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An Equivalence between Categories of Modules for Generalized Kac-Moody Lie Algebras

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Abstract. An equivalence between categories of modules for a generalized Kac-Moody algebra and modules for an appropriate parabolic subalgebra is shown. In particular, properties such as the irreducibility and complete reducibility of a module whose weights satisfy certain conditions can be determined by restriction to a subalgebra.

1. Introduction

Generalized Kac-Moody algebras are Lie algebras determined by certain symmetrizable matrices as in [11], and [1]. Let $\mathfrak{g}$ be a generalized Kac-Moody algebra. In [10] some examples of irreducible highest weight modules for $\mathfrak{g}$ which are generalized Verma modules are given. These generalized Verma modules are modules that are induced from an irreducible highest weight module for a "parabolic" subalgebra $\mathfrak{p} = (\mathfrak{g}_S + \mathfrak{h}) \oplus \mathfrak{u}^+$, where $\mathfrak{g}_S$ is a generalized Kac-Moody subalgebra of $\mathfrak{g}$ (often $\mathfrak{g}_S$ can be chosen to be a Kac-Moody or semi-simple Lie algebra). In this paper, we show that the converse is also true, that in fact any irreducible $\mathfrak{g}$ module whose weights satisfy the appropriate conditions is a generalized Verma module for a parabolic subalgebra. More generally, it is shown that a subcategory of the category $\mathcal{O}$ of $\mathfrak{g}$-modules is equivalent to a subcategory of the category $\mathcal{O}$ of the Lie algebra $\mathfrak{r} = \mathfrak{g}_S + \mathfrak{h}$.

This result is most useful in cases where the subalgebra $\mathfrak{g}_S$ can be chosen to be a semi-simple or Kac-Moody Lie algebra, so that some of the representation theory for the generalized Kac-Moody algebra can be reduced to that of the semi-simple or Kac-Moody subalgebra.

In §2 of this paper, we review the definition of generalized Kac-Moody algebra, introduce notation and define the category of $\mathfrak{g}$ modules that we use. Section 3 contains the statement and proof of the main theorem. The equivalence of categories is proven by using the functor determined by inducing a module for the parabolic subalgebra $\mathfrak{p}$ to $\mathfrak{g}$, and the functor determined by considering the set of elements of a $\mathfrak{g}$-module V annihilated by $\mathfrak{u}^+$. The method of proof is similar to that in [4] (see also [5] and [12]). Section 4 contains some applications.

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2. The Setting

We recall the definition of a generalized Kac-Moody algebra. The definition given here is similar to Borcherds' original definition [1] and agrees with the definition in [9]. A variety of other definitions exist, for example [2], [11], [13]. We refer the reader to [11] and [13] for general definitions involving Lie algebras defined by symmetrizable matrices, of which generalized Kac-Moody algebras are an example. Definitions and proofs regarding the basic structure of generalized Kac-Moody algebras are presented in [1],[8], [9] and [10]. For the convenience of the reader, we include the definitions that differ from the definitions for the more familiar Kac-Moody algebras.

We define a generalized Kac-Moody algebra over $\mathbb{C}$ by specifying generators and relations. Let $\mathbb{Z}_+$ denote the nonnegative integers. Let I be a countable index set, $A = (a_{ij})_{i,j\in I}$ a matrix with entries in $\mathbb{R}$ satisfying the conditions:

1. A is symmetrizable.
2. If $a_{ii} > 0$, then $a_{ii} = 2$.
3. For all $k \neq j$ $a_{jk} \leq 0$ and for all $i \in I$ such that $a_{ii} > 0$, $a_{ij} \in -\mathbb{Z}_+$ for all $j \in I$.

Let $\mathfrak{g}(A)'$ be the Lie algebra with generators $e_i, f_i, h_i, i \in I$ and defining relations: For all $i, j \in I$

$$[h_i, h_j] = 0, [e_i, f_j] = \delta_{ij} h_i$$
$$[h_i, e_j] = a_{ij} e_j, [h_i, f_j] = -a_{ij} f_j.$$

For all $i \in I$ such that $a_{ii} > 0$

$$(\operatorname{ad} e_i)^{-a_{ij}+1} e_j = 0, (\operatorname{ad} f_i)^{-a_{ij}+1} f_j = 0.$$

Finally, for all $i, j \in I$ such that $a_{ij} = a_{ji} = 0$

$$[e_i, e_j] = 0, [f_i, f_j] = 0.$$

Define degree derivations ∂_i, $i \in I$, on $\mathfrak{g}(A)'$ by taking $\partial_i(e_j) = \delta_{ij}$, $\partial_i(f_j) = -\delta_{ij}$ and $\partial_i(h_j) = 0$. Let $\mathfrak{d}$ be the abelian Lie algebra generated by the ∂_i, $i \in I$.

Definition 1. The generalized Kac-Moody algebra $\mathfrak{g} = \mathfrak{g}(A)$ is defined to be the semidirect product $\mathfrak{g}(A)' \ltimes \mathfrak{d}$.

We also call any Lie algebra of the form $\mathfrak{g}(A)/\mathfrak{c}$ where $\mathfrak{c}$ is a central ideal a generalized Kac-Moody algebra. Borcherds' Monster Lie algebra is an example of this kind.

The Lie algebra $\mathfrak{g}$ has a triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$, where $\mathfrak{n}^{\pm}$ is the subalgebra generated by the e_i (resp. the f_i), and the Cartan subalgebra $\mathfrak{h}$ is the abelian subalgebra spanned by the h_i and the ∂_i. There is a symmetric invariant bilinear form on $\mathfrak{g}$, and on $\mathfrak{h}^*$, which will be denoted by $(\cdot,\cdot)$ in both cases.

The *roots* of $\mathfrak{g}$ are the nonzero elements φ of $\mathfrak{h}^*$ such that $\mathfrak{g}^\varphi = \{x \in \mathfrak{g} | [h,x] = \varphi(h)x \text{ for all } h \in \mathfrak{h}\}$ is nonzero. Denote the set of roots of $\mathfrak{g}$ by Δ. There is a system of positive roots Δ_+ and $\Delta = \Delta_+ \cup -\Delta_+$. Roots with positive square norm are called real, and roots with non-positive square norm are called imaginary. Denote the set of real roots by Δ_R. The set of simple roots, denoted $\{\alpha_i\}_{i\in I} \subset \mathfrak{h}^*$, are defined by the conditions

$$[h, e_i] = \alpha_i(h)e_i \text{ for all } h \in \mathfrak{h}.$$

Note that $\alpha_j(h_i) = a_{ij}$, and $\alpha_j(\partial_i) = \delta_{ij}$ for all $i,j \in I$ and that the α_i for $i \in I$ are linearly independent. The bilinear form is chosen so that $(\alpha_i, \alpha_j) = a_{ij}$ for all $i,j \in I$. Note that for $i \in I$ $(\alpha_i,\alpha_i) = a_{ii}$ may be non-positive, so that simple roots may be imaginary.

Given an element $\mu \in \mathfrak{h}^*$ of the form $\lambda - \sum_{i\in I} n_i\alpha_i$, $n_i \in \mathbb{Z}_+$, we define the *depth* $d_\lambda(\mu) = \sum_{i\in I} n_i$.

Given a $\mathfrak{g}$-module and $\lambda \in \mathfrak{h}^*$ let

$$M^\lambda = \{v \in M | h \cdot v = \lambda(h)v, \text{for all } h \in \mathfrak{h}\}.$$

The elements $\lambda \in \mathfrak{h}^*$ for which $M^\lambda \neq 0$ are called the *weights* of M. Given a $\mathfrak{g}$-module M let $P(M)$ denote the set of weights of M. A weight $\lambda \in (\mathfrak{h})^*$ is *dominant* if $(\lambda, \alpha_i) \in \mathbb{R}$ and $(\lambda,\alpha_i) \geq 0$ for all $i \in I$. A weight λ is called *integral* if $\lambda(h_i) \in \mathbb{Z}_+$ for all $i \in I$ such that $a_{ii} > 0$. Denote by P_+ the set of dominant integral weights.

Definition 2. A $\mathfrak{g}$-module M is a *standard* module if M is a highest weight module with highest weight $\mu \in P_+$ and highest weight vector v such that:

1. for $i \in I$, if $(\mu,\alpha_i) = 0$ then $f_i \cdot v = 0$;
2. if α_i $(i \in I)$ is real then $f_i^{n_i+1} \cdot v = 0$, where $n_i = 2(\mu,\alpha_i)/(\alpha_i,\alpha_i)$ (necessarily a nonnegative integer).

Given any $\mu \in P_+$ there is a unique (up to isomorphism) standard irreducible highest weight $\mathfrak{g}$-module of highest weight μ, denote this module by $L_{\mathfrak{g}}(\mu)$ [9].

Fix S a distinguished subset of I containing $\{i \in I | \alpha_i \in \Delta_R\}$. Denote by $\mathfrak{g}_S$ the generalized Kac-Moody subalgebra of $\mathfrak{g}$ associated to the matrix $(a_{ij})_{i,j\in S}$. Assume that S is chosen so that the resulting matrix $(a_{ij})_{i,j\in S}$ is indecomposible. Note that if $S = \{i \in I | \alpha_i \in \Delta_R\}$ then $\mathfrak{g}_S$ is a Kac-Moody algebra. In the case of the monster Lie algebra we can choose $\mathfrak{g}_S = \mathfrak{sl}_2$; (see §4 below).

We make the following definitions: $\Delta^S = \Delta \cap \coprod_{i\in S} \mathbb{Z}\alpha_i$, $\Delta^S_+ = \Delta_+ \cap \Delta^S$ and $\Delta^S_- = \Delta_- \cap \Delta^S$. Denote by $\mathfrak{h}_S$ the span of the h_i, $i \in S$. There is a root space decomposition

$$\mathfrak{g}_S = \coprod_{\varphi\in\Delta^S_+} \mathfrak{g}^\varphi \oplus \mathfrak{h}_S \oplus \coprod_{\varphi\in\Delta^S_-} \mathfrak{g}^\varphi.$$

Define the following subalgebras of $\mathfrak{g}$:

$$\mathfrak{u}^+ = \coprod_{\varphi \in \Delta_+ \backslash \Delta_+^S} \mathfrak{g}^\varphi; \quad \mathfrak{u}^- = \coprod_{\varphi \in \Delta_- \backslash \Delta_-^S} \mathfrak{g}^\varphi; \quad \mathfrak{r} = \mathfrak{g}_S + \mathfrak{h}.$$

Let $\mathfrak{p} = \mathfrak{r} \oplus \mathfrak{u}^+$, the "parabolic" subalgebra of $\mathfrak{g}$ determined by S. Then

$$\mathfrak{g} = \mathfrak{u}^- \oplus \mathfrak{r} \oplus \mathfrak{u}^+ = \mathfrak{u}^- \oplus \mathfrak{p}.$$

If M is a $\mathfrak{r}$-module we define a $\mathfrak{p}$-module which we also call M by letting $\mathfrak{u}^+$ act as zero on M. Note that if M is irreducible, or completely reducible as a $\mathfrak{g}_S$ or $\mathfrak{p}$-module, then the $\mathfrak{p}$-module M is irreducible, or completely reducible as well.

We recall the definition of generalized Verma module of [6]. Let $\lambda \in P_+$, and consider the standard (irreducible) highest weight $\mathfrak{r}$-module $L(\lambda)$ associated to λ (this is a standard module for $\mathfrak{g}_S$). The highest weight space of $L(\lambda)$ (as a $\mathfrak{g}_S$-module) is a weight space for $\mathfrak{h}$, with weight λ. Let $\mathfrak{u}^+$ act trivially on $L(\lambda)$; this gives $L(\lambda)$ the structure of an irreducible $\mathfrak{p}$-module. Define the generalized Verma module $V^{L(\lambda)}$ to be the induced module $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} L(\lambda)$.

Definition 3. Let $\mathcal{O}_\mathfrak{g}$ denote the category of $\mathfrak{g}$-modules that are weight modules whose weight spaces are finite dimensional, and whose set of weights lies in a finite union of sets of the form

$$D(\lambda) = \{\lambda - \sum_{i \in I} n_i \alpha_i | \lambda \in \mathfrak{h}^*\}.$$

The morphisms are $\mathfrak{g}$-module homomorphisms.

Definition 4. Let $\mathcal{O}_\mathfrak{g}^S$ be the full subcategory of $\mathcal{O}_\mathfrak{g}$ whose objects are $\mathfrak{g}$-modules $V \in \mathcal{O}_\mathfrak{g}$ such that every weight $\mu \in P(V)$ satisfies $(\mu, \alpha_i) > 0$ for all $i \in I \backslash S$.

Let $M \in \mathcal{O}_\mathfrak{g}$. Recall that a highest weight series for M is an increasing filtration

$$(0) \subset M_1 \subset M_2 \subset \cdots$$

of submodules of M satisfying $\bigcup_{i=0}^\infty M_i = M$. For $M_{i+1} \neq M_i$ the module M_{i+1}/M_i is a highest weight module. We extend a well known result of [6] to modules in $\mathcal{O}_\mathfrak{g}$, where the index set I may be countably infinite.

Proposition 2.1. *Let $M \in \mathcal{O}_\mathfrak{g}$. Then M has a highest weight series $\{M_i\}_{i \in \mathbb{Z}}$.*

Proof. Let $\lambda_1, \lambda_2, \ldots \lambda_r$ be a set of weights of such that

$$P(M) \subset D(\lambda_1) \cup D(\lambda_2) \ldots \cup D(\lambda_r).$$

With out loss of generality we may assume that the sets $D(\lambda_i)$ are distinct. Then any $\mu \in P(M)$ is of the form $\mu = \lambda_j - \beta$ for some j $1 \leq j \leq r$, with

$\beta = \sum n_i \alpha_i$ where $n_i \in \mathbb{Z}_+$. Define $d(\mu) = \sum i n_i$. Since $d(\mu) = N$ implies $n_i = 0$ for sufficiently large i the set $\{\mu \in P(M) | d(\mu) = N\}$ is finite for all $N \in \mathbb{Z}_+$. Now a filtration of M can be constructed using the same argument as in the case where I is finite [6]: Let μ_1 be a weight in $P(M)$ with minimal $d(\mu_1)$. Choose a nonzero weight vector $x_1 \in M^{\mu}$. By the minimality of $d(\mu_1)$ the vector x_1 is a highest weight vector. Denote by M_1 the highest weight module generated by x_1. To construct a module M_2 with $(0) \subset M_1 \subset M_2$ and M_2/M_1 a highest weight module repeat the above argument for the module M/M_1. The series $d(\mu_k)$ is increasing, and can only remain at one value for finitely many k. Thus eventually $M^{\mu} = M_n^{\mu} = M_{n+1}^{\mu} \dots$ for every $\mu \in P(M)$. This construction gives the desired highest weight series. ■

In order to prove the main theorem we will use the following result from category theory [7].

Lemma 2.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be categories. Let F be a functor from $\mathcal{A}$ to $\mathcal{B}$. The categories are equivalent if and only if*

1. *F is full, that is, for all objects V_1, V_2 of $\mathcal{A}$ the mapping*

$$Hom_{\mathcal{A}}(V_1, V_2) \to Hom_{\mathcal{B}}(F(V_1), F(V_2))$$

   described by $f \mapsto F(f)$ is surjective.
2. *F is faithful, that is, the above mapping is injective.*
3. *For every object M of $\mathcal{B}$ there is an object V of $\mathcal{A}$ such that M is isomorphic to $F(V)$.* ■

3. The equivalence of categories

We will show that the category $\mathcal{O}_{\mathfrak{g}}^S$ is equivalent to the category $\mathcal{O}_{\mathfrak{r}}^S$.

If $V \in \mathcal{O}_{\mathfrak{g}}^S$ let $V^{\mathfrak{u}^+}$ denote the the set of elements of V annihilated by the action of $\mathfrak{u}^+$. The space $V^{\mathfrak{u}^+}$ is an $\mathfrak{g}_s$-module since $U(\mathfrak{g}_s)V^{\mathfrak{u}^+} \subset V^{\mathfrak{u}^+}$. Likewise $V^{\mathfrak{u}^+}$ is an $\mathfrak{r}$-module and a $\mathfrak{p}$-module. Note that the subspace $V^{\mathfrak{u}^+}$ considered as an $\mathfrak{r}$-module is in $\mathcal{O}_{\mathfrak{r}}^S$.

The functors between our categories $\mathcal{O}_{\mathfrak{g}}^S$ and $\mathcal{O}_{\mathfrak{r}}^S$ are simply restriction and induction.

Definition 5. The functor $R : \mathcal{O}_{\mathfrak{g}}^S \to \mathcal{O}_{\mathfrak{r}}^S$ is defined as follows: $R(V) = V^{\mathfrak{u}^+}$ for $V \in \mathcal{O}_{\mathfrak{g}}^S$ and given a pair V, U of modules in $\mathcal{O}_{\mathfrak{g}}^S$ and a module homomorphism $f : V \to U$ define $R(f) = \bar{f}$ to be the restriction of the homomorphism f to $V^{\mathfrak{u}^+}$.

Definition 6. The functor $\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} : \mathcal{O}_{\mathfrak{r}}^S \to \mathcal{O}_{\mathfrak{g}}^S$ is defined by $\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}}(N) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} N$ for N an object of $\mathcal{O}_{\mathfrak{r}}^S$, considered as a $\mathfrak{p}$-module with $\mathfrak{u}^+$ acting as multiplication by 0, and $\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}}(f) = 1 \otimes f$ for f a morphism in $\mathcal{O}_{\mathfrak{r}}^S$, considered as a morphism in $\mathcal{O}_{\mathfrak{p}}^S$.

Lemma 3.1. *Let $M \in \mathcal{O}_{\mathfrak{r}}^S$. Then $(Ind_{\mathfrak{p}}^{\mathfrak{g}} M)^{\mathfrak{u}^+} = 1 \otimes M$.*

Proof. By definition M is annihilated by $\mathfrak{u}^+$ as a $\mathfrak{p}$-module, so $1 \otimes M \subset (\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} M)^{\mathfrak{u}^+}$.

Using the Poincare-Birkhoff-Witt theorem, we have an isomorphism $\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} M \simeq U(\mathfrak{u}^-) \otimes_{\mathbb{C}} M$. Consider $U(\mathfrak{u}^-) \otimes_{\mathbb{C}} M = (U(\mathfrak{u}^-)\mathfrak{u}^- \otimes_{\mathbb{C}} M) \oplus (1 \otimes M)$.

Let $w \in U(\mathfrak{u}^-)\mathfrak{u}^- \otimes_{\mathbb{C}} M$ be nonzero. Since M is a weight module, we may assume w is a linear combination of vectors of the form $f_{i_1} f_{i_2} \dots f_{i_j} \otimes v$, where $v \in M$ is a weight vector of weight λ, and the $f_{i_1} f_{i_2} \cdots f_{i_j}$, $i_1 \le i_2 \le \cdots \le i_j$ is an element of the Poincare-Birkhoff-Witt basis of $U(\mathfrak{u}^-) \subset U(\mathfrak{n}^-)$. We will consider the elements $f_{i_1} f_{i_2} \cdots f_{i_j} \otimes v$, $i_1 \le i_2 \le \cdots \le i_j$ to be basis vectors of the vector space $U(\mathfrak{u}^-)\mathfrak{u}^- \otimes v$. Let $w = f_{i_1} f_{i_2} \dots f_{i_j} \otimes v$. Not all of the indices $i_1, i_2, \dots i_j$ are in S, by the definition of $\mathfrak{u}^-$. Assume that $i_k \in I \backslash S$, we will show that $e_{i_k} w \ne 0$. The vector w can be rewritten as $w = f_{i_1} f_{i_2} \cdots f_{i_{k-1}} f_{i_k}^n f_{i_{k+1}} \cdots f_{i_j} \otimes v$ where $n \ge 1$, $i_i \le i_2 \le \cdots \le i_{k-1} < i_k < i_{k+1} \le \cdots \le i_j$, renaming the indices if necessary. Using the commutation relations of the Lie algebra,

$$
\begin{aligned}
e_{i_k} w &= f_{i_1} \cdots f_{i_{k-1}} \left(e_{i_k} f_{i_k}^n \right) \cdots f_{i_j} \otimes v \\
&= f_{i_1} \cdots f_{i_{k-1}} \left(f_{i_k}^n e_{i_k} + n f_{i_k}^{n-1} h_{i_k} + -a_{i_k i_k} \frac{(n-1)n}{2} f_{i_k}^{n-1} \right) f_{i_{k+1}} \cdots f_{i_j} \otimes v \\
&= f_{i_1} \cdots f_{i_k}^n e_{i_k} f_{i_{k+1}} \cdots f_{i_j} \otimes v + n f_{i_1} \cdots f_{i_k}^{n-1} h_{i_k} f_{i_{k+1}} \cdots f_{i_j} \otimes v \\
&\quad + -a_{i_k i_k} \frac{(n-1)n}{2} f_{i_1} \cdots f_{i_k}^{n-1} f_{i_{k+1}} \cdots f_{i_j} \otimes v.
\end{aligned}
$$

Since v is in M it is annihilated by $e_{i_k} \in \mathfrak{u}^+$, and one has

$$
f_{i_1} \cdots f_{i_k}^n e_{i_k} f_{i_{k+1}} \cdots f_{i_j} \otimes v = f_{i_1} \cdots f_{i_k}^n f_{i_{k+1}} \cdots f_{i_j} e_{i_k} \otimes v = 0.
$$

It is also true that

$$
h_{i_k} f_{i_{k+1}} \cdots f_{i_j} \otimes v = f_{i_{k+1}} \cdots f_{i_j} h_{i_k} \otimes v + \sum_{s=1}^{j-k} -a_{i_k i_{k+s}} f_{i_{k+1}} \cdots f_{i_j} \otimes v.
$$

Thus

$$
\begin{aligned}
& e_{i_k} w \\
&= n f_{i_1} \cdots f_{i_k}^{n-1} f_{i_{k+1}} \cdots f_{i_j} h_{i_k} \otimes v + n \sum_{s=1}^{j-k} -a_{i_k i_{k+s}} f_{i_1} \cdots f_{i_k}^{n-1} f_{i_{k+1}} \cdots f_{i_j} \otimes v \\
&\quad - a_{i_k i_k} \frac{(n-1)n}{2} f_{i_1} \cdots f_{i_k}^{n-1} f_{i_{k+1}} \cdots f_{i_j} \otimes v \\
&= n\lambda(h_{i_k}) f_{i_1} \cdots f_{i_k}^{n-1} f_{i_{k+1}} \cdots f_{i_j} \otimes v - n \sum_{s=1}^{j-k} a_{i_k i_{k+s}} f_{i_1} \cdots f_{i_k}^{n-1} f_{i_{k+1}} \cdots f_{i_j} \otimes v \\
&\quad - a_{i_k i_k} \frac{(n-1)n}{2} f_{i_1} \cdots f_{i_k}^{n-1} f_{i_{k+1}} \cdots f_{i_j} \otimes v \\
&= \left[n\lambda(h_{i_k}) + n \sum_{s=1}^{j-k} -a_{i_k i_{k+s}} + -a_{i_k i_k} \frac{(n-1)n}{2} \right] f_{i_1} \cdots f_{i_k}^{n-1} f_{i_{k+1}} \cdots f_{i_j} \otimes v. \\
&\ne 0.
\end{aligned}
$$

The above expression is nonzero because $\lambda(h_{i_k})>0$, $a_{i_k i_{k+s}}\leq 0$ for all $1\leq s\leq j-k$ and $a_{i_k i_k}\leq 0$ for $i_k\in I\backslash S$. Let $w'=f_{i'_1}\cdots f_{i'_l}\otimes v$, $i'_1\leq i'_2\leq\cdots i'_l$ be a basis vector not equal to w. If f_{i_k} does not occur in w' we have $e_{i_k}w'=0$, and if f_{i_k} does occur in w' then the lists $i_1\leq i_2\leq\cdots\leq i_j$ and $i'_1\leq i'_2\leq\cdots\leq i'_l$ do not agree, so by the above computation $e_{i_k}w$ is not in the linear span of $e_{i_k}w'$. Now consider an element $w\in U(\mathfrak{u}^-)\mathfrak{u}^-\otimes v$ that is a linear combination of the basis vectors $f_{i_1}f_{i_2}\dots f_{i_j}\otimes v$, $i_1\leq i_2\cdots\leq i_j$, choose an index $i_k\in I\setminus S$ that appears in the first term of w. Then $e_{i_k}w$ is a linear combination of the basis vectors, with nonzero first term, so is nonzero. We conclude that w is not annihilated by $\mathfrak{u}^+$, that $(\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}}M)^{\mathfrak{u}^+}\subset 1\otimes M$, and finally that $(\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}}M)^{\mathfrak{u}^+}=1\otimes M$. ∎

Proposition 3.2. *Let $V\in\mathcal{O}_{\mathfrak{g}}^S$ be a highest weight module. Then $V=U(\mathfrak{g})\otimes_{U(\mathfrak{p})}V^{\mathfrak{u}^+}$.*

Proof. Let λ be the highest weight of V, with highest weight vector v_0. The induced module $U(\mathfrak{g})\otimes_{U(\mathfrak{p})}V^{\mathfrak{u}^+}$ is also a highest weight module for $\mathfrak{g}$. There is a natural map $\mu: U(\mathfrak{g})\otimes_{U(\mathfrak{p})}V^{\mathfrak{u}^+}\to U(\mathfrak{g})V^{\mathfrak{u}^+}=V$, which we will show is an isomorphism. The map μ is surjective because $V^{\mathfrak{u}^+}$ generates the module V, as $V^{\mathfrak{u}^+}$ contains the highest weight vector v_0. Let $W=\mathrm{Ker}\,\mu$. If W is nontrivial then W contains a weight vector $w\neq 0$ whose weight is of minimal depth. Applying an element of $\mathfrak{u}^+$ to $w\in W$ reduces the depth of w, so that w is annihilated by $\mathfrak{u}^+$. By Lemma 3.1 $w\in 1\otimes V^{\mathfrak{u}^+}$, which contradicts the fact that μ is injective when restricted to $1\otimes V^{\mathfrak{u}^+}$. ∎

Proposition 3.3. *Let $V\in\mathcal{O}_{\mathfrak{g}}^S$, then $V=U(\mathfrak{g})\otimes_{U(\mathfrak{p})}N$ for some $N\in\mathcal{O}_{\mathfrak{r}}^S$.*

Proof. Consider the highest weight series

$$(0)=V_0\subset V_1\subset V_2\subset\cdots$$

where $\bigcup_{i=0}^{\infty}V_i=V$. Note $\bigcup_{i=0}^{\infty}V_i^{\mathfrak{u}^+}=(\bigcup_{i=0}^{\infty}V_i)^{\mathfrak{u}^+}=V^{\mathfrak{u}^+}$.

We will prove the existence of an isomorphism by induction on $i\in\mathbb{Z}_+$. The module V_1 is by assumption a highest weight module so Proposition 3.2 shows the proposition holds in this case. Similarly, the result holds for the highest weight module V_{i+1}/V_i any $i\in\mathbb{Z}_+$. Assume that $V_i=U(\mathfrak{g})\otimes V_i^{\mathfrak{u}^+}$. Applying the five lemma to the diagram

$$\begin{tikzcd}
0 \arrow[r] & U(\mathfrak{g})\otimes V_i^{\mathfrak{u}^+} \arrow[r] \arrow[d, "\approx"'] & U(\mathfrak{g})\otimes V_{i+1}^{\mathfrak{u}^+} \arrow[r] \arrow[d] & U(\mathfrak{g})\otimes(V_{i+1}/V_i)^{\mathfrak{u}^+} \arrow[r] \arrow[d, "\approx"'] & 0 \\
0 \arrow[r] & V_i \arrow[r] & V_{i+1} \arrow[r] & V_{i+1}/V_i \arrow[r] & 0
\end{tikzcd}$$

implies $V_{i+1}=U(\mathfrak{g})\otimes V_{i+1}^{\mathfrak{u}^+}$, so by induction $V_i=U(\mathfrak{g})\otimes V_i^{\mathfrak{u}^+}$ for all $i\in\mathbb{Z}_+$. Thus

$$V=\bigcup U(\mathfrak{g})\otimes V_i^{\mathfrak{u}^+}=U(\mathfrak{g})\otimes\bigcup V_i^{\mathfrak{u}^+}=U(\mathfrak{g})\otimes V^{\mathfrak{u}^+}.$$ ∎

Theorem 3.4. *The category $\mathcal{O}_{\mathfrak{g}}^S$ is equivalent to the category $\mathcal{O}_{\mathfrak{r}}^S$.*

Proof. First we show that the functor $\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}}:\mathcal{O}_{\mathfrak{r}}^S\to\mathcal{O}_{\mathfrak{g}}^S$ is full. All objects V_1,V_2 of $\mathcal{O}_{\mathfrak{g}}^S$ are, by Proposition 3.2, of the form $V_i=U(\mathfrak{g})\otimes_{U(\mathfrak{p})}V_i^{\mathfrak{u}^+}$ and

$V_2 = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} V_2^{\mathfrak{u}^+}$. Then any $\mathfrak{g}$-homomorphism $g : V_1 \to V_2$ is determined by its action on $V_1^{\mathfrak{u}^+}$

$$g(U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} V_1^{\mathfrak{u}^+}) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \bar{g}(V_1^{\mathfrak{u}^+}).$$

Hence $g = 1 \otimes \bar{g}$.

Since $1 \otimes f = 1 \otimes g$ if and only if $f = g$ the functor $\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}}$ is faithful.

Finally, Proposition 3.3 says that for every object V of $\mathcal{O}_{\mathfrak{g}}^S$ there is an $N \in \mathcal{O}_{\mathfrak{r}}^S$ with $\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}}(N) = V$. The conditions of Lemma 2.2 are satisfied and the categories are equivalent. ■

4. Applications

Our first corollary includes Proposition 4.2 of [10].

Corollary 4.1. *Fix $S \subset I$. For $\mu \in P_+$ satisfying $(\mu, \alpha_i) > 0$ for $i \in I \backslash S$ the irreducible highest weight module $L_{\mathfrak{g}}(\mu)$ is a generalized Verma module:*

$$L_{\mathfrak{g}}(\mu) = Ind_{\mathfrak{p}}^{\mathfrak{g}} L(\mu),$$

here $L(\mu)$ is the irreducible highest weight $\mathfrak{p}$-module of highest weight μ, which is a standard irreducible module for the generalized Kac-Moody algebra $\mathfrak{g}_S$. Conversely, any generalized Verma module $V^{L(\mu)} = Ind_{\mathfrak{p}}^{\mathfrak{g}} L(\mu)$ is an irreducible highest weight $\mathfrak{g}$-module. ■

The following complete reducibility result appears in [14](where he uses the additional assumption that I is finite). For a Kac-Moody algebra $\mathfrak{l}$, $\mu \in \mathfrak{h}^*$, and an $\mathfrak{l}$-module M in $\mathcal{O}_{\mathfrak{l}}$ define the multiplicity $[M : L(\mu)]$ of the irreducible $\mathfrak{l}$-module $L(\mu)$ in M to be the number of proper factors of type $L(\mu)$ in any local composition series of M at μ see [13].

Corollary 4.2. *Let $\mathfrak{g}$ be a generalized Kac-Moody algebra. Take $S = \{i \in I | a_{ii} > 0\}$, so that $\mathfrak{g}_S$ is a Kac-Moody algebra. Let V be a module in the category $\mathcal{O}_{\mathfrak{g}}^S$. Then V is completely reducible and*

$$V = \bigoplus_{\mu \in P_+} [V^{\mathfrak{u}^+}, L(\mu)] L_{\mathfrak{g}}(\mu).$$

Proof. Since $\mathfrak{g}_S$ is a Kac-Moody algebra and $V^{\mathfrak{u}^+}$ is an integrable $\mathfrak{g}_S$-module the complete reducibility of integrable modules of a Kac-Moody algebra implies

$$V^{\mathfrak{u}^+} = \oplus_{\mu \in P_+} [V^{\mathfrak{u}^+}, L_{\mathfrak{r}}(\mu)] L_{\mathfrak{r}}(\mu).$$

Applying the main theorem gives

$$\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} V^{\mathfrak{u}^+} = \oplus_{\mu \in P_+} [V^{\mathfrak{u}^+}, L_S(\mu)] \mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} L_{\mathfrak{r}}(\mu)$$

So by Corollary 4.1 and Proposition 3.2

$$V = \oplus_{\mu \in P_+}[V^{\mathfrak{u}^+}, L_{\mathfrak{r}}(\mu)]L_{\mathfrak{g}}(\mu). \qquad \blacksquare$$

For a module $V \in \mathcal{O}_{\mathfrak{g}}$ recall the formal character is defined as

$$\mathrm{ch}(V) = \sum_{\lambda \in \mathfrak{h}^*} \mathrm{dim} V_\lambda e^\lambda.$$

Using Theorem 3.4, we can compute the character of a module $V \in \mathcal{O}_{\mathfrak{g}}^S$ by computing the product of the $\mathfrak{h}$ weight modules $U(\mathfrak{u}^-)$ and $V^{\mathfrak{u}^+}$.

Corollary 4.3. *Let $V \in \mathcal{O}_{\mathfrak{g}}^S$. Then*

$$ch(V) = ch(U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} V^{\mathfrak{u}^+}) = ch(U(\mathfrak{u}^-)) \cdot ch(V^{\mathfrak{u}^+}) \qquad \blacksquare$$

Depending upon the structure of the Lie algebra, $\mathrm{ch}(U(u^-))$ and $V^{\mathfrak{u}^+}$ may be computed more readily than $\mathrm{ch}(V)$. For example, choose S so that $a_{ij} \neq 0$ for all $i, j \in I \backslash S$. The subalgebra $\mathfrak{g}_S$ of $\mathfrak{g}$ and hence its universal enveloping algebra $\mathcal{U}(\mathfrak{g}_S)$ act via the adjoint action on $\mathfrak{g}$. Identify the root $-\alpha_j \in -\Delta_+$ of $\mathfrak{g}$ with the weight of the highest weight vector f_j of the standard $\mathfrak{g}_S$-module $L_{\mathfrak{g}_S}(-\alpha_i) = \mathcal{U}(\mathfrak{g}_S) \cdot f_j \subset \mathfrak{g}$. By Theorem 5.1 in [9] $\mathfrak{g} = \mathfrak{u}^+ \oplus (\mathfrak{g}_S + \mathfrak{h}) \oplus \mathfrak{u}^-$, where $\mathfrak{u}^-$ is the free Lie algebra over the vector space $\sum_{j \in I \backslash S} L(-\alpha_i)$.

Then

$$\mathrm{ch}(U(\mathfrak{u}^-)) = 1 - \sum_{j \in I \backslash S} \mathrm{ch}(L_S(-\alpha_j)) = \prod_{\varphi \in \Delta_+ \backslash \Delta_+^S} (1 - e^{-\varphi})^{\dim F(V)^{-\varphi}}$$

and

$$\mathrm{ch}(V) = \prod_{\varphi \in \Delta_+ \backslash \Delta_+^S} (1 - e^{-\varphi})^{\dim F(V)^{-\varphi}} \mathrm{ch}(V^{\mathfrak{u}^+}). \tag{1}$$

In the case of the monster Lie algebra, one can choose S so that $\mathfrak{g}_S = \mathfrak{sl}_2$. Then each $L(-\alpha_i)$ is a finite dimensional $\mathfrak{sl}_2$-module of highest weight $-\alpha_1$. For this case equation (1) specializes to Borcherds' product formula for the modular function $j(\tau)$ [2]. Formula (1) appears in [8],[9] and is used in [10] to further elucidate a portion of the monstrous moonshine phenomenon already studied in [2].

Computing the homology $H_n(\mathfrak{g}, V)$ for $V \in \mathcal{O}_{\mathfrak{g}}^S$ can also be reduced to computing the homology groups for modules for the smaller Lie algebra $\mathfrak{g}_S$.

Corollary 4.4. *For a g-module V in $\mathcal{O}_{\mathfrak{g}}^S$*

$$H_n(\mathfrak{g}, V^t) \simeq H_n(\mathfrak{p}, (V^{u^+})^t).$$

Proof. This corollary follows from Theorem 3.4 and Proposition 4.2 of [3]. $\blacksquare$

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