradation Lie algebra

This family generalizes examples introduced in subsection B.

Let $K = \prod_{j=1}^{\infty} M_j$ be an $\mathbf{N}$-gradation Lie algebra and consider the induced B-C-H group $\exp(K)$ with $\mathbf{R}$-multiplication (2.4). Then by Proposition 2.4(b), $E(\exp(K)) = M_1$. Let ϵ be a subset of the linear space M_1 closed under scalar multiplication by real numbers. Let $S_\epsilon(K)$ be the subgroup of $\exp(K)$ generated by ϵ. Clearly $S_\epsilon(K)$ is a group of strings.

D. Free groups of strings.

Definition 3.4. A pointed set (E, e) with a map $* : \mathbf{R} \times E \to E$ is said to be an $\mathbf{R}$-*set* if in (2.1) the conditions (a),(b) hold true and moreover for each $\phi \in E$, $\phi \neq e$, the map $j_\phi : \mathbf{R} \to E$ defined by $j_\phi(t) = t * \phi$ is injective.

Definition 3.5. Let E be an $\mathbf{R}$-set and P be a group of strings. A map $\alpha : E \to P$, is said to be *exponential* if for $s, t \in \mathbf{R}$ and $\phi \in E$,

$$\alpha(s * \phi) = s * \alpha(\phi)$$

$$\alpha((s + t) * \phi) = (s * \alpha(\phi))(t * \alpha(\phi)).$$

For an $\mathbf{R}$-set (E,e) let $\bar{E} = E \setminus \{e\}$ and let the equivalence relation '$\sim$' in $\bar{E}$ be defined by

$$\phi_1 \sim \phi_2 \qquad \text{iff} \qquad \phi_1 = s * \phi_2 \qquad \text{for some} \quad s \neq 0.$$

The equivalence classes of '$\sim$' will be called '*lines*' and will be denoted by $\tilde{\phi}$. The family of all lines will be denoted by $\tilde{E}$. Let $\pi : \bar{E} \to \tilde{E}$ be the quotient map and let $j : \tilde{E} \to \bar{E}$ be a right inverse to π, i.e. $\pi \circ j = \mathrm{id}$. Then $\phi \sim j \circ \pi(\phi)$ for $\phi \in \bar{E}$ and thus there is a unique mapping $\gamma : \bar{E} \to \mathbf{R}$ determined by the condition $\phi = \gamma(\phi) * j \circ \pi(\phi)$.

Let X_E be the linear space over $\mathbf{R}$ with the basis consisting of all the elements $\tilde{f} \in \tilde{E}$. The map $r : E \to X_E$ is defined by $r(\phi) = \gamma(\phi)\tilde{\phi}$ for $\phi \neq e$ and $r(e) = 0$.

Clearly r depends on the choice of the selector j but all the maps obtained in this way are in a sense equivalent. They are exponential and injective. We shall refer to r as the *rectifying representation* of E.

Definition 3.6. Let (E,e) be an $\mathbf{R}$-set. *The free group of strings over E* is a group of strings $F_E^{\mathbf{R}}$ together with exponential map $i_E^{\mathbf{R}} : E \to F_E^{\mathbf{R}}$ such that for each group of strings P and an exponential map $\alpha : E \to P$, there exists a unique $\mathbf{R}$-homomorphism $\beta : F_E^{\mathbf{R}} \to P$ with $\alpha = \beta \circ i_E^{\mathbf{R}}$.

Proposition 3.7. *For each $\mathbf{R}$-set (E,e) there exist a unique free group of strings $F_E^{\mathbf{R}}$ over E. The map $i_E^{\mathbf{R}} : E \to F_E^{\mathbf{R}}$ is injective.*

Proof. Let $F_{\bar{E}}$ be the free group over $\bar{E}$. Let the map $i : E \to F_{\bar{E}}$ be defined by $i(e) = \theta$, where θ is the unit of $F_{\bar{E}}$ and $i(\phi) = \phi$ for $\phi \neq e$.

For $s \in R$ and $f = \prod_{i=1}^{n} \phi_i^{\epsilon_i}$ with $\phi_i \in \bar{E}$ and ϵ_i equal $+1$ or -1 define the product $s * f = \displaystyle\prod_{i=1}^{n} (s * \phi_i)^{\epsilon_i}$ and put additionally $s * f = \theta$ for $f = \theta$. Observe that this product satisfies (2.1)(a),(b),(c).

To obtain also the condition (2.2) for the elements of $i(\bar{E})$, consider the normal subgroup $G(W)$ of $F_{\bar{E}}$ generated by the subset

$$W = \{(s*\phi)(t*\phi)((s+t)*\phi)^{-1} : s,t \in \mathbf{R} \text{ and } \phi \in \bar{E}\}$$

and define $F_E^{\mathbf{R}} = F_{\bar{E}}/G(W)$ and $j = \pi \circ i$, where $\pi : F_{\bar{E}} \to F_E^{\mathbf{R}}$ is the quotient homomorphism.

We claim that $F_E^{\mathbf{R}}$ is a group of strings with the required properties, moreover j is exponential and injective.

First observe that W is invariant with respect to the above defined $\mathbf{R}$-product thus the quotient group may be equipped with the induced $\mathbf{R}$-product which by construction of W satisfies also (2.2) for $\phi \in j(E)$. In particular j is exponential. Also note that $j(E)$ generates $F_E^{\mathbf{R}}$.

Let a group of strings P and an exponential map $\alpha : E \to P$ be given. Because $F_{\bar{E}}$ is a free group, there exists a homomorphism $\delta : F_{\bar{E}} \to P$ such that $\alpha = \delta \circ i$, i.e. $\delta(f) = \prod_{i=1}^{n} (\alpha(\phi_i))^{\epsilon_i}$ for $f = \prod_{i=1}^{n} \phi_i^{\epsilon_i} \in F_{\bar{E}}$.

Since α is an exponential $\mathbf{R}$-map, δ is an $\mathbf{R}$-map as well and $G(W) \subset \ker \delta$. Thus δ induces the required $\mathbf{R}$-homomorphism β. Taking X_E for P and the rectifying representation r for α, by the injectivity of r we deduce that j is injective as well. ∎

4. Topological groups of strings

Definition 4.1. A topological group of strings P is a topological group and a group of strings, such that the operation

$$\mathbf{R} \times P \ni (s, f) \to s * f \in P$$

is jointly continuous.

Examples 4.2. As in Section 3, we leave it to the reader to verify that the algebraic structures of groups of strings introduced there, enriched by the topologies indicated below, yield topological groups of strings in the sense of Definition 4.1.

(a) Let G be a topological group. Then $S(G)$ with the compact-open topology (i.e. the topology of uniform convergence on compact sets) is a topological group of strings.

(b) Let g be a topological Lie algebra. Equip $g^{\mathbf{N}}$ and the associated group $\exp(g^{\mathbf{N}})$ with the product topology. Viewed as a topological subgroup of $\exp(g^{\mathbf{N}})$, $S(g)$ becomes a topological group of strings.

(c) For $K = \prod_{j=1}^{\infty} M_j$ a topological $\mathbf{N}$-gradation Lie algebra an analogous procedure yields the structure of a topological group of strings on each of the groups $S_\epsilon(K)$ (see Section 3.C).

In the structure of a topological group of strings P a special role is played by the closed central descending sequence $\left(\bar{P}_j\right)_{j\geq 1}$.

The *central descending sequence* $(P_j)_{j\geq 1}$ of a group P is the sequence of its normal subgroups such that the j-th subgroup is generated by all the commutators $\{f_1, \ldots, f_k\}$ with $k \geq j$ (for a full discussion of central descending series see e.g. [6]).

The *closed central descending sequence* of a topological group P is the sequence of normal subgroups which are the closures of the subgroups in the central descending sequence of P.

In the following two propositions let P be a topological group of strings with the central descending series $(P_j)_{j\geq 1}$ and the closed central descending series $\left(\bar{P}_j\right)_{j\geq 1}$. We omit the simple proof of the first proposition:

Proposition 4.3. (a) *The members of the central descending sequence as well as of the closed central descending series are* $\mathbf{R}$*-subgroups.*

(b) *Each* P_j *is generated by the elements* $\{\phi_1, \ldots, \phi_k\}$ *with* $k \geq j$ *where* ϕ_i *'s are exponential elements.*

Proposition 4.4. *Let* $\phi_1, \phi_2, \ldots, \phi_j \in E(P)$.

(a) *For each positive integer* n,

$$n * \{\phi_1, \phi_2, \ldots, \phi_j\} = (\{\phi_1, \phi_2, \ldots, \phi_j\})^{n^j} \mod P_{j+1}. \tag{4.1}$$

(b) *For each* $f \in \bar{P}_j$ *and each* $n \in \mathbf{N}$ *there exists* $\Delta_{j,n}(f) \in \bar{P}_{j+1}$ *such that*

$$f = (\frac{1}{n} * f^{n^j}) \cdot \Delta_{j,n}(f). \tag{4.2}$$

Proof. (a) It is known (cf. [10] and [12] Lemma 12) that for arbitrary $f, g \in P$ and $h \in P_{j-1}$ we have $\{f \cdot g, h\} = \{f, h\} \cdot \{g, h\}$ mod P_{j+1}. It follows that for every positive integer n one gets the following mod P_{j+1} equalities:

$$n * \{\phi_1, \phi_2, \ldots, \phi_j\} = \{n * \phi_1, n * \phi_2, \ldots, n * \phi_j\} =$$

$$= \{\phi_1^n, \phi_2^n, \ldots, \phi_j^n\} = \{\phi_1, \{\phi_2^n, \ldots, \phi_j^n\}\}^n.$$

Hence applying an induction argument we get (4.1).

(b) P_j being commutative mod P_{j+1} , from (4.1) we infer that for each $f \in P_j$,

$$f = (\frac{1}{n} * f^{n^j}) \text{ mod } P_{j+1}.$$

Since both sides of the above equality depend continuously on f, we may extend it to a mod $\bar{P}_{j+1}$ equality valid for $f \in \bar{P}_j$, which is equivalent to (4.2). ∎

Let us observe that part (a) of the Proposition 4.4 as well as the statement obout the central descending series in the Proposition 4.3 work just as well for groups of strings which are not topological.

Definition 4.5. A topological group of strings P is said to be *analytic* if

$$\bar{P}_\infty = \bigcap_{j=1}^{\infty} \bar{P}_j = \{e\}. \tag{4.3}$$

Proposition 4.6. *Let* P *be a topological group of strings and let* $T = P/\bar{P}_\infty$. *Then* T *is analytic.*

Proof. We claim that $\bar{P}_j = \pi^{-1}(\bar{T}_j)$.

Indeed, since $\pi : P \to T$ is surjective, $\pi(P_j) = T_j$. The continuity of π implies that $\pi(\bar{P}_j) \subset \bar{T}_j$ and $\pi(\bar{P}_j)$ is dense in $\bar{T}_j$. To prove that

$$\pi(\bar{P}_j) = \bar{T}_j \tag{4.4}$$

we only need to show that $\pi(\bar{P}_j)$ is closed. Since $\bar{P}_\infty \subset \bar{P}_j$ one has $\pi(\bar{P}_j) = T \setminus \pi(P \setminus \bar{P}_j)$ and since $P \setminus \bar{P}_j$ is open, $\pi(P \setminus \bar{P}_j)$ is also open. So $\pi(\bar{P}_j)$ is closed.

Now (4.4) implies that $\pi^{-1}(\bar{T}_j) = \bar{P}_j \cdot \bar{P}_\infty$. But $\bar{P}_\infty \subset \bar{P}_j$ and the claim follows.

To finish the proof suppose that $F \in \bar{T}_j$ for $j = 1, 2, \ldots$. Let $h \in \pi^{-1}(F)$. Then $h \in \bar{P}_j$ for $j = 1, 2, \ldots$, hence $h \in \bar{P}_\infty = \ker\pi$. Thus $F = \pi(h) = e$, ∎

5. The N-gradation Lie algebra of a topological group of strings

Throughout this section P will be a topological group of strings with the closed central descending series $\left(\bar{P}_j\right)_{j\geq 1}$. We shall follow the convention that $(\sim(\cdot))$ denotes the quotient class of $(\cdot)$.

Let $M_j = \bar{P}_j/\bar{P}_{j+1}$. The group M_j is abelian. Since $\bar{P}_j$ and $\bar{P}_{j+1}$ are $\mathbf{R}$-subgroups, M_j is an $\mathbf{R}$-group as well.

We shall change the multiplication in M_j (denoted by '*') introducing the new one by :

$$s \star (\sim f) = \left(\sqrt[j]{|s|}\right) * (\sim f^{sgn(s)}) \text{ for } s \in \mathbf{R} \text{ and } f \in \bar{P}_j \tag{5.1}$$

(the new multiplication will be referred to as *the modified multiplication*).

(2.1) (a)-(c) hold for the $*$ -multiplication on M_j. Since M_j is abelian the modified multiplication satisfies conditions (2.1) (a)-(c) as well.

Now observe that for $(\sim f) = (\sim \{\phi_1, \phi_2, \ldots, \phi_j\}) \in M_j$ where each $\phi_i \in E(P)$, the following condition holds:

$$m \star (\sim f) = (\sim f)^m \text{ for } m = sgn(n)|n|^j \text{ where } n \in \mathbf{Z} \tag{5.2}$$

Indeed, by (4.1)

$$m \star (\sim f) = \sqrt[j]{|n|^j} * (\sim f^{sgn(m)}) = |n| * (\sim f^{sgn(n)}) = (\sim f^{|n|^j sgn(n)}) = (\sim f)^m.$$

Since M_j is abelian (5.2) holds also for products of elements of the considered form and their inverses. The both sides of (5.2) depending continuously on f and all such products forming a dense subgroup of M_j (c.f also Proposition 5.5 below) we infer that (5.2) holds true on M_j.

In this case the following criterion which applies the Waring-Hilbert theorem cf [4] works:

Proposition 5.1. *Let M be a topological $\mathbf{R}$-group. M is a linear topological space if and only if there exists $j \in \mathbf{N}$ such that for each $f \in M$ and m of the form $m = sgn(n)|n|^j$ with $n \in \mathbf{Z}$*

$$m \star f = f^m \tag{5.3}$$

Proof. For the proof see [12] Lemma 13. ■

Corollary 5.2. *Each of M_j $j = 1, 2, \ldots,$ equipped with the modified multiplication, is a linear topological space.* ■

Consider now the (topological) product

$$L(P) = \prod_{j=1}^{\infty} M_j. \tag{5.6}$$

It is known ([1],[5],[8]) that $L(P)$ has a structure of Lie ring whose additive part is given by coordinatewise abelian multiplication and the Lie bracket is introduced as the common extension of the family of biadditive maps

$$[\cdot,\cdot]_{k,m} : M_k \times M_m \to M_{k+m} \qquad k, m \in \mathbf{N}$$

where $[(\sim f), (\sim g)]_{k,m} = (\sim \{f, g\})$ and $(\sim (\cdot))$ denotes the corresponding quotient class of $(\cdot)$.

Theorem 5.3. (cf [12] Section 7). *Let P be a topological group of strings. Equip $L(P)$ with the two coordinatewise defined $\mathbf{R}$-products:*

'the quotient product' - defined for each M_j as the quotient R-product .
'the modified product' - defined for each M_j by the formula (5.1).
Then

(a) *The topological Lie ring structure on $L(P)$ is compatible with the modified product yielding together an $\mathbf{N}$-gradation topological Lie algebra structure on $L(P)$.*

(b) *The topological group structure on $\exp(L(P))$ is compatible with the quotient product yielding together an $\mathbf{R}$-group structure on $\exp(L(P))$ with*

$$E(\exp L(P)) = (M_1, 0, 0, 0, \ldots). \tag{5.7}$$

Proof. (a) By Corollary 5.2 the modified multiplication defines a linear structure on $L(P)$. This structure is compatible with the Lie bracket. Indeed since the modified multiplication is compatible with the addition , and Lie bracket is biadditive $[n\star a, b] = n\star[a, b]$ for $n \in \mathbf{N}$. Hence $[s\star a, b] = s\star[a, b]$ for rational s and by continuity of the bracket this property extends to each real s. Similarly one proves homogeneity for the second argument.

(b) Let us observe that for $f \in \bar{P}_j$ and $s \in R$

$$\sim (s * f) = s * (\sim f) = s^j \star (\sim f) \tag{5.8}$$

It follows by Proposition 2.4 that the quotient multiplication induces on $\exp(L(P))$ an $\mathbf{R}$-group structure and (5.7) holds. ■

Proposition 5.4. *Let P be a topological group of strings. For each $j \in N$*

$$\bar{P}_j = P_j \cdot \bar{P}_{j+1}. \tag{5.9}$$

Proof. Let $\delta_j : \bar{P}_j \to M_j$ be the quotient $\mathbf{R}$-homomorphism. Because M_j is a linear space for each $f \in \bar{P}_j$ and each integer k

$$\delta_j(f^k) = k \star \delta_j(f). \tag{5.10}$$

Using δ_j (5.8) may be written in the form

$$\delta_j(s * f) = s^j \star \delta_j(f) \tag{5.11}$$

Observe that to show (5.9) it will be suficient associate to each $f \in \bar{P}_j$ an element $f_j \in P_j$, such that

$$\delta_j(f) = \delta_j(f_j) \tag{5.12}$$

We shall define such f_j by the following inductive procedure: Let $\Delta_1(f) = f$ and let $\Delta_{i+1}(f) = (2 * \Delta_i(f))(\Delta_i(f))^{-2^i}$, for $i = 1, 2 \ldots, j-1$.

Put finally $f_j = \lambda_j * \Delta_j(f)$ where λ_j is a properly chosen real number.

We claim that $\Delta_i(f) \in P_i$ for each i. Indeed, by (4.1) $2 * h = h^{2^i} \mod P_{i+1}$ for $h \in P_i$. This inductively implies our claim. In particular $f_j \in P_j$. Let us observe next that if $f \in \bar{P}_j$ then for $i = 1, 2, \ldots$ $\Delta_i(f) \in \bar{P}_j$ as well, so by (5.10) and (5.11) we get

$$\delta_j(f_j) = \delta_j(\lambda_j * \Delta_j(f)) = \lambda_j^j \star \delta_j((2 * \Delta_{j-1}(f))(\Delta_{j-1}(f))^{-2^{j-1}}) =$$

$$= \lambda_j^j \cdot (2^j - 2^{j-1}) \star \delta_j(\Delta_{j-1}(f)) = \ldots = (\lambda_j^j \cdot \prod_{i=1}^{j-1}(2^j - 2^i)) \star \delta_j(f).$$

It follows that for $\lambda_j = (\prod_{i=1}^{j-1}(2^j - 2^i))^{\frac{-1}{j}}$ the condition (5.12) holds true. ■

Proposition 5.5. *For P a topological group of strings, with the attached topological $\mathbf{N}$-gradation Lie algebra $L(P)$ define*

$$\tau : E(P) \ni \phi \to ((\sim \phi), 0, 0, \ldots) \in L(P). \tag{5.13}$$

Then

(a) $\tau(E(P))$ generates the direct sum subalgebra $(L(P))_0$.

(b) If P is analytic τ is injective.

Proof. (a) By Proposition 5.4 the elements $\{\phi_1, \phi_2, \ldots, \phi_j\}$ where $\phi_i \in E(P)$ generate $\bar{P}_j$ modulo $\bar{P}_{j+1}$. By definition of the sum operation and Lie bracket in $L(P)$ this means that the corresponding brackets $[\tau(\phi_1), \tau(\phi_2), \ldots, \tau(\phi_j)]$ span linearly M_j.

(b) We have to show that for P analytic and $\phi_1, \phi_2 \in E(P)$, $\phi_1 \bar{P}_2 = \phi_2 \bar{P}_2$ implies $\phi_1 = \phi_2$. This results from the more general fact : for $\phi_1, \phi_2 \in E(P)$

$$(\phi_1 \cdot \phi_2 \in \bar{P}_2) \Rightarrow (\phi_1 \cdot \phi_2 \in \bigcap_{j=1}^{\infty} \bar{P}_j). \tag{5.14}$$

For (5.14) it suffices to show that for each $j \in \mathbf{N}$, $j \geq 2$

$$(\phi_1 \cdot \phi_2 \in \bar{P}_j) \Rightarrow (\phi_1 \cdot \phi_2 \in \bar{P}_{j+1}).$$

Let $\delta_j : \bar{P}_j \to M_j$ be the quotient R-homomorphism. Proceeding by induction we shall show that $(\phi_1 \cdot \phi_2 \in \bar{P}_j)$ implies $\delta_j(\phi_1 \cdot \phi_2) = 0$.

For this observe that by (4.2) $\phi_1^2 \cdot \phi_2^2 = 2 * (\phi_1 \cdot \phi_2) = (\phi_1 \cdot \phi_2)^{2^j} \cdot \Delta_j$ where $\Delta_j \in \bar{P}_{j+1}$. Thus $\delta_j(\phi_1^2 \cdot \phi_2^2) = \delta_j(\phi_1 \cdot \phi_2)^{2^j} = 2^j \delta_j(\phi_1 \cdot \phi_2)$.

Simultaneously $\phi_1^2 \cdot \phi_2^2 = \phi_1 \cdot \phi_2 \cdot (\phi_2^{-1} \cdot \phi_1 \cdot \phi_2)\phi_2$ and $\delta_j((\phi_2^{-1} \cdot \phi_1 \cdot \phi_2)\phi_2) = \delta_j(\phi_1 \cdot \phi_2)$. Thus $2\delta_j(\phi_1 \cdot \phi_2) = 2^j \delta_j(\phi_1 \cdot \phi_2)$ and (b) follows. ■

Before presenting examples of $L(P)$ attached to various P , let us discuss 'the integration problem' for topological N-gradation Lie algebra.

For $K = \prod_{j=1}^{\infty} M_j$ a topological $\mathbf{N}$-gradation Lie algebra, let $\exp(K)$ be its associated B-C-H group. For $k \in \mathbf{N}$ consider the map

$$d_k : \exp(K) \ni f = (a_1, a_2, \ldots) \to a_k \in M_k$$

and observe that d_k restricted to $\prod_{j=k}^{\infty} M_j$ is given by the limit $d_k(f) = \lim_{n\to\infty} \frac{1}{n} * f^{n^k}$.

In the following we shall need

Lemma 5.6. *Let $f = \{\phi_1, \phi_2, \ldots, \phi_j\} \in \exp(K)$ where $j \geq 2$ and $\phi_i \in M_1$ for each i. Then*

$$d_i(f) = 0 \qquad \text{for} \quad 1 \leq i < j \tag{5.15}$$

$$d_j(f) = [\phi_1, \ldots, \phi_j]. \tag{5.16}$$

Proof. The proof is straightforward (with the use of the induction argument) and it will be left to the reader. ■

Consider now $K = \prod_{j=1}^{\infty} M_j$ a topological N-gradation Lie algebra such that the set of j-fold commutators $[M_1, \ldots, M_1]$ spans M_j for each j. Let ϵ be an $\mathbf{R}$-subset of M_1 which spans linearly M_1, and let S_ϵ be the topological group of strings generated by ϵ (c.f Section 3C)

Proposition 5.7. $L(S_\epsilon) = K$.

Proof. Observe that $d_k(f) = 0$ for $k \in \mathbf{N}$ and $f \in (\bar{S}_\epsilon)_{k+1}$ by (5.15). Note also that Lemma 5.6 and the B-C-H formula imply that for $f_1, f_2 \in (\bar{S}_\epsilon)_k$

$$d_k(f_1 \cdot f_2) = d_k(f_1) + d_k(f_2). \tag{5.17}$$

Hence d_k for each $k \in \mathbf{N}$ induces a linear map $\tilde{d}_k : (\bar{S}_\epsilon)_k/(\bar{S}_\epsilon)_{k+1} \ni (\sim f) \to d_k(f) \in M_k$.

Note that $d_k((\sim \{\phi_1, \phi_2, \ldots, \phi_k\})) = [\phi_1, \phi_2, \ldots, \phi_k]$ for $\phi_1, \phi_2, \ldots, \phi_k \in \epsilon$ and since ϵ spans linearly M_1, $\tilde{d}_k$ is surjective.

We claim that $\tilde{d}_k$ is injective as well. For this representing d_k in the form $d_k(f) = \lim_{n\to\infty} \frac{1}{n} * f^{n^k}$, and using (4.2) we get a decomposition $f = d_j(f) \cdot \Delta(f)$, where $\Delta(f)$ is in the closure of $(S_\epsilon)_{k+1}$ in $\exp(K)$, (Note that neither $d_k(f)$ nor $\Delta(f)$ have to be in S_ϵ). Nevertheless if $f \in (\bar{S}_\epsilon)_k$ and $d_k(f) = 0$ the decomposition implies that $f = \Delta(f) \in (\bar{S}_\epsilon)_{k+1}$ which proves the claim.

To finish the proof observe that the bijective linear map

$$\alpha : L(S_\epsilon) \ni ((\sim f_1), (\sim f_2), \ldots) \to (d_1(f_1), d_2(f_2), \ldots) \in K$$

is also an $\mathbf{N}$-gradation Lie algebra homomorphism. Indeed, the B-C-H formula implies that for $f \in (\bar{S}_\epsilon)_j$ and $g \in (\bar{S}_\epsilon)_k$, $d_{j+k}(\sim \{f, g\}) = [d_j((\sim f)), d_k((\sim g))]$ what is just needed (cf. (2.6)). ■

Corollary 5.8. *Let a topological group of strings P be given with the $\mathbf{N}$-gradation Lie algebra $L(P)$ and let τ be defined by (5.13). Let $\Delta = \tau(E(P))$ and let S_Δ be the topological group of strings generated by Δ. Then $L(P) = L(S_\Delta)$.*

Comment: In the both Proposition 5.7 and Corollary 5.8 the equalities $L(S_\epsilon) = K$ and $L(P) = L(S_\Delta)$ have only algebraic meaning. This also applies to the equality $L(S(a)) = \prod_{j=1}^{\infty} a_j$ in the following example.

Examples 5.9. (a) Let a be a topological Lie algebra and let $S(a)$ be the group of strings over a considered with the product topology restricted from $\exp(a^{\mathbf{N}})$. Proposition 5.7 implies that $L(S(a)) = \prod_{j=1}^{\infty} a_j$, where $(a_j)_{j\geq 1}$ is the central descending series of the Lie algebra a.

(b) Let G be a Lie group with the Lie algebra g. Identifying $S(G)$ with $S(g)$ according to Proposition 3.3 and endowing $S(g)$ with the product topology like in (a) we get $L(S(G)) = \prod_{j=1}^{\infty} g_j$, where $(g_j)_{j\geq 1}$ is the central descending series of the Lie algebra g of G.

6. The Lie functor

In this section we follow the notation of Section 5. Let P and Q be topological groups of strings. Let $M_j = \bar{P}_j/\bar{P}_{j+1}$ and $N_j = \bar{Q}_j/\bar{Q}_{j+1}$.

Proposition 6.1. *For a continuous $\mathbf{R}$-homomorphism $\Psi : P \to Q$ the formulas*

$$M_j \ni (\sim f) \to (\sim \Psi(f)) \in N_j \tag{6.1}$$

define an $\mathbf{N}$-gradation Lie algebra homomorphism $L(\Psi)$.

Proof. Since Ψ is a continuous homomorphism it maps subgroups of the closed central descending series in P into the corresponding subgroups in Q. It follows that for each $j \in \mathbf{N}$ the maps (6.1) are properly defined. The family of maps (6.1) may then be extended by linearity to $L(P)_0$ providing, due to the form of the Lie structure of $L(P)_0$ and $L(Q)_0$ a homomorphism preserving gradation. This homomorphism may then be extended by continuity (in the product topology) to the whole of $L(P)$, yielding an $\mathbf{N}$-gradation Lie algebra homomorphism. ∎

In the following example we are only suggesting the situation leaving the verification of the details to the reader.

Example 6.2. (a) Let G, H be Lie groups with Lie algebras g and h. Let $\psi : G \to H$ be a Lie group homomorphism and let $\tilde{\psi} : g \to h$ be the corresponding homomorphism of Lie algebras. Put $\Psi : S(G) \ni f \to \psi \circ f \in S(H)$. Clearly Ψ is an $\mathbf{R}$-homomorphism of groups of strings.

Representing $L(S(G))$ and $L(S(H))$ as in Example 5.9 (b), we may describe $L(\Psi)$ in the form

$$L(\Psi)(x_1, x_2, \ldots) = (\tilde{\psi}(x_1), \tilde{\psi}(x_2), \ldots) \tag{6.2}$$

where $(g_j)_{j\geq 1}$ is the central descending series of the Lie algebra g and $x_j \in g_j$ for $j = 1, 2, \ldots$

(b) Let g and h be topological Lie algebras and $S(g)$, $S(h)$ be the corresponding groups of strings considered in the product topology. For a homomorphism $\tilde{\psi} : g \to h$ the formula

$$\tilde{\Psi} : g^{\mathbf{N}} \ni (\sum_{j=1}^{\infty} x_j t^j) \to \sum_{j=1}^{\infty} \tilde{\psi}(x_j) t^j \in h^{\mathbf{N}} \tag{6.3}$$

defines a homomorphism of $\mathbf{N}$-gradation Lie algebras. The same formula may be understood as giving an $\mathbf{R}$-homomorphism of groups $\exp(g^{\mathbf{N}})$ and $\exp(h^{\mathbf{N}})$. Restricting this homomorphism to $S(g)$ we obtain the homomorphism $\tilde{\Psi} : S(g) \to S(h)$ corresponding to $\tilde{\psi}$. The formula (6.3) restricted to the Lie subalgebra appearing in (6.2) also describes the homomorphism $L(\tilde{\Psi})$.

Concerning the integration of Lie algebra homomorphisms the following necessary condition is to be noted:

Proposition 6.3. *Given an $\mathbf{N}$-gradation Lie algebra homomorphism $\psi : L(P) \to L(Q)$, if $\psi = L(\Psi)$ for some continuous $\mathbf{R}$-homomorphism $\Psi : P \to Q$ then the following 'initial condition' holds:*

($\star$) *There exists an $\mathbf{R}$-map $\Psi_0 : E(P) \to E(Q)$ such that*

$$((\sim \Psi_0(\phi)), 0, 0, \ldots) = \psi((\sim \phi), \ldots). \tag{6.4}$$

Proof. Define Ψ_0 as the restriction of Ψ to $E(P)$. ∎

The following example shows that for a given $\mathbf{N}$-gradation Lie algebra homomorphism there may be no map satisfying (6.4).

Example 6.4. Let $P = S_\Omega(G)$ (cf. Section 3A) for $G = SO(3, \mathbf{R})$ (i.e for the group of rotations of $\mathbf{R}^3$) where Ω is composed of 3 families of one-parameter subgroups of G - of all rotations about 3 fixed pairwise orthogonal axes. It is well-known that the exponential map $\exp : S(G) \to G$ restricted to $S_\Omega(G)$ is surjective (the construction of 'Euler angles'). It may also be proved (cf. Proposition 5.7) that in this case $L(S(G)) = L(P)$. Picking an automorphism $L(\Psi)$ of $L(P)$ of the form (6.2) where $\Psi : S(G) \ni f \to \psi \circ f \in S(G)$ and ψ is an automorphism of G which fails to preserve the chosen set of 3 one-parameter subgroups of G one finds that $L(\Psi)$ fails to admit any 'initial condition' (with respect to $E(P)$).

Comment: We silently assume here that $E(P) = \Omega$. The precise arguments for that are given in the proof of Theorem 6.10 below, where Ω is changed by Δ.

We shall show that 'the initial condition' is sufficient for the existence of Ψ provided the group Q is analytic. First let us examine the problem for free groups of strings.

For P and Q topological groups of strings consider $E(P)$ and $E(Q)$ with the structures of $\mathbf{R}$-sets (cf. Definition 3.4) induced from P and Q and let $F^{\mathbf{R}}_{E(P)}$ and $F^{\mathbf{R}}_{E(Q)}$ be free groups of strings over $E(P)$ and $E(Q)$ respectively. Denote by $i^{\mathbf{R}}_P : E(P) \to F^{\mathbf{R}}_{E(P)}$ and $i^{\mathbf{R}}_Q : E(Q) \to F^{\mathbf{R}}_{E(Q)}$ the canonical exponential embeddings.

Since the inclusion maps $i_P : E(P) \to P$ and $i_Q : E(Q) \to Q$ are exponential (in the sense of Definition 3.5) they induce surjective $\mathbf{R}$-homomorphisms $\pi^P : F^{\mathbf{R}}_{E(P)} \to P$ and $\pi^Q : F^{\mathbf{R}}_{E(Q)} \to Q$.

Proposition 6.5. (a) *Every $\mathbf{R}$-map $\Psi_0 : E(P) \to E(Q)$ extends to a unique $\mathbf{R}$-homomorphism $\gamma : F^{\mathbf{R}}_{E(P)} \to F^{\mathbf{R}}_{E(Q)}$.*

(b) *An $\mathbf{R}$-map $\Psi_0 : E(P) \to E(Q)$ extends to an $\mathbf{R}$-homomorphism $P \to Q$ provided $\gamma(\ker(\pi^P)) \subset \ker(\pi^Q)$ or equivalently if for $f \in F^{\mathbf{R}}_{E(P)}$,*

$$\big(\pi^P(f) = 0\big) \Rightarrow \big((\pi^Q \circ \gamma)(f) = 0\big). \tag{6.5}$$

Proof. (a) Since the composite $\mathbf{R}$-map $i^{\mathbf{R}}_Q \circ \Psi_0 : E(P) \to F^{\mathbf{R}}_{E(Q)}$ is exponential it induces by Definition 3.6 the required $\mathbf{R}$-homomorphism γ.

(b) Since by (a), γ extending Ψ_0 exists and moreover $P = F^{\mathbf{R}}_{E(P)}/\ker(\pi^P)$ and $Q = F^{\mathbf{R}}_{E(Q)}/\ker(\pi^Q)$, the condition $\gamma(\ker(\pi^P)) \subset \ker(\pi^Q)$ implies (b). ∎

To apply the above algebraic result to the topological case we shall need the following two lemmas, the first one specifying the result of Proposition 5.4 to the particular situation considered.

Lemma 6.6. *Let $\delta^P_j : \bar{P}_j \to M_j$ and $\delta^Q_j : \bar{Q}_j \to N_j$ be the quotient homomorphisms. Define $(\overline{F^{\mathbf{R}}_{E(P)}})_j = (\pi^P)^{-1}(\bar{P}_j)$ and $(\overline{F^{\mathbf{R}}_{E(Q)}})_j = (\pi^Q)^{-1}(\bar{Q}_j)$.*

For each $j \geq 2$ there exist maps $p^P_j : (\overline{F^{\mathbf{R}}_{E(P)}})_j \to (F^{\mathbf{R}}_{E(P)})_j$ and $p^Q_j : (\overline{F^{\mathbf{R}}_{E(Q)}})_j \to (F^{\mathbf{R}}_{E(Q)})_j$ such that

$$\begin{aligned}
&(a) \qquad \gamma \circ p^P_j = p^Q_j \circ \gamma \quad \text{on} \quad (\overline{F^{\mathbf{R}}_{E(P)}})_j,\\
(6.6) \qquad &(b) \qquad \delta^P_j \circ \pi^P = \delta^P_j \circ \pi^P \circ p^P_j \quad \text{on} \quad (\overline{F^{\mathbf{R}}_{E(P)}})_j,\\
&(c) \qquad \delta^Q_j \circ \pi^Q = \delta^Q_j \circ \pi^Q \circ p^Q_j \quad \text{on} \quad (\overline{F^{\mathbf{R}}_{E(Q)}})_j.
\end{aligned}$$

Proof. For fixed $j \geq 2$ proceeding as in the proof of Proposition 5.4 we associate with a given $f \in (\overline{F^{\mathbf{R}}_{E(P)}})_j$ (respectively $f \in (\overline{F^{\mathbf{R}}_{E(Q)}})_j$ the element $f_j \in (F^{\mathbf{R}}_{E(P)})_j$ (resp. $f_j \in (F^{\mathbf{R}}_{E(Q)})_j$).

Since the construction of f_j makes use only of the group multiplication and of the $\mathbf{R}$-product, (6.6) (a) holds. The properties (6.6)(b) and (c) translate (5.12) into the present context. ∎

Lemma 6.7. *Let* $f \in (F^{\mathbf{R}}_{E(P)})_j$. *Then*

$$\delta^Q_j \circ \pi^Q \circ \gamma(f) = \psi \circ \delta^P_j \circ \pi^P(f), \tag{6.7}$$

where ψ *is the homomorphism from the Proposition 6.3.*

Proof. $f \in (F^{\mathbf{R}}_{E(P)})_j$ implies $\gamma(f) \in (F^{\mathbf{R}}_{E(Q)})_j$, thus $\pi^Q \circ \gamma(f) \in Q_j \subset \bar{Q}_j$ and both sides of (6.7) are well defined.

Since the maps in (6.7) are homomorphisms, it suffices to consider f of the form $\{\phi_1, \ldots, \phi_j\}$ where ϕ_i's are in $E(P)$. Then the left hand side of (6.7) takes the form:

$$\begin{aligned}
&\delta^Q_j \circ \pi^Q \circ \gamma(\{\phi_1, \ldots, \phi_j\}) = \delta^Q_j \circ \pi^Q(\{\gamma(\phi_1), \ldots, \gamma(\phi_j)\}) = \\
&= (\sim \{\Psi_0(\phi_1), .., \Psi_0(\phi_j)\}) = [(\sim \Psi_0(\phi_1)), \ldots, (\sim \Psi_0(\phi_j))]
\end{aligned}$$

Similarly for the right hand side:

$$\psi((\sim \{\phi_1, \ldots, \phi_j\})) = \psi(\{(\sim \phi_1), \ldots, (\sim \phi_j)\}) = [\psi((\sim \phi_1)), \ldots, \psi((\sim \phi_1))]$$

The final expressions for the left and right hand sides are equal due to the 'initial condition'. ■

Theorem 6.8. *Let* P *and* Q *be topological groups of strings and* Q *be analytic. Let an* $\mathbf{N}$*-gradation Lie algebra homomorphism* $\psi : L(P) \to L(Q)$ *be given, satisfying the 'initial condition'* (6.4). *Then:*

(a) *There exists unique* $\mathbf{R}$*-homomorphism* $\Psi : P \to Q$ *extending* Ψ_0.

(b) *The homomorphism* Ψ *satisfies* $\Psi(\bar{P}_j) \subset \bar{Q}_j$ for $j = 1, 2, 3, \ldots$. *The induced homomorphism of* $\mathbf{N}$*-gradation Lie algebras is equal to* ψ.

(c) *If* ψ *is injective and* P *is analytic then also* Ψ *is injective.*

Proof. (a) Observe first that if Ψ exists, it is determined for $f = \phi_1 \cdot \phi_2 \cdot \ldots \cdot \phi_n$, with ϕ_i's in $E(P)$, by the formula $\Psi(f) = \Psi_0(\phi_1) \cdot \Psi_0(\phi_2) \cdot \ldots \cdot \Psi_0(\phi_n)$. We claim that (due to the 'initial condition') the right hand side does not depend on the representation of f as a product of exponential elements. As indicated in Proposition 6.5 (b) this amounts to (6.5). Let us observe next, that for analytic Q one may replace (6.5) by the following sequence of implications holding for $j = 1, 2, 3, \ldots$ and $f \in F^{\mathbf{R}}_{E(P)}$:

$$\big(\pi^P(f) \in \bar{P}_{j+1}\big) \wedge \big((\pi^Q \circ \gamma)(f) \in \bar{Q}_j)\big) \Rightarrow \big((\pi^Q \circ \gamma)(f) \in \bar{Q}_{j+1}\big). \tag{6.8}$$

We show (6.8) by induction on j.

j=1: Let $f = \phi_1 \cdot \ldots \cdot \phi_n$ and $\delta^P_1 \circ \pi^P(f) = 0$. We claim that $(\delta^Q_1 \circ \pi^Q \circ \gamma)(f) = 0$. Indeed, with the 'initial condition' in the form $\delta^Q_1 \circ \pi^Q(\Psi_0(\phi)) = \psi \circ \delta^P_1 \circ \pi^P(\phi)$ one gets

$$\begin{aligned}
\delta^Q_1 \circ \pi^Q \circ \gamma(\phi_1 \cdot \ldots \cdot \phi_n) &= (\delta^Q_1 \circ \pi^Q)(\Psi_0(\phi_1) \cdot \ldots \cdot \Psi_0(\phi_n)) \\
&= (\delta^Q_1 \circ \pi^Q)(\Psi_0(\phi_1)) + \ldots + (\delta^Q_1 \circ \pi^Q)(\Psi_0(\phi_n)) \\
&= \psi \circ \delta^P_1 \circ \pi^P(\phi_1)) + \ldots + \psi \circ \delta^P_1 \circ \pi^P(\phi_n) \\
&= \psi(\delta^P_1 \circ \pi^P(\phi_1) + \ldots + \delta^P_1 \circ \pi^P(\phi_n)) \\
&= \psi \circ (\delta^P_1) \circ \pi^P)(f) = 0.
\end{aligned}$$

Induction step: For $j \geq 2$, $f \in F^{\mathbf{R}}_{E(P)}$ let $\delta^P_j \circ \pi^P(f) = 0$ and let $\delta^Q_{j-1} \circ \pi^Q \circ \gamma(f) = 0$. We claim that $\delta^Q_j \circ \pi^Q \circ \gamma(f) = 0$. By the induction assumption $\gamma(f) \in (\overline{F^{\mathbf{R}}_{E(Q)}})_j$, thus consecutive use of (6.6) (c), (6.6) (a), Lemma 6.7 and (6.6) (b), taking into account that $\delta^P_j \circ \pi^P(f) = 0$, yields

$$\delta^Q_j \circ \pi^Q \circ \gamma(f) = \delta^Q_j \circ \pi^Q \circ p^Q_j \circ \gamma(f) = \delta^Q_j \circ \pi^Q \circ \gamma \circ p^P_j(f) =$$

$$= \psi \circ \delta^P_j \pi^P \circ p^P_j(f) = \psi \circ \delta^P_j \circ \pi^P(f) = 0$$

(b) The conditions (6.8) mean that $\Psi(\bar{P}_j) \subset \bar{Q}_j$ for $j = 1, 2, 3, \ldots$. Moreover if $F = \pi^P(f) \in \bar{P}_j$ the equality $\delta^Q_j \circ \pi^Q \circ \gamma(f) = \psi \circ \delta^P_j \pi^P(f) = 0$ shown in (a) means that $\sim (\Psi(F)) = \psi(\sim (F))$ i.e. that the homomorphism ψ is induced by Ψ.

(c) To show that Ψ is injective provided ψ is injective and P is analytic we may procede like in (a) but in the opposite direction, i.e we shall show by induction on j the sequence of implications

$$\big((\pi^Q \circ \gamma(f) \in \bar{Q}_{j+1}) \wedge (\pi^P(f) \in \bar{P}_j)\big) \Rightarrow \big((\pi^P \circ \gamma)(f) \in \bar{P}_{j+1}\big)\,. \tag{6.9}$$

$j = 1$: Let $f = \phi_1 \cdot \ldots \cdot \phi_n$ and $\Psi_0(\phi_1) \cdot \ldots \cdot \Psi_0(\phi_n) \in \bar{Q}_2$. We claim that $f \in \bar{P}_2$ i.e. that $(\delta^P_1 \circ \pi^P)(\phi_1 \cdot \ldots \cdot \phi_n) = 0$. Since ψ is injective this is the same as to show that $(\psi \circ \delta^P_1 \circ \pi^P)(\phi_1 \cdot \ldots \cdot \phi_n) = 0$. With the 'initial condition' in the form $\psi \circ \delta^P_1 \circ \pi^P(\phi) = \delta^Q_1 \circ \pi^Q(\Psi_0(\phi))$ one gets

$$\psi \circ \delta^P_1 \circ \pi^P(\phi_1 \cdot \ldots \cdot \phi_n) = (\delta^Q_1 \circ \pi^Q)(\Psi_0(\phi_1) \cdot \ldots \cdot \Psi_0(\phi_n))$$

$$= (\psi \circ \delta^P_1 \circ \pi^P)(\phi_1) + \ldots + (\psi \circ \delta^P_1 \circ \pi^P)(\phi_n)) = \delta^Q_1 \circ \pi^Q(\Psi_0(\phi_1)) + \ldots +$$

$$+ \delta^Q_1 \circ \pi^Q(\Psi_0(\phi_n)) = \delta^Q_1 \circ \pi^Q(\Psi_0(\phi_1) \cdot \ldots \cdot \Psi_0(\phi_n)) = 0$$

Induction step: For $j \geq 2$, $f \in F^{\mathbf{R}}_{E(P)}$ let $(\delta^Q_j \pi^Q \circ \gamma)(f) = 0$ and let $\delta^P_{j-1} \circ \pi^P(f) = 0$. We claim that $\delta^P_j \circ \pi^P(f) = 0$ and since ψ is injective this amounts to showing that $\psi \circ \delta^P_j \circ \pi^P(f) = 0$.

By the induction assumption $(f) \in (\overline{F^{\mathbf{R}}_{E(P)}})_j$, thus consecutive use of (6.6) (b), Lemma 6.7, (6.6)(a) and (6.6)(c), taking into account that $(\delta^Q_j \circ \pi^Q \circ \gamma)(f) = 0$ yields

$$\psi \circ \delta^P_j \circ \pi^P(f) = \psi \circ \delta^P_j \circ \pi^P \circ p^P_j(f) =$$

$$\delta^Q_j \circ \pi^Q \circ \gamma \circ p^P_j(f) = \delta^Q_j \circ \pi^Q \circ p^Q_j \circ \gamma(f) = \delta^Q_j \circ \pi^Q \circ \gamma(f) = 0 \qquad \blacksquare$$

Remark 6.9. Note that in the case of analytic Q there is at most one function $\Psi_0 : E(P) \to E(Q)$ satisfying (6.4).

Indeed, by Proposition 5.5 (b) each equivalence class mod $\bar{Q}_2$ contains at most one representative from $E(Q)$.

Theorem 6.10. *Let P be a topological group of strings with Lie algebra $L(P)$. Let S_Δ be the subgroup of $\exp(L(P))$ introduced in Corollary 5.8. There exists a surjective $\mathbf{R}$-homomorphism $\pi : P \to S_\Delta$ with $ker(\pi) = \bar{P}_\infty$, i.e. the following sequence is exact*

$$0 \quad \to \quad \bar{P}_\infty \quad \stackrel{j}{\to} \quad P \quad \stackrel{\pi}{\to} \quad S_\Delta \quad \to \quad 0\,. \tag{6.9}$$

Proof. First observe that $L(P) = L(P/\bar{P}_\infty)$. Also note that by Proposition 4.6, $P/\bar{P}_\infty$ is analytic. It follows that the proof reduces to showing that for P analytic, P is isomorphic to S_Δ. Assume thus that P is analytic and let

$$\alpha : L(S_\Delta) \ni ((\sim f_1), (\sim f_2), \ldots) \to (d_1(f_1), d_2(f_2), \ldots) \in L(P)$$

be the isomorphism introduced in Proposition 5.7 and let τ be defined by (5.13). Since $\Delta \subset E(S_\Delta)$, we may regard τ as an $E(S_\Delta)$-valued map and note that it provides an initial condition for the isomorphism α^{-1}. Indeed, for $f \in S_\Delta$ of the form $((\sim \phi), 0, 0, \ldots)$, we have $d_1(f) = (\sim \phi)$. Thus by Theorem 6.8 (a) there exists a homomorphism $\Psi : P \to S_\Delta$ extending τ and induced by α^{-1}.

We claim that $\Delta = E(S_\Delta)$. Indeed, since P is analytic and α^{-1} is injective also Ψ is injective by Theorem 6.8 (b). Let $\phi \in E(S_\Delta)$. Then $\phi = \Psi(f)$ and for $s_1, s_2 \in \mathbf{R}$:

$$\Psi((s_1 + s_2) \star f) = (s_1 + s_2) \star \Psi(f) = (s_1 \star \Psi(f))(s_2 \star \Psi(f) =$$

$$= \Psi(s_1 \star f)\Psi(s_2 \star f) = \Psi((s_1 \star f)(s_2 \star f)).$$

Since Ψ is injective this implies $f \in E(P)$ and thus $\phi \in \tau(E(P)) = \Delta$.

Note now that τ is surjective by the definition of Δ, and by Proposition 5.5 (b) it is injective as well. Thus τ and its inverse provide the initial conditions for α^{-1} and α respectively. Since both groups P and S_Δ are analytic we obtain (applying Theorem 6.8) two homomorphisms: $\Psi : P \to S_\Delta$ induced by the initial condition τ and $\bar{\Psi} : S_\Delta \to P$ induced by the initial condition τ^{-1}. Clearly $\bar{\Psi}$ has to be the inverse of Ψ. ■

Comment: Theorem 6.10 corresponds (in a different context) to the Theorem 28 of [12].

Remark 6.11. The exact sequence (6.9) where S_Δ is equipped with the product topology restricted from $\exp(L(P))$ need not be an extension of topological groups. A posteriori S_Δ may be considered with the quotient topology resulting from the identification with $P/\bar{P}_\infty$. This gives to the extension (6.9) also topological meaning.

7. Concluding remarks

The axioms defining a module M over a ring Q necessarily imply that the underlying group of M is abelian. It appears that relaxation of the distributivity condition

$$(q_1 + q_2) \cdot m = (q_1 \cdot m)(q_2 \cdot m)$$

to the one assumed only for m in some generating subset of the underlying group of M furnishes (in the case $Q = \mathbf{R}$) a family of interesting noncommutative models - the groups of strings.

As pointed out in Section 3, there are two parallel important classes of examples of groups of strings : those connected with topological groups and those connected with Lie algebras. The interplay of these two kinds, suggested by Proposition 3.3, is the potential source of a new approach to Lie group theory. It constitutes the main motivation for studying groups of strings.

Groups of strings fit into the existing body of Lie theory, e.g. they explain the algebraic background of some classical formulas. For instance the formulas (4.2) (for $P = S(G)$ equipped with the compact-open topology, G a Lie group, $j = 1$ and f being the product of two one-parameter subgroups of G) is connected with the classical Trotter formula defining the sum of two one-parameter subgroups:

$$\phi_{x+y}(t) = \lim_{n\to\infty} \left(\phi_x \left(\frac{t}{n} \right) \phi_y \left(\frac{t}{n} \right) \right)^n$$

where

$$i : g \ni x \to \phi_x \in \Lambda(G)$$

is the mapping assigning to a left invariant vector field x its integral curve ϕ_x satisfying $\phi_x(0) = e$.

Similarly for $j = 2$ and $f = \{\phi_1, \phi_2\}$, (4.2) explains another Trotter formula defining the Lie bracket of one-parameter subgroups.

Groups of strings give a continuous transition from combinatorial to differential-geometric parts of Lie theory. We shall illustrate this claim by the following example (since the full explanation would demand too much of space we shall leave it without further comments):

Example 7.1. Let (E, e) be an $\mathbf{R}$-set and let $F_E^{\mathbf{R}}$ be the free group of strings over E (cf. Definition 3.6). Moreover let $F_E^{\mathbf{R}}$ be equipped with a suitable topology (e.g. the pull back topology resulting from the injective homomorphism $\beta : F_E^{\mathbf{R}} \to MG_{\tilde{E}}$ into the Magnus group). Then the $\mathbf{N}$-gradation Lie algebra of $F_E^{\mathbf{R}}$ is isomorphic to the free Lie algebra over (E, e).

It is our belief that nonanalyticity (understood as irregular behaviour of the Lie functor) observed for some important models of 'infinite dimensional Lie groups' like C^∞ diffeomorphisms of smooth compact manifolds (cf [7]) has as background nonanalyticity of the corresponding topological groups of strings. We illustrate this by the following conjecture:

For M be a compact smooth manifold, let G be the group of all smooth diffeomorphisms of M and let g be the Lie algebra of all smooth vector fields on M. Let us consider each of these objects with the suitable C^∞ topology. Let $S(G)$ and $S(g)$ be the corresponding topological groups of strings.

Conjecture. The group of strings $S(g)$ is isomorphic to the maximal string subgroup of $\exp(L(S(G)))$. In particular (by Theorem 6.10) the following sequence is exact:

$$0 \;\to\; (\overline{S(G)})_\infty \;\xrightarrow{j}\; S(G) \;\xrightarrow{\pi}\; S(g) \;\to\; 0.$$

Note that by Proposition 3.3, for Banach-Lie groups, $S(G)$ and $S(g)$ are isomorphic.

The groups of strings, as presented here, evolved from its initial status described in the paper [12] where the idea of abstract group of strings and their connection with **N**-gradation Lie algebras already appeared. The lack of some important steps (e.g. Proposition 5.5 (b) and Proposition 5.7 of the present paper) left [12] with no final conclusions such as Theorems 6.8 and 6.10 of this note. The present treatment is more algebraic. It puts an emphasis on the B-C-H formula which is of algebraic nature, whereas in [12] the proofs are based on the Trotter formulas which demand a topology. The actual presentation gives much better insight into the subject and possibilities of applications.

Aknowledgement: The author is deeply indebted to the referee. His remarkable efford enabled to remove errors and mistakes and resulted in many improvements of the text. In particular the previous form of Theorem 6.8 and of the proof of Theorem 6.10 has been changed.

References

[1] Boseck, H., G. Czichowski, and K.-P. Rudolph, "Analysis on Topological Groups - General Lie Theory," Teubner, Leipzig 1981.

[2] Bourbaki, N., "Elements of Mathematics, Lie Groups and Lie Algebras," Part I, Ch. 1-3. Herman Publishers, 1971, English edition Addison Wesley 1975.

[3] Dynkin, E. B., *Normirovannye algebry Lie i analticeskye gruppy*, Uspehi Mat. Nauk **5** (1950), 134–186 [in Russian].

[4] Hilbert, D., *Beweis für die Darstellbarkeit der ganzen Zahlen durch eine Anzahl n-ter Potenzen (Waringsches Problem)*, Math. Ann. **67** (1909), 281–300.

[5] Hausner, M., and J. Schwartz, "Lie Groups Lie Algebras," Gordon and Breach Science Publishers, 1969.

[6] Magnus, W., A. Karrass, and D. Solitar "Combinatorial Group Theory: Presentation of Groups in Terms of Generators and Relations," John Wiley & Sons, 1966.

[7] Milnor, J., *Remarks on infinite dimensional Lie groups*, Proc. Summer School on Quantum Gravity, B. DeWitt Ed., Les Houches, 1983.

[8] Palis, J., *Vector fields generate few diffeomorphisms*, Bull. Am. Math. Soc. **80** (1973), 503–505.

[9] Riss, J., *Élémentes de calcul différentiel et théorie des distributions sur les groupes abéliens localement compacts*, Acta Math. **89** (1953), 45–105.

[10] Serre, J.-P., "Lie Algebras and Lie Groups," Benjamin, 1965.

[11] Su-Ching Chen, and R. Yoh, *The category of generalized Lie groups*, Trans. Amer. Math. Soc. **199** (1974), 281–294.

[12] Wojtyński, W., *One-parameter subgroups and the B-C-H formula*, Studia Math. **111** (1994), 163–185.

[13] —, *Quasinilpotent Banach-Lie algebras are Baker–Campbell–Hausdorff*, J. Funct. Anal. **153**, (1998) 405–413.

[14] —, *An introduction to string Lie theory,* in: Martin Peinador, E., and J. Núñez Garcia (Eds.) "Nuclear Groups and Lie Groups," Heldermann-Verlag, 2001; 227–238.

[15] —, *Effective integration of Lie algebras*, submitted.

Wojciech Wojtyński
Institute of Mathematics,
Warsaw University,
02-097 Banacha 2,
Warsaw, Poland
wowo@mimuw.edu.pl
and
Mathematical Institute
Polish Academy of Sciences
00-950, Śniadeckich 8
Warsaw, Poland

Received August 18, 2001
and in final form December 20, 2002