Journal of Lie Theory
Volume **14** (2004) 271–286

The Lie Algebra Cohomology of Jets[1]

Yunhyong Kim

Communicated by K.-H. Neeb

Abstract. Let $\mathfrak{g}$ be a finite-dimensional complex semi-simple Lie algebra. We present a new calculation of the continuous cohomology of the Lie algebra $z\mathfrak{g}[[z]]$. In particular, we shall give an explicit formula for the Laplacian on the Lie algebra cochains, from which we can deduce that the cohomology in each dimension is a finite-dimensional representation of $\mathfrak{g}$ which contains any irreducible representation of $\mathfrak{g}$ at most once.
MSC subject classification: 17B56, 17B65.

1. Introduction

Let $\mathfrak{g}$ be a complex semi-simple Lie algebra. In this paper, we shall calculate the cohomology of the Lie algebra $z\mathfrak{g}[[z]]$ of formal power series (with vanishing constant term) by an infinite dimensional analog of the method described in the paper by B. Kostant [5].

The Lie algebras of interest in Kostant's paper are nilpotent Lie subalgebras $\mathfrak{n}$ of a finite dimensional semi-simple Lie algebra $\mathfrak{g}$. Kostant identifies the cohomology of $\mathfrak{n}$ with the kernel of the Laplace operator on the cochains of $\mathfrak{n}$. The cochains of $\mathfrak{n}$ can be identified with a summand in the cochains of the larger algebra $\mathfrak{g}$, and Kostant defines an operator $\tilde{L}$ on the cochains of $\mathfrak{g}$ which restricts to the Laplace operator on the cochains of $\mathfrak{n}$. Calculating the kernel of $\tilde{L}$, which turns out to be easier than a direct calculation of the Laplace operator on the Lie algebra cochains of the nilpotent Lie algebra, yields the cohomology of the Lie subalgebra.

The Lie algebra $\mathfrak{a} = z\mathfrak{g}[[z]]$ with which we are concerned, is an infinite dimensional topologically nilpotent subalgebra of the algebra $\mathfrak{g}[[z]][z^{-1}]$ of formal loops in $\mathfrak{g}$. We would like to emulate Kostant's method in the following way. First we will define and describe a graded complex of "semi-infinite forms". On this complex, we will define an operator $\tilde{L}$. The Lie algebra cochains of $\mathfrak{a}$ will be shown to be a subcomplex of the semi-infinite forms. It will be proved that the operator $\tilde{L}$ restricts to the Laplace operator on the subcomplex. We will then give an explicit formula for $\tilde{L}$, which will finally enable us to calculate its kernel and give a description of the cohomology of $\mathfrak{a}$.

[1]Written while holding a research fellowship at the Abdus Salam International Centre for Theoretical Physics, Trieste, Italy and the Max-Planck Institut für Mathematik, Bonn, Germany.

ISSN 0949–5932 / $2.50

The final result of this paper follows already from the theorem of H. Garland and J. Lepowsky [3]. However, they make use of the weak Bernstein-Gelfand-Gelfand resolution and do not concern themselves with an explicit description of the Laplace operator on the Lie algebra cochains. The calculation in this paper gives an explicit formula. This formula is useful in connection with the smooth cochain cohomology of loop groups, which will hopefully be discussed in a separate paper. The discussion also relates the cohomology to semi-infinite cohomology which is of independent interest. Finally, the translation of Kostant's result into an infinite dimensional setting is appealing in itself, because it illustrates the power of his method.

To be precise, we will describe the Lie algebra $\mathfrak{a}$ in the following way: let G be a connected semi-simple real Lie group. Let $\mathfrak{g}$ denote the Lie algebra of G. Consider the Lie algebra $\mathfrak{A}$ consisting of Laurent polynomials of the form

$$\sum_{p\in\mathbf{Z}} A_p z^p$$

where p runs over the integers, A_p is in the complexification $\mathfrak{g}_{\mathbf{C}}$ of $\mathfrak{g}$, and such that $A_p = 0$ for all but a finite number of p's. Given $A = \sum_{p\in\mathbf{Z}} A_p z^p$ and $B = \sum_{p\in\mathbf{Z}} B_p z^p$, the Lie bracket is

$$[A, B] = \sum_{p,q\in\mathbf{Z}} [A_p, B_q] z^{p+q}.$$

Note that $\mathfrak{A}$ can be decomposed as

$$\mathfrak{A} = \overline{\mathfrak{a}} \oplus \mathfrak{g}_{\mathbf{C}} \oplus \mathfrak{a}$$

where $\overline{\mathfrak{a}}$ is the Lie algebra consisting of elements of the form $\sum_{k<0} A_k z^k$, and $\mathfrak{a}$ is the Lie algebra consisting of elements of the form $\sum_{k>0} A_k z^k$. We would like to calculate the cohomology of $\mathfrak{a}$ where we consider the pth degree cochains $A^*(\mathfrak{a})$ of the Lie algebra to be complex multilinear alternating continuous maps

$$\mathfrak{a} \times \cdots \times \mathfrak{a} \longrightarrow \mathbf{C}.$$

The Lie algebra $\mathfrak{a}$ can be related to the real Lie algebra quotient $\mathfrak{J}$ of the Lie algebra $\mathfrak{L}_0\mathfrak{g}$ of based loops in $\mathfrak{g}$ by those whose derivatives vanish to infinite order. The Lie algebra $\mathfrak{J}$ can be identified with the Lie algebra $z\mathfrak{g}[[z]]$. The cochains on the complexification of $\mathfrak{J}$ is a subspace of the cochains on $\mathfrak{a}$ and the inclusion induces an isomorphism on the level of cohomology. We will come back to all this in some more detail at the end of this paper.

In Section 2 we will describe the semi-infinite forms on $\mathfrak{A}$ and introduce an operator $\tilde{L}$ on these forms. We will also prove that $\tilde{L}$ restricts to the Laplacian on the cochains of $\mathfrak{a}$. Then, in Section 3 we will write down an explicit formula for $\tilde{L}$, which will enable us to calculate the kernel of the Laplacian. Section 4 will summarise the results following from the formula for the Laplacian. Finally, in Section 5, we will discuss the relationship between $\mathfrak{a}$ and the loop group of G.

Before we move on to the next section, note the following convention with regard to notation. There will be many infinite sums in this paper. To avoid ambiguity, every effort has been made to keep track of indices over which each sum is to be taken. However, it is to be understood that, as a rule, repeated indices will be summed over all the integers unless a restriction has been specified.

2. Semi-infinite forms

We will first define the cochain complex of "semi-infinite forms" on $\mathfrak{A}$. This definition follows the one for general graded Lie algebras found in [2] (Section 1).

Let $\mathfrak{g}$ and $\mathfrak{A}$ be as in the previous section and let $\mathfrak{g}_{\mathbf{C}}$ be the complexification of $\mathfrak{g}$ like before. Let c be the coxeter number of $\mathfrak{g}_{\mathbf{C}}$ and let $\langle\cdot,\cdot\rangle$ be $\frac{1}{2c}$ times the Killing form. Let the dimension of $\mathfrak{g}_{\mathbf{C}}$ be n. Choose an ordered orthonormal basis $\{\alpha_i\}_i^n$ of $\mathfrak{g}_{\mathbf{C}}$ with respect to $\langle\cdot,\cdot\rangle$. Let $e_{i,k} = \alpha_i z^k$. Then $\{e_{i,k}\}_{i,k\in\mathbf{Z}}$ form a basis of $\mathfrak{A}$. Denote by $\{e^{i,k}\}_{i,k\in\mathbf{Z}}$ the corresponding dual basis elements in $\mathfrak{A}^*$. Define the space $\wedge^d_\infty(\mathfrak{A})$ of semi-infinite forms of degree d as the complex linear space spanned by formal symbols of the form $\omega = e^{i_0,k_0}\wedge e^{i_1,k_1}\wedge\cdots\wedge e^{i_p,k_p}\wedge\cdots$ such that there exists $N(\omega)\in\mathbf{Z}$ for each ω, so that, for all $p > N(\omega)$, $k_p n + i_p - n = d - p$ and such that

$$e^{i_0,k_0}\wedge e^{i_1,k_1}\wedge\cdots\wedge e^{i_{p-1},k_{p-1}}\wedge e^{i_{p+1},k_{p+1}}\wedge e^{i_p,k_p}\wedge e^{i_{p+2},k_{p+2}}\wedge\cdots$$

$$= -e^{i_0,k_0}\wedge e^{i_1,k_1}\wedge\cdots\wedge e^{i_{p-1},k_{p-1}}\wedge e^{i_p,k_p}\wedge e^{i_{p+1},k_{p+1}}\wedge e^{i_{p+2},k_{p+2}}\cdots$$

Given any $x = \sum_q x_q e_{j_q,l_q}$ in $\mathfrak{A}$ and $x' = \sum_q x_q e^{j_q,l_q}$ in $\mathfrak{A}^*$, there are operators $\iota(x)$ and $\epsilon(x')$ on $\wedge^*_\infty(\mathfrak{A})$ given by

$$\iota(x)(\omega) = \sum_{p,q}(-1)^p x_q e^{i_p,k_p}(e_{j_q,l_q}) e^{i_0,k_0}\wedge\cdots\wedge\widehat{e^{i_p,k_p}}\wedge\cdots$$

where $\widehat{e^{i_p,k_p}}$ means that the term will be omitted, and

$$\epsilon(x')(\omega) = \sum_q x_q(e^{j_q,l_q}\wedge\omega).$$

In order to simplify the notation, we will write $\iota(e_{i,k})$ and $\epsilon(e^{i,k})$ as $\iota_{i,k}$ and $\epsilon^{i,k}$ respectively. These operators serve to define $\wedge^*_\infty(\mathfrak{A})$ as a module of the Clifford algebra on $\mathfrak{A}\oplus\mathfrak{A}^*$ associated to the pairing $\langle x, x'\rangle$ for $x\in\mathfrak{A}$ and $x'\in\mathfrak{A}^*$. That is, the anti-commutator $[\iota_{i,k},\epsilon^{j,m}]_+ = \delta_{i,j}\delta_{k,m}$, where $\delta_{i,j} = 0$ if $i\neq j$ and $\delta_{i,i} = 1$. Also note that $\wedge^*_\infty(\mathfrak{A}) = \oplus_d \wedge^d_\infty$ is bi-graded: apart from the degree there is a second grading by *energy* where the energy $E(\omega)$ of ω above is defined as $\Sigma_{k>0} k\epsilon_{i,k}\iota_{i,k}(\omega) - \Sigma_{k\leq 0} k\iota_{i,k}\epsilon_{i,k}(\omega) = E(\omega)\omega$. Note that this makes sense and that $E(\omega)$ is an integer.

Now let

$$:\iota_{i,k}\epsilon^{j,m} := \begin{cases} \iota_{i,k}\epsilon^{j,m} & \text{if } k\leq 0, \\ -\epsilon^{j,m}\iota_{i,k} & \text{if } k>0 \end{cases} \tag{1}$$

For each $x\in\mathfrak{A}$, there is an operator $\mathcal{L}(x)$ on $\wedge^*_\infty(\mathfrak{A})$, defined by

$$\mathcal{L}(e_{i,k}) = \sum C^p_{iq} : \iota_{p,s}\epsilon^{q,s-k} :$$

where C_{iq}^p are the structure constants with respect to the basis $\{\alpha_p\}$, i.e.,

$$[\alpha_i, \alpha_q] = \sum_p C_{iq}^p \alpha_p.$$

Note that this operator makes sense on semi-infinite forms. Although this is an infinite sum as it is written, only a finite number of terms are non-zero on any semi-infinite form. Since we had taken an orthonormal basis with respect to the Killing form, the structure constants C_{iq}^p are anti-symmetric in the three indices i, p, q. This, along with the identity $[\iota_{i,k}, \epsilon^{j,m}]_+ = \delta_{i,j}\delta_{k,m}$, implies that we could just as well have written $\mathcal{L}(e_{i,k}) = \sum C_{iq}^p \iota_{p,s}\epsilon^{q,s-k}$. Again, we will simplify $\mathcal{L}(e_{i,k})$ to $\mathcal{L}_{i,k}$.

Remark 2.1. Given a finite sum $\Sigma_i^k a_i b_i$ it is clear that $[\Sigma a_i b_i, c] = \Sigma[a_i, c]_+ b_i - \Sigma a_i [b_i, c]_+$ and that $[\Sigma a_i b_i, c] = \Sigma[a_i, c]b_i + \Sigma a_i[b_i, c]$. However, we will be dealing with infinite sums. In this case the above identities only hold if all sums and terms make sense as operators on the semi-infinite forms. It is for this reason that in Proposition 2.2 we will write simply $\mathcal{L}(e_{i,k}) = \sum C_{iq}^p \iota_{p,s}\epsilon^{q,s-k}$, but in Lemma 2.4 the ordering (1) will be used again.

Proposition 2.2. *The commutator $[\iota_{j,m}, \mathcal{L}_{i,k}]$ is given by*

$$[\iota_{j,m}, \mathcal{L}_{i,k}] = -\sum_p C_{ij}^p \iota_{p,m+k}.$$

Proof. Note that

$$\begin{aligned}
[\iota_{j,m}, \mathcal{L}_{i,k}] &= [\iota_{j,m}, \sum_{q,p,s} C_{iq}^p \iota_{p,s}\epsilon^{q,s-k}] \\
&= \sum_{p,q,s} C_{iq}^p [\iota_{j,m}, \iota_{p,s}]_+ \epsilon^{q,s-k} - \sum_{s,p,q} C_{iq}^p \iota_{p,s}[\iota_{j,m}, \epsilon^{q,s-k}]_+ \\
&= -\sum_{s\le 0,p,q} C_{iq}^p \iota_{p,s}\delta_{j,q}\delta_{m,(s-k)} - \sum_{s>0,p,q} C_{iq}^p \iota_{p,s}\delta_{j,q}\delta_{m,(s-k)} \\
&= -\sum_p C_{ij}^p \iota_{p,k+m},
\end{aligned}$$

concluding the proof. Likewise, it is just as easy to see that $[\epsilon^{j,m}, \mathcal{L}_{i,k}] = \sum_q C_{iq}^j \epsilon^{q,m-k}$.

Proposition 2.3. *The operators $\mathcal{L}_{i,k}$ define a projective representation of $\mathfrak{A}$ on $\wedge_\infty^*(\mathfrak{A})$.*

This follows directly from the following lemma.

Lemma 2.4. *The commutator $[\mathcal{L}_{i,k}, \mathcal{L}_{j,m}]$ is given by*

$$[\mathcal{L}_{i,k}, \mathcal{L}_{j,m}] = \mathcal{L}([e_{i,k}, e_{j,m}]),$$

if $m \neq -k$, and

$$[\mathcal{L}_{i,k}, \mathcal{L}_{j,-k}] = \mathcal{L}([e_{i,k}, e_{j,-k}]) + 2c \cdot \delta_{i,j} k.$$

Proof. Assume that $m \geq 0$(if $m \leq 0$, we merely need to replace m by $-m$). Note that

$$
\begin{aligned}
[\mathcal{L}_{i,k}, \mathcal{L}_{j,m}] &= [\sum_{q,p,s} C_{iq}^{p} : \iota_{p,s}\epsilon^{q,s-k} :, \mathcal{L}_{j,m}] \\
&= -[\sum_{s>0} C_{iq}^{p}\epsilon^{q,s-k}\iota_{p,s}, \mathcal{L}_{j,m}] + [\sum_{s\leq 0} C_{iq}^{p}\iota_{p,s}\epsilon^{q,s-k}, \mathcal{L}_{j,m}] \\
&= -\sum_{s>0} C_{iq}^{p}[\epsilon^{q,s-k}, \mathcal{L}_{j,m}]\iota_{p,s} - \sum_{s>0} C_{iq}^{p}\epsilon^{q,s-k}[\iota_{p,s}, \mathcal{L}_{j,m}] \\
&\qquad + \sum_{s\leq 0} C_{iq}^{p}[\iota_{p,s}, \mathcal{L}_{j,m}]\epsilon^{q,s-k} + \sum_{s\leq 0} C_{iq}^{p}\iota_{p,s}[\epsilon^{q,s-k}, \mathcal{L}_{j,m}] \\
&= -\sum_{s>0} C_{iq}^{p}C_{jn}^{q}\epsilon^{n,s-k-m}\iota_{p,s} + \sum_{s>0} C_{iq}^{p}C_{jp}^{n}\epsilon^{q,s-k}\iota_{n,s+m} \\
&\qquad - \sum_{s\leq 0} C_{iq}^{p}C_{jp}^{n}\iota_{n,s+m}\epsilon^{q,s-k} + \sum_{s\leq 0} C_{iq}^{p}C_{jn}^{q}\iota_{p,s}\epsilon^{n,s-k-m} \\
&= \sum_{p,q,n} C_{iq}^{p}C_{jn}^{q} : \iota_{p,s}\epsilon^{n,s-k-m} : - \sum_{p,q,n} C_{in}^{q}C_{jq}^{p} : \iota_{p,s}\epsilon^{n,s-k-m} : \\
&\qquad - \sum_{0<s\leq m} C_{in}^{q}C_{jq}^{p}[\iota_{p,s}, \epsilon^{n,s-k-m}]_{+}
\end{aligned}
$$

Since $[e_{i,k}, e_{j,m}] = [\alpha_i, \alpha_j]z^{k+m}$, we have

$$
\mathcal{L}([e_{i,k}, e_{j,m}]) = \sum_{q} C_{ij}^{q}\mathcal{L}_{q,k+m} = \sum_{p,q,s} C_{ij}^{q}C_{qn}^{p} : \iota_{p,s}\epsilon^{n,s-k-m} : .
$$

In terms of structure constants, the Jacobi identity for $\mathfrak{g}_{\mathbf{C}}$ translates into

$$
\sum_{q}(C_{ij}^{q}C_{qn}^{p} + C_{jn}^{q}C_{qi}^{p} + C_{ni}^{q}C_{qj}^{p}) = 0.
$$

Hence

$$
\begin{aligned}
[\mathcal{L}_{i,k}, \mathcal{L}_{j,m}] - \mathcal{L}([e_{i,k}, e_{j,m}]) &= \sum_{p,q,n} C_{iq}^{p}C_{jn}^{q} : \iota_{p,s}\epsilon^{n,s-k-m} : - \sum_{p,q,n} C_{in}^{q}C_{jq}^{p} : \iota_{p,s}\epsilon^{n,s-k-m} : \\
&\qquad - \sum_{0<s\leq m} C_{in}^{q}C_{jq}^{p}[\iota_{p,s}, \epsilon^{n,s-k-m}]_{+} \\
&\qquad - \sum_{p,q,n} C_{ij}^{q}C_{qn}^{p} : \iota_{p,s}\epsilon^{n,s-k-m} : \\
&= -\sum_{0<s\leq m} C_{in}^{q}C_{jq}^{p}[\iota_{p,s}, \epsilon^{n,s-k-m}]_{+}.
\end{aligned}
$$

This shows that, unless $k = -m$ and $p = n$,

$$
[\mathcal{L}_{i.k}, \mathcal{L}_{j,m}] - \mathcal{L}([e_{i,k}, e_{j,m}]) = 0,
$$

and, when $k = -m$ and $p = n$, we have

$$
[\mathcal{L}_{i,k}, \mathcal{L}_{j,-k}] - \mathcal{L}([e_{i,k}, e_{j,-k}]) = k \cdot \sum_{q,n} C_{in}^{q}C_{jq}^{n} = \sum_{q,n} C_{in}^{q}C_{jq}^{n} = 2c\langle \alpha_i, \alpha_j \rangle.
$$

Since we had chosen an orthonormal basis with respect to the Killing form, the last term is only non-zero when $i = j$, and

$$\mathcal{L}([e_{i,k}, e_{i,-k}]) = 0.$$

Thus

$$[\mathcal{L}_{i,k}, \mathcal{L}_{i,-k}] = 2c \cdot k.$$

The identity obviously does not depend on the assumption that $m \geq 0$, hence this concludes the proof of Lemma 2.4. ∎

Remark 2.5. Note that the projective representation of $\mathfrak{A}$, when restricted to $\mathfrak{g}$, becomes a genuine representation which determines an action of G. There is also a natural rotation action of the circle $\mathbf{T}$ on loops which defines an action of $\mathbf{T} \times G$ on the semi-infinite forms.

Define $d : \wedge^*_\infty(\mathfrak{A}) \to \wedge^*_\infty(\mathfrak{A})$ which increases degree by 1 by

$$d = \frac{1}{2} \sum_{i,k} \mathcal{L}_{i,k} \epsilon^{i,k}. \tag{2}$$

Although d is expressed as a sum over all integers k, as an operator on any element of $\wedge^*_\infty(\mathfrak{A})$ only a finite number of its terms will be non-zero, because there is an integer N for any element ω in $\wedge^*_\infty(\mathfrak{A})$ such that ω will be annihilated by $\epsilon^{i,k}$ for $k < N$. Also note that because of our choice of basis, $\mathcal{L}_{i,k}$ and $\epsilon^{i,k}$ commute, hence it doesn't matter in which order we write it. Consider the *twisted* operator

$$\tilde{d} = \frac{1}{2} \sum_{i,k} s_k \mathcal{L}_{i,k} \epsilon^{i,k},$$

where $s_k = 1$ when $k > 0$ and $s_k = -1$ when $k \leq 0$. Take the adjoint $\tilde{d}^*$ of $\tilde{d}$ and let $\tilde{L} = \tilde{d}^* d + d\tilde{d}^*$. Now define

$$\Omega = e^{n,0} \wedge e^{n-1,0} \wedge \cdots e^{1,0} \wedge e^{n,-1} \wedge \cdots$$

(recall that n is the dimension of $\mathfrak{g}_{\mathbf{C}}$). Call this *the vacuum vector* of $\wedge^*_\infty(\mathfrak{A})$. Note that the energy and degree of the vacuum vector is zero. The Lie algebra cochains $A^*(\mathfrak{a})$ can be identified with a subspace of $\wedge^*_\infty(\mathfrak{A})$ by the map $a \mapsto \epsilon(a)\Omega$. The main statement of this section is the following.

Proposition 2.6. *The operator $\tilde{L}$ restricts to the ordinary Laplacian $L = d^*d + dd^*$ on $A^*(\mathfrak{a})$.*

The proposition follows directly from the following two lemmas.

Lemma 2.7. *The operator d restricts to the ordinary Lie algebra differential (which we will also denote d) on the subspace $A^*(\mathfrak{a}) \subset \wedge^*_\infty(\mathfrak{A})$.*

Lemma 2.8. *The adjoint $\tilde{d}^*$ of $\tilde{d}$ restricts to the adjoint d^* of the ordinary Lie algebra differential d on $A^*(\mathfrak{a}) \subset \wedge^*_\infty(\mathfrak{A})$.*

Proof. [Lemma 2.7] We need only prove two things. First we will prove that

$$d(\alpha \wedge \omega) = d(\alpha) \wedge \omega \pm \alpha \wedge d(\omega) \tag{3}$$

for $\alpha \in A^*(\mathfrak{a})$ and $\omega \in \wedge^*_\infty(\mathfrak{A})$, where the sign depends on the degree of α. Then we will prove that

$$d\Omega = 0. \tag{4}$$

This will give us Lemma 2.7 since it will prove that, for $\alpha \in A^*(\mathfrak{a})$,

$$d(\epsilon(\alpha)\Omega) = \epsilon(d(\alpha))\Omega.$$

Identity 3 follows, with a bit of calculation, from the fact that $\epsilon^{i,k}$ anti-commutes with any other $\epsilon^{j,m}$ and the fact that $[\epsilon^{j,m}, \mathcal{L}_{i,k}] = \sum_q C^j_{iq}\epsilon^{q,m-k}$. On the other hand,

$$d\Omega = \frac{1}{2}\sum_{i,k} \mathcal{L}_{i,k}\epsilon^{i,k}\Omega.$$

By the definition of Ω, the only possible non-zero terms are the ones for which $k > 0$. Recall that $\mathcal{L}_{i,k}$ and $\epsilon^{i,k}$ commute, so all we need to show is that $\mathcal{L}_{i,k}\Omega = 0$ for $k > 0$. But

$$\mathcal{L}_{i,k} = \sum_{p,q,s} C^p_{iq} : \iota_{p,s}\epsilon^{q,s-k} := \sum_{s\leq 0,p,q} C^p_{iq}\iota_{p,s}\epsilon^{q,s-k} - \sum_{s>0,p,q} C^p_{iq}\epsilon^{q,s-k}\iota_{p,s}$$

is zero on Ω, since $\epsilon^{q,s-k}$ is zero on Ω if $s \leq 0$ and $\iota_{p,s}$ is zero on Ω if $s > 0$. This concludes the proof of Lemma 2.7. ∎

Proof. [Lemma 2.8] For $c_1 \in A^*(\mathfrak{a})$ and $c_2 \in \wedge^*_\infty(\mathfrak{A})$,

$$\begin{aligned}\langle \tilde{d}^*(c_1 \wedge \Omega), c_2\rangle &= \langle c_1 \wedge \Omega, \tilde{d}(c_2)\rangle \\ &= \langle c_1 \wedge \Omega, -\frac{1}{2}\sum_{k\leq 0} \mathcal{L}_{i,k}\epsilon^{i,k}(c_2) + \frac{1}{2}\sum_{k>0}\mathcal{L}_{i,k}\epsilon^{i,k}(c_2)\rangle.\end{aligned}$$

Since all other terms will be killed, we may assume that c_2 is a linear combination of elements of type $c_3 \wedge \Omega$ and elements of type $c_4 \wedge \Omega_{j,m}$ where $\Omega_{j,m}$ is the vacuum vector Ω with $e^{j,m}$ missing and $c_3, c_4 \in A^*(\mathfrak{a})$. It is enough to show that

$$\langle \tilde{d}^*(c_1 \wedge \Omega), c_3 \wedge \Omega\rangle = \langle d^*(c_1) \wedge \Omega, c_3 \wedge \Omega\rangle. \tag{5}$$

and that

$$\langle \tilde{d}^*(c_1 \wedge \Omega), c_4 \wedge \Omega_{j,m}\rangle = 0.$$

For $k \leq 0$, $\mathcal{L}_{i,k}\epsilon^{i,k}(c_3 \wedge \Omega) = 0$ and

$$\frac{1}{2}\sum_{k>0}\mathcal{L}_{i,k}\epsilon^{i,k}(c_3 \wedge \Omega) = \frac{1}{2}\sum_{k>0} C^p_{iq} : \iota_{p,s}\epsilon^{q,s-k} : \epsilon^{i,k}(c_3 \wedge \Omega)$$

The sum is over all p, q, s, i as well as $k > 0$. Since $k > 0$, note that the terms for which $s \leq 0$ are zero (the operator $\epsilon^{q,s-k}$ is zero on Ω). So,

$$\frac{1}{2}\sum_{k>0}\mathcal{L}_{i,k}\epsilon^{i,k}(c_3 \wedge \Omega) = d(c_3 \wedge \Omega).$$

Hence,

$$\langle \tilde{d}^*(c_1 \wedge \Omega), c_3 \wedge \Omega \rangle = \langle d^*(c_1 \wedge \Omega), c_3 \wedge \Omega \rangle = \langle d^*(c_1) \wedge \Omega, c_3 \wedge \Omega \rangle$$

(the last equality follows from identity 3 and identity 4), which proves the identity 5. On the other hand,

$$\begin{aligned}
\tilde{d}(c_4 \wedge \Omega_{j,m}) &= -\frac{1}{2}\sum_{k\leq 0} \mathcal{L}_{i,k}\epsilon^{i,k}(c_4 \wedge \Omega_{j,m}) + \frac{1}{2}\sum_{k>0} \mathcal{L}_{i,k}\epsilon^{i,k}(c_4 \wedge \Omega_{j,m}) \\
&= -\frac{1}{2}\sum_{k\geq 0} \mathcal{L}_{i,-k}\epsilon^{i,-k}(c_4 \wedge \Omega_{j,m}) + \frac{1}{2}\sum_{k>0} \mathcal{L}_{i,k}\epsilon^{i,k}(c_4 \wedge \Omega_{j,m}) \\
&= -\frac{1}{2}\sum_{k\geq 0, s>0} C^p_{iq}\iota_{p,s}\epsilon^{q,s+k}\epsilon^{i,-k}(c_4 \wedge \Omega_{j,m}) \\
&\qquad\qquad + \frac{1}{2}\sum_{k>0, s>0} C^p_{iq}\iota_{p,s}\epsilon^{q,s-k}\epsilon^{i,k}(c_4 \wedge \Omega_{j,m}).
\end{aligned}$$

If $s \leq 0$, note that the only non-zero terms in $\sum_{k>0} C^p_{iq}\iota_{p,s}\epsilon^{q,s+k}\epsilon^{i,-k}(c_4 \wedge \Omega_{j,m})$ or in $\sum_{k>0} C^p_{iq}\iota_{p,s}\epsilon^{q,s-k}\epsilon^{i,k}(c_4 \wedge \Omega_{j,m})$ which lie in $A^*(\mathfrak{a})$ are the ones where $p = q$ or $p = i$. Since we have chosen an orthonormal basis of $\mathfrak{g}$, these terms are zero. On the other hand

$$\sum_{k>0, s>0} C^p_{iq}\iota_{p,s}\epsilon^{q,s-k}\epsilon^{i,k}(c_4 \wedge \Omega_{j,m}) = \sum_{s>0}\sum_{k>-s} C^p_{iq}\iota_{p,s}\epsilon^{q,-k}\epsilon^{i,s+k}(c_4 \wedge \Omega_{j,m}).$$

Therefore,

$$\tilde{d}(c_4 \wedge \Omega_{j,m}) = \frac{1}{2}\sum_{s>0}\sum_{0>k>-s} C^p_{iq}\iota_{p,s}\epsilon^{q,-k}\epsilon^{i,s+k}(c_4 \wedge \Omega_{j,m}).$$

But $\iota_{p,s}\epsilon^{q,-k}\epsilon^{i,s+k}(c_4 \wedge \Omega_{j,m})$ can not be contained in $A^*(\mathfrak{a})$ when $-s < k \leq 0$, because $c_4 \wedge \Omega_{j,m}$ is missing $e^{i,k}$ for some $k < 0$ and $\iota_{p,s}\epsilon^{q,-k}\epsilon^{i,s+k}$ can not replace this missing element. Hence,

$$\langle \tilde{d}^*(c_1 \wedge \Omega), c_4 \wedge \Omega_{j,m} \rangle = 0,$$

concluding the proof of Lemma 2.8. ■

The results in this section show that, to calculate the Lie algebra cohomology of $\mathfrak{a}$, we need only find the kernel of $\tilde{L}$. Since the semi-infinite forms are acted on by $\mathbf{T} \times G$ (Remark 2.5), we know that the cochains of $\mathfrak{a}$ are acted on by $\mathbf{T} \times G$. In fact we will shortly see that $\mathcal{L}_{i,0}$, for each i, commutes with the operator d and hence, the action of $\mathbf{T} \times G$ on the cochains induces an action on the cohomology. It follows that the cohomology can be written as a sum of irreducible representations of $\mathbf{T} \times G$. The exact nature of the decomposition will follow from the explicit formula for $\tilde{L}$ which will be given in the next section.

3. The calculation for the operator $\tilde{L}$

The main aim of this section is to find a convenient expression of $\tilde{L}$ which will enable us to calculate its kernel.

Proposition 3.1. *The Laplacian $[d, \tilde{d}^*]_+$ is given by*

$$[d, \tilde{d}^*]_+ = -\sum_{k>0} c \cdot k \epsilon^{i,k} \iota_{i,k} - \sum_{k<0} c \cdot k \iota_{i,k} \epsilon^{i,k} + \frac{1}{2} \sum_i \mathcal{L}_{i,0}^2.$$

Proof. [Proposition 3.1] First note that

$$\begin{aligned}[d, \tilde{d}^*]_+ &= [d, -\frac{1}{2} \sum_{i,k} s_k \iota_{i,k} \mathcal{L}_{i,k}]_+ \\ &= [d, -\frac{1}{2} (\sum_{k>0,i} \iota_{i,k} \mathcal{L}_{i,-k} - \sum_{k<0,i} \iota_{i,k} \mathcal{L}_{i,-k} - \sum_i \iota_{i,0} \mathcal{L}_{i,0})]_+\end{aligned}$$

This is equal to

$$\begin{aligned}&-\frac{1}{2}(\sum_{k>0,i} [d, \iota_{i,k}]_+ \mathcal{L}_{i,-k} - \sum_{k>0,i} \iota_{i,k} [d, \mathcal{L}_{i,-k}]) \\ &+\frac{1}{2}(\sum_{k<0,i} [d, \iota_{i,k}]_+ \mathcal{L}_{i,-k} - \sum_{k<0,i} \iota_{i,k} [d, \mathcal{L}_{i,-k}]) \\ &+\frac{1}{2}(\sum_i [d, \iota_{i,0}]_+ \mathcal{L}_{i,0} - \sum_i \iota_{i,0} [d, \mathcal{L}_{i,0}]).\end{aligned} \tag{6}$$

To resolve this equation, we need to identify $[d, \iota_{i,k}]_+$ and $[d, \mathcal{L}_{i,-k}]$. Given a Lie group and its Lie algebra, the infinitesimal action of the Lie algebra on the Lie group is the Lie derivative which can be written as the anti-commutator of the differential and the interior product with respect to the vector fields in the Lie algebra. The operator $\mathcal{L}_{i,k}$ on the semi-infinite forms (with the given basis) can likewise be expressed as $\mathcal{L}_{i,k} = [\iota_{i,k}, d]_+$. This easily follows from the definition of d and the anti-commutators and commutators calculated in the previous section. Substituting this into $[\mathcal{L}_{i,k}, d]$, and using the Jacobi identity, it is an easy calculation to see that

$$[\mathcal{L}_{i,k}, d] = [\iota_{i,k}, d^2].$$

We claim that

Lemma 3.2. *The square d^2 can be expressed as*

$$d^2 = \sum_{k>0,i} 2c \cdot k \epsilon^{i,k} \epsilon^{i,-k}.$$

Lemma 3.2 would imply that

$$[\mathcal{L}_{i,k}, d] = 2c \cdot k \epsilon^{i,-k}.$$

From this, we see that

$$[d, \mathcal{L}_{i,-k}] = 2c \cdot k\epsilon^{i,k}. \tag{7}$$

Substituting (7) and the identity for $[\iota_{i,k}, d]_+$ in (6), we have

$$[d, \tilde{d}^*]_+ = -\frac{1}{2}\sum_{k>0}(\mathcal{L}_{i,k}\mathcal{L}_{i,-k} - 2c \cdot k\iota_{i,k}\epsilon^{i,k}) + \frac{1}{2}\sum_{k<0}(\mathcal{L}_{i,k}\mathcal{L}_{i,-k} - 2c \cdot k\iota_{i,k}\epsilon^{i,k}) + \frac{1}{2}\sum_{i}\mathcal{L}_{i,0}\mathcal{L}_{i,0}.$$

Using the commutation rules for $\mathcal{L}_{i,k}, \iota_{i,k}$ and $\epsilon_{i,k}$, we have

$$\begin{aligned}-\frac{1}{2}\sum_{k>0}(\mathcal{L}_{i,k}\mathcal{L}_{i,-k} - 2c \cdot k\iota_{i,k}\epsilon^{i,k}) &= -\frac{1}{2}\sum_{k>0}(2c \cdot k + \mathcal{L}_{i,-k}\mathcal{L}_{i,k} - 2c \cdot k + 2c \cdot k\epsilon^{i,k}\iota_{i,k}) \\ &= -\frac{1}{2}\sum_{k>0}(\mathcal{L}_{i,-k}\mathcal{L}_{i,k} + 2c \cdot k\epsilon^{i,k}\iota_{i,k}).\end{aligned}$$

Hence,

$$\begin{aligned}[d, \tilde{d}^*]_+ &= \frac{1}{2}\sum_{k<0}\mathcal{L}_{i,k}\mathcal{L}_{i,-k} - \frac{1}{2}\sum_{k>0}\mathcal{L}_{i,-k}\mathcal{L}_{i,k} - \sum_{k>0}c \cdot k\epsilon^{i,k}\iota_{i,k} - \sum_{k<0}c \cdot k\iota_{i,k}\epsilon^{i,k} \\ &\quad + \frac{1}{2}\sum_{i}\mathcal{L}_{i,0}\mathcal{L}_{i,0} \\ &= -\sum_{k>0}c \cdot k\epsilon^{i,k}\iota_{i,k} - \sum_{k<0}c \cdot k\iota_{i,k}\epsilon^{i,k} + \frac{1}{2}\sum_{i}\mathcal{L}_{i,0}^2.\end{aligned}$$

■

To complete the proof of Proposition 3.1, we need only prove Lemma 3.2.

Proof. [Lemma 3.2] First of all, we will prove that d^2 is a homomorphism on the $\wedge^*(\mathfrak{a}^*)$-module $\wedge^*_\infty(\mathfrak{A})$. To see this, all we need to show is that $[d^2, \epsilon(\alpha)] = 0$ for any $\alpha \in \wedge^*(\mathfrak{a}^*)$. First note that

$$[d^2, \epsilon(\alpha)] = [d, [d, \epsilon(\alpha)]]_+ = [d, [d, \epsilon(\alpha)]_+]. \tag{8}$$

If α is of odd degree then $[d, \epsilon(\alpha)]_+$ is multiplication by $d(\alpha)$. If α is of even degree then $[d, \epsilon(\alpha)]$ is multiplication by $d(\alpha)$. The operator d increases the degree of α by one. Hence, Identity (8) implies that, regardless of the degree of α, $[d^2, \epsilon(\alpha)]$ is multiplication by $d(d(\alpha))$. However $d^2(\alpha) = 0$. Hence $[d^2, \epsilon(\alpha)] = 0$. Now let Ω_{-k} be the element in $\bigwedge_\infty = \wedge^*_\infty(\mathfrak{A})$ given by

$$\Omega_{-k} = e^{1,-k} \wedge \cdots \wedge e^{n,-k} \wedge e^{1,-k-1} \wedge \cdots \wedge e^{n,-k-1} \wedge \cdots$$

(n is the dimension of $\mathfrak{g}_{\mathbf{C}}$) and let Λ denote the exterior algebra $\wedge(\mathfrak{A}^*)$. Then

$$\bigwedge_\infty = \bigcup_k \Lambda\Omega_{-k},$$

where $\Lambda\Omega_{-k}$ denotes all elements of the form $\epsilon(\alpha)\Omega_{-k}$ for $\alpha \in \Lambda$. Let I_{-k} be ideal of Λ generated by the elements $e^{a,-m}$ for $m \geq k$. Then $\Lambda\Omega_{-k}$ is a Λ/I_{-k}-module. Hence, $\bigwedge_\infty$ is actually a $\hat{\Lambda}$-module where $\hat{\Lambda}$ is the direct limit of Λ/I_{-k} as k runs over the positive integers. We will prove the following lemma.

Lemma 3.3. *If T is an even degree homomorphism of Λ-modules then T is multiplication by an element $\alpha \in \hat{\Lambda}$ of even degree.*

Proof. If T is even degree and if T is multiplication by $\alpha \in \hat{\Lambda}$ then α must be of even degree, so it is enough to show that T is multiplication by α. First note that $\Lambda\Omega_{-k}$ consists of all the elements ξ of $\bigwedge_\infty$ such that $\epsilon(\alpha)\xi = 0$ for all $\alpha \in I_{-k}$. This shows that $T(\Lambda\Omega_{-k}) \subset \Lambda\Omega_{-k}$. In particular, $T(\Omega_{-k}) \subset \Lambda\Omega_{-k}$. That is,

$$T(\Omega_{-k}) = \alpha_{-k}\Omega_{-k},$$

for $\alpha_{-k} \in \Lambda/I_{-k}$. But

$$\Omega_{-k} = \omega_{-k}\Omega_{-k-1},$$

where $\omega_{-k} = e^{1,-k} \wedge \cdots \wedge e^{n,-k}$. So

$$\alpha_{-k}\Omega_{-k} = T(\Omega_{-k}) = T(\omega_{-k}\Omega_{-k-1}) = \omega_{-k}T(\Omega_{-k-1}) = \omega_{-k}\alpha_{-k-1}\Omega_{-k-1}$$

$$= \alpha_{-k-1}\omega_{-k}\Omega_{-k-1} = \alpha_{-k-1}\Omega_{-k}.$$

This means that $\alpha_{-k} = \alpha_{-k-1}$ in Λ/I_{-k}. So the collection $\{\alpha_{-k}\}$, as k runs over the integers, defines an element of $\hat{\Lambda}$, concluding the proof of the lemma. ∎

The lemma proves that d^2 is multiplication by some element $\tilde{\omega} \in \Lambda^2$. However

$$\begin{aligned}
[\iota_{i,k}, [\iota_{j,m}, d^2]]_+ &= [\iota_{i,k}, \mathcal{L}_{j,m}d - d\mathcal{L}_{j,m}] \\
&= [\iota_{i,k}, \mathcal{L}_{j,m}]d + \mathcal{L}_{j,m}\mathcal{L}_{i,k} - \mathcal{L}_{i,k}\mathcal{L}_{j,m} + d[\iota_{i,k}, \mathcal{L}_{j,m}] \\
&= \sum_p C_{ij}^p \mathcal{L}_{p,k+m} - [\mathcal{L}_{i,k}, \mathcal{L}_{j,m}] \\
&= -\delta_{i,j}\delta_{k,-m} 2c \cdot k.
\end{aligned}$$

Suppose $\tilde{\omega} = \sum_{s,t,u,v} f_{s,t,u,v} e^{s,u} e^{t,v}$. A short calculation shows that

$$[\iota_{i,k}, [\iota_{j,m}, \epsilon(\tilde{\omega})]]_+ = -2f_{ijkm}.$$

So,

$$\tilde{\omega} = \sum c \cdot k e^{i,k} e^{i,-k} = \sum_{k>0} 2c \cdot k e^{i,k} e^{i,-k}.$$ ∎

The formula of Proposition 3.1 shows that $\mathcal{L}_{i,0}$ commutes with the Laplacian. This proves that on an irreducible representation of $\mathfrak{g}$ the Laplacian acts by a scalar. Hence, we need only check how it acts on lowest weight vectors, which brings us to the main theorem of the paper.

Theorem 3.4. *The twisted Laplacian $\tilde{L} = [d, \tilde{d}^*]_+$ defined above acts on a irreducible representation $V \subset A^q(\mathfrak{a})$ of $\mathbf{T} \times G$ ($\mathbf{T}$ is the group of rotations) with lowest weight λ by*

$$-\langle \rho, \lambda \rangle + \frac{1}{2} \parallel \lambda \parallel^2 - c \cdot k$$

where k is the energy of the lowest weight vector, and ρ is the half sum of the positive roots of G.

Proof. Given the identity (3), to determine how $[d, \tilde{d}^*]$ acts on $A^*(\mathfrak{a})$, it is enough to calculate it on a vector of the form $v = e^{l,k} \wedge \Omega$. We know from Proposition 3.1 that

$$\tilde{L}(v) = \frac{1}{2}\sum_i \mathcal{L}_{i,0}^2(v) - \sum_{m>0,i} c \cdot m\epsilon^{i,m}\iota_{i,m}(v) - \sum_{m<0,i} c \cdot m\iota_{i,m}\epsilon^{i,m}(v).$$

It is obvious that the third term is zero on v. The second term is also easily seen to be

$$-c \cdot k(v).$$

To calculate the first term, note that

$$\mathcal{L}_{i,0} = \sum_{p,q,s} C_{iq}^p : \iota_{p,s}\epsilon^{q,s} : .$$

Then, since $\sum_{p,q,s\leq 0} C_{iq}^p \iota_{p,s}\epsilon^{q,s}$ kills anything of the form $\beta \wedge \Omega$,

$$\mathcal{L}_{i,0}^2(v) = \mathcal{L}_{i,0}\{-\sum_{p,q,s>0} C_{iq}^p \epsilon^{q,s}\iota_{p,s}(v)\} = \mathcal{L}_{i,0}\{-\sum_q C_{iq}^l e^{q,k} \wedge \Omega\} = \sum_{q,r} C_{iq}^l C_{ir}^q e^{r,k} \wedge \Omega.$$

But,

$$\sum_{q,r} C_{iq}^l C_{ir}^q = \sum_{q,r} C_{ir}^q \langle[\alpha_i, \alpha_q], \alpha_l\rangle = \sum_r \langle[\alpha_i, [\alpha_i, \alpha_r]], \alpha_l\rangle = \sum_r \langle[\alpha_i, [\alpha_i, \alpha_l]], \alpha_r\rangle,$$

where the last equality holds because of the Jacobi identity and the fact that $\langle[x, y], z\rangle = \langle x, [y, z]\rangle$. So $\frac{1}{2}\sum \mathcal{L}_{i,0}^2$ acts as $-\frac{1}{2}\sum_i \alpha_i^2$, which is how the Casimir of $\mathfrak{g}$ acts with respect to the chosen basis (the minus sign comes in because we are acting on the dual space). The action of the Casimir on a lowest weight vector has been already worked out (e.g. Section 9.4 [6])and has the form

$$\frac{1}{2}\sum_i \mathcal{L}_{i,0}^2(v) = \{-\langle \rho, \lambda\rangle + \frac{1}{2} \parallel \lambda \parallel^2\}(v),$$

where λ is the lowest weight for the representation and ρ is the half sum of all positive roots of G. This concludes the proof of Theorem 3.4. ■

From Theorem 3.4 it immediately follows that the cohomology of $\mathfrak{a}$ in any degree is finite dimensional. To see this, first note that the cochain complex $A^*(\mathfrak{a})$ of $\mathfrak{a}$ can be decomposed according to the energy grading, so that $A^p(k)(\mathfrak{a})$ denotes the space of pth cochains of energy k. The differential does not change the energy of the cochain, hence, $A^*(k)(\mathfrak{a})$ is a subcomplex of $A^*(\mathfrak{a})$. If $p > k$, then $A^p(k)(\mathfrak{a}) = 0$. That is, the cohomology $H^*(k)(\mathfrak{a})$ of $A^*(k)(\mathfrak{a})$ is finite dimensional. The cohomology $H^*(k)(\mathfrak{a})$ is the part of $H^*(\mathfrak{a})$ which is of energy k. On the other hand, the energy level of the cochains of any one degree is bounded. To prove this, let us write $H^*(k)(\mathfrak{a}; \mathbf{C})$ as a sum of irreducible representations V_λ of G with lowest weight λ. Theorem 3.4 shows us that the twisted Laplacian acts as $P(\lambda) - c \cdot k$ on V_λ where $P(\lambda)$ is the Casimir operator of G on V_λ and k is the energy of the lowest weight vector. Since the Laplacian is zero on the cohomology and the twisted Laplacian restricts to the Laplacian on the cohomology, this implies that $P(\lambda) = c \cdot k$ on V_λ. The Casimir operator has been shown to depend only on the lowest weight λ (in Section 9.4 [6]) which is a sum of p roots of G. This means that only a finite number of irreducible representations of G occur in the cohomology of degree p. We conclude that $H^p(k)(\mathfrak{a}; \mathbf{C}) = 0$ if $c \cdot k > \sup P(\lambda)$ where the supremum is taken over all the lowest weights λ of irreducible representations that might occur in $H^p(\mathfrak{a}; \mathbf{C})$.

The above argument does not show that there is only one copy of any irreducible representation in the sum. But this follows from the results of the next section.

4. Conclusion

Based on the results in the previous sections, we will summarise the conclusion of this paper in the following theorem.

Theorem 4.1. *The pth degree cohomology of $\mathfrak{a}$ can be written as a direct sum of irreducible representations of $\mathbf{T} \times G$,*

$$H^p(\mathfrak{a}) = \bigoplus_w V_w,$$

where the sum ranges over elements w of length p in the the quotient $\frac{\mathcal{W}_{\text{aff}}}{\mathcal{W}}$ of the affine Weyl group $\mathcal{W}_{\text{aff}}$ by the Weyl group of the finite dimensional Lie group G. By the length of an element in the quotient we mean the length of the shortest representative.

Proof. If $\mathfrak{g}_{\mathbf{C}}$ is semi-simple, $\mathfrak{g}_{\mathbf{C}} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_p$ for some p and $\mathfrak{g}_i$ simple. Then

$$\mathfrak{a} = \mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_p$$

where $\mathfrak{a}_i$ is the corresponding Lie subalgebra of $\mathfrak{A}$ associated to $\mathfrak{g}_i$. Then

$$H^*(\mathfrak{a}) = H^*(\mathfrak{a}_1) \otimes \cdots \otimes H^*(\mathfrak{a}_p).$$

Hence, it is enough to calculate the cohomology when $\mathfrak{g}_{\mathbf{C}}$ is simple. In this case the formula for the Laplacian given in Theorem 3.4 can be re-written in a much tidier and familiar form.

Recall Remark 2.5. There is an action of $\mathbf{T} \times G$ and any weight of $\mathbf{T} \times G$ can be described by a triple $\lambda = (n_1, \lambda, 0)$, where n_1 is the weight of the rotation action of $\mathbf{T}$, λ is a weight of the G action and the last component is the weight of the central extension corresponding to the projective representation of $\mathfrak{A}$. Recall that c denotes the coxeter number of $\mathfrak{g}$. And, let $\boldsymbol{\rho}$ denote the weight $(0, \rho, -c)$ where ρ is a half sum of the positive roots of G. Define

$$\langle (n_1, \lambda_1, b_1), (n_2, \lambda_2, b_2) \rangle = -n_2 b_1 - n_1 b_2 + \langle \lambda_1, \lambda_2 \rangle,$$

where $\langle \cdot, \cdot \rangle$ on the right-hand side is the inner product we have chosen for $\mathfrak{g}_{\mathbf{C}}$. Then the formula in Theorem 3.4 for the Laplacian translates into

$$\tilde{L} = \frac{1}{2}(|| \boldsymbol{\lambda} - \boldsymbol{\rho} ||^2 - || \boldsymbol{\rho} ||^2), \tag{9}$$

on a irreducible representation with lowest weight $\boldsymbol{\lambda} = (k, \lambda, 0)$ where λ is a lowest weight of G and k denotes the energy of the lowest weight vector. The rest of the proof for Theorem 4.1 follows from Lemma 4.3 and the remark below. ■

Remark 4.2. As the lowest weight of a representation of $\mathfrak{g}$ on $A^*(\mathfrak{a})$, λ can be expressed as a sum of roots (not necessarily all negative or positive) of $\mathfrak{g}_{\mathbf{C}}$. Since $\mathfrak{a}$ is spanned by $e_{i,k}$ where $k > 0$, the circle acts non-trivially and the energy k must always be positive. Hence, $\boldsymbol{\lambda}$ has to be a sum of positive affine roots which are not roots of $\mathfrak{g}_{\mathbf{C}}$.

Proposition 4.3. *Let $\boldsymbol{\lambda}$ be a sum of positive affine roots. The expression (9) is zero if and only if the positive affine roots in the sum $\boldsymbol{\lambda}$ are exactly the set of positive affine roots turned negative by the shortest representative of a coset in $\frac{\mathcal{W}_{\text{aff}}}{\mathcal{W}}$.*

Proof. First assume that Equation (9) is zero. Let $\mathcal{P}$ denote the positive alcove. We can choose w in the affine Weyl group $\mathcal{W}_{\text{aff}}$ so that $w(\boldsymbol{\rho} - \boldsymbol{\lambda}) \in \mathcal{P}$. Since $\boldsymbol{\lambda}$ can be written as a sum of positive affine roots, $\boldsymbol{\rho} - w(\boldsymbol{\rho} - \boldsymbol{\lambda})$ is also a sum of positive affine roots, i.e., $\boldsymbol{\rho} - w(\boldsymbol{\lambda} - \boldsymbol{\rho})$ is positive or zero on anything in the positive alcove. Since $\boldsymbol{\rho} \in \mathcal{P}$ and $w(\boldsymbol{\rho} - \boldsymbol{\lambda}) \in \mathcal{P}$, $\boldsymbol{\rho} + w(\boldsymbol{\rho} - \boldsymbol{\lambda}) \in \mathcal{P}$. In fact, because $\boldsymbol{\rho}$ is in the interior of the positive alcove $\mathcal{P}$, so is $\boldsymbol{\rho} + w(\boldsymbol{\rho} - \boldsymbol{\lambda})$. Hence,

$$\| \boldsymbol{\rho} \|^2 - \| \boldsymbol{\rho} - \boldsymbol{\lambda} \|^2$$

$$= \langle \boldsymbol{\rho} - w(\boldsymbol{\rho} - \boldsymbol{\lambda}), \boldsymbol{\rho} + w(\boldsymbol{\rho} - \boldsymbol{\lambda}) \rangle \geq 0.$$

This is equal to zero if and only if $\boldsymbol{\rho} - w(\boldsymbol{\rho} - \boldsymbol{\lambda}) = 0$, which in turn implies $\boldsymbol{\lambda} = \boldsymbol{\rho} - w^{-1}\boldsymbol{\rho}$. It is already known that $\boldsymbol{\rho} - w^{-1}\boldsymbol{\rho}$ is the sum $s(\boldsymbol{\lambda})$ of all the positive affine roots which become negative under the action of w^{-1} (see [6] p280). Furthermore, no other sum of positive affine roots can equal $\boldsymbol{\lambda}$. To see this, suppose such a sum $\boldsymbol{\alpha}_1 + \cdots + \boldsymbol{\alpha}_k$ existed. Then $w^{-1}(s(\boldsymbol{\lambda})) = w^{-1}(\boldsymbol{\alpha}_1) + \cdots + w^{-1}(\boldsymbol{\alpha}_k)$. The term $w^{-1}(s(\hat{\lambda}))$ is a sum of negative roots by construction, but some of of the $w^{-1}(\boldsymbol{\alpha}_i)$ would be positive roots. Those which have turned negative are in the sum $w^{-1}(s(\boldsymbol{\lambda}))$ and can be cancelled from each side, so we will be left with an identity for which the left-hand side is a sum of negative roots and the right-hand side is a sum of positive roots. This is not possible. Hence, the positive affine roots in $\boldsymbol{\lambda}$ have to be exactly the set of positive affine roots which turn negative by some w^{-1} in $\mathcal{W}_{\text{aff}}$. Recall that $\boldsymbol{\lambda}$ is a sum of positive affine roots which are not roots of G (see Remark 4.2). Hence, w^{-1} has to be a representative of a coset in $\frac{\mathcal{W}_{\text{aff}}}{\mathcal{W}}$. Since any other element of $\mathcal{W}_{\text{aff}}$ which belong to the same coset would turn roots of G negative, w^{-1} is the shortest representative.

On the other hand, assume that $\boldsymbol{\lambda} = \boldsymbol{\alpha}_1 + \cdots + \boldsymbol{\alpha}_k$ for positive affine roots $\boldsymbol{\alpha}_i$ and that there exists an element $w \in \mathcal{W}$ such that $\boldsymbol{\alpha}_1, \cdots, \boldsymbol{\alpha}_k$ are exactly the positive roots which become negative by w, then

$$\boldsymbol{\rho} - \boldsymbol{\lambda} = w(\boldsymbol{\rho}),$$

that is,

$$\| \boldsymbol{\rho} - \boldsymbol{\lambda} \|^2 = \| w(\boldsymbol{\rho}) \|^2 = \| \boldsymbol{\rho} \|^2 .$$

In other words, $P(\boldsymbol{\lambda}) = 0$, concluding the proof of Proposition 4.3. ∎

5. Comment

Given a finite dimensional Lie group G, the loop group LG of G is the infinite dimensional Lie group consisting of smooth maps from the circle to G. The Lie algebra $\mathfrak{L}\mathfrak{g}$ of the loop group is a vector space of maps from the circle to the Lie algebra $\mathfrak{g}$ of G. In this section we wish to make a short comment on the relationship between $\mathfrak{a}$ and $\mathfrak{L}\mathfrak{g}$. Choose a base point 0 on the circle S^1. Let $\mathfrak{L}_0\mathfrak{g}$ denote the Lie algebra of loops in $\mathfrak{g}$ which vanish at the base point 0. Given an element in $\mathfrak{L}_0\mathfrak{g}$, we can associate to it its Taylor series at 0, which is represented by an infinite formal series

$$a_1 t + \frac{1}{2} a_2 t^2 + \cdots + \frac{1}{N!} a_N t^N + \cdots ,$$

where a_i represents the ith derivative of the loop at 0. Let $\mathfrak{J}$ denote the vector space spanned by all formal series with coefficients in $\mathfrak{g}$ and vanishing constant term. It is a Lie algebra. If we let $\mathfrak{L}_\infty\mathfrak{g}$ denote the loops whose derivatives vanish to infinite order, it is clear that the Taylor series map is an injective map from the quotient $\frac{\mathfrak{L}_0\mathfrak{g}}{\mathfrak{L}_\infty\mathfrak{g}}$ to $\mathfrak{J}$. It is a known but a non-trivial fact that this map is also surjective (see [7], p.390, Theorem 38.1). Let $\mathfrak{J}_N$ denote the vector space of polynomials with coefficients in $\mathfrak{g}$

$$a_1 t + \frac{1}{2} a_2 t^2 + \cdots + \frac{1}{N!} a_N t^N,$$

endowed with the product topology. Take the topology of $\mathfrak{J}$ to be the inverse limit topology induced by the topology on $\mathfrak{J}_N$. The Taylor series map takes the quotient Lie algebra isomorphically, as topological vector spaces, to $\mathfrak{J}$. If we take the complexification $\mathfrak{J}_{\mathbf{C}}$ of $\mathfrak{J}$, there is a map $\psi : \mathfrak{a} \longrightarrow \mathfrak{J}_{\mathbf{C}}$ which induces a map

$$H^*(\mathfrak{J}_{\mathbf{C}}; \mathbf{C}) \xrightarrow{\psi^*} H^*(\mathfrak{a}; \mathbf{C})$$

in cohomology. Since $\mathfrak{J}$ is an inverse limit of the $\mathfrak{J}_N$, this map is injective. Each Lie algebra cochain has an energy level. This energy level is not changed by the differential, hence the total cohomology is a sum over the cohomology at each energy level. Restricted to any one energy level, ψ is an isomorphism. We showed earlier that, for each cohomology degree, only finite number of energy levels are involved. Hence, ψ is an isomorphism.

References

[1] Bröcker, T., and T. tom Dieck, "Representations of Compact Lie Groups," Graduate Texts in Mathematics **98**, Springer-Verlag, New York, 1985.

[2] Frenkel, I. B., H. Garland, and G. J. Zuckerman, *Semi-infinite cohomology and string theory*, Proc. Natl. Acad. Sci. **83** (1986), 8442–8446.

[3] Garland, H., and J. Lepowsky, *Lie algebra homology and the Macdonald-Kac formulas*, Invent. Math. **34** (1976), 37–76.

[4] Humphreys, J., "Introduction to Lie Algebras and Representation Theo ry," Graduate Texts in Mathematics **9**, Springer-Verlag, New York, 1980.

[5] Kostant, B., *Lie algebra cohomology and the generalized Borel-Weil theorem*, Ann. of Math. **74** (1961), 329–87.

[6] Pressley, A., and G. B. Segal, "Loop Groups," Oxford Mathematical Monographs, Oxford Science Publications, 1986.

[7] Treves, F., "Topological Vector Spaces, Distributions and Kernels," Academic Press, 1967.

Yunhyong Kim
Max-Planck Institut für Mathematik
Vivatsgasse 7
D-53111 Bonn
Germany
kimy@mpim-bonn.mpg.de

Received March 14, 2003
and in final form August 18, 2003