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On the Nilpotency of Certain Subalgebras of Kac-Moody Lie Algebras

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Abstract. Let $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ be an indecomposable Kac-Moody Lie algebra associated with the generalized Cartan matrix $A = (a_{ij})$ and W be its Weyl group. For $w \in W$, we study the nilpotency index of the subalgebra $S_w = \mathfrak{n}_+ \cap w(\mathfrak{n}_-)$ and find that it is bounded by a constant $k{=}k(A)$ which depends only on A but not on w for all $A = (a_{ij})$ finite, affine of type other than E or F and indefinite type with $|a_{ij}| \geq 2$. In each case we find the best possible bound k. In the case when $A = (a_{ij})$ is hyperbolic of rank two we show that the nilpotency index is either 1 or 2.

Introduction

Let $A = (a_{ij})_{i,j\in I}$ be an indecomposable generalized Cartan matrix and $\mathfrak{g} = \mathfrak{g}(A)$ denote the associated Kac-Moody Lie algebra over the field of complex numbers [2]. Following the usual convention we will take the index set I to be $\{0, 1, \cdots, n\}$ when A is of affine type and I to be $\{1, 2, \cdots, n\}$ otherwise. Let $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ be the triangular decomposition of $\mathfrak{g}$ with respect to the Cartan subalgebra $\mathfrak{h}$ and let $\Delta = \Delta_+ \cup \Delta_-$ denote the set of roots with Δ_+ and Δ_- denoting the set of positive and negative roots respectively. Let $\Pi = \{\alpha_i | i \in I\}$ denote the set of simple roots and $\check{\Pi} = \{h_i | i \in I\}$ be the set of simple coroots. Note that $\alpha_i(h_j) = a_{ji}$ for $i, j \in I$. Let $\mathfrak{g}_\alpha$ denote the α-root space, and $Q = \sum_{i\in I} \mathbb{Z}\alpha_i$ denote the root lattice. For $\alpha, \beta \in Q$, we define $\alpha > \beta$ if $\alpha - \beta \in Q_+ = \sum_{i\in I} \mathbb{Z}_{\geq 0}\alpha_i$ and $\alpha \neq \beta$. For $\alpha = \sum_{i\in I} k_i\alpha_i \in Q_+$ define $ht(\alpha) = \sum_{i\in I} k_i$ to be the height of α. Let W be the Weyl group of $\mathfrak{g}$ generated by the simple reflections $\{r_i | i \in I\}$. For $w \in W$ we denote $\Delta_+(w) = \{\alpha \in \Delta_+ | w^{-1}\alpha < 0\}$ and $w(\mathfrak{n}_\pm) = \bigoplus_{\alpha \in \Delta_\pm} \mathfrak{g}_{w\alpha}$. In this paper, for $w \in W$ we study the nilpotency index of the nilpotent subalgebras

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$$S_w = \mathfrak{n}_+ \cap w(\mathfrak{n}_-) = \bigoplus_{\alpha \in \Delta_+(w)} \mathfrak{g}_\alpha .$$

In [3], Billig and Pianzola conjectured that the nilpotency index of the subalgebra S_w is bounded by a constant $k = k(A)$ which depends only on the Cartan matrix A and not on w. This information helps in determining the existence of certain subroot system within the root system of a Kac-Moody Lie algebra. In this paper we prove the following theorem which settles the conjecture in most cases. Although we believe the conjecture to be true in the remaining few cases we have not been able to prove it using our approach.

Main Theorem. For $w \in W$ the nilpotency index of the subalgebras S_w is bounded by $k = k(A)$ where

1. k is the height of the highest long root when A is of finite type.
2. $k = h - 1$, h being the Coxeter number, when A is of affine type 1 other than E, F or A is of type $A_{2n}^{(2)}$ and $k = \check{h} - 1$, $\check{h}$ being the dual Coxeter number, when A is of affine type 2 or 3 other than $A_{2n}^{(2)}$ or $E_6^{(2)}$.
3. $k = 1$ when A is of indefinite type with $|a_{ij}| \geq 2$ and $k = 2$ when $A = \begin{pmatrix} 2 & -a \\ -1 & 2 \end{pmatrix}$, $a > 4$.

Furthermore, in each case k is the best bound possible.

In the following three sections we prove the three parts of our main theorem in Theorem 1.2, Theorem 2.5, Theorem 3.5 and Theorem 3.6.

1. A of finite type

Let A be of finite type and $\mathfrak{g} = \mathfrak{g}(A)$ be the associated simple Lie algebra. Let $\mathfrak{h}$ be the Cartan subalgebra and $\Pi = \{\alpha_1, \cdots, \alpha_n\}$ be the set of simple roots. Let θ be the highest long root. Since $-\Pi = \{-\alpha_1, \cdots, -\alpha_n\}$ is also a root basis and since W acts transitively on the root bases [1], there exists $w_0 \in W$ such that $w_0^{-1}(\Pi) = -\Pi$. Note that in this case clearly the nilpotency index has a bound. Our interest is to find the best possible bound k.

Proposition 1.1. *Each positive root α can be written as $\alpha_{i_1} + \cdots + \alpha_{i_k}$, $\alpha_{i_j} \in \Pi$, such that each partial sum $\alpha_{i_1} + \cdots + \alpha_{i_j}$ is also a root.*

Theorem 1.2. *Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra. If $k = ht(\theta)$, then $S_w^{(k)} = 0$ for all $w \in W$. Moreover, k is the least such integer.*

Proof. Let $w_0 \in W$ such that $w_0(\Pi) = -\Pi$. Then $S_{w_0} = \bigoplus_{\alpha \in \Delta_+} \mathfrak{g}_\alpha$. By Proposition 1.1 there is a sequence of positive roots $\beta_1, \cdots, \beta_k = \theta$ with $\mathrm{ht}(\beta_i) = i$. Then $S_{w_0}^{(t)} = \sum_{ht(\alpha) > t} \mathfrak{g}_\alpha$. Hence $S_{w_0}^{(k)} = 0$. On the other hand, if $s < k$, then $[\mathfrak{g}_{\alpha_i}, \mathfrak{g}_{\beta_s}] = \mathfrak{g}_{\beta_s + \alpha_i} \neq 0$ for some $\alpha_i \in \Pi$ with $\beta_s + \alpha_i = \beta_{s+1}$. Hence $S_{w_0}^{(k-1)} \neq 0$. Since $w_0^{-1}(\Pi) = -\Pi$ and $\Delta_+(w_0) = \Delta_+$, it follows that $S_{w_0}^{(t)} \supseteq S_w^{(t)}$ for $t = 1, \cdots, k$, and all $w \in W$. Hence $S_w^{(k)} = 0$ for all $w \in W$. ■

2. A of affine type

Let $A = (a_{ij})_{i,j\in I}$, with $I = \{0,1,\cdots,n\}$, be of affine type, $\mathfrak{g} = \mathfrak{g}(A)$ be the associated affine Lie algebra, $\overset{\circ}{A} = (a_{ij})$, $1 \le i,\ j \le n$, and $\overset{\circ}{\mathfrak{g}} = \mathfrak{g}(\overset{\circ}{A})$ be the corresponding simple Lie algebra with Cartan matrix $\overset{\circ}{A}$. Let Δ and $\overset{\circ}{\Delta}$ denote the sets of roots, W and $\overset{\circ}{W}$ the Weyl groups, and $\Pi = \{\alpha_0,\cdots,\alpha_n\}$ and $\overset{\circ}{\Pi} = \{\alpha_1,\cdots,\alpha_n\}$ the sets of simple roots of $\mathfrak{g}$ and $\overset{\circ}{\mathfrak{g}}$ respectively. Let the subscripts $+$ and $-$ stand for positive and negative roots and let the superscripts 're' and 'im' stand for real and imaginary roots respectively. Thus Δ_+^{re} denotes the set of real positive roots in Δ. Let $\delta = \sum_{i=0}^{n} a_i\alpha_i$ where each a_i is a positive integer, $gcd(a_0,\cdots,a_n) = 1$ and $A(a_0,\cdots,a_n)^t = 0$. Then $\Delta_+^{im} = \{j\delta | j \in \mathbb{Z}_{>0}\}$. Since Δ_+^{im} is W-invariant, $\Delta_+(w) = \{\alpha \in \Delta_+ | w^{-1}(\alpha) < 0\} \subseteq \Delta_+^{re}$ for each $w \in W$.

It is known that $a_0 = 1$ unless $A = A_{2n}^{(2)}$ in which case $a_0 = 2$. Furthermore, if $\mathfrak{g}$ is of type 1 or of type $A_{2n}^{(2)}$, then $\delta - a_0\alpha_0$ is the highest long root in $\overset{\circ}{\Delta}_+$. Otherwise, it is the highest short root.

The real roots for $\mathfrak{g} = \mathfrak{g}(A)$, where A is of affine type are described in [2]. They are of the form $\gamma = \beta + j\delta$ where $\beta \in \overset{\circ}{\Delta}$ and j is an integer, and when $A = A_{2n}^{(2)}$, we also have roots of the form $\gamma = \frac{1}{2}(\beta + (2j-1)\delta)$ where β is a long root in $\overset{\circ}{\Delta}$ and j is an integer.

Let θ denote the highest long root in $\overset{\circ}{\Delta}_+$, $k = ht(\theta)$ and $w_0 \in \overset{\circ}{W}$ such that $w_0^{-1}(\overset{\circ}{\Pi}) = -\overset{\circ}{\Pi}$. By the description of real roots in Δ and Theorem 1.2 it follows that $S_{w_0}^{(k-1)} \ne 0$. It is known that k is related to the Coxeter number h and the dual Coxeter number $\check{h}$ as follows. If A is of type 1, then $k = h - 1$. If A is of type $A_{2n}^{(2)}$, then $k = h - 2$ and $k = \check{h} - 1$ otherwise. For the algebras in the Main Theorem, we will show that $S_w^{(k)} = 0$ for all $w \in W$ ($S_w^{(k+1)} = 0$ for $\mathfrak{g} = A_{2n}^{(2)}$) and that $S_w^{(k)} \ne 0$ for some $w \in W$ when $\mathfrak{g} = A_{2n}^{(2)}$.

For $w \in W$, define $\mathbb{N}_0 = \{0,1,2,\ldots\}$ and

$$\begin{aligned} X &= \{\beta \in \overset{\circ}{\Delta}_+ | (\exists j \in \mathbb{N}_0)\ \beta + j\delta \in \Delta_+(w) \text{ or } \tfrac{1}{2}(\beta + j\delta) \in \Delta_+(w)\} \\ Y &= \{\beta \in \overset{\circ}{\Delta}_- | (\exists j \in \mathbb{N}_0)\ \beta + j\delta \in \Delta_+(w) \text{ or } \tfrac{1}{2}(\beta + j\delta) \in \Delta_+(w)\} \end{aligned}$$

and $-X = \{-\alpha | \alpha \in X\}$, $-Y = \{-\alpha | \alpha \in Y\}$. Let $\{\beta_i\}$ be a sequence in $X \cup Y$ such that $s_m = \sum_{l=1}^{m} \beta_i \in X \cup Y$ for each m. The β_i are not necessarily distinct.

Proposition 2.1. *Let $\mathfrak{g} = \mathfrak{g}(A)$ be an affine Lie algebra with Weyl group W and let $w \in W$. Then*

1. *$X \cap (-Y) = \emptyset$*
2. *If $j \ne r$, then $s_j \ne \pm s_r$*
3. *No partial sum of the β_i is 0.*

Proof. (1) Let $\beta \in X \cap (-Y)$. Then $\beta + n_1\delta$ or $\frac{1}{2}(\beta + n_1\delta)$ is in $\Delta_+(w)$ for some non-negative integer n_1 and $-\beta + n_2\delta$ or $\frac{1}{2}(-\beta + n_2\delta)$ is in $\Delta_+(w)$ for some positive integer n_2. Suppose $\gamma_1 = \frac{1}{2}(\beta + n_1\delta) \in \Delta_+(w)$ and $\gamma_2 = (-\beta + n_2\delta) \in \Delta_+(w)$ for $n_1 \geq 0$, $n_2 > 0$. Then $w^{-1}(-\beta + n_2\delta) < 0$ and $w^{-1}(\frac{1}{2}(\beta + n_1\delta)) < 0$. Hence either $0 < (2n_1 + n_2)\delta < 0$ or $0 < (n_1 + n_2)\delta < 0$, a contradiction. The other cases are treated in the same manner. ■

(2) Suppose that $s_j = s_r$ for $j > r$. Then $s_r = s_r + \beta_{r+1} + \cdot + \beta_j$ and $0 = \beta_{r+1} + \cdots + \beta_j$. Thus $\beta_{r+1} = -(\beta_{r+2} + \cdots + \beta_j)$. But $w^{-1}(\beta_{r+i} + n_i\delta) < 0$, $n_i \geq 0$ for $i = 2, \cdots, j - r$. Therefore $w^{-1}((\beta_{r+2} + \cdots + \beta_j) + (n_2 + \cdots + n_{j-r})\delta) < 0$. If $\beta_{r+1} \in X$, then $\beta_{r+2} + \cdots + \beta_j \in Y$ and $\beta_{r+1} \in X \cap (-Y)$, a contradiction. Similarly, if $\beta_{r+1} \in Y$, then $-\beta_{r+1} \in X \cap (-Y)$, a contradiction. Now assume that $s_j = -s_r$ for some $j > r$. Then $s_r + \beta_{r+1} + \cdots + \beta_j = -s_r$ and $\beta_{r+1} = -(2s_r + \beta_{r+2} + \cdot + \beta_j)$. Thus either $\beta_{r+1} \in X \cap (-Y)$ or $-\beta_{r+1} \in X \cap (-Y)$. Either case is a contradiction.

(3) Suppose that $s_{i_p} = \sum_{j=1}^{p} \beta_{i_j} = 0$. Then $\beta_{i_p} = -\sum_{j=1}^{p-1} \beta_{i_j}$ and $\beta_{i_p} \in X \cap (-Y)$ or $-\beta_{i_p} \in X \cap (-Y)$. Both cases yield contradictions.

For each $\overset{\circ}{\mathfrak{g}}$ we associate a graph using the highest long root θ. For $\mathfrak{g} \neq D_n^{(1)}$ the graph has vertices $\theta_0 = 0, \theta_1, \cdots, \theta_k = \theta$ where $\theta_j = \alpha_{i_1} + \cdot + \alpha_{i_j}$, $\alpha_{i_l} \in \overset{\circ}{\Pi}$, each $\theta_j \neq 0$ for all $j > 0$ and $k = \mathrm{ht}(\theta)$. As we have seen, $S_{w_0}^{(k-1)} \neq 0$ where $w_0(\overset{\circ}{\Pi}) = -\overset{\circ}{\Pi}$. For $D_n^{(1)}$, there is an extra vertex since both $\alpha_1 + \cdots + \alpha_{n-1}$ and $\alpha_1 + \cdots + \alpha_{n-2} + \alpha_n$ are roots. The graphs and associated θ_j are as follows:

$A = A_n^{(1)}$

$$\theta_0 - \theta_1 - \theta_2 \ \cdots \ \theta_{n-1} - \theta_n$$

$$\theta_i = \alpha_1 + \cdots + \alpha_i, \quad 1 \leq i \leq n, \quad \theta = \alpha_1 + \cdots + \alpha_n$$

$A = B_n^{(1)}$ or $D_{n+1}^{(2)}$

$$\theta_0 - \theta_1 - \theta_2 \ \cdots \ \theta_{2n-2} - \theta_{2n-1}$$

$$\theta_i = \alpha_1 + \cdots + \alpha_i, \quad 1 \leq i \leq n, \quad \theta = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_n$$

$$\theta_{n+i+1} = \alpha_1 + \cdots + \alpha_n + \alpha_n + \cdots + \alpha_{n-i} \quad 0 \leq i \leq n-2 \ .$$

$A = C_n^{(1)}$ or $A_{2n}^{(2)}$ or $A_{2n-1}^{(2)}$

$$\theta_0 - \theta_1 \ \cdots \ \theta_{2n-2} - \theta_{2n-1}$$

$$\theta_i = \alpha_1 + \cdots + \alpha_i \ , \quad 1 \leq i \leq n, \quad \theta = 2\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-1} + \alpha_n$$

$$\theta_{n+i+1} = \alpha_1 + \cdots + \alpha_n + \cdots + \alpha_{n-(i+1)}, \ 0 \leq i \leq n-2.$$

$A = D_n^{(1)}$

$$\begin{array}{ccccc} & & \theta_{n-1} & & \\ \theta_0 - \theta_1 - \theta_2 \cdots - & \diagup & & \diagdown & - \cdots \theta_{2n-2} - \theta_{2n-1} \\ & \diagdown & & \diagup & \\ & & \hat{\theta}_{n-1} & & \end{array}$$

$$\theta_i = \alpha_1 + \cdots + \alpha_i, \quad 1 \leq i \leq n-1, \quad \hat{\theta}_{n-1} = \alpha_1 + \cdots + \alpha_{n-2} + \alpha_n$$

$\theta_n = \alpha_1 + \cdots + \alpha_n, \quad \theta_{n+i} = \alpha_1 + \cdots + \alpha_{n-1} + \alpha_n + \alpha_{n-2} + \cdots + \alpha_{n-i-1} \quad 1 \leq i \leq n-3,$

$\theta = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$

$A = G_2^{(1)}$

$$\theta_0 \text{ --- } \theta_1 \text{ --- } \theta_2 \text{ --- } \theta_3 \text{ --- } \theta_4 \text{ --- } \theta_5$$

$\theta_1 = \alpha_1, \quad \theta_2 = \alpha_1 + \alpha_2, \quad \theta_3 = \alpha_1 + \alpha_2 + \alpha_2$

$\theta_4 = \alpha_1 + \alpha_2 + \alpha_2 + \alpha_2, \quad \theta_5 = \alpha_1 + \alpha_2 + \alpha_2 + \alpha_2 + \alpha_1 = \theta.$

$A = D_4^{(3)}$

$$\theta_0 \text{ --- } \theta_1 \text{ --- } \theta_2 \text{ --- } \theta_3 \text{ --- } \theta_4 \text{ --- } \theta_5$$

$\theta_1 = \alpha_2, \ \theta_2 = \alpha_2 + \alpha_1, \ \theta_3 = \alpha_2 + \alpha_1 + \alpha_1$

$\theta_4 = \alpha_2 + \alpha_1 + \alpha_1 + \alpha_1, \ \theta_5 = \alpha_2 + \alpha_1 + \alpha_1 + \alpha_1 + \alpha_2 = \theta.$

Note that the Coxeter numbers and dual Coxeter numbers for these algebras are listed on page 80 of [2] and correspond to the respective heights as previously noted.

Each root in $\mathring{\Delta}$ gives a connected directed subgraph as is seen by checking each case using the results on p. 64-65 of [1].

Example 2.2. Let $A = A_5^{(1)}$ and let $\theta_i = \alpha_1 + \cdots + \alpha_i$ for $i = 1, \cdots, 5$. Let $\theta = \theta_5$. Then $ht(\theta) = 5$ and the graph is

$$\theta_0 \text{ --- } \theta_1 \text{ --- } \theta_2 \text{ --- } \theta_3 \text{ --- } \theta_4 \text{ --- } \theta_5$$

A typical root is $\beta = \alpha_3 + \alpha_4 = \theta_4 - \theta_2$ and the subgraph is

$$\theta_2 \text{ --- } \theta_3 \text{ --- } \theta_4$$

β uses two vertices. Generally if $\beta_1 + \cdots + \beta_s$ is a root for $s = 1, \cdots, t$, then $t+1$ vertices are used. Since cycles are not permitted, only 5 β's may occur in a string. These remarks hold in the general case.

Suppose that $\mathfrak{g} \neq A_{2n}^{(2)}$ and that $S_w^{(t-1)} \neq 0$. Then there exists t roots of the form $\beta_i + m_i\delta$ in $\Delta_+(w)$, $\beta_i \in \mathring{\Delta}$ and $m_i \in Z$, whose sum is a root and the corresponding $\sum_{i=1}^{t} \beta_i \in X \cup Y$. In case $\mathfrak{g} \neq D_n^{(1)}$, $t < k+1$ where $k = ht(\theta)$ which is seen as follows (see also the example). $s_1 = \beta_1$ contains two vertices on the graph. Then $s_2 = \beta_1 + \beta_2$ forms a connection of one of the original vertices with a new vertex since the graph is connected with no cycles using Proposition 2.1. Continuing in the manner, s_t uses $t+1$ vertices. Hence $t < k+1$ and $S_w^{(k)} = 0$ for all $w \in W$.

Suppose that $\mathfrak{g} = D_n^{(1)}$. The number of vertices is $k+2$. As in the preceding paragraph, $t \leq k+1$. Suppose that $t = k+1$. Then there are roots $\beta_1, \cdots, \beta_t \in X \cup Y$ with $s_j = \sum_{i=1}^{j} \beta_i \in X \cup Y$ for $j = 1, \cdots, t$.

Let z_0 be the initial vertex of s_t. Hence z_0 is the initial vertex of some β_{i_1}, with terminal vertex z_1. Then z_1 is the initial vertex of some β_{i_2} with terminal vertex z_2. Continue this process to obtain the sequence $S = \{z_0, z_1, \cdots, z_t\}$ with β_{i_j} connecting z_{j-1} to z_j and all vertices on the graph are listed in S. To simplify the notation we will write that β_j connects z_{j-1} to z_j.

Two vertices will be called symmetric if a subgraph joining them is of the form

$$B: \ \pm(2\alpha_i + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n) \ \text{ or } \ C: \ \pm(\alpha_{n-1} - \alpha_n) \ .$$

We denote symmetric vertices by x_i and y_i.

Lemma 2.3. *Adjacent vertices in S are not symmetric.*

Proof. Otherwise a root would be of the form B or C, a contradiction. ∎

Define $z_i < z_j$ in S if $i < j$. Thus $\beta_{i+1} + \cdots + \beta_j$ connects z_i to z_j.

Lemma 2.4. *$x_i < x_j$ if and only if $y_j < y_i$.*

Proof. Suppose that $x_i < x_j$ and $y_i < y_j$. Then some $\beta_{k_1} + \cdots + \beta_{k_b}$ connects y_i to y_j. Then $\beta_{i+1} + \cdots + \beta_j = -(\beta_{k_1} + \cdots + \beta_{k_b})$ which yields a partial sum of β equal to 0. This contradicts Proposition 2.1. ∎

The following Corollary is an immediate consequence of Lemma 2.4.

Corollary 2.5. *$x_i < x_j < x_l$ if and only if $y_l < y_j < y_i$.*

Now suppose that $z_0 = 0$ and let $z_1 = x_i$. By Lemma 2.3, y_i is the final vertex in S. Let $z_2 = x_j$. Then y_j is the next to last vertex in S. Continue this process and finally we obtain adjacent symmetric vertices, a contradiction. If the final vertex in S is 0, this same process yields another contradiction.

Therefore, suppose that neither the initial nor final vertex in S is 0. Then the initial and final vertices are symmetric by Lemma 2.4. Thus s_t has the form B or C and is not a root. But $s_t \in X \cup Y$ and hence is a root. Therefore $t \neq k+1$. Consequently, $t < k+1$ and only k roots are allowed in a non-zero sum. Hence $S_w^{(k)} = 0$ for each $w \in W$.

Now we consider the case $\mathfrak{g} = \mathfrak{g}(A_{2n}^{(2)})$. The existence of half roots makes this case special. The real roots divide into the cases

1. $\Delta_s = \Delta_s^{re} = \{\frac{1}{2}(\alpha + (2j-1)\delta) | \alpha \in \mathring{\Delta}_l, \ j \in \mathbb{Z}\}$
2. $\Delta_m = \Delta_m^{re} = \{\alpha + j\delta | \alpha \in \mathring{\Delta}_s, \ j \in \mathbb{Z}\}$
3. $\Delta_l = \Delta_l^{re} = \{\alpha + 2j\delta | \alpha \in \mathring{\Delta}_l, \ j \in \mathbb{Z}\}$

where $\mathring{\Delta}_s = \{\pm(\alpha_i + \cdots + \alpha_n + \cdots + \alpha_j) | 1 \le i < j \le n\} \cup \ \{\pm(\alpha_i + \cdots + \alpha_j) | 1 \le i < j \le n-1\}$ and $\mathring{\Delta}_l = \{\pm(\alpha_i + \cdots + \alpha_n + \cdots + \alpha_i) | 1 \le i \le n\}$.

Define $\Delta_i + \Delta_j = \Delta_k$ if there exist $\gamma_i \in \Delta_i, \gamma_j \in \Delta_j$ such that $\gamma_i + \gamma_j \in \Delta_k$. If $\gamma_i + \gamma_j$ is never a root when $\gamma_i \in \Delta_i$ and $\gamma_j \in \Delta_j$, then let $\Delta_i + \Delta_j = 0$. We obtain the following addition table.

	Δ_s	Δ_m	Δ_l
Δ_s	$\Delta_m \cup \Delta_l$	Δ_s	0
Δ_m	Δ_s	$\Delta_m \cup \Delta_l$	Δ_m
Δ_l	0	Δ_m	0

This table is verified case by case. The material in [1, p. 64] is helpful.

Let $\gamma_i = \frac{1}{2}(\beta_i + (2n_i - 1)\delta) \in \Delta_s$ for $i = 1, 2$. Then $\gamma_1 + \gamma_2 = \frac{1}{2}(\beta_1 + \beta_2 + 2(n_1 + n_2 - 1)\delta)$. If $\gamma_i + \gamma_j$ is a root, then either $\frac{1}{2}(\beta_i + \beta_j) \in \mathring{\Delta}_s$ and $\gamma_i + \gamma_j \in \Delta_m$ or $\frac{1}{2}(\beta_i + \beta_j) \in \mathring{\Delta}_l$ and $\gamma_i + \gamma_j \in \Delta_l$. Hence $\Delta_s + \Delta_s = \Delta_m \cup \Delta_l$.

Let $\gamma_1 = \frac{1}{2}(\beta_1 + (2n_1 - 1)\delta) \in \Delta_s$ and $\gamma_2 = \beta_2 + n_2\delta \in \Delta_m$. Then $\gamma_1 + \gamma_2 = \frac{1}{2}[(\beta_1 + 2\beta_2) + (2n_1 + 2n_2 - 1)\delta]$. If $\gamma_1 + \gamma_2$ is a root, then $\beta_1 + 2\beta_2 \in \mathring{\Delta}_l$ and $\gamma_1 + \gamma_2 \in \Delta_s$. Hence $\Delta_s + \Delta_m = \Delta_s$.

Let $\gamma_1 = \frac{1}{2}(\beta_1 + (2n_1 - 1)\delta) \in \Delta_s$ and $\gamma_2 = \beta_2 + 2n_2\delta \in \Delta_l$. Then $\gamma_1 + \gamma_2 = \frac{1}{2}[(\beta_1 + 2\beta_2) + (2n_1 - 1 + 4n_2)\delta]$. But $\beta_1 + 2\beta_2$ is not a root since $\beta_1, \beta_2 \in \mathring{\Delta}_l$. Hence $\Delta_s + \Delta_l = 0$.

Let $\gamma_i = \beta_i + n_i\alpha \in \Delta_m$ for $i = 1, 2$. Then $\gamma_1 + \gamma_2 = (\beta_1 + \beta_2) + (n_1 + n_2)\delta$. It is possible that $\beta_1 + \beta_2 \in \mathring{\Delta}_l$ or in $\mathring{\Delta}_s$. Hence $\gamma_1 + \gamma_2 \in \Delta_m \cup \Delta_l$. Therefore, $\Delta_m + \Delta_m = \Delta_m \cup \Delta_l$.

Let $\gamma_1 = \beta_1 + 2n_1\delta \in \Delta_l$, $\gamma_2 = \beta_2 + n_2\delta \in \Delta_m$. Then $\gamma_1 + \gamma_2 = (\beta_1 + \beta_2) + (2n_1 + n_2)\delta$. If $\gamma_1 + \gamma_2$ is a root, then $\beta_1 + \beta_2 \in \mathring{\Delta}_s$. Hence $\gamma_1 + \gamma_2 \in \Delta_m$. So $\Delta_m + \Delta_l = \Delta_m$.

Let $\gamma_i = \beta_i + 2n_i\delta \in \Delta_l$, for $i = 1, 2$. Then $\gamma_1 + \gamma_2 = \beta_1 + \beta_2 + 2(n_1 + n_2)\delta$. But $\beta_1 + \beta_2$ is not a root. Hence $\gamma_1 + \gamma_2$ is not a root and $\Delta_l + \Delta_l = 0$.

In the graph of $A_{2n}^{(2)}$ there are $2n$ vertices and $k = ht(\theta) = 2n - 1$. As in previous cases no more than k β_i's can exist in a given sum that is a non-zero root. We proceed by considering how the number of summands in a sum of $\gamma_i \in \Delta^+(w)$ translates into the number of summands in the corresponding β_i. From the results in the table and their proofs, adding a γ adds one β unless

$$\begin{aligned}
&D: \ \gamma_1 + \cdots + \gamma_j \in \Delta_s \quad \text{and} \quad \gamma_{j+1} \in \Delta_s \\
\text{or} \quad &E: \ \gamma_1 + \cdots + \gamma_j \in \Delta_m \quad \text{and} \quad \gamma_{j+1} \in \Delta_s \\
\text{or} \quad &F: \gamma_1 + \cdots + \gamma_j \in \Delta_s \quad \text{and} \quad \gamma_{j+1} \in \Delta_m.
\end{aligned}$$

In the first case, the number of β's do not decrease. In the other cases, the number of β's increase by at least two.

Again consider $\gamma_1 + \cdots + \gamma_t$. Let n_t be the number of corresponding β's and set $n_0 = 0$. Then $n_1 = 1$. Checking cases it is seen that $n_2 < 2$ implies $n_2 = 1$ and $\gamma_1 + \gamma_2 \neq \Delta_s$.

We claim that $n_j < j$ implies $n_j = j - 1$ and $\gamma_1 + \cdots + \gamma_j \notin \Delta_s$. Assume the result holds for j and consider $\gamma_1 + \cdots + \gamma_{j+1}$. Suppose that $n_j = j - 1$. By assumption $\gamma_1 + \cdots + \gamma_j \in \Delta_m \cup \Delta_l$. If $\gamma_{j+1} \in \Delta_s$, then $n_{j+1} \geq j + 1$. If $\gamma_{j+1} \notin \Delta_s$, then $n_{j+1} = j$. Now suppose that $n_j \neq j - 1$. Then $n_j \geq j$ implies $n_{j+1} \geq j$. If $n_{j+1} = j = n_j$, then $\gamma_1 + \cdots + \gamma_j$, $\gamma_{j+1} \in \Delta_s$ and $\gamma_1 + \cdots + \gamma_{j+1} \neq \Delta_s$. Hence the result holds.

Now if $S_w^{(t-1)} \neq 0$, then $t-1 \leq n_t < k+1$ and $t < k+2$. Hence $S_w^{(k+1)} = 0$.

It remains to show that $S_w^{(k)} \neq 0$ for some $w \in W$. We will work with the graph for $A_{2n}^{(2)}$ and the discussion of the roots of C_n in [1, p. 64]. The roots are of the form $\pm 2e_i$ and $\pm(e_i \pm e_j)$, $i \neq j$ with base $\alpha_j = e_{j+1} - e_j$, $j = 1, \cdots, n-1$ and $\alpha_n = 2e_n$. Let $w_0 \in \overset{\circ}{W}$ such that $w_0^{-1}(\overset{\circ}{\Pi}) = -\overset{\circ}{\Pi}$. Let $\gamma_1 = \frac{1}{2}(\theta + (2j-1)\delta)$, $\gamma_2 = \frac{1}{2}(\theta + (2j+1)\delta)$, $\gamma_3 = \frac{1}{2}(-\alpha_1 + j_1\delta), \cdots, \gamma_{k+2} = (-\alpha_k + j_k\delta)$ where $k = ht(\theta) = 2n-1$. The first two of these roots are in Δ_s and $\gamma_1 + \gamma_2 = \theta + 2j\delta \in \Delta_l$. The remaining roots are in Δ_m and $\gamma_1 + \cdots + \gamma_j \in \Delta_m$ for all $j > 2$. On the graph, θ goes from θ_0 to θ_k and the second θ does not move from vertex θ_k. Each remaining γ_i moves one vertex to the left. Hence $\gamma_1 + \cdots + \gamma_{k+1}$ is a root and $S_{w_0}^{(k)} \neq 0$ but $\gamma_1 + \cdots + \gamma_{k+2}$ is a cycle and $S_{w_0}^{(k+1)} = 0$.

Due to the relation between the heights of the highest long roots with the Coxeter numbers and dual Coxeter numbers pointed out earlier in the paper, we have completed the proof of the following theorem.

Theorem 2.6. *Let $\mathfrak{g} = \mathfrak{g}(A)$ be any affine Lie algebra other than type E or F. Then there is a smallest positive integer m such that $S_w^{(m)} = 0$ for all $w \in W$. Furthermore,*

1. if A is of affine type 1 or $A_{2n}^{(2)}$ then $m = h-1$ where h is the Coxeter number, and

2. if A is of affine type 2 or 3 and $A \neq A_{2n}^{(2)}, E_6^{(2)}$, then $m = \check{h}-1$ where $\check{h}$ is the dual Coxeter number.

3. A of indefinite type

Let $A = (a_{ij})$, $1 \leq i,j \leq n$ be an indecomposable generalized Cartan matrix of indefinite type and $\mathfrak{g} = \mathfrak{g}(A)$ be the associated Kac-Moody Lie algebra. Let $\Pi = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ denote the set of simple roots and $W = \langle r_1, r_2, \cdots, r_n \rangle$ denote the Weyl group of $\mathfrak{g}$, where $r_i = r_{\alpha_i}$ are the simple reflections. The following lemma is well known.

Lemma 3.1. 2 . *For $w \in W$, let $w = r_{i_1} \cdots r_{i_t}$ be a reduced expression. Then $\Delta_+(w) = \{\beta_1, \beta_2, \cdots, \beta_t\}$, where $\beta_p = r_{i_1} \cdots r_{i_{p-1}}(\alpha_{i_p})$, $1 \leq p \leq t$, are distinct real positive roots.*

Proposition 3.2. *Let $w = r_{i_1} \cdots r_{i_t} \in W$ be reduced and*

$$\beta_p = r_{i_1} \cdots r_{i_{p-1}}(\alpha_{i_p}),\ 1 \leq p \leq t$$

as above. Then for $1 \leq k < j \leq t$, $1 \leq l \leq t$, we have $\beta_k + \beta_j = \beta_l \Rightarrow k < l < j$.

Proof. Clearly $l \neq k, j$. Suppose $l < k$. Then

$$r_{i_1} \cdots r_{i_{k-1}} \alpha_{i_k} + r_{i_1} \cdots r_{i_{j-1}} \alpha_{i_j} = r_{i_1} \cdots r_{i_{l-1}} \alpha_{i_l}$$

implies

$$r_{i_l} \cdots r_{i_{k-1}} \alpha_{i_k} + r_{i_l} \cdots r_{i_{j-1}} \alpha_{i_j} = \alpha_{i_l} .$$

Since $r_{i_l} \cdots r_{i_{k-1}} r_{i_k}$ and $r_{i_l} \cdots r_{i_{j-1}} r_{i_j}$ are reduced by Lemma 3.1 we have

$$r_{i_l} \cdots r_{i_{k-1}} \alpha_{i_k} > 0 \text{ and } r_{i_l} \cdots r_{i_{j-1}} \alpha_{i_j} > 0 \ ,$$

which implies that

$$\begin{aligned} 1 = ht(\alpha_{i_l}) &= ht(r_{i_l} \cdots r_{i_{k-1}} \alpha_{i_k} + r_{i_l} \cdots r_{i_{j-1}} \alpha_{i_j}) \\ &= ht(r_{i_l} \cdots r_{i_{k-1}} \alpha_{i_k}) + ht(r_{i_l} \cdots r_{i_{j-1}} \alpha_{i_j}) \geq 1 + 1 = 2 \ , \end{aligned}$$

a contradiction. Now suppose $j < l$. Then $\beta_k + \beta_j = \beta_l$ implies

$$\alpha_{i_j} = r_{i_j} \cdots r_{i_{l-1}} \alpha_{i_l} + r_{i_{j-1}} \cdots r_{i_{k+1}} \alpha_{i_k} \ .$$

As before, since $r_{i_j} \cdots r_{i_{l-1}} r_{i_l}$ and $r_{i_{j-1}} \cdots r_{i_{k+1}} r_{i_k}$ are reduced, by Lemma 3.1 we have

$$ht(r_{i_j} \cdots r_{i_{l-1}} \alpha_{i_l}) \geq 1 \text{ and } ht(r_{i_{j-1}} \cdots r_{i_{k+1}} \alpha_{i_k}) \geq 1 \ ,$$

which gives a contradiction. Hence $k < l < j$. ■

Corollary 3.3. *Let $w = r_{i_1} \cdots r_{i_t} \in W$ be reduced and $\beta_p = r_{i_1} \cdots r_{i_{p-1}} \alpha_{i_p}$, $1 \leq p \leq t$ as before. If $\beta_1 < \beta_2 < \cdots < \beta_t$, then $S_w^{(1)} = 0$.*

Proof. Since $\{\beta_p\}$ is an increasing sequence, it follows from Proposition 3.2 that $\beta_i + \beta_j \notin \Delta^+(w)$, $1 \leq i, \ j \leq t$. Hence $S_w^{(1)} = [S_w, \ S_w] = [\mathfrak{g}_{\beta_1} \oplus \cdots \oplus \mathfrak{g}_{\beta_t}, \ \mathfrak{g}_{\beta_1} \oplus \cdots \oplus \mathfrak{g}_{\beta_t}] = 0$. ■

Lemma 3.4. *Let $A = (a_{ij})$ with $|a_{ij}| \geq 2, \ 1 \leq i, \ j \leq n$. Then for $\beta \in Q_+$, $r_i\beta \geq \beta \Rightarrow r_k(r_i\beta) \geq r_i\beta$ for all $k \neq i$.*

Proof. Let $\beta = \sum_{j=1}^n x_j \alpha_j \in Q^+$. Then $r_i\beta \geq \beta$ implies $\beta(h_i) \leq 0$, which implies $2x_i \leq \sum_{j \neq i}(-a_{ij})x_j$. To show that $r_k(r_i\beta) \geq r_i\beta$, we need to show that $(\beta - \beta(h_i)\alpha_i)(h_k) = \beta(h_k) - \beta(h_i)a_{ki} \leq 0$. We have

$$\begin{aligned} \beta(h_i)a_{ki} - \beta(h_k) &= (2x_i + \textstyle\sum_{j \neq i}(a_{ij})x_j)a_{ki} - (2x_k + \sum_{j \neq k}(a_{kj})x_j) \\ &= x_i a_{ki} + (a_{ik}a_{ki} - 2)x_k + \textstyle\sum_{j \neq k,i}(a_{ij}a_{ki} - a_{kj})x_j \\ &\geq \tfrac{1}{2}\textstyle\sum_{j \neq i}(-a_{ij})a_{ki}x_j + (a_{ik}a_{ki} - 2)x_k + \sum_{j \neq k,i}(a_{ij}a_{ki} - a_{kj})x_j \\ &= \tfrac{1}{2}\left[(a_{ik}a_{ki} - 4)x_k + \textstyle\sum_{j \neq k,i}(a_{ij}a_{ki} - 2a_{kj})x_j\right] \\ &\geq 0 \ , \end{aligned}$$

since $|a_{ij}| \geq 2$, hence $(a_{ik}a_{ki} - 4) \geq 0$ and $(a_{ij}a_{ki} - 2a_{kj}) \geq 0$. Therefore, the result follows. ■

Theorem 3.5. *Let $A = (a_{ij})$ with $|a_{ij}| \geq 2, \ 1 \leq i, j \leq n$. Then $S_w^{(1)} = 0$ for all $w \in W$.*

Proof. Let $w = r_{i_1} r_{i_2} \cdots r_{i_t} \in W$ be reduced. Then $\Delta^+(w) = \{\beta_1, \beta_2, \cdots, \beta_t\}$ where $\beta_p = r_{i_1} \cdots r_{i_{p-1}}(\alpha_{i_p})$,
$1 \le p \le t$. Consider

$$\begin{aligned} \beta_{p+1} - \beta_p &= r_{i_1} \cdots r_{i_p} \alpha_{i_{p+1}} - r_{i_1} \cdots r_{i_{p-1}} \alpha_{i_p} \\ &= r_{i_1} \cdots r_{i_{p-1}} \beta \ , \end{aligned}$$

where $\beta = (-a_{i_p i_{p+1}} - 1)\alpha_{i_p} + \alpha_{i_{p+1}} \in Q_+$ since $a_{i_p i_{p+1}} \le -2$. Now

$$\begin{aligned} r_{i_{p-1}}\beta - \beta &= -\beta(h_{i_{p-1}})\alpha_{i_{p-1}} \\ &= [(-a_{i_p i_{p+1}} - 1)(-a_{i_{p-1} i_p}) + (-a_{i_{p-1} i_{p+1}})]\alpha_{i_{p-1}}. \end{aligned}$$

Since $-a_{ij} \ge 2$ for $i \ne j$, we have $r_{i_{p-1}}\beta - \beta \ge 0$, hence $r_{i_{p-1}}\beta \ge \beta$. Therefore, by Lemma 3.4, we have $\beta_{p+1} - \beta_p = r_{i_1} \cdots r_{i_{p-1}} \beta \ge \beta > 0$ for $1 \le p \le t$. This implies that $\beta_1 < \beta_2 < \cdots < \beta_t$ and hence by Corollary 3.3 we have $S_w^{(1)} = 0$. ■

Now consider the case $A = \begin{pmatrix} 2 & -a \\ -b & 2 \end{pmatrix}$, $ab > 4$, $a, b \ge 0$, any rank two hyperbolic GCM. If $a \ge 2$ and $b \ge 2$, then by Theorem 3.5 we have $S_w^{(1)} = 0$ for all $w \in W$. Hence without loss of generality assume $b = 1$ and $a > 4$. In this case the Weyl group W is an infinite dihedral group with presentation $W = \{(r_1 r_2)^j, \ r_2(r_1 r_2)^j | j \in \mathbb{Z}\}$. We define the sequence of integers $\{A_j\}_{j \in \mathbb{Z}}$ by the recurrence relations

$$A_0 = 0, \ A_1 = 1, \ A_{2j+2} = A_{2j+1} - A_{2j} \text{ and } A_{2j+1} = aA_{2j} - A_{2j-1} \ .$$

Note that, $A_{-j} = -A_j$ for $j \in \mathbb{Z}$ and

$$r_2(\alpha_1) = A_1\alpha_1 + A_2\alpha_2 \ .$$

Assume that

$$r_2(r_1 r_2)^j(\alpha_1) = A_{2j+1}\alpha_1 + A_{2j+2}\alpha_2 \ .$$

Then

$$\begin{aligned} &r_2(r_1 r_2)^{j+1}(\alpha_1) = (r_2 r_1) r_2 (r_1 r_2)^j (\alpha_1) \\ &= (r_2 r_1)(A_{2j+1}\alpha_1 + A_{2j+2}\alpha_2) \\ &= -A_{2j+1}(\alpha_1 + \alpha_2) + A_{2j+2}(a\alpha_1 + a\alpha_2 - \alpha_2) \\ &= (aA_{2j+2} - A_{2j+1})\alpha_1 + (aA_{2j+2} - A_{2j+1} - A_{2j+2})\alpha_2 \\ &= A_{2j+3}\alpha_1 + (A_{2j+3} - A_{2j+2})\alpha_2 = A_{2j+3}\alpha_1 + A_{2j+4}\alpha_2 \ . \end{aligned}$$

Hence by induction for $j \ge 0$ we have

$$r_2(r_1 r_2)^j(\alpha_1) = A_{2j+1}\alpha_1 + A_{2j+2}\alpha_2 \ .$$

Similarly,

$$(r_1 r_2)^j(\alpha_1) = A_{2j+1}\alpha_1 + A_{2j}\alpha_2 \ .$$

Note that

$$\begin{aligned} r_2(r_1 r_2)^{-j}(\alpha_1) &= -(r_1 r_2)^{j-1}(\alpha_1) = -A_{2j-1}\alpha_1 - A_{2j-2}\alpha_2 \\ &= A_{2(-j)+1}\alpha_1 + A_{2(-j)+2}\alpha_2 \ , \end{aligned}$$

and

$$\begin{aligned} (r_1 r_2)^{-j}(\alpha_1) &= -r_2(r_1 r_2)^{j-1}(\alpha_1) \\ &= -A_{2j-1}\alpha_1 - A_{2j}\alpha_2 = A_{2(-j)+1}\alpha_1 + A_{2(-j)}\alpha_2 \ . \end{aligned}$$

Therefore, for all $j \in \mathbb{Z}$ we have

$$\begin{aligned} r_2(r_1r_2)^j(\alpha_1) &= A_{2j+1}\alpha_1 + A_{2j+2}\alpha_2 \ , \\ (r_1r_2)^j(\alpha_1) &= A_{2j+1}\alpha_1 + A_{2j}\alpha_2 \ . \end{aligned}$$

Similarly,

$$\begin{aligned} r_2(r_1r_2)^j(\alpha_2) &= -aA_{2j}\alpha_1 - A_{2j+1}\alpha_2 \ , \\ (r_1r_2)^j(\alpha_2) &= -aA_{2j}\alpha_1 - A_{2j-1}\alpha_2 \ . \end{aligned}$$

Theorem 3.6. *Let* $A = \begin{pmatrix} 2 & -a \\ -1 & 2 \end{pmatrix}$, $a > 4$. *Then* $S_w^{(1)} \neq 0$ *and* $S_w^{(2)} = 0$ *for all* $w \in W$.

Proof. In this case $W = \{r_2(r_1r_2)^j,\ (r_1r_2)^j | j \in \mathbb{Z}\}$. We consider the case $w = r_2(r_1r_2)^j$, $j \geq 0$. The proof for the other cases are similar. For $w = r_2(r_1r_2)^j$, $l(w) = 2j+1$ and

$$\Delta_+(w) = \{\beta_1, \beta_2, \cdots, \beta_{2j+1}\}$$

where

$$\begin{aligned} \beta_{2p-1} &= (r_2r_1)^{p-1}\alpha_2 = (r_1r_2)^{-p+1}\alpha_2 = -aA_{-2p+2}\alpha_1 - A_{-2p+1}\alpha_2 \\ &= aA_{2p-2}\alpha_1 + A_{2p-1}\alpha_2 \ , \end{aligned}$$

for $p = 1, 2, \cdots, j+1$ and for $p = 1, 2, \cdots, j$,

$$\beta_{2p} = (r_2r_1)^{p-1}r_2(\alpha_1) = r_2(r_1r_2)^{p-1}(\alpha_1) = A_{2p-1}\alpha_1 + A_{2p}\alpha_2 \ .$$

Note that since $\beta_p \in \Delta_+$, $1 \leq p \leq 2j+1$, we have $A_p \geq 0$. Observe that since $a > 4$, we have

$$\beta_4 = A_3\alpha_1 + A_4\alpha_2 = (a-1)\alpha_1 + (a-2)\alpha_2 > \alpha_1 + \alpha_2 = \beta_2 \ .$$

Assume

$$\beta_{2p} = A_{2p-1}\alpha_1 + A_{2p}\alpha_2 > \beta_{2p-2} = A_{2p-3}\alpha_1 + A_{2p-2}\alpha_2 \ .$$

Hence

$$A_{2p-1} \geq A_{2p-3} \text{ and } A_{2p} \geq A_{2p-2} \ .$$

Then

$$\begin{aligned} A_{2p+1} - A_{2p-1} &= aA_{2p} - 2A_{2p-1} = aA_{2p} - 2(A_{2p} + A_{2p-2}) \\ &= (a-2)A_{2p} - 2A_{2p-2} > 2(A_{2p} - A_{2p-2}) \geq 0 \ , \end{aligned}$$

and

$$\begin{aligned} A_{2p+2} - A_{2p} &= A_{2p+1} - 2A_{2p} = A_{2p+1} - \tfrac{2}{a}(A_{2p+1} + A_{2p-1}) \\ &= \tfrac{1}{a}\left((a-2)A_{2p+1} - 2A_{2p-1}\right) \\ &> \tfrac{2}{a}(A_{2p+1} - A_{2p-1}) \geq 0. \end{aligned}$$

Hence,

$$\beta_{2p+2} - \beta_{2p} = (A_{2p+1} - A_{2p-1})\alpha_1 + (A_{2p+2} - A_{2p})\alpha_2 > 0$$

Therefore, by induction the subsequence $\{\beta_{2p}\}_{p=1}^{j}$ is an increasing sequence. Similarly, the subsequence $\{\beta_{2p-1}\}_{p=1}^{j+1}$ is also an increasing sequence. Also since $a > 4$

$$\beta_4 - \beta_1 \;=\; (a-1)\alpha_1 + (a-3)\alpha_2 > 0,$$

and $(a-1)(a-3) > (a-1)$. Note that $\beta_{2p+2} - \beta_{2p-1} = (r_2 r_1)(\beta_{2p} - \beta_{2p-3})$. Hence using induction we have $\beta_{2p+2} - \beta_{2p-1} > 0$.

$$\begin{aligned}\beta_{2p} + \beta_{2p+2} &= (A_{2p-1} + A_{2p+1})\alpha_1 + (A_{2p} + A_{2p+2})\alpha_2 \\ &= aA_{2p}\alpha_1 + A_{2p+1}\alpha_2 = \beta_{2p+1}\end{aligned}$$

Therefore, $S_w^{(1)} \neq 0$ and also $\beta_{2p-1} < \beta_{2p+2} < \beta_{2p+1}$ for $p = 1, 2, \cdots, j-1$. Observe that

$$\begin{aligned}\beta_{2p-1} + \beta_{2p+1} &= a(A_{2p-2} + A_{2p})\alpha_1 + (A_{2p-1} + A_{2p+1})\alpha_2 \\ &= aA_{2p-1}\alpha_1 + aA_{2p}\alpha_2 = a\beta_{2p} \notin \Delta_+(w).\end{aligned}$$

Furthermore, since $\{\beta_{2p}\}$ and $\{\beta_{2p-1}\}$ are increasing sequences and $\beta_{2p-1} < \beta_{2p+2} < \beta_{2p+1}$ for $p = 1, 2, \cdots, j-1$, by Proposition 3.2 we have $\beta_{2k-1} + \beta_{2p+1} \notin \Delta_+(w)$, $\beta_{2k} + \beta_{2p+1} \notin \Delta_+(w)$, $\beta_{2k-1} + \beta_{2p} \notin \Delta_+(w)$ and $\beta_{2k} + \beta_{2p} \neq \beta_{2m}$ for $k < m < p$. For example, if $\beta_{2k-1} + \beta_{2p+1} = \beta_{2m}$ then $\beta_{2m} > \beta_{2p+1} > \beta_{2p+2}$ which is a contradiction since $\{\beta_{2p}\}$ is an increasing sequence. If $\beta_{2k-1} + \beta_{2p} = \beta_{2m-1}$ for some $k < m \leq p$ then since $\beta_{2p} > \beta_{2p-3}$ the only possibility is $m = p$. However, $\beta_{2k-1} + \beta_{2p} = \beta_{2p-1} = \beta_{2p-2} + \beta_{2p}$ will imply $\beta_{2k-1} = \beta_{2p-2}$ which is not true. Therefore,

$$S_w^{(1)} \subseteq \bigoplus_{p=1}^{j+1} \mathfrak{g}_{\beta_{2p-1}}$$

and hence by Corollary 3.3, $S_w^{(2)} = [S_w, S_w^{(1)}] = 0$, since $\{\beta_{2p-1}\}$ is an increasing sequence. ∎

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