Journal of Lie Theory
Volume **14** (2004) 227–242

The Helgason–Fourier Transform for Symmetric Spaces II

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Communicated by M. Cowling

Abstract. We formulate analogues of the Hausdorff–Young and Hardy–Littlewood–Paley inequalities, the Wiener Tauberian theorem, and some uncertainty theorems on Riemannian symmetric spaces of noncompact type using the Helgason–Fourier transform.

1. Introduction

In this article, we continue the study of the Helgason–Fourier transform of arbitrary L^1 and L^p functions on Riemannian symmetric spaces X, initiated in [16] and [24]. In section 3 we summarize the results from [16] and [24]. In section 4 we consider the Hausdorff–Young and Hardy–Littlewood–Paley inequalities for the Helgason–Fourier transform, which so far seem to have been studied only for spherical functions or K-finite functions. In section 5 we take up analogues of the Wiener Tauberian Theorem. While this has been studied extensively in [22], the results there are stated in terms of the partial Fourier transforms of the K-finite components of the functions involved. Here, on the other hand, we express our results in terms of the Helgason–Fourier transform, which by analogy with the Euclidean case, seems to be the natural vehicle to express such results. In particular Theorem 5.5 is completely new. Finally in section 6 we describe some uncertainty principles for the Helgason–Fourier transform.

The authors thank Professor S. C. Bagchi for some valuable comments.

2. Notation and Preliminaries

Let G be a connected noncompact semisimple Lie group with finite centre and let K be a fixed maximal compact subgroup of G. Let $X = G/K$ be the corresponding Riemannian symmetric space of the noncompact type. Let $G = KAN$ be an Iwasawa decomposition of G and let $\mathfrak{a}$ be the Lie algebra of A. Let $\mathfrak{a}^*$ be the real dual of $\mathfrak{a}$ and $\mathfrak{a}^*_{\mathbb{C}}$ be its complexification. Then for any $g \in G$, $g = K(g) \exp H(g) N(g)$,where $K(g) \in K$, $N(g) \in N$ and $H(g) \in \mathfrak{a}$. Let M be the centralizer of A in K. Let db be a K-invariant normalized measure on

ISSN 0949–5932 / \$2.50

K/M. Let $C_c^\infty(X)$ be the set of compactly supported smooth functions on X. As usual we can consider functions on X as right K-invariant functions on G. For $f \in C_c^\infty(X)$, $b \in K$ and $\lambda \in \mathfrak{a}^*$ we define the Helgason–Fourier transform by

$$\tilde{f}(\lambda, b) = \int_G f(x) e^{(-i\lambda+\rho)(A(bx))}\, dx, \tag{2.1}$$

where $A(g) = -H(g^{-1})$. In the above, ρ is the half sum of the positive restricted roots of G/K, dx an element of the Haar measure on G. (Actually $\tilde{f}(\lambda, \cdot)$ is a function on K which is right invariant under M, i.e., a function on K/M; however we choose to ignore this point in this exposition.)

We will assume throughout that X is of rank 1, and hence $\dim \mathfrak{a}^* = 1$. Therefore we can identify $\mathfrak{a}^*_{\mathbb{C}}$ with $\mathbb{C}$ by identifying $\lambda\rho$ with $\lambda \in \mathbb{C}$. Under this identification, $\mathfrak{a}^* = \mathbb{R}$ and $\rho = 1$. Thus in various integrals below, by $A(x)$ and $H(x)$, we actually mean $\rho(A(x))$ and $\rho(H(x))$.

For each $\lambda \in \mathbb{C}$, let ϕ_λ be the elementary spherical function given by:

$$\phi_\lambda(g) = \int_K e^{(-i\lambda+1)A(kg)}\, dk.$$

For $\lambda \in \mathbb{C}$ one has the spherical principal series representation π_λ of G on $L^2(K/M)$ defined by:

$$(\pi_\lambda(x)V)(b) = e^{(i\lambda-1)H(x^{-1}b)} V(K(x^{-1}b)), \qquad \forall b \in K$$

(see [12] for details). One also knows that π_λ is unitary if λ is real. For $\lambda \in \mathbb{R}$, π_λ is also irreducible. It can be shown that for $f \in L^1(X)$ and $\lambda \in \mathbb{R}$:

$$\pi_\lambda(f) = \int_G f(g) \pi_\lambda(g)\, dg$$

is a bounded linear operator on $L^2(K/M)$ which is given by

$$(\pi_\lambda(f)V)(b) = \left(\int_K V(u)\, du\right) \int_G e^{(i\lambda-1)H(x^{-1}b)} f(x)\, dx.$$

That is,

$$(\pi_\lambda(f))V(b) = \left(\int_K V(u)\, du\right) \tilde{f}(\lambda, b).$$

The integral above is over K/M, but as mentioned above, we slur over the difference.

We conclude this section with the following two important formulae, due to Helgason. For sufficiently nice functions, say for $f \in C_c^\infty(X)$, one has the inversion formula:

$$f(x) = |\mathcal{W}|^{-1} \int_{\mathfrak{a}^*} \int_K \tilde{f}(\lambda, k) e^{(i\lambda-\rho)(H(x^{-1}k))}\, dk\, \mu(\lambda)\, d\lambda \tag{2.2}$$

and the Plancherel formula:

$$\int_G |f(x)|^2\, dx = |\mathcal{W}|^{-1} \int_{\mathfrak{a}^*} \int_K |\tilde{f}(\lambda, k)|^2\, dk\, \mu(\lambda)\, d\lambda. \tag{2.3}$$

Here dx is an element of a *suitably normalized* Haar measure, $\mu(\lambda)\,d\lambda$ is essentially Harish-Chandra's Plancherel measure restricted to the spherical principal series, and $|\mathcal{W}|$ is the order of the Weyl group of G/K. Helgason proved ([14]) that the Fourier transform $f \mapsto \tilde{f}$ extends to an isometry of $L^2(X)$ onto
$$L^2(K/M \times \mathfrak{a}^*_+,\, dk\, \mu(\lambda)\, d\lambda).$$

It is well known that $\mu(\lambda) = |c(\lambda)|^{-2}$, where $c(\lambda)$ is Harish-Chandra's c-function. We shall use the symbols $\mu(\lambda)$ and $|c(\lambda)|^{-2}$ interchangeably.

For any function space $F(X)$ of X, $F(G//K)$ will denote the subspace of K-invariant functions, or, what is the same, of K-biinvariant functions on G.

For unexplained notation and terminology, the reader is referred to [13].

3. The Helgason–Fourier transform of L^1 and L^p functions

Suppose that $\mathcal{S}_1$ and $\mathcal{S}_p$ (where $p \in (1,2)$) are the Helgason–Johnson strip ([15]) and the corresponding p-strip, i.e., when $p \in [1,2)$ (including 1),

$$\mathcal{S}_p = \{\lambda \in \mathbb{C} \mid |\operatorname{Im}\lambda| \le (\frac{2}{p} - 1)\}.$$

Let $\mathcal{S}_p^\circ$ be the interior $\mathcal{S}_p$.

It is well known that the elementary spherical function ϕ_λ is bounded if and only if $\lambda \in \mathcal{S}_1$ ([15]). It is also known that if $\lambda \in \mathcal{S}_p^\circ$, then $\phi_\lambda \in L^q$, where $\frac{1}{p} + \frac{1}{q} = 1$. This easily follows from the estimates for ϕ_0 and ϕ_λ given in [12] (Proposition 4.6.1 and Theorem 4.6.4).

There seems to be some misunderstanding regarding the domain of definition of the Helgason–Fourier transform for L^1-functions on X ([4], p. 319). However it is observed in [16] that for L^1 functions on the symmetric space X, the *natural domain* of definition of the Helgason–Fourier transform is the strip $\mathcal{S}_1$. In fact, more precisely:

Theorem 3.1. [16] *Let $f \in L^1(X)$. Then there exists a subset B of K of full measure, depending on f, such that $\tilde{f}(\lambda, k)$ exists for all $k \in B$ and $\lambda \in \mathcal{S}_1$. For each fixed $k \in B$, $\tilde{f}(\cdot, k)$ is holomorphic in $\mathcal{S}_1^\circ$ and continuous on $\mathcal{S}_1$. Further, $\tilde{f}(\lambda,\cdot) \in L^1(K)$ and $\|\tilde{f}(\lambda,\cdot)\|_{L^1(K)} \le \|f\|_1$, for all $\lambda \in \mathcal{S}_1$. Finally, $\|\tilde{f}(\lambda,\cdot)\|_{L^1(K)} \longrightarrow 0$ as $|\lambda| \longrightarrow \infty$, uniformly in $\mathcal{S}_1$.*

Remark 3.2. Since $\pi_\lambda(f)e_0(\cdot) = \tilde{f}(\lambda,\cdot)$ as L^2 functions on K, it is clear that, for $f \in L^1(X)$ and $\lambda \in \mathfrak{a}^* = \mathbb{R}$,

$$\tilde{f}(\lambda,\cdot) \in L^2(K) \qquad \text{and} \qquad ||\tilde{f}(\lambda,\cdot)||_{L^2(K)} \le ||f||_{L^1(X)},$$

where e_0 is the constant function 1 on K/M.

Remark 3.3. For an idea of the proof of Theorem 3.1, see [24], where the above theorem was generalized to L^p where $1 \le p < 2$, using the techniques of [16].

We thus have:

Theorem 3.4. *Let $f \in L^p(X)$, where $1 < p < 2$. Then there exists a subset B of K of full Haar measure, such that $\tilde{f}(\lambda, k)$ exists for all $k \in B$ and $\lambda \in \mathcal{S}_p^\circ$, and $\tilde{f}(\lambda, \cdot)$ is in $L^1(K)$ for all $\lambda \in \mathcal{S}_p^\circ$. Further for each fixed $k \in B$, $\tilde{f}(\cdot, k)$ is holomorphic in $\mathcal{S}_p^\circ$. Finally, if*

$$\mathcal{F} = \{\lambda \in \mathbb{C} \mid |\operatorname{Im}\lambda| \leq \delta\},$$

where $\delta < \frac{2}{p} - 1$, then $\|\tilde{f}(\lambda, \cdot)\|_{L^1(K)} \longrightarrow 0$ as $|\lambda| \longrightarrow \infty$, uniformly in $\mathcal{F}$.

Remark 3.5. (i) The above results are actually true without any restriction on the rank of X. Although we shall work out everything for rank 1 symmetric spaces, many results of section 3 are true for arbitrary rank. However in section 5 we will obtain analogues of the Wiener Tauberian theorem for symmetric spaces and here the fact that the rank of X is 1 will be crucial. Therefore, to maintain uniformity we prefer to assume throughout that the rank of X is 1.
(ii) For $(\tau, V_\tau) \in \widehat{K}$, let $C_c^\infty(G, V_\tau)$ be the set of V_τ-valued, compactly supported, C^∞ functions on G such that $f(xk) = \tau(k^{-1})f(x)$ for $k \in K$ and $x \in G$. Camporesi ([5]) defined the Helgason–Fourier transform for $f \in C_c^\infty(G, V_\tau)$ by

$$\tilde{f}(\lambda, k) = \int_G e^{(i\lambda-\rho)(H(x^{-1}k)}\tau(K(x^{-1}k))f(x)\,dx$$

for all $k \in K$ and $\lambda \in \mathfrak{a}^*$. Here $K(x)$ is the K-part of x in its Iwasawa decomposition. For $1 \leq p \leq 2$, we define $L^p(G, V_\tau)$ to be the set of all $f\colon G \longrightarrow V_\tau$ such that $f(xk) = \tau(k^{-1})f(x)$ for all $k \in K$ and $x \in G$, and

$$\int_G \|f(x)\|_{V_\tau}^p\,dx < \infty\}.$$

Then as τ is unitary it is not hard to see that we can use the arguments in the proofs of theorems stated above to establish that for $f \in L^p(G, V_\tau)$, there is a set $B(f) \subset K$ of full Haar measure such that $\tilde{f}(\lambda, k)$ exists for all $k \in B(f)$ and every $\lambda \in \mathcal{S}_p^\circ$ when $p > 1$, and every $\lambda \in \mathcal{S}_1$ when $p = 1$.

In [28] Stanton and Tomas have an inversion formula when $f \in L^p(X)$, $1 \leq p < 2$ and $f * \phi_\lambda \in L^1(\mathfrak{a}^*, |c(\lambda)|^{-2})\,d\lambda$. At the end this paper the authors also remark that the inversion formula can also be formulated in terms of the Helgason–Fourier transform. However, as observed earlier the Helgason–Fourier transform was defined at that time only for sufficiently rapidly decreasing functions and not for general L^p-functions. But now that the elementary theory of the Helgason–Fourier transform has been worked out for the L^p-functions, we can use the result in [28] to obtain the following inversion formula:

Theorem 3.6. *If $f \in L^p(X)$, $1 \leq p < 2$, and $\tilde{f} \in L^1(\mathfrak{a}^* \times K, |c(\lambda)|^{-2}\,d\lambda\,dk)$, then,*

$$f(x) = \frac{1}{|\mathcal{W}|}\int_K \int_{\mathfrak{a}^*} \tilde{f}(\lambda, k) e^{(i\lambda-\rho)H(x^{-1}k)}\,|c(\lambda)|^{-2}\,d\lambda\,dk,$$

for almost every $x \in X$, in particular for all Lebesgue points of f.

We note that

$$(f * \phi_\lambda)(x) = \int_K \tilde{f}(\lambda, k) e^{-(i\lambda+\rho)H(x^{-1}k)}\, dk \qquad \forall f \in C_c^\infty(X).$$

In view of the earlier remarks, we can show that the above formula extends to $L^p(X)$ when $1 \leq p < 2$. Also, the condition on the integrability of $\tilde{f}$ ensures the integrability of $f * \phi_\lambda(x)$ as a function of λ, for fixed x.

4. Hausdorff–Young and Hardy–Littlewood–Paley Theorems

In [8] and [7], analogues of the Hausdorff-Young and the Hardy–Littlewood–Paley inequalities were proved for left K-finite functions on X and for bi-K-invariant functions on X. We shall show that easy interpolation arguments provide versions of these theorems involving the Helgason–Fourier transforms of functions on X which are not necessarily K-finite. First, let us take up the Hausdorff–Young theorem.

Theorem 4.1. *Let $1 < p < 2$ and $\frac{1}{p} + \frac{1}{p'} = 1$. If $f \in L^p(X)$, then*

$$\left(\frac{1}{|\mathcal{W}|} \int_{\mathfrak{a}^*} \|\tilde{f}(\lambda, \cdot)\|_{L^1(K)}^{p'} \, |c(\lambda)|^{-2}\, d\lambda \right)^{\frac{1}{p'}} \leq \|f\|_p.$$

Proof. We consider the measure spaces (X, dx) and $(\mathfrak{a}^*, |\mathcal{W}|^{-1} |c(\lambda)|^{-2}\, d\lambda)$ and define a (sublinear) operator T on simple functions on X as

$$Tf(\lambda) = \|\tilde{f}(\lambda, \cdot)\|_{L^1(K)}.$$

Then $\|Tf(\lambda)\|_\infty \leq \|f\|_1$, as $\|\tilde{f}(\lambda, \cdot)\|_{L^1(K)} \leq \|f\|_1$. Again,

$$\begin{aligned}
\|Tf(\lambda)\|_2 &= \frac{1}{|\mathcal{W}|} \int_{\mathfrak{a}^*} \|\tilde{f}(\lambda, \cdot)\|_{L^1(K)}^2 \, |c(\lambda)|^{-2}\, d\lambda \\
&\leq \frac{1}{|\mathcal{W}|} \int_{\mathfrak{a}^*} \int_K |\tilde{f}(\lambda, k)|^2\, dk\, |c(\lambda)|^{-2}\, d\lambda \\
&= \|f\|_2.
\end{aligned}$$

The last equality is the Plancherel theorem for the Helgason–Fourier transform (2.3). The theorem now follows from the Riesz Convexity theorem ([29]). ∎

Next we proceed towards the Hardy–Littlewood–Paley inequality. Let m_γ and $m_{2\gamma}$ be the multiplicities of the positive roots γ and 2γ.

Theorem 4.2. *Let $1 < p < 2$. If $f \in L^p(X)$, then*

$$\int_{\mathfrak{a}^*} \|\tilde{f}(\lambda, \cdot)\|_{L^1(K)}^p |\lambda|^{(p-2)} (1 + |\lambda|)^{-(m_\gamma + m_{2\gamma})} \, |c(\lambda)|^{-2}\, d\lambda \leq C \|f\|_p^p. \tag{4.1}$$

Proof. We write β for $m_\gamma + m_{2\gamma}$. Let us consider the measure spaces (X, dx) and $(\mathfrak{a}^*, |\mathcal{W}|^{-1}(1+|\lambda|)^{-\beta+1}|\lambda|^{-4}\,|c(\lambda)|^{-2}\,d\lambda)$. We define an operator T as

$$Tf(\lambda) = \|\tilde{f}(\lambda,\cdot)\|_{L^1(K)}|\lambda|^2,$$

where f is defined on (X, dx) and Tf on $(\mathfrak{a}^*, |\mathcal{W}|^{-1}(1+|\lambda|)^{-\beta+1}|\lambda|^{-4}\,|c(\lambda)|^{-2}\,d\lambda)$.

Now $\beta \geq 1$, so $(1+|\lambda|)^{-\beta+1} \leq 1$ and thus $\|Tf(\lambda)\|_2 \leq \|f\|_2$ by the Plancherel theorem, i.e., T is of strong type $(2,2)$.

For $t > 0$, let $E_t = \{\lambda \in \mathfrak{a}^* \mid Tf(\lambda) > t\}$ and $A = \{\lambda \in \mathfrak{a}^* \mid |\lambda| > (\frac{t}{\|f\|_1})^{\frac{1}{2}}\}$, and let $a = (\frac{t}{\|f\|_1})^{\frac{1}{2}}$. Since $\|\tilde{f}(\lambda,\cdot)\|_{L^1(K)} \leq \|f\|_1$, we see that $E_t \subset A$. Hence

$$\begin{aligned}|E_t| \leq |A| &= \frac{1}{|\mathcal{W}|}\int_A |\lambda|^{-4}(1+|\lambda|)^{-\beta+1}\,|c(\lambda)|^{-2}\,d\lambda\\ &= \frac{2}{|\mathcal{W}|}\int_a^\infty (1+|\lambda|)^{-\beta+1}|\lambda|^{-4}\,|c(\lambda)|^{-2}\,d\lambda,\end{aligned}$$

where $|E_t|$ and $|A|$ are the measures of the corresponding sets. We have the following estimate of $\mu(\lambda) = |c(\lambda)|^{-2}$ (see [1], p. 394):

$$|c(\lambda)|^{-2} \asymp \langle\lambda,\gamma\rangle^2(1+|\langle\lambda,\gamma\rangle|)^{m_\gamma+m_{2\gamma}-2} \qquad \forall \lambda \in \mathfrak{a}^*. \tag{4.2}$$

Using this estimate, we see that

$$\begin{aligned}|E_t| &\leq \frac{2}{|\mathcal{W}|}\int_a^\infty (1+|\lambda|)^{-\beta+1}|\lambda|^{-4}|\lambda|^2(1+|\lambda|)^{\beta-2}\,d\lambda\\ &\leq 2B\int_a^\infty |\lambda|^{-3}\,d\lambda = B\frac{\|f\|_1}{t},\end{aligned}$$

for some constant B, so T is of weak type $(1,1)$. By the Marcinkiewicz interpolation theorem ([29]), T is of strong type (p,p), because $(1+|\lambda|) \geq |\lambda|^{2-p}$. ■

It is clear from the proof that (4.1) can be replaced by the following slightly stronger inequality:

$$\int_{\mathfrak{a}^*} \|\tilde{f}(\lambda,\cdot)\|^p_{L^1(K)}|\lambda|^{2(p-2)}(1+|\lambda|)^{-(m_\gamma+m_{2\gamma})+1}\,|c(\lambda)|^{-2}\,d\lambda \leq C\|f\|_p^p. \tag{4.3}$$

An analogue of the Hardy–Littlewood–Paley inequality for $q > 2$ will follow from the more general Hausdorff–Young theorem for *Lorentz spaces*. Our proof will be guided mainly by the remarks of Cowling (MR 88e:43006). First we recall some terminology and results. We consider the space $L^{(q)}(X)$ (see [7]), defined to be the set of all measurable functions f on X such that $\|f\|_{(q)} < \infty$, where

$$\|f\|_{(q)} = \left(\int_G |f(x)|^q d(xK,K)^{(q-2)}J(x)^{(q-2)}\,dx\right)^{\frac{1}{q}}$$

and $J(x)$ is the Jacobian of the KA^+K decomposition of G. For a function $f \in L^{(q)}(X)$, it can be shown using arguments similar to those of Theorem 3.4 that there exists a subset $B(f)$ of full Haar measure in K such that $\tilde{f}(\lambda,k)$ exists for every $\lambda \in \mathcal{S}^\circ_{q'}$ and $k \in B(f)$ where $\frac{1}{q}+\frac{1}{q'} = 1$.

Let $(\mathcal{X},\mu)$ be a σ-finite measure space and let $1<p<\infty$. Define

$$\|f\|^*_{pq} = \begin{cases} \left(\dfrac{q}{p}\int_0^\infty t^{\frac{q}{p}} f^*(t)^q \dfrac{dt}{t}\right)^{\frac{1}{q}} & \text{when } q<\infty \\ \sup_{t>0} t\lambda_f(t)^{\frac{1}{p}} & \text{when } q=\infty. \end{cases}$$

Here λ_f is the distribution function of f and f^* is the *non-increasing rearrangement* of f. The Lorentz space $L^{(p,q)}(\mathcal{X})$ is defined to be the set of all measurable functions f on $\mathcal{X}$ such that $\|f\|^*_{pq}<\infty$. It is well known that $L^{(p,p)}=L^p$ and if $q_1\le q_2$, then $\|f\|^*_{pq_2}\le\|f\|^*_{pq_1}$ and consequently $L^{(p,q_1)}\subseteq L^{(p,q_2)}$. For details about Lorentz spaces $L^{(p,q)}$, we refer to [29].

We need the following theorem of O'Neil:

Theorem 4.3. [18] *Let $2<q<\infty$ and $r=\frac{q}{q-2}$, and take $g\in L^q$ and $h\in L^{(r,\infty)}$. Then $f=gh\in L^{(q',q)}$, with $\|f\|^*_{q'q}\le\|g\|_q\|h\|^*_{r\infty}$.*

We also need the following technical lemma, which can be proved in a standard way using integration by parts.

Lemma 4.4. *Let $\psi(x)=d(xK,K)J(x)$. Then $m\{x\mid|\psi(x)|\le t\}\le t$ for $t>0$, where m is the Haar measure of the group.*

With the above ingredients we can prove the following theorem by yet another application of interpolation. We shall include a brief sketch of the proof.

Theorem 4.5. *Let $f\in L^{(q)}(X)$, where $q>2$. Then there is a constant C such that*

$$\frac{1}{|\mathcal{W}|}\int_{\mathfrak{a}^*}\|\tilde f(\lambda,\cdot)\|^q_{L^1(K)}\,|c(\lambda)|^{-2}\,d\lambda\le C\|f\|^q_{(q)}.$$

Proof. For simple functions f on X, we define $Tf(\lambda)=\|f(\lambda,\cdot)\|_{L^1(K)}$. Then by the properties of the Lorentz space discussed above and from the definition of Tf, it follows that

$$\|Tf\|^*_{\infty\infty}=\|Tf\|_\infty\le C\|f\|_1=\|f\|^*_{11}.$$

Also by the Plancherel theorem (2.3),

$$\|Tf\|^*_{2\infty}\le\|Tf\|^*_{22}\le\|f\|_2\le\|f\|^*_{21}.$$

Hence by the interpolation theorem for Lorentz spaces ([29], Theorem 3.5 p. 197),

$$\|Tf\|^*_{qq}\le\|f\|^*_{q'q}. \tag{4.4}$$

Let us define $g(x)=f(x)\psi(x)^{(1-\frac{2}{q})}$ where as above $\psi(x)=d(xK,K)J(x)$. Then $g\in L^q(G)$, by the hypothesis of the theorem.

From Lemma 4.4 it follows that, for $t>0$,

$$\begin{aligned} m\{x\mid|\psi(x)|^{(\frac{2}{q}-1)}>t\} &= m\{x\mid|\psi(x)|^{(1-\frac{2}{q})}<\frac{1}{t}\} \\ &\le Ct^{-\frac{q}{q-2}}. \end{aligned}$$

Therefore $\psi^{(\frac{2}{q}-1)} \in L^{(r,\infty)}(X)$ where $r = \frac{q}{q-2}$. From (4.4) and Theorem 4.3,

$$\begin{aligned}\frac{1}{|\mathcal{W}|}\int_{\mathfrak{a}^*} \|\tilde{f}(\lambda,\cdot)\|^q_{L^1(K)}\,|c(\lambda)|^{-2}\,d\lambda &\le \|f\|^*_{pq} \\ &\le \|g\|_q\|\psi\|^*_{r\infty} \\ &\le C\int_G |f(x)|^q d(xK,K)^{(q-2)}J(x)^{(q-2)}\,dx.\end{aligned}$$

This completes the proof. ∎

Remark 4.6. The methods of this section can be used to prove corresponding theorems when X is of general rank.

5. L^p-versions of Wiener's theorem for symmetric spaces revisited

Let $\{e_0, e_1, \dots\}$ be an orthonormal basis of $L^2(K/M)$, such that e_0 is the constant function 1 on K/M and $e_1, e_2, \dots$ are K-finite functions in $L^2(K/M)$, each of which transforms according to some K-type in $\widehat{K}$ which is class 1 with respect to M. Let $\phi_\lambda^{(i)} = \langle \pi_\lambda(x)e_0, e_i\rangle$. By comparing these with ϕ_λ, it can be easily shown that $\phi_\lambda^{(i)}$ is bounded if $\lambda \in \mathcal{S}_1$, and in $L^q(X)$ if $\lambda \in \mathcal{S}_p^\circ$ where $\frac{1}{p}+\frac{1}{q}=1$.

It is well known that the matrix coefficients of the principal series representations $\phi_\lambda^{(i)}$ are eigenfunctions of the Laplace–Beltrami operator Δ of X with eigenvalue (λ^2+1). Thus, from the self-adjointness of Δ, for sufficiently nice f,

$$(\Delta f)_i\widehat{\ }(\lambda) = (\lambda^2+1)\widehat{f_i}(\lambda) \qquad \text{and} \qquad (\Delta f)\tilde{\ }(\lambda,\cdot) = (\lambda^2+1)\tilde{f}(\lambda,\cdot). \tag{5.1}$$

Here $\widehat{f_i}(\lambda)$ is actually $\int_G f(x)\phi_\lambda^{(i)}(x)\,dx$.

We have noted earlier that if $f \in L^1(X)$ and $\lambda \in \mathbb{R}$, then $\tilde{f}(\lambda,\cdot) \in L^2(K)$. For each fixed $\lambda \in \mathbb{R}$, we have the Fourier series:

$$\tilde{f}(\lambda,\cdot) =_{L^2} \sum_i \widehat{f_i}(\lambda)e_i \qquad \text{and} \qquad \|\tilde{f}(\lambda,\cdot)\|^2_{L^2(K)} = \sum_i |\widehat{f_i}(\lambda)|^2. \tag{5.2}$$

Notice that $\widehat{f_0}(\lambda)$, usually denoted by $\widehat{f}(\lambda)$, is just the spherical Fourier transform of f, given by $\widehat{f}(\lambda) = \int_G f(x)\phi_\lambda(x)\,dx$.

For some $\varepsilon > 0$, let $\mathcal{T}_\varepsilon$ be the strip $\mathcal{T}_\varepsilon = \{\lambda \in \mathbb{C} \mid |\operatorname{Im}\lambda| \le 1+\varepsilon\}$ and $\mathcal{T}_\varepsilon^\circ$ be its interior. Let us define $\mathcal{L}_\varepsilon(X)$ to be the set of measurable functions such that $\int_G |f(x)|e^{\varepsilon d(xK,K)}\,dx < \infty$. Then

(i) $C_c^\infty(X) \subset \mathcal{L}_\varepsilon(X) \subset L^1(X)$, and hence $\mathcal{L}_\varepsilon(X)$ is a dense subset of $L^1(X)$.

(ii) for $f \in \mathcal{L}_\varepsilon(X)$, $\tilde{f}(\lambda,k)$ exists for all $(\lambda,k) \in \mathcal{T}_\varepsilon \times B(f)$, is holomorphic in $\mathcal{T}_\varepsilon^\circ$ and continuous in $\mathcal{T}_\varepsilon$ in λ for each $k \in B(f)$, where $B(f) \subset K$ is a set of full Haar measure, as in Theorem 3.1.

(iii) for f as above, $\tilde{f}(\lambda,\cdot)$ is in $L^1(K)$ for $\lambda \in \mathcal{T}_\varepsilon$ and $\|\tilde{f}(\lambda,\cdot)\|_{L^1(K)} \longrightarrow 0$ as $|\lambda| \longrightarrow \infty$, uniformly in $\mathcal{T}_\varepsilon$.

(iv) if $\varepsilon = 0$, then $\mathcal{L}_\varepsilon(X) = L^1(X)$ and $\mathcal{T}_\varepsilon$ reduces to $\mathcal{S}_1$.

For $p \in [1,2)$, let $L^p(G//K)$ be the set of K-biinvariant functions in $L^p(G)$. The following analogue of the Wiener Tauberian theorem for K-biinvariant functions on X was proved in [22], [23] extending a result of [3]:

Theorem 5.1. *Let $\mathcal{F} \subset L^p(G//K)$ where $p \in [1,2)$. Suppose that for some $\varepsilon > 0$, the spherical Fourier transform of every element f of $\mathcal{F}$ can be extended holomorphically on an augmented strip $\mathcal{S}_p^\varepsilon = \{\lambda \in \mathbb{C} \mid |\operatorname{Im}\lambda| \le (\frac{2}{p}-1)+\varepsilon\}$ and $\lim_{|\lambda|\longrightarrow\infty} |\widehat{f}(\lambda)| = 0$ on $\mathcal{S}_p^\varepsilon$. Assume further that $\{\widehat{f} \mid f \in \mathcal{F}\}$ does not have a common zero on $\mathcal{S}_p^\varepsilon$ and that there exists $f^0 \in \mathcal{F}$ which satisfies*

$$\limsup_{|t|\longrightarrow\infty} |\widehat{f^0}(t)| e^{\alpha e^{|t|}} > 0 \qquad \forall \alpha > 0.$$

Then the $L^1(G//K)$ module generated by $\mathcal{F}$ is dense in $L^p(G//K)$.

Remark 5.2. For $SL_2(\mathbb{R})$, the necessity of some kind of *not too rapidly decreasing condition* on the Fourier transform was established in [9]. Even in general, as far back in 1972, Gangolli predicted that some sort of a not too rapidly decreasing condition would be necessary. For details, see pages 87–90 of [11].

With this preparation we can now offer the following versions of the Wiener Tauberian theorem for symmetric spaces.

Theorem 5.3. *Let $\mathcal{F} \subset \mathcal{L}_\varepsilon(X)$ for some $\varepsilon > 0$. Let Z denote the set of all $\lambda \in \mathcal{T}_\varepsilon$ such that $\tilde{f}(\lambda,\cdot) \equiv 0$ for all $f \in \mathcal{F}$. If Z is empty and there exists $f^0 \in \mathcal{F}$ such that*

$$\limsup_{\lambda\in\mathbb{R}, |\lambda|\longrightarrow\infty} \|\tilde{f^0}(\lambda,\cdot)\|^2_{L^2(K)} e^{\alpha e^{|\lambda|}} > 0 \qquad \forall \alpha > 0, \tag{5.3}$$

then the left G-translates of the functions in $\mathcal{F}$ span a dense subspace of $L^1(X)$.

Remark 5.4. (i) In [22] and [26], it was assumed that the Fourier transforms of the functions in $\mathcal{F}$ exist in an augmented strip. This virtually amounts to demanding that the functions are in a suitable weighted L^1-space.
(ii) The condition (5.3) says that $\tilde{f}$ does not go to zero *too rapidly* at infinity. See Remark 5.2 above.

Before giving the proof, let us investigate the necessity of the hypothesis of the theorem.

For $f \in L^1(X)$, let W be the closed span of the left G-translates of f. Suppose that $\tilde{f}(\lambda,\cdot) \equiv 0$ on K for some $\lambda \in \mathcal{S}_1$. That is,

$$\int_G f(x) e^{(i\lambda-1)H(x^{-1}k)}\, dx = 0$$

for all $k \in K$. This implies that, for any $y \in G$,

$$\int_K e^{-(i\lambda+1)H(y^{-1}k)} \int_G f(x) e^{(i\lambda-1)H(x^{-1}k)}\, dx\, dk = 0.$$

We can use Fubini's theorem and the following *symmetry of the elementary spherical function* ([13], Theorem 1.1, p. 224)

$$\phi_\lambda(x^{-1}y) = \int_K e^{(i\lambda-1)H(x^{-1}k)} e^{-(i\lambda+1)H(y^{-1}k)}\, dk \tag{5.4}$$

to conclude that

$$\int_G f(x)\phi_\lambda(x^{-1}y)\,dx = \int_G {}^y f(x)\phi_\lambda(x^{-1})\,dx = 0 \qquad \forall y \in G,$$

where ${}^y f$ is the left translate of f by $y \in G$. This amounts to saying that $\int_G g(x)\phi_\lambda(x^{-1})\,dx = 0$ for all $g \in W$. But ϕ_λ is bounded when $\lambda \in \mathcal{S}_1$, so it defines a linear functional on $L^1(X)$. Therefore W is a proper subspace of $L^1(X)$. Thus the necessity of the nonvanishing condition on the Helgason–Johnson strip $\mathcal{S}_1$ is established.

Proof. If h is a left K-invariant function in $\mathcal{L}_\varepsilon(X)$, it is not hard to show that $h^* * h$, which is K-biinvariant, is also in $\mathcal{L}_\varepsilon(X)$. Here $h^*(x) = \overline{h(x^{-1})}$.

Let $g = f^{0*} * f^0$. Then g is a K-biinvariant function in $\mathcal{L}_\varepsilon$ which is in the left $L^1(G)$-module generated by f^0.

Also, for every $\lambda \in \mathbb{C}$ for which $\widehat{f^0}$ can be defined, $\widehat{f^{0*}}_i(\overline{\lambda}) = \overline{\widehat{f^0}_i(\lambda)}$. As noted earlier, for $\lambda \in \mathbb{R}$,

$$\tilde{f^0}(\lambda,\cdot) =_{L^2} \sum_i \widehat{f^0}_i(\lambda)e_i \qquad \text{and} \qquad \|\tilde{f^0}(\lambda,\cdot)\|^2_{L^2(K)} = \sum_i |\widehat{f^0}_i(\lambda)|^2.$$

On the other hand,

$$\widehat{g}(\lambda) = \sum_i |\widehat{f^0}_i(\lambda)|^2 \qquad \forall \lambda \in \mathbb{R},$$

where $\widehat{g}(\lambda)$ is the spherical Fourier transform of g. Therefore g satisfies

$$\limsup_{\lambda\in\mathbb{R},|\lambda|\longrightarrow\infty} |\widehat{g}(\lambda)|e^{\alpha e^{|\lambda|}} > 0 \qquad \forall \alpha > 0. \tag{5.5}$$

The following is essentially proved in [22] and [23]: given $\lambda \in \mathcal{T}_\varepsilon$ and $f \in \mathcal{F}$ with $\tilde{f}(\lambda,\cdot) \not\equiv 0$, there exists a K-biinvariant function $f_\lambda \in \mathcal{L}_\varepsilon(X)$ in the $L^1(G)$-module generated by f with $\widehat{f_\lambda}(\lambda) \neq 0$. Also note that for a K-biinvariant function its Helgason–Fourier transform is independent of $k \in K$ and hence it collapses to its spherical Fourier transform. Therefore by (iii) in the discussion preceding Theorem 5.1, $\widehat{f_\lambda}(\nu) \longrightarrow 0$ as $|\nu| \longrightarrow \infty$ uniformly for $\nu \in \mathcal{T}_\varepsilon$.

Consequently, the family $\{f_\lambda\} \cup \{g\}$ satisfies the conditions of Theorem 5.1, and so the $L^1(G//K)$-module generated by the above family is dense in $L^1(G//K)$. Since $\{f_\lambda\}$ and g are contained in the $L^1(G)$-module generated by $\mathcal{F}$ in $L^1(X)$, it follows that the closed span of the left translates of the functions in $\mathcal{F}$ contains $L^1(G//K)$. As the smallest such left translation invariant closed subspace of $L^1(X)$ is $L^1(X)$ itself, the theorem is proved. ■

We also have the following alternative version of the theorem where the growth condition on the Fourier transform is substituted by a condition requiring that at least one of the functions be not "too smooth".

Theorem 5.5. *Let $\mathcal{F} \subset \mathcal{L}_\varepsilon(X)$ for some $\varepsilon > 0$. Let Z denote the set of all $\lambda \in \mathcal{T}_\varepsilon$ such that $\tilde{f}(\lambda,\cdot) \equiv 0$ for all $f \in \mathcal{F}$. If Z is empty and if there exists f^0 in $\mathcal{F}$ which is not equal to a real analytic function almost everywhere, then the left G-translates of the functions in $\mathcal{F}$ span a dense subspace of $L^1(X)$.*

Proof. If we show that f^0 satisfies condition (5.3), then this theorem will follow from the previous one.

If f^0 does not satisfy (5.3), then

$$\|\tilde{f}^0(\lambda,\cdot)\|^2_{L^2(K)} \le Ce^{-\alpha e^{|\lambda|}} \qquad \forall \lambda \in \mathbb{R} \tag{5.6}$$

for some constant C and some $\alpha > 0$. Then by appealing to Helgason's Plancherel Theorem (2.3) and by observing that the Plancherel measure $|c(\lambda)|^{-2}$ is at most of polynomial growth on $\mathbb{R}$, f^0 is clearly in $L^2(X)$.

By remarks made earlier, if g is a sufficiently nice function, for example in $C_c^\infty(X)$, then $(\Delta g)\tilde{}(\lambda,\cdot) = (\lambda^2+1)\tilde{g}(\lambda,\cdot)$.

We consider Δf^0 in the sense of distributions. If h is an L^2-function on X, such that $(1+\lambda^2)\tilde{h}(\lambda,\cdot)$ is also in $L^2(\mathfrak{a}^* \times K, \mu(\lambda)\,d\lambda\,dk)$, then it is not hard to show that Δh, which is a priori only defined as a distribution, is actually in $L^2(X)$ and $(\Delta h)\tilde{}(\lambda,\cdot) = (\lambda^2+1)\tilde{h}(\lambda,\cdot)$.

Now $\tilde{f}^0(\lambda,\cdot)$ is very rapidly decreasing in λ, so $\Delta f^0 \in L^2(X)$ and

$$(\Delta f^0)\tilde{}(\lambda,\cdot) = (\lambda^2+1)\tilde{f}^0(\lambda,\cdot).$$

By repeated application of the same argument, we see that $\Delta^m f^0$ is in $L^2(X)$ for any positive integer m and

$$(\Delta^m f^0)\tilde{}(\lambda,\cdot) = (\lambda^2+1)^m\tilde{f}^0(\lambda,\cdot). \tag{5.7}$$

As Δ is elliptic, Sobolev theory implies that f^0 can be taken to be C^∞.

From the Plancherel theorem (2.3), for all positive integers m,

$$\|\Delta^m f^0\|_2^2 = \frac{1}{|\mathcal{W}|}\int_{\mathbb{R}} \|(\Delta^m f^0)\tilde{}(\lambda,\cdot)\|_{L^2(K)}\,\mu(\lambda)\,d\lambda.$$

Using (5.6) and (5.7), we see that

$$\|\Delta^m f^0\|_2^2 \le C\int_0^\infty (\lambda^2+1)^{2m} e^{-\alpha e^\lambda}\,\mu(\lambda)\,d\lambda.$$

We further use the estimate (4.2) of $\mu(\lambda)$ to get

$$\|\Delta^m f^0\|_2^2 \le C\int_0^\infty \lambda^{4m-1} e^{-\alpha\lambda^2}\,d\lambda,$$

where the constant C is independent of m. Therefore

$$\|\Delta^m f^0\|_2^2 \le C_1^{2m}(2m)! \tag{5.8}$$

for some constant C_1 and for all positive integers m. From an elliptic regularity theorem of Kotaké and Narasimhan ([17], Theorem 3.8.9), f^0 is real analytic, a contradiction. Thus f^0 satisfies (5.3) and the theorem is proved. ■

Remark 5.6. To try to understand the nature of functions which satisfy the growth condition (5.3) it was pointed out in [22], [23] that if a function f^0 in $\mathcal{F}$ is in $C_c^\infty(X)$, then the Phragmén–Lindelöff theorem guarantees that it satisfies the required not-too-rapidly-decreasing condition. More generally if $|f^0(x)| \le Ce^{-\alpha d(xK,K)^2}$, then we can apply an analogue of Hardy's theorem due to Sitaram and Sundari ([27]) to get the same conclusion about f^0. In view of Theorem 5.4, we can also replace the condition (5.3) in Theorem 5.2 by the hypothesis that at least one function in $\mathcal{F}$ vanishes on a set of positive Haar measure.

Let us now take up the L^p-version of Wiener's theorem.

Theorem 5.7. *Let $1 < p < 2$ and $\mathcal{F} \subset L^p(X) \cap L^1(X)$. For some ε in $(0, p-1)$, let Z denote the set of $\lambda \in \mathcal{S}_{p-\varepsilon}$ such that $\tilde{f}(\lambda, \cdot) \equiv 0$ for all f in $\mathcal{F}$. If Z is empty and there exists $f^0 \in \mathcal{F}$ which satisfies (5.3), then the left G-translates of the functions in$\mathcal{F}$ span a dense subspace of $L^p(X)$.*

To show the necessity of the nonvanishing condition on $\mathcal{S}_p^\circ$, we can argue as in the L^1-case. Our hypothesis however concerns λ on a slightly larger set $\mathcal{S}_{p-\varepsilon}$.

Proof. Since $f^0 \in L^1(X)$, as in Theorem 5.3 we shall consider the K-biinvariant function $g = f^{0*} * f^0$, which is in the $L^1(G)$-module generated by $\mathcal{F}$. Then as $\widehat{g}(\lambda) = \|\tilde{f}(\lambda, \cdot)\|^2_{L^2(K)}$ for $\lambda \in \mathbb{R}$,

$$\limsup_{\lambda \in \mathbb{R}, |\lambda| \longrightarrow \infty} \|\widehat{g}(\lambda)\| e^{\alpha e^{|\lambda|}} > 0 \qquad \forall \alpha > 0. \tag{5.9}$$

As in Theorem 5.3 we can produce K-biinvariant functions out of the left G-translates of elements of $\mathcal{F}$ which do not have a common zero in $\mathcal{S}_{p-\varepsilon}$. Since $\mathcal{F} \subset L^1(X)$, these G-translates of elements of $\mathcal{F}$ will continue to be in $L^1 \cap L^p(X)$ and hence their spherical Fourier transform will vanish uniformly at infinity on the strip $\mathcal{S}_{p-\varepsilon}$. Therefore we can again use the Wiener Tauberian Theorem for K-biinvariant L^p functions (Theorem 5.1) to complete the proof. ∎

6. Some uncertainty principles for the Helgason–Fourier transform

The results in this section illustrate the following meta-theorem: A function and its Fourier transform cannot be simultaneously "small" (see section 5 of [10]). The significance of this section rests mainly on the hope that one can generalize these results to eigenfunction expansions with respect to an arbitrary elliptic operator. After all, Theorem 3.6 and the Euclidean inversion formula can be viewed as eigenfunction expansions (see [19]).

Let ν be the measure $d\nu(\lambda, k) = |c(\lambda)|^{-2}\, d\lambda\, dk$. Let $\sigma(x) = d(xK, K)$ and $A(r) = m\{x \mid \sigma(x) \leq r\}$, where m is the Haar measure of G. By B_R we denote the geodesic ball of radius R in X centered at the origin $o = eK$. M. Benedicks ([2]) proved a qualitative uncertainty theorem on $\mathbb{R}^d$, which was generalized to symmetric spaces in [20]. In terms of the Helgason–Fourier transform, the theorem in [20] is the following:

Theorem 6.1. *If $f \in L^1(X)$ satisfies $m(\{x \in G \mid f(x) \neq 0\}) < \infty$ and $\mu(\{\lambda \in \mathfrak{a}^* \mid \tilde{f}(\lambda, \cdot) \not\equiv 0\}) < \infty$, then $f = 0$ almost everywhere.*

In view of the analyticity of $\tilde{f}(\lambda, k)$ in λ for almost every $k \in K$, when f is in $L^p(X)$, where $p \in [1, 2)$, we can now prove the following stronger version:

Theorem 6.2. *Let $f \in L^2(X)$. If $m(\{x \in X \mid f(x) \neq 0\} < \infty$ and $\nu\{(\lambda, k) \in \mathfrak{a}^* \times K \mid \tilde{f}(\lambda, k) \neq 0\} < \infty$, then $f = 0$ almost everywhere.*

We also have the following result on *local uncertainty*:

Theorem 6.3. *Let $\theta \in (0, \frac{1}{2})$. Then*

$$\left(\int_E |\tilde{f}(\lambda, k)|^2 \, d\nu(\lambda, k)\right)^{\frac{1}{2}} \le e^{CR} \nu(E)^\theta \|A(\sigma(\cdot))^\theta f\|_2$$

for all $f \in L^2(X)$ which vanish almost everywhere outside B_R, and for all measurable E in $\mathfrak{a}^* \times K$. (The constant C depends only on the underlying group G of the symmetric space X.)

We omit the proof, as the arguments are along the lines of the proof of Theorem 3.1 in [21], where a similar result for the group theoretic Fourier transform of K-finite functions is proved. Here however we can deal with functions on X with no K-finite restriction. This is the exact analogue of the Euclidean case, except that in view of the unboundedness of the basic eigenfunctions $x \mapsto e^{(i\lambda+\rho)H(x^{-1}k)}$ as opposed to the Euclidean case where one considers $x \mapsto e^{i\lambda(\omega,x)}$, we have to impose a restriction on the support of f. Note that this restriction does not oppose the spirit of the principle of local uncertainty. An important difference between Theorem 6.3 and Theorem 3.1 of [21] is that here we look at $\tilde{f}(\lambda, k)$ instead of $\left(\int_K |\tilde{f}(\lambda, k)|^2 \, dk\right)^{1/2}$ (the same can be said of Theorem 6.2 and Theorem 6.1).

We shall conclude this section with a theorem on *approximate concentration.* Let $f \in L^2(X)$. Suppose that $U \subset X$ and $V \subset \mathfrak{a}_+^* \times K$, and that $\varepsilon, \delta > 0$. We say that f is *ε-concentrated* on U if

$$\int_{X \setminus U} |f|^2 \le \varepsilon^2 \int_X |f|^2.$$

Similarly, we say that $\tilde{f}$ is *δ-concentrated* on V if

$$\int_{(a_+^* \times K) \setminus V} |\tilde{f}|^2 \le \delta^2 \|\tilde{f}\|_2^2,$$

where the L^2 norm on the right side is with respect to ν.

Theorem 6.4. *Suppose that $U \subset B_R$ and V are sets of positive finite measure of X and $\mathfrak{a}_+^* \times K$ respectively. If a nonzero $f \in L^2(X)$ is ε-concentrated on U and $\tilde{f}$ is δ-concentrated on V, then*

$$m(U)\nu(V) \ge e^{-CR}(1 - \varepsilon - \delta)^2,$$

where the constant C depends only on the underlying group G.

For a Gelfand pair (G, K), a version of this theorem is proved in [30]. There the author considers approximate concentration of the Fourier transform of f with respect to the spherical representations on a set $V \subset \mathfrak{a}_+^*$. In terms of the Helgason–Fourier transform, this amounts to demanding that the function $g(\lambda) = (\int_K |\tilde{f}(\lambda, k)|^2 \, dk)^{1/2}$ is approximately concentrated on a set $V \subset \mathfrak{a}_+^*$ of finite measure with respect to $|c(\lambda)|^{-2} \, d\lambda$. Instead we consider the approximate concentration of $\tilde{f}$ on $V \subset \mathfrak{a}_+^* \times K$ with respect to the measure ν. The proof of Theorem 6.4 proceeds along essentially the same lines as the result of Donoho and Stark [6]. However we need the following technical lemma, which may be of independent interest. We omit the proof since it involves fairly standard techniques.

Lemma 6.5. *Let $h \in L^2(\mathfrak{a}_+^* \times K, \nu)$ and let h be right M-invariant in the K-variable, such that $\nu\{(\lambda, k) \in \mathfrak{a}_+^* \times K \mid h(\lambda, k) \neq 0\} < \infty$. Then*

$$g(x) = \int_{\mathfrak{a}_+^* \times K} h(\lambda, k) e^{(i\lambda - \rho)(H(x^{-1}k))} \, d\nu(\lambda, k)$$

is a well defined L^2 function and $\tilde{g} = h$.

For reasons similar to those explained for the previous theorem here also we have to look at functions supported inside a ball of some fixed radius R. For sets U and V in Theorem 6.4, we define the projections P_U and Q_V on $L^2(X)$ by

$$P_U f = f \chi_U \qquad \text{and} \qquad (Q_V f)\tilde{} = \tilde{f} \chi_V.$$

Note that

$$Q_V f(x) = \int_{\mathfrak{a}_+^* \times K} \tilde{f}(\lambda, k) \chi_V(\lambda, k) e^{(i\lambda - \rho)(H(x^{-1}k))} \, d\nu(\lambda, k)$$

is in $L^2(X)$ by the above lemma. We show that

$$|P_U Q_V f(x)| \leq \chi_U(x) e^{CR} \|f\|_2 \, \nu(V)^{\frac{1}{2}}$$

and

$$\|P_U Q_V f\|_2 \geq (1 - \varepsilon - \delta) \|f\|_2.$$

From these two inequalities we have the required result.

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Received December 28, 2002
and in final form June 12, 2003