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Abelian Complex Structures on Solvable Lie Algebras

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Abstract. We obtain a characterization of the Lie algebras admitting abelian complex structures in terms of certain affine Lie algebras $\mathfrak{aff}(A)$, where A is a commutative algebra.

1. Introduction

An *abelian* complex structure on a real Lie algebra $\mathfrak{g}$ is an endomorphism of $\mathfrak{g}$ satisfying

$$J^2 = -I, \qquad [Jx, Jy] = [x, y], \quad \forall x, y \in \mathfrak{g}. \tag{1}$$

If G is a Lie group with Lie algebra $\mathfrak{g}$ these conditions imply the vanishing of the Nijenhuis tensor on the invariant almost complex manifold (G, J), that is, J is integrable on G.

Our interest arises from properties of the complex manifolds obtained by considering this class of complex structures on Lie algebras. For instance, an abelian hypercomplex structure on $\mathfrak{g}$, that is, is a pair of anticommuting abelian complex structures, gives rise to an invariant weak HKT structure (see [7] and [9]).

Abelian complex structures on Lie algebras were first considered in [1] where a construction is given starting with a 2-step nilpotent Lie algebra and applying successively a "doubling" procedure. It follows from results of [3] that $\mathfrak{aff}(\mathbb{C})$, the Lie algebra of the affine motion group of $\mathbb{C}$, is the unique 4-dimensional Lie algebra carrying an abelian hypercomplex structure. In [4] the particular class of H-type Lie algebras was studied in detail and a precise answer was given to the question of when such an algebra admits an abelian complex structure.

It was proved in [8] that a real Lie algebra admitting an abelian complex structure is necessarily solvable. In the present article we give a characterization of the solvable Lie algebras admitting an abelian complex structure in terms of certain affine Lie algebras $\mathfrak{aff}(A)$, A a commutative algebra (Theorem 5.3). These affine Lie algebras are natural generalizations of $\mathfrak{aff}(\mathbb{C})$ and the corresponding Lie groups are complex affine manifolds. It turns out, using the classification given in [13], that all 4-dimensional Lie algebras carrying abelian complex structures are central extensions of affine Lie algebras.

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In § 6. we study obstructions to the existence of abelian complex structures.

2. Complex structures on affine Lie algebras

A complex structure on a real Lie algebra $\mathfrak{g}$ is an endomorphism J of $\mathfrak{g}$ satisfying

$$J^2 = -Id, \qquad J[x,y] - [Jx,y] - [x,Jy] - J[Jx,Jy] = 0, \quad \forall x,y \in \mathfrak{g}. \tag{2}$$

Note that complex Lie algebras are those for which the endomorphism J satisfies the stronger condition

$$J^2 = -Id, \qquad J[x,y] = [x,Jy], \quad \forall x,y \in \mathfrak{g} \tag{3}$$

By a hypercomplex structure we mean a pair of anticommuting complex structures.

A rich family of Lie algebras carrying complex structures is obtained by considering a finite dimensional real associative algebra A and $\mathfrak{aff}(A)$ the Lie algebra $A \oplus A$ with Lie bracket given as follows:

$$[(a,b),(a',b')] = (aa' - a'a, ab' - a'b), \qquad a,b,a',b' \in A.$$

Let J be the endomorphism of $\mathfrak{aff}(A)$ defined by

$$J(a,b) = (b,-a), \qquad a,b \in A. \tag{4}$$

A computation shows that J defines a complex structure on $\mathfrak{aff}(A)$. Note that when A is a vector space with the trivial product structure $ab = 0$, $a,b \in A$ one obtains the abelian Lie algebra $\mathbb{R}^n \oplus \mathbb{R}^n$ with the standard complex structure $J(a,b) = (b,-a)$. Furthermore, if one assumes the algebra A to be a complex associative algebra, this extra assumption allows us to equip $\mathfrak{aff}(A)$ with a pair of anti-commuting complex structures. Indeed, the endomorphism K on $\mathfrak{aff}(A)$ defined by $K(a,b) = (-ia, ib)$ for $a,b \in A$ satisfies (2) and since $JK = -KJ$, J and K define a hypercomplex structure.

Proposition 2.1. $\mathfrak{aff}(A)$ *carries a natural hypercomplex structure for any complex associative algebra* A.

The Lie groups having Lie algebras $\mathfrak{aff}(A)$ carry invariant complex affine structures. Indeed, the bilinear map ∇ given by $\nabla_{(a,b)}(c,d) = (ac, ad)$ satisfies

$$\nabla_{(a,b)} J(c,d) = J\nabla_{(a,b)}(c,d), \qquad \nabla_{(a,b)}(c,d) - \nabla_{(c,d)}(a,b) = [(a,b),(c,d)]$$

and $R((a,b),(c,d)) = 0$ where

$$R((a,b),(c,d)) = \nabla_{[(a,b),(c,d)]} - [\nabla_{(a,b)}, \nabla_{(c,d)}]$$

is the curvature tensor. In particular, using results of Boyom [5], any such simply connected Lie group can be embedded as leaf of a left invariant lagrangian foliation in a symplectic Lie group.

3. Abelian complex structures

An *abelian* complex structure on a real Lie algebra $\mathfrak{g}$ is an endomorphism of $\mathfrak{g}$ satisfying

$$J^2 = -I, \qquad [Jx, Jy] = [x, y], \quad \forall x, y \in \mathfrak{g}. \tag{5}$$

By an abelian hypercomplex structure we mean a pair of anticommuting abelian complex structures.

We observe that one can rewrite condition (2) as follows

$$J([x, y] - [Jx, Jy]) = [Jx, y] - [x, Jy] \quad \forall x, y \in \mathfrak{g}. \tag{6}$$

Thus, abelian complex structures are integrable. Moreover, from (6) one has that if $[x, y] - [Jx, Jy] \neq 0$ for some x, y then the commutator subalgebra has dimension ≥ 2. In particular, if $\mathfrak{g}$ is a real Lie algebra with 1-dimensional commutator $[\mathfrak{g}, \mathfrak{g}]$ then every complex structure on $\mathfrak{g}$ is abelian (compare with Proposition 4.1 in [2]).

There exist algebraic restrictions to the existence of abelian complex structures. We recall the following result

Proposition 3.1. [8] *Let $\mathfrak{g}$ be a real Lie algebra admitting an abelian complex structure. Then $\mathfrak{g}$ is solvable.*

Given a complex structure J on a Lie algebra $\mathfrak{g}$, the endomorphism J extends to the complexification $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \oplus i\mathfrak{g}$ giving a splitting

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}^{1,0} \oplus \mathfrak{g}^{0,1}$$

where

$$\mathfrak{g}^{1,0} = \{X - iJX : X \in \mathfrak{g}\} \text{ and } \mathfrak{g}^{0,1} = \{X + iJX : X \in \mathfrak{g}\}$$

are complex Lie subalgebras of $\mathfrak{g}^{\mathbb{C}}$. Using (5) one verifies that abelian complex structures are those for which the subalgebras $\mathfrak{g}^{1,0}$ and $\mathfrak{g}^{0,1}$ are abelian, and conversely.

In order to give another characterization of abelian complex structures we need first to consider the following general class of complex structures on matrix algebras.

Let V be a real vector space, $\dim V = 2n$, and fix a complex endomorphism I of V (i.e. $I^2 = -Id$). Let us denote by L_I, (resp. R_I) the endomorphism of $\mathfrak{gl}(V)$ defined as $L_I(u) = I \circ u$ (resp. $R_I(u) = u \circ I$), $u \in \mathfrak{gl}(V)$. It is straightforward to show that L_I (resp. R_I) defines a complex structure on $\mathfrak{gl}(V)$, that is, it satisfies (2). Moreover, the subalgebra $\mathfrak{gl}_{\mathbb{C}}(V)$ of endomorphisms of V commuting with I is L_I and R_I invariant and the restriction of L_I or R_I to this subalgebra satisfies (3).

Consider next an arbitrary Lie algebra $\mathfrak{g}$ and assume that J is an endomorphism of $\mathfrak{g}$ satisfying $J^2 = -Id$. In particular, $\dim \mathfrak{g} = 2n$. Consider on $\mathfrak{g}^*$ the induced endomorphism, that we denote also by J, given by $J\alpha = -\alpha J$, $\alpha \in \mathfrak{g}^*$. According to the previous observation, R_{-J} is integrable on $\mathfrak{gl}(\mathfrak{g})$ and L_J is integrable on $\mathfrak{gl}(\mathfrak{g}^*)$. It follows after a computation that J is an abelian complex structure on $\mathfrak{g}$ if and only if the adjoint representation

$$\text{ad} : (\mathfrak{g}, J) \to (\mathfrak{gl}(\mathfrak{g}), R_{-J})$$

is holomorphic, that is, $\operatorname{ad}(Jx) = R_{-J}(\operatorname{ad}(x))$ for all $x \in \mathfrak{g}$. Equivalently, the coadjoint representation

$$\operatorname{ad}^* : (\mathfrak{g}, J) \to (\mathfrak{gl}(\mathfrak{g}^*), L_J)$$

is holomorphic, that is, $\operatorname{ad}^*(Jx) = L_J(\operatorname{ad}^*(x))$ for all $x \in \mathfrak{g}$. This paragraph can be summarized as follows:

Theorem 3.2. *Let J be a complex structure on the real Lie algebra $\mathfrak{g}$. Then the following conditions are equivalent:*

i) *J is abelian.*

ii) *The complex subalgebras $\mathfrak{g}^{1,0}$ and $\mathfrak{g}^{0,1}$ of $\mathfrak{g}^{\mathbb{C}}$ are abelian.*

iii) *The adjoint representation ad: $(\mathfrak{g}, J) \to (\mathfrak{gl}(\mathfrak{g}), R_{-J})$ is holomorphic.*

iv) *The coadjoint representation ad^* : $(\mathfrak{g}, J) \to (\mathfrak{gl}(\mathfrak{g}^*), L_J)$ is holomorphic.*

The simplest examples of non abelian Lie algebras carrying abelian complex structures are provided by:

Examples 3.3. i) $\mathfrak{aff}(\mathbb{R})$, the Lie algebra of the affine motion group of $\mathbb{R}$ (the bidimensional non-abelian Lie algebra), $\mathfrak{aff}(\mathbb{R}) = span\{x, y\}$, with bracket $[x, y] = x$ and J given by $Jx = y$.

ii) $\mathbb{R} \times \mathfrak{h}_n$, where $\mathfrak{h}_n$ stands for the $2n+1-$dimensional Heisenberg Lie algebra, $\mathbb{R} \times \mathfrak{h}_n = span\{w, z, x_i, y_i, i = 1, \dots, n\}$, with non zero bracket $[x_i, y_i] = z$ and J given by $Jz = w$, $Jx_i = y_i, i = 1, \dots, n$.

The Lie algebras introduced in i) and ii) have one dimensional commutator. Moreover, every Lie algebra with one dimensional commutator is a trivial central extension of one of these (see Theorem 4.1 in [2]). Hence we obtained the following result:

Proposition 3.4. *Every even dimensional Lie algebra with one dimensional commutator carries an abelian complex structure.*

The next family of examples will play a crucial role in the characterization given in Theorem 5.3.

Example 3.5. Consider the Lie algebra $\mathfrak{aff}(A)$ defined in §2. where A is a commutative algebra. Let J be the complex structure on $\mathfrak{aff}(A)$ defined by equation (4). Then one verifies that J is an abelian complex structure. We note that when $A = \mathbb{R}$ or $A = \mathbb{C}$, we obtain the Lie algebra of the affine motion group of either $\mathbb{R}$ or $\mathbb{C}$. Moreover, if A is a complex commutative algebra then the complex structure $K(a, b) = (ia, -ib)$ which anticommutes with J is also abelian, hence in this case we obtain an abelian hypercomplex structure.

Proposition 3.6. *If A is a complex commutative algebra then the natural hypercomplex structure on $\mathfrak{aff}(A)$ is abelian.*

The 4-dimensional Lie algebras admitting abelian complex structures are essentially affine algebras $\mathfrak{aff}(A)$ for some commutative algebra A (see Proposition 5.1). In the general situation these algebras are also involved as building blocks (Theorem 5.3).

A particular case of the construction just considered occurs when one assumes A to be the set of complex matrices of the form

$$\begin{pmatrix} 0 & a_1 & a_2 & \dots & a_{k-1} & a_k \\ 0 & 0 & a_1 & \dots & a_{k-2} & a_{k-1} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a_1 & a_2 \\ 0 & 0 & 0 & \dots & 0 & a_1 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

A is commutative and $\mathfrak{aff}(A)$ is k-step nilpotent, therefore existence of abelian complex structures imposes no restriction on the degree of nilpotency (compare with [8]).

Proposition 3.7. *For any positive integer k there exists a k-step nilpotent Lie algebra carrying an abelian hypercomplex structure.*

We observe that all known examples of Lie algebras carrying abelian complex structures are two-step solvable, but we do not know if this holds in general.

4. Main theorem

In this section we give a characterization of solvable Lie algebras admitting abelian complex structures. It is our aim to show that the building blocks of such algebras are the affine algebras considered in Example 3.5).

Proposition 4.1. *Let $\mathfrak{s}$ be a solvable Lie algebra with an abelian complex structure J admitting a decomposition $\mathfrak{s} = \mathfrak{u} + J\mathfrak{u}$ with $\mathfrak{u}$ an abelian ideal. Then $(\mathfrak{s}/\mathfrak{z}, J)$ is holomorphically isomorphic to $\mathfrak{aff}(A)$ for some commutative algebra A.*

Proof. We note first that if $\mathfrak{s}$ is as in the statement then $\mathfrak{u} \cap J\mathfrak{u} \subset \mathfrak{z}$, $\mathfrak{z}$ the center of $\mathfrak{s}$. Indeed, if $x = Jx' \in \mathfrak{u} \cap J\mathfrak{u}$ then $[x, u] = 0, u \in \mathfrak{u}$, and

$$[x, Ju] = [Jx', Ju] = [x', u] = 0, \qquad u \in \mathfrak{u}$$

showing that $x \in \mathfrak{z}$.

Let $A = \{\mathrm{ad}\,(Jx) \ : \ x \in \mathfrak{u}\}$ and let $f : \mathfrak{s} \to \mathfrak{aff}(A)$ be defined by

$$f(x + Jy) = (\mathrm{ad}\,(Jy)\ , \mathrm{ad}\,(Jx)).$$

If $x' + Jy' = x + Jy$ then both, $J(x' - x)$ and $J(y' - y)$, belong to $\mathfrak{z}$, hence f is well defined. Clearly, $\mathfrak{z}$ is contained in the kernel of f, since $x + Jy \in \mathfrak{z}$ implies that x and Jy are in $\mathfrak{z}$. Conversely, if $\mathrm{ad}\,(Jy) = 0 = \mathrm{ad}\,(Jx)$, then x and Jy are in $\mathfrak{z}$ since J is abelian. We verify next that f is a Lie algebra homomorphism. If $x + Jy,\ x' + Jy' \in \mathfrak{s}$, then

$$f[x + Jy, x' + Jy'] = (0, \mathrm{ad}\,(J([x, Jy'] + [Jy, x']))).$$

On the other hand,

$$[(\mathrm{ad}\,(Jy), \mathrm{ad}\,(Jx)), (\mathrm{ad}\,(Jy'), \mathrm{ad}\,(Jx'))] = (0, \mathrm{ad}\,(Jy)\,\mathrm{ad}\,(Jx')|_{\mathfrak{u}} - \mathrm{ad}\,(Jy')\,\mathrm{ad}\,(Jx)).$$

Now,

$$\begin{aligned} \mathrm{ad}\,(J([x,Jy']+[Jy,x']))|_{\mathfrak{u}} &= -\mathrm{ad}\,([x,Jy']+[Jy,x'])J|_{\mathfrak{u}} \\ &= \mathrm{ad}\,(Jy)\,\mathrm{ad}\,(Jx')|_{\mathfrak{u}} - \mathrm{ad}\,(Jy')\,\mathrm{ad}\,(Jx)|_{\mathfrak{u}} \end{aligned}$$

since $\mathrm{ad}\,(Jx)J|_{\mathfrak{u}} = 0$ for $x \in \mathfrak{u}$, and

$$\mathrm{ad}\,(J([x,Jy']+[Jy,x']))|_{J\mathfrak{u}} = 0 = \mathrm{ad}\,(Jy)\,\mathrm{ad}\,(Jx')|_{J\mathfrak{u}} - \mathrm{ad}\,(Jy')\,\mathrm{ad}\,(Jx)|_{J\mathfrak{u}}.$$

Therefore,

$$\mathrm{ad}\,(J([x,Jy']+[Jy,x'])) = \mathrm{ad}\,(Jy)\,\mathrm{ad}\,(Jx') - \mathrm{ad}\,(Jy')\,\mathrm{ad}\,(Jx)$$

showing that f induces a Lie algebra isomorphism between $\mathfrak{s}/\mathfrak{z}$ and $\mathfrak{aff}(A)$. Moreover, f is holomorphic since

$$fJ(x+Jy) = f(-y+Jx) = (\mathrm{ad}\,(Jx)\ ,-\mathrm{ad}\,(Jy)).$$

We show next, using a case by case verification, that the 4-dimensional Lie algebras admitting abelian complex structures are fully described by the previous proposition, that is, they are central extensions of affine algebras.

5. The 4-dimensional case

The 4-dimensional solvable Lie algebras $\mathfrak{s}$ carrying complex structures were classified in [13] when $\dim[\mathfrak{s},\mathfrak{s}] \leq 2$ and in [11] when dimension of $\dim[\mathfrak{s},\mathfrak{s}] = 3$. ¿From this classification one verifies that the complex structures such that $\mathfrak{s}^{1,0}$ and $\mathfrak{s}^{0,1}$ are abelian occur only when $\dim[\mathfrak{s},\mathfrak{s}] \leq 2$ (this also follows from Proposition 6.1 below). They all appear in the classification given in [13] and are denoted by $S_1, S_2, S_8, S_9, S_{10}, S_{11}$ there. We list them below:

1. S_0 : $\mathfrak{s} = \mathbb{R}^4$.
2. S_1 : $\mathfrak{s} = \mathfrak{h}_1 \oplus \mathbb{R}$, a direct sum of ideals, where $\mathfrak{h}_1$ is the 3-dimensional Heisenberg algebra (see example 3.3 ii).
3. S_2 : $\mathfrak{s} = \mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R}^2$, a direct sum of ideals.
4. S_8 : $\mathfrak{s} = \mathfrak{aff}(\mathbb{R}) \oplus \mathfrak{aff}(\mathbb{R})$, a direct sum of ideals.
5. S_9 : $\mathfrak{s} = \mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R}^2$, a semidirect sum (adjoint representation)
6. S_{10} :$\mathfrak{s} = \mathfrak{aff}(\mathbb{R}) \oplus \mathfrak{aff}(\mathbb{R})$, a semidirect product of algebras (adjoint representation).
7. S_{11} : $\mathfrak{s} = \mathfrak{aff}(\mathbb{C})$, the complexification of $\mathfrak{aff}(\mathbb{R})$.

The above Lie algebras, modulo their center, coincide with $\mathfrak{aff}(A)$ for certain commutative algebras A which are listed below:

1. S_0 : $A = 0$ with the trivial structure.
2. S_1 : $A = \mathbb{R}$, $\mathbb{R}$ with the trivial structure.

3. S_2 : $A = \mathbb{R}$, $\mathbb{R}$ with the standard structure.

4. S_8 : $A = \{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, a, b \in \mathbb{R}\}$.

5. S_9 : $A = \{\begin{pmatrix} a & 0 \\ b & a \end{pmatrix}, a, b \in \mathbb{R}\}$.

6. S_{10} : $A = \{\begin{pmatrix} a & b \\ b & a \end{pmatrix}, a, b \in \mathbb{R}\}$.

7. S_{11} : $A = \mathbb{C}$, $\mathbb{C}$ with the standard structure.

The above paragraph can be summarized as follows:

Proposition 5.1. *Let $\mathfrak{s}$ be a 4-dimensional Lie algebra admitting an abelian complex structure.*
Then $\mathfrak{s}/\mathfrak{z}$ is isomorphic to $\mathfrak{aff}(A)$ for some commutative algebra A.

The next example shows that Proposition 4.1 does not exhaust the class of Lie algebras carrying abelian complex structures.

Example 5.2. Let $\mathfrak{s} = \mathbb{R}x_1 \oplus \mathbb{R}y_1 \oplus \cdots \oplus \mathbb{R}x_k \oplus \mathbb{R}y_k \oplus \mathfrak{v}$ with $\dim \mathfrak{v} = 2n$, $k, n \geq 1$. Fix a real endomorphism J of $\mathfrak{s}$ such that $J^2 = -I$ and

$$Jx_j = y_j, \quad j = 1, \ldots, k, \qquad\qquad J\mathfrak{v} \subset \mathfrak{v}. \tag{7}$$

Let $T_1, \ldots, T_k$ be a commutative family of endomorphisms of $\mathfrak{v}$ satisfying

$$T_i T_j = -T_i\, J|_{\mathfrak{v}}\, T_j\, J|_{\mathfrak{v}} \quad \text{for all } i, j.$$

This condition is satisfied, for instance, if T_i commutes with $J|_{\mathfrak{v}}$ for all $i = 1, \ldots, k$. Define a bracket on $\mathfrak{s}$ as follows

$$[x_j, v] = T_j J v, \qquad\qquad [y_j, v] = T_j v \quad \text{for all } v \in \mathfrak{v} \tag{8}$$

and extend it by skew-symmetry. It turns out that $\mathfrak{s}$ equipped with this bracket is a Lie algebra and J becomes an abelian complex structure on $\mathfrak{s}$. Observe that $\mathfrak{s}$ is not in general an affine algebra, but it has the following property: there exists a J-stable ideal $\mathfrak{s}_1 = \mathfrak{v}$ isomorphic to $\mathfrak{aff}(\mathbb{R}^n)$ such that $\mathfrak{s}/\mathfrak{s}_1$ is isomorphic to $\mathfrak{aff}(\mathbb{R}^k)$, where both, $\mathbb{R}^n$ and $\mathbb{R}^k$, are equipped with the trivial algebra structure. The general situation is described by the following theorem.

Theorem 5.3. *Let $\mathfrak{s}$ be a real Lie algebra and let J be an abelian complex structure on $\mathfrak{s}$. Then there exists an increasing sequence $\{0\} = \mathfrak{s}_0 \subset \mathfrak{s}_1 \subset \cdots \subset \mathfrak{s}_{r-1} \subset \mathfrak{s}_r = \mathfrak{s}$ of J-stable ideals of $\mathfrak{s}$ such that $\mathfrak{s}_j/\mathfrak{s}_{j-1}$ is holomorphically isomorphic to a central extension of $\mathfrak{aff}(A_j)$ with the abelian complex structure given by equation (4), for some commutative algebra A_j, $1 \leq j \leq r$.*

Proof. We proceed by induction on $\dim \mathfrak{s}$. The theorem is trivially satisfied if $\dim \mathfrak{s} = 2$. If $\dim \mathfrak{s} > 2$, we assume that the theorem is true for all Lie algebras of dimension strictly less than $\dim \mathfrak{s}$. Since J is abelian, $\mathfrak{s}$ must be solvable (Proposition 3.1). Let $\mathfrak{u}$ be a non zero abelian ideal in $\mathfrak{s}$, then $\mathfrak{s}_1 = \mathfrak{u} + J\mathfrak{u}$ is a solvable Lie algebra satisfying the hypothesis of Proposition 4.1. Hence $\mathfrak{s}_1$ is holomorphically isomorphic to a central extension of $\mathfrak{aff}(A_1)$ with the abelian complex structure given by equation (4) for some commutative algebra A_1. If $\mathfrak{s}_1 = \mathfrak{s}$ we are done. Otherwise, since $\mathfrak{s}_1$ is a J-invariant ideal of $\mathfrak{s}$, the inductive hypothesis applies to the Lie algebra $\mathfrak{s}/\mathfrak{s}_1$ with the induced abelian complex structure. ■

6. Some obstructions

As a consequence of Proposition 1.5 in [12], if $\mathfrak{n}$ is a nilpotent Lie algebra admitting an abelian complex structure then $[\mathfrak{n},\mathfrak{n}]$ must have codimension ≥ 3. On the other hand, we exhibited in §5.2 solvable Lie algebras $\mathfrak{s}$ with $[\mathfrak{s},\mathfrak{s}]$ of codimension $2k$, $k \geq 1$, admitting abelian complex structures. The following result implies that if $[\mathfrak{s},\mathfrak{s}]$ has codimension 1 and $\dim \mathfrak{s} > 2$ then abelian complex structures do not exist on $\mathfrak{s}$.

Proposition 6.1. *Let $\mathfrak{s}$ be a solvable Lie algebra such that $[\mathfrak{s},\mathfrak{s}]$ has codimension 1 in $\mathfrak{s}$. If $\mathfrak{s}$ admits an abelian complex structure then $\mathfrak{s}$ is isomorphic to $\mathfrak{aff}(\mathbb{R})$.*

Proof. Let J be an abelian complex structure on $\mathfrak{s}$ and set

$$\mathfrak{s} = \mathbb{R}a \oplus \mathfrak{n}$$

where $\mathfrak{n} = [\mathfrak{s},\mathfrak{s}]$ and a can be chosen so that $Ja \in \mathfrak{n}$. Then

$$\begin{aligned} \mathfrak{n} &= \operatorname{Im}\operatorname{ad}(a) + [\mathfrak{n},\mathfrak{n}] = \operatorname{Im}\operatorname{ad}(Ja) + [\mathfrak{n},\mathfrak{n}] \\ &= \mathbb{R}[a,Ja] + \operatorname{Im}\operatorname{ad}(Ja)|_{\mathfrak{n}} + [\mathfrak{n},\mathfrak{n}] = \mathbb{R}[a,Ja] + [\mathfrak{n},\mathfrak{n}] \end{aligned} \tag{9}$$

and we get $\mathfrak{n}' \subset [\mathfrak{n},\mathfrak{n}']$, hence $\mathfrak{n}' = [\mathfrak{n},\mathfrak{n}']$. Now, $\mathfrak{n}$ is nilpotent, so we must have $\mathfrak{n}' = \{0\}$ and therefore $\mathfrak{n} = \mathbb{R}[a,Ja]$. This implies the result. ■

As a consequence of the above proposition we obtain a large family of Lie algebras which do not carry abelian complex structures. In fact, consider a nilpotent Lie algebra $\mathfrak{n}$, $\dim \mathfrak{n} > 1$, admitting a non-singular derivation D and set $\mathfrak{s} = \mathbb{R}a \oplus \mathfrak{n}$ where the action of a on $\mathfrak{n}$ is given by D. It follows from the proposition that there is no abelian complex structure on $\mathfrak{s}$. A particular case of this construction is given by Damek-Ricci extensions of H-type Lie algebras (see [6]). In particular, the solvable Lie algebras corresponding to the rank one symmetric spaces of non-compact type [10] do not admit abelian complex structures, though it is well known that they do admit complex structures (equation (2)).

Abelian complex structures are frequent on two-step nilpotent Lie algebras (see [1] and [4]), but even in this case the following restriction is encountered:

Proposition 6.2. *Let $\mathfrak{n}$ be a two-step nilpotent Lie algebra such that*

$$2\dim[\mathfrak{n},\mathfrak{n}] = n(n-1), \ \text{where } n = \dim\mathfrak{n} - \dim\mathfrak{z} \geq 3$$

and $\mathfrak{z}$ is the center of $\mathfrak{n}$. Then $\mathfrak{n}$ does not admit an abelian complex structure.

Proof. We assume that $\mathfrak{n}$ admits an abelian complex structure J. Fix a Hermitian inner product $\langle\,,\rangle$ on $\mathfrak{n}$ and consider the orthogonal decomposition $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v}$. Being J abelian, it follows that both, $\mathfrak{z}$ and $\mathfrak{v}$, are J-stable. Define a linear map $j : \mathfrak{z} \to \text{End}\,(\mathfrak{v})$, $z \mapsto j_z$, where j_z is determined as follows:

$$\langle j_z v, w\rangle = \langle z, [v,w]\,\rangle, \quad \forall v, w \in \mathfrak{v}. \tag{10}$$

Observe that j_z, $z \in \mathfrak{z}$, are skew-symmetric so that $z \to j_z$ defines a linear map $j : \mathfrak{z} \to \mathfrak{so}(\mathfrak{v})$ and the restriction of j to $[\mathfrak{n},\mathfrak{n}]$ is injective. It follows from Lemma 1.1 in [4] that J commutes with j_z for all $z \in \mathfrak{z}$, which is a contradiction. In fact, our assumption on $\dim[\mathfrak{n},\mathfrak{n}]$ says that the map $j : \mathfrak{z} \to \mathfrak{so}(\mathfrak{v})$ is surjective. Therefore, since $n \geq 3$, the only endomorphisms of $\mathfrak{v}$ commuting with all j_z, $z \in \mathfrak{z}$, are real multiples of the identity. ■

Recall that a two-step nilpotent Lie algebra $\mathfrak{n}$ is said to be free, of rank n, when $\mathfrak{z} = [\mathfrak{n},\mathfrak{n}]$ and $2\dim\mathfrak{z} = n(n-1)$, where $n = \dim\mathfrak{n} - \dim\mathfrak{z}$. The above result says that the free two-step nilpotent Lie algebras of rank $n \geq 3$ do not admit abelian complex structures.

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