Journal of Lie Theory
Volume **14** (2004) 111–140

Algorithmic Construction of Hyperfunction Solutions to Invariant Differential Equations on the Space of Real Symmetric Matrices

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Communicated by J. Hilgert

Abstract. This is the second paper on invariant hyperfunction solutions of invariant linear differential equations on the vector space of $n \times n$ real symmetric matrices. In the preceding paper [22], we proved that every invariant hyperfunction solution is expressed as a linear combination of Laurent expansion coefficients of the complex power of the determinant function with respect to the parameter. Fundamental properties of the complex power have been investigated in [19]. In this paper, we give algorithms to determine the space of invariant hyperfunction solutions and apply the algorithms to some examples. These algorithms enable us to compute in a fully constructive way all the invariant hyperfunction solutions for all the invariant differential operators in terms of Laurent expansion coefficients of the complex power of the determinant function.
Keywords: invariant hyperfunctions, symmetric matrix spaces, linear differential equations,
2000 Mathematics Subject Classification: Prim.: 22E45, 58J15; Sec.: 35A27

Introduction.

Let $\boldsymbol{V} := \mathrm{Sym}_n(\mathbb{R})$ be the space of $n \times n$ symmetric matrices over the real field $\mathbb{R}$ and let $\mathbf{SL}_n(\mathbb{R})$ be the special linear group over $\mathbb{R}$ of degree n. Then the group $\boldsymbol{G} := \mathbf{SL}_n(\mathbb{R})$ acts on the vector space $\boldsymbol{V}$ by the representation

$$\rho(g) : x \longmapsto g \cdot x := gx^t g, \tag{1}$$

with $x \in \boldsymbol{V}$ and $g \in \boldsymbol{G}$. Let $D(\boldsymbol{V})$ be the algebra of linear differential operators on $\boldsymbol{V}$ with polynomial coefficients and let $\mathcal{B}(\boldsymbol{V})$ be the space of hyperfunctions on $\boldsymbol{V}$. We denote by $D(\boldsymbol{V})^{\boldsymbol{G}}$ and $\mathcal{B}(\boldsymbol{V})^{\boldsymbol{G}}$ the subspaces of $\boldsymbol{G}$-invariant linear differential operators and of $\boldsymbol{G}$-invariant hyperfunctions on $\boldsymbol{V}$, respectively. For a given invariant differential operator $P(x, \partial) \in D(\boldsymbol{V})^{\boldsymbol{G}}$ and an invariant hyperfunction $v(x) \in \mathcal{B}(\boldsymbol{V})^{\boldsymbol{G}}$, we consider the linear differential equation

$$P(x, \partial)u(x) = v(x) \tag{2}$$

* Supported in part by The Mitsubishi Foundation and by the Grant-in-Aid for Scientific Research (C)(2)11640161 and (C)(2)13640163 , Ministry of Education, Culture, Sports, Science and Technology, Japan

ISSN 0949–5932 / \$2.50

where the unknown function $u(x)$ is in $\mathcal{B}(\boldsymbol{V})^{\boldsymbol{G}}$.

This paper is the sequel of the author's paper [22]. In the preceding paper [22], under some suitable conditions, the author proved that every invariant hyperfunction solution to (2) is expressed as a linear combination of Laurent expansion coefficients of the complex power of the determinant function with respect to the complex parameter. On the other hand, the author [19] has made clear which Laurent expansion coefficients appear from the Laurent expansion of the complex power of the determinant function. In this paper, we give algorithms to construct *all* invariant hyperfunction solutions to (2) in terms of the Laurent expansion coefficients by using the results of [22] and [19]. *The aim of this paper is not only to give solution spaces in an abstract form but also to give algorithms to construct all the solutions for given differential equations of the form* (2) *using the Laurent expansion coefficients of the complex power of the determinant function.*

Our algorithms are described as procedures with a differential equation of the form (2) as input and its solution space or its solution as output. However, they fully depend on the roots of b_P-functions of $P(x, \partial) \in D(\boldsymbol{V})^{\boldsymbol{G}}$ and the exact orders of the poles of the complex power functions. The author has given an algorithm to compute the b_P-function of a given invariant differential operator in [20]. On the other hand, the exact orders of poles of the complex power functions are completely determined by the author in [19]. By combining the results in [22] and [19], our algorithms works well in practical computations of invariant hyperfunction solutions.

We explain the organization of this paper. In §1, we introduce some notions and notations and review the results obtained in [22]. We state Theorem 1.3, Corollary 1.7, Theorem 1.8 and Theorem 1.9, which are main theoretical results of the paper [22]. They guarantee that every $\boldsymbol{G}$-invariant hyperfunction solution for $P(x, \partial)u(x) = 0$ or $P(x, \partial)u(x) = v(x)$ can be written as a finite sum of the Laurent expansion coefficients of $|\det(x)|^s$ and that the solution space is determined by the b_P-function of $P(x, \partial)$ (see Definition 1.1). In §2, we give a method to determine the order of pole of $P^{[\vec{a},s]}(x)$ as an application of the author's results in [19], and introduce "standard basis". The standard basis will be used in the algorithms in the later sections. Then, we give the algorithms to construct $\boldsymbol{G}$-invariant hyperfunction solutions in §3 and §4 to given $\boldsymbol{G}$-invariant differential equations. Some examples of typical $\boldsymbol{G}$-invariant differential equations and their invariant solutions are given in §5.

The main results of this paper are the algorithms in §3 and §4 and the examples in §5. They are not abstract existence theorems but practical algorithms to construct all the invariant solutions. For example, we prove in Proposition 5.2 that every $\mathbf{SL}_n(\mathbb{R})$-invariant hyperfunction solution to the differential equation $\det(x)u(x) = 0$ on $\boldsymbol{V} = \mathrm{Sym}_n(\mathbb{R})$ is a linear sum of $\mathbf{SL}_n(\mathbb{R})$-invariant measures on the $\mathbf{SL}_n(\mathbb{R})$-orbits in the set $\boldsymbol{S} := \{x \in \mathrm{Sym}_n(\mathbb{R}) \mid \det(x) = 0\}$. This is proved by carrying out the procedure of Algorithm 3.2. At the same time, our algorithm guarantees that there are no other solutions except for the invariant measures. This is a natural extension of the fact that the hyperfunction solution to the differential equation $xu(x) = 0$ on the real line is only a constant multiple of the delta function $u(x) = c \cdot \delta(x)$. This may be rather a well-known example, but our algorithm can be applied to all the invariant differential equations if the b_P-function of the invariant differential operator can be computed.

Methods to construct explicit solutions to linear partial differential equations which are invariant under the action of Lie groups has been studied for a long time. However, they are supposed to be a special type of differential operators. For example, Riesz [27] considered the Cauchy problem for d'Alembertian, which is an invariant differential equation by the Lorentz group action. Methée [13], [15] dealt with the invariant distribution solutions for d'Alembertian, which was applied to the study of quantum field theory by Bogoliubov et al [4]. Gårding [7] studied the Cauchy problem for the differential operator given of determinant type, which is considered as an invariant differential operator under the group action of the special linear group. These studies can be regarded as investigation on group-invariant linear differential equations. Modern method that exploit micro-local analysis actively enable us to construct explicit hyperfunction solutions algorithmically for a wider range of invariant differential operators. The author considered invariant differential operators on the real symmetric matrix space to show a sample of the algorithm. We can calculate *all the invariant hyperfunction solutions to all the invariant linear differential equations* under some additional conditions — the homogeneity of differential operators and the the condition on b_P-function — on the real symmetric matrix space.

The author thinks that the method employed in this paper is rather orthodox. Namely, we are only analyzing the distributions defined by the complex powers of polynomials of special type, which had been developed by Riesz [27] and Gårding [7] before the concept of Schwartz's distributions or Sato's hyperfunctions was produced. Old analysis is sufficiently great but seems to be insufficient now since we obtain more precise results like the outcomes of this paper — which could not be achieved by old technique. We can analyze a wider range of invariant differential equations, especially on prehomogeneous vector spaces, and they will be analyzed in the future papers. Refer to Sato and Shintani [28] for typical prehomogeneous vector spaces and see Kashiwara, Kawai, and Kimura [12] for micro-local analysis.

Our work has close connections with the analysis on homogeneous cones and the invariant distributions on them. A systematic monograph [6] has been written by Faraut and Korányi. Other recent references on invariant distributions are Blind[3] and Ricci and Stein[26]. On the other hand, we can find many interesting works about invariant differential equations on some kinds of spaces where Lie groups are acting. For example, see the recent works Ames[2] and Bouaziz and Kamoun[5]. Concerning invariant differential operators, Nomura's works [24] and [25] are of value for reference.

Notations: In this paper, for a square matrix x, we denote by ${}^t x$, $\mathrm{tr}(x)$ and $\det(x)$ the transpose of x, the trace of x and the determinant of x, respectively. The complex numbers, the real numbers and the integers are denoted by $\mathbb{C}$, $\mathbb{R}$ and $\mathbb{Z}$, respectively. $\mathbb{Z}_{\geq 0}$ denotes the non-negative integers and $\mathbb{Z}_{>0}$ denotes the positive integers.

Acknowledgement: The author expresses his sense of gratitude to the referee for the careful reading of the manuscript and kindhearted advice.

1. Fundamental definitions and theorems.

In this section we review the fundamental definitions and theorems given in the preceding paper [22]. For the precise definitions and the complete proofs of theorems, refer to the paper [22].

Let $\boldsymbol{V}$ be a finite dimensional real vector space of dimension m with linear coordinate $(x_1,\ldots,x_m)$. Then a polynomial with complex coefficients on $\boldsymbol{V}$ is given as a complex finite linear combination of monomials $x^\alpha := x_1^{\alpha_1}\cdots x_m^{\alpha_m}$ with $\alpha := (\alpha_1,\ldots,\alpha_m) \in \mathbb{Z}_{\geq 0}^m$. We denote by ∂_i the partial derivative $\frac{\partial}{\partial x_i}$. We define a monomial of $\frac{\partial}{\partial x_i}$'s by $\partial^\beta := \partial_1^{\beta_1}\cdots\partial_m^{\beta_m}$ with $\beta := (\beta_1,\ldots,\beta_m) \in \mathbb{Z}_{\geq 0}^m$. We define the degrees of multi-index by $|\alpha| := \alpha_1+\cdots+\alpha_m$ and $|\beta| := \beta_1+\cdots+\beta_m$. For a given monomial differential operator $a_{\alpha\beta}x^\alpha\partial^\beta$, we call $|\alpha|-|\beta|$ (resp. $|\beta|$) *homogeneous degree* (resp. *order*) of the monomial differential operator $a_{\alpha\beta}x^\alpha\partial^\beta$. A *homogeneous differential operator of homogeneous degree* k in $D(\boldsymbol{V})$ is a differential operator given as a finite linear combination of monomial differential operators of homogeneous degree k. We denote by $D_k(\boldsymbol{V})$ the $\mathbb{C}$-vector space of homogeneous differential operators of homogeneous degree k. We call a differential operator invariant under the action of all $g \in \boldsymbol{G}$ a $\boldsymbol{G}$*-invariant differential operator* on $\boldsymbol{V}$. We denote $D(\boldsymbol{V})^{\boldsymbol{G}}$ the totality of $\boldsymbol{G}$-invariant differential operators on $\boldsymbol{V}$. We can easily check that $D(\boldsymbol{V})^{\boldsymbol{G}}$ is a subalgebra of $D(\boldsymbol{V})$ and that $D(\boldsymbol{V})^{\boldsymbol{G}} = \bigoplus_{k\in\mathbb{Z}} D_k(\boldsymbol{V})^{\boldsymbol{G}} := \bigoplus_{k\in\mathbb{Z}} D_k(\boldsymbol{V}) \cap D(\boldsymbol{V})^{\boldsymbol{G}}$ is a natural gradation induced from the gradation $D(\boldsymbol{V}) = \bigoplus_{k\in\mathbb{Z}} D_k(\boldsymbol{V})$.

Let $\mathcal{B}(\boldsymbol{V})$ be the space of hyperfunctions on $\boldsymbol{V}$ and let $\mathcal{B}(\boldsymbol{V})^{\boldsymbol{G}}$ be the space of $\boldsymbol{G}$-invariant hyperfunctions on $\boldsymbol{V}$. We say that $v(x) \in \mathcal{B}(\boldsymbol{V})$ is *quasi-homogeneous* provided that there exist a complex number $\lambda \in \mathbb{C}$ and a non-negative integer $k \in \mathbb{Z}_{\geq 0}$ satisfying

$$\underbrace{F_{r,\lambda}\circ F_{r,\lambda}\circ\cdots\circ F_{r,\lambda}}_{(k+1)\text{-times}}(v) = 0 \tag{3}$$

for all $r \in \mathbb{R}_{>0}$ with $F_{r,\lambda}(v) := v(r\cdot x) - r^\lambda v(x)$. We call $\lambda \in \mathbb{C}$ the *homogeneous degree* (or simply the *degree*) of $v(x)$ and call $k \in \mathbb{Z}_{\geq 0}$ the *quasi-degree* of $v(x)$. It is easily checked that (3) is equivalent to

$$(\vartheta - \lambda)^{k+1} v(x) = 0 \tag{4}$$

with $\vartheta := \sum_{i=1}^m x_i\partial_i$. In particular, when a quasi-homogeneous function $v(x)$ is of quasi-degree k and not $k-1$, we say that $v(x)$ is quasi-homogeneous of *proper quasi-degree* k.

We use the following notations in this paper.

1. $QH(\lambda) := \{u(x) \in \mathcal{B}(\boldsymbol{V}) \mid u(x)$ is quasi-homogeneous of degree $\lambda \in \mathbb{C}\}$.

2. $QH(\lambda)^{\boldsymbol{G}} := QH(\lambda) \cap \mathcal{B}(\boldsymbol{V})^{\boldsymbol{G}}$.

3. $QH := \bigoplus_{\lambda\in\mathbb{C}} QH(\lambda)$.

4. $QH^{\boldsymbol{G}} := \bigoplus_{\lambda\in\mathbb{C}} QH(\lambda)^{\boldsymbol{G}}$.

Proposition 1.1. *Let $P(x, \partial) \in D(\boldsymbol{V})$ (resp. $\in D(\boldsymbol{V})^{\boldsymbol{G}}$) be a non-zero homogeneous differential operator of homogeneous degree μ. If $f(x) \in \mathcal{B}(\boldsymbol{V})$ (resp. $\in \mathcal{B}(\boldsymbol{V})^{\boldsymbol{G}}$) is quasi-homogeneous of degree $\lambda \in \mathbb{C}$, then $P(x, \partial) f(x) \in \mathcal{B}(\boldsymbol{V})$ (resp. $\in \mathcal{B}(\boldsymbol{V})^{\boldsymbol{G}}$) is quasi-homogeneous of degree $\lambda + \mu \in \mathbb{C}$.*

The proof of Proposition 1.1 is found in [22, Proposition1.1].

¿From now on, let $\boldsymbol{V} := \mathrm{Sym}_n(\mathbb{R})$ be the space of $n \times n$ symmetric matrices over the real field $\mathbb{R}$ and let $\boldsymbol{G} := \mathbf{SL}_n(\mathbb{R})$ be the special linear group over $\mathbb{R}$ of degree n. Then the group $\boldsymbol{G}$ acts on the vector space $\boldsymbol{V}$ by the representation

$$\rho(g) : x \longmapsto g \cdot x := gx\,{}^t g,$$

with $x \in \boldsymbol{V}$ and $g \in \boldsymbol{G}$. The vector space $\boldsymbol{V}$ decomposes into a finite number of $\mathbf{GL}_n(\mathbb{R})$-orbits:

$$\boldsymbol{V} := \bigsqcup_{\substack{0 \le i \le n \\ 0 \le j \le n-i}} \boldsymbol{S}_i^j \tag{5}$$

where

$$\boldsymbol{S}_i^j := \{x \in \mathrm{Sym}_n(\mathbb{R}) \mid \mathrm{sgn}(x) = (j, n-i-j)\} \tag{6}$$

with integers $0 \le i \le n$ and $0 \le j \le n-i$. Here, $\mathrm{sgn}(x)$ for $x \in \mathrm{Sym}_n(\mathbb{R})$ is the signature of the quadratic form $q_x(\vec{v}) := {}^t\vec{v} \cdot x \cdot \vec{v}$ on $\vec{v} \in \mathbb{R}^n$. The subset $\boldsymbol{S}_i := \{x \in \boldsymbol{V} \mid \mathrm{rank}(x) = n-i\}$ is the set of elements of rank $n-i$.

We set $P(x) := \det(x)$ and $\boldsymbol{S} := \{x \in \boldsymbol{V} | \det(x) = 0\}$. We call the set $\boldsymbol{S}$ the *singular set* and call the $\boldsymbol{G}$-orbits in $\boldsymbol{S}$ the *singular orbits.* The subset $\boldsymbol{V} - \boldsymbol{S}$ decomposes into $(n+1)$ connected components,

$$\boldsymbol{V}_i := \{x \in \mathrm{Sym}_n(\mathbb{R}) \mid \mathrm{sgn}(x) = (i, n-i)\} \tag{7}$$

with $i = 0, 1, \ldots, n$.

We define the complex power function of $P(x)$ by

$$|P(x)|_i^s := \begin{cases} |P(x)|^s & \text{if } x \in \boldsymbol{V}_i, \\ 0 & \text{if } x \notin \boldsymbol{V}_i, \end{cases} \tag{8}$$

for a complex number $s \in \mathbb{C}$. We can regard $|P(x)|_i^s$ as a tempered distribution — and hence a hyperfunction — with a meromorphic parameter $s \in \mathbb{C}$. We consider a linear combination of the hyperfunctions $|P(x)|_i^s$:

$$P^{[\vec{a}, s]}(x) := \sum_{i=0}^{n} a_i \cdot |P(x)|_i^s \tag{9}$$

with $s \in \mathbb{C}$ and $\vec{a} := (a_0, a_1, \ldots, a_n) \in \mathbb{C}^{n+1}$. Then $P^{[\vec{a}, s]}(x)$ is a hyperfunction with a meromorphic parameter $s \in \mathbb{C}$ and it depends on $\vec{a} \in \mathbb{C}^{n+1}$ linearly.

A homogeneous differential operator of degree $k \in \mathbb{Z}$ is given by

$$P(x, \partial) = \sum_{\substack{\alpha, \beta \in \mathbb{Z}_{\ge 0}^m \\ |\alpha| - |\beta| = k}} a_{\alpha\beta} x^\alpha \partial^\beta$$

with $m := n(n+1)/2$. The notations here are written as

$$x = (x_{ij})_{n \ge j \ge i \ge 1}, \quad \partial = (\partial_{ij}) = \left(\frac{\partial}{\partial x_{ij}} \right)_{n \ge j \ge i \ge 1},$$

$$x^\alpha = \prod_{n \geq j \geq i \geq 1} x_{ij}^{\alpha_{ij}}, \quad \partial^\beta = \prod_{n \geq j \geq i \geq 1} \partial_{ij}^{\beta_{ij}}$$

with

$$\alpha = (\alpha_{ij}) \in \mathbb{Z}_{\geq 0}^m, \quad |\alpha| = \sum_{n \geq j \geq i \geq 1} \alpha_{ij},$$

$$\beta = (\beta_{ij}) \in \mathbb{Z}_{\geq 0}^m, \quad |\beta| = \sum_{n \geq j \geq i \geq 1} \beta_{ij}.$$

We define ∂^* by

$$\partial^* = (\partial_{ij}^*) = \left(\epsilon_{ij} \frac{\partial}{\partial x_{ij}}\right), \text{ with } \epsilon_{ij} := \begin{cases} 1 & \text{if } i = j, \\ 1/2 & \text{if } i \neq j. \end{cases} \tag{10}$$

For an invariant differential operator $P(x, \partial) \in D(\boldsymbol{V})^{\boldsymbol{G}}$, we have the following proposition.

Proposition 1.2. *Let $P(x, \partial) \in D(\boldsymbol{V})^{\boldsymbol{G}}$ be a homogeneous differential operator.*

1. *The homogeneous degree of $P(x, \partial)$ is in $(n \cdot \mathbb{Z})$. If the homogeneous degree of $P(x, \partial)$ is nk, then $P(g \cdot x, {}^t g^{-1} \cdot \partial) = \det(g)^{2k} P(x, \partial)$.*

2. *If the homogeneous degree of $P(x, \partial)$ is nk with $k \in \mathbb{Z}$ and $x \in \mathrm{Sym}_n(\mathbb{R})$ is positive definite, then we have*

$$P(x, \partial)(\det x)^s = b_P(s)(\det x)^{s+k} \tag{11}$$

where $b_P(s)$ is a polynomial in $s \in \mathbb{C}$. We have also

$$\begin{aligned} P(x, \partial) P^{[\vec{a}, s]}(x) &= b_P(s) \det(x)^k P^{[\vec{a}, s]}(x) \\ &= b_P(s) \mathrm{sgn}(\det(x))^k P^{[\vec{a}, s+k]}(x) \\ &= b_P(s) P^{[\vec{a}^{\#k}, s+k]}(x) \end{aligned} \tag{12}$$

for all $x \in \boldsymbol{V} - \boldsymbol{S}$. Here we put

$$\vec{a}^{\#k} := ((-1)^{nk} a_0, (-1)^{(n-1)k} a_1, \ldots, a_n) \in \mathbb{C}^{n+1}. \tag{13}$$

3. *If the homogeneous degree of $P(x, \partial)$ is nk with $k < 0$, then $b_P(s)$ is divisible by $b^{\underline{-k}}(s-1)$ where $b^{\underline{-k}}(s-1) := b(s-1)b(s-2)\cdots b(s-(-k))$ with $b(s) := \prod_{i=1}^n (s + \frac{i+1}{2})$.*

The proof of Proposition 1.2 is given by [22, Proposition 3.1].

Now we give the definition of b_P*-function* for a given $\boldsymbol{G}$-invariant differential operator $P(x, \partial)$.

Definition 1.1. **(b_P-function)** *Let $P(x, \partial) \in D(\boldsymbol{V})^{\boldsymbol{G}}$ be a homogeneous differential operator of homogeneous degree k. We call $b_P(s)$ in* (11) *the b_P*-function *of $P(x, \partial)$.*

Then we have the following theorem, whose proof is given in [22, Theorem 4.1].

Theorem 1.3. *Let $P(x,\partial) \in D(\boldsymbol{V})^{\boldsymbol{G}}$ be a non-zero homogeneous differential operator with homogeneous degree kn. We suppose that*

$$\text{degree of } b_P(s) \; = \text{order of } P(x,\partial). \tag{14}$$

The space of $\boldsymbol{G}$-invariant hyperfunction solutions to $P(x,\partial)u(x) = 0$ is finite dimensional. The solutions $u(x)$ are given as finite linear combinations of quasi-homogeneous $\boldsymbol{G}$-invariant hyperfunctions.

The following proposition is [22, Proposition 5.1]. Its proof is easy.

Proposition 1.4. *Let ∂^* be the symmetric matrix of differential operators defined by* (10).

1. *We have*

$$\begin{aligned}(\det(\partial^*))P^{[\vec{a},s+1]}(x) &= b(s)\cdot P^{[\vec{a}^{\#},s]}(x)\\ &= b(s)\cdot(\det(x))\cdot P^{[\vec{a},s-1]}(x)\end{aligned} \tag{15}$$

   with $\vec{a}^{\#} = \vec{a}^{\#1} := ((-1)^n a_0, \ldots, -a_{n-1}, a_n)$ and

$$b(s) = c\cdot(s+1)(s+\frac{3}{2})\cdots(s+\frac{n+1}{2}), \tag{16}$$

   where c is a constant.

2. *$P^{[\vec{a},s]}(x)$ is holomorphic with respect to $s \in \mathbb{C}$ except for poles at $s = -(k+1)/2$ with $k \in \mathbb{Z}_{\geq 0}$. The possible highest order of the pole of $P^{[\vec{a},s]}(x)$ at $s = -(k+1)/2$ is*

$$\begin{cases} \lfloor\frac{k+1}{2}\rfloor & \text{if } k = 1,2.\ldots,n-1,\\ \lfloor\frac{n}{2}\rfloor & \text{if } k = n,n+1.\ldots, \text{ and } k+n \text{ is odd},\\ \lfloor\frac{n+1}{2}\rfloor & \text{if } k = n,n+1.\ldots, \text{ and } k+n \text{ is even}.\end{cases} \tag{17}$$

We give here two definitions, [22, Definition 5.1] and [22, Definition 5.2], which concern the order of poles and the Laurent expansion coefficients of $P^{[\vec{a},s]}(x)$.

Definition 1.2. (Possible highest order) *Let $\lambda \in \mathbb{C}$ be a fixed complex number.*

1. *We denote by $PHO(\lambda)$ the possible highest order of the pole of $P^{[\vec{a},s]}(x)$ at $s = \lambda$. Namely we define*

$$PHO(\lambda) := \begin{cases} \lfloor\frac{k+1}{2}\rfloor & \text{if } \lambda = -\frac{k+1}{2}\ (k = 1,2.\ldots,n-1),\\ \lfloor\frac{n}{2}\rfloor & \text{if } \lambda = -\frac{k+1}{2}\ (k = n,n+1.\ldots, \text{ and } k+n \text{ is odd}),\\ \lfloor\frac{n+1}{2}\rfloor & \text{if } \lambda = -\frac{k+1}{2}\ (k = n,n+1.\ldots, \text{ and } k+n \text{ is even}),\\ 0 & \text{otherwise}.\end{cases} \tag{18}$$

2. *Let $q \in \mathbb{Z}$. We define a vector subspace $A(\lambda,q)$ of $\mathbb{C}^{n+1}$ by*

$$A(\lambda,q) := \{\vec{a} \in \mathbb{C}^{n+1} \mid P^{[\vec{a},s]}(x) \text{ has a pole of order} \leq q \text{ at } s = \lambda\}. \tag{19}$$

   Then we have $A(\lambda,q-1) \subset A(\lambda,q)$ by definition. We define $\overline{A(\lambda,q)}$ by

$$\overline{A(\lambda,q)} := A(\lambda,q)/A(\lambda,q-1). \tag{20}$$

It is easily verified that $\overline{A(\lambda,q)}=\{0\}$ if $q>PHO(\lambda)$ or $q<0$. We have

$$\bigoplus_{q\in\mathbb{Z}}\overline{A(\lambda,q)}=\bigoplus_{0\le q\le PHO(\lambda)}\overline{A(\lambda,q)}\simeq\mathbb{C}^{n+1}. \tag{21}$$

In particular, $\vec{a}=0$ if $\vec{a}\in A(\lambda,q)$ for some $q<0$ since $A(\lambda,q)=\{0\}$ for $q<0$. However, when $q<0$, a pole of order q means a zero of order $-q$.

Definition 1.3. (Laurent expansion coefficients) *Let $\lambda\in\mathbb{C}$ be a fixed complex number.*

1. *We define $o(\vec{a},\lambda)\in\mathbb{Z}$ by*

$$o(\vec{a},\lambda):=\text{ order of pole of } P^{[\vec{a},s]}(x) \text{ at } s=\lambda. \tag{22}$$

   Then $o(\vec{a},\lambda)\in\mathbb{Z}_{\ge 0}$. We have $p=o(\vec{a},\lambda)$ if and only if $\vec{a}\in A(\lambda,p)$ and $[\vec{a}]\in\overline{A(\lambda,p)}$ is not zero.

2. *Let $\vec{a}\in\mathbb{C}^{n+1}$ and let $r=o(\vec{a},\lambda)\in\mathbb{Z}_{\ge 0}$. Then we have the Laurent expansion of $P^{[\vec{a},s]}(x)$ at $s=\lambda$,*

$$P^{[\vec{a},s]}(x)=\sum_{w=-r}^{\infty}P_w^{[\vec{a},\lambda]}(x)\,(s-\lambda)^w. \tag{23}$$

It is easily checked that the Laurent expansion coefficient $P_w^{[\vec{a},\lambda]}(x)$ is linear with respect to $\vec{a}\in\mathbb{C}^{n+1}$.

Next, we give a definition of a standard basis of $\mathbb{C}^{n+1}$, following [22, Definition 5.3].

Definition 1.4. (Standard basis) *Let*

$$SB:=\{\vec{a}_0,\vec{a}_1,\ldots,\vec{a}_n\} \tag{24}$$

be a basis of $\mathbb{C}^{n+1}$. We say that SB is a standard basis of $\mathbb{C}^{n+1}$ at $s=\lambda$ if the following property holds: there exists an increasing integer sequence

$$0<k(0)<k(1)<\cdots<k(PHO(\lambda))=n \tag{25}$$

such that $SB_q:=\{\vec{a}_0,\vec{a}_1,\ldots,\vec{a}_{k(q)}\}$ is a basis of $A(\lambda,q)$ for each q in $0\le q\le PHO(\lambda)$.

It is easily seen that the representatives of SB_q-SB_{q-1} form a basis of the quotient vector space $\overline{A(\lambda,q)}:=A(\lambda,q)/A(\lambda,q-1)$.

Proposition 1.5. *Let $SB:=\{\vec{a}_0,\vec{a}_1,\ldots,\vec{a}_n\}$ be a standard basis of $\mathbb{C}^{n+1}$ at $s=\lambda$ and let $r_j:=o(\vec{a}_j,\lambda)\in\mathbb{Z}_{\ge 0}$. Then the Laurent expansion coefficients at $s=\lambda$*

$$\{P_{-r_j+i}^{[\vec{a_j},\lambda]}(x)\mid i\in\mathbb{Z}_{\ge 0}\text{ and } j=0,1,2,\ldots,n\} \tag{26}$$

are linearly independent.

The proof of Proposition 1.5 is given in [22, Proposition 5.3].

Theorem 1.6. *Let $r := o(\vec{a}, \lambda) \in \mathbb{Z}_{\geq 0}$ be the order of the pole of $P^{[\vec{a},s]}(x)$ at $s = \lambda$ defined by* (22)*.*

1. *Then the Laurent expansion coefficient of $P^{[\vec{a},s]}(x)$ at $s = \lambda$ defined by* (23) *$P_w^{[\vec{a},\lambda]}(x)$ is a quasi-homogeneous hyperfunction of degree $n\lambda$ of quasi-degree $r+w$. Conversely, let $v(x) \in QH(n\lambda)^{\boldsymbol{G}}$, the space of $\boldsymbol{G}$-invariant quasi-homogeneous hyperfunctions. Then $v(x)$ is written as a linear combination of Laurent expansion coefficients of $|P(x)|_i^s$ at $s = \lambda$.*

2. *Let*

$$LC(\lambda, w) := \text{ vector space generated by } \{P_w^{[\vec{a},\lambda]}(x) \mid \vec{a} \in \mathbb{C}^{n+1}\}. \tag{27}$$

Then we have the direct sum decomposition:

$$QH(n\lambda)^{\boldsymbol{G}} = \bigoplus_{\substack{w \in \mathbb{Z}, \\ w \geq -PHO(\lambda)}} LC(\lambda, w). \tag{28}$$

The proof of Theorem 1.6 is given in [22, Theorem 5.3]. By combining Theorem 1.3 and Theorem 1.6, we have the following corollary.

Corollary 1.7. *Let $P(x, \partial) \in D(\boldsymbol{V})^{\boldsymbol{G}}$ be a non-zero homogeneous differential operator with homogeneous degree kn satisfying the condition* (14)*. Then the $\boldsymbol{G}$-invariant hyperfunction solutions $u(x)$ to the differential equation $P(x, \partial)u(x) = 0$ are written as finite linear combinations of Laurent expansion coefficients of $|P(x)|_i^s$ at a finite number of points.*

Then we have the following two theorems, which are the main results of [22].

Theorem 1.8. *Let $P(x, \partial) \in D(\boldsymbol{V})^{\boldsymbol{G}}$ be a non-zero homogeneous differential operator with homogeneous degree kn and let $v(x)$ be a quasi-homogeneous $\boldsymbol{G}$-invariant hyperfunction of homogeneous degree $n\lambda$. We suppose that*

$$b_P(s) \not\equiv 0. \tag{29}$$

Then

1. *We can construct a $\boldsymbol{G}$-invariant hyperfunction solution $u(x) \in \mathcal{B}(\boldsymbol{V})^{\boldsymbol{G}}$ to the differential equation $P(x, \partial)u(x) = v(x)$ which is given as a sum of Laurent expansion coefficients of $|P(x)|_i^s$ at $s = \lambda - k$ and hence is quasi-homogeneous of degree $n(\lambda - k)$.*

2. *Any $\boldsymbol{G}$-invariant hyperfunction solution $u(x)$ is given as finite linear combinations of quasi-homogeneous $\boldsymbol{G}$-invariant hyperfunctions, and hence it is written as a finite linear combinations of Laurent expansion coefficients of $|P(x)|_i^s$ at a finite number of points in $\mathbb{C}$.*

The proof of Theorem 1.8 is given in [22, Theorem 6.1].

Theorem 1.9. *Let $P(x,\partial) \in D(\boldsymbol{V})^{\boldsymbol{G}}$ be a non-zero homogeneous differential operator of homogeneous degree kn satisfying the condition* (14). *Then we can construct the $\boldsymbol{G}$-invariant quasi-homogeneous hyperfunction solution of homogeneous degree $n\lambda$ to the differential equation $P(x,\partial)u(x) = 0$ as a finite linear combination of Laurent expansion coefficients of $|P(x)|_i^s$ $(i = 0,\dots,n)$ at $s = \lambda$. It is determined by the homogeneous degree kn and $b_P(s)$ and does not depend on $P(x,\partial)$ itself.*

The proof of Theorem 1.9 is given in [22, Theorem 6.2].

2. Orders of poles of complex powers of determinant functions.

In the preceding section, we have reviewed that the solutions of (2) can be constructed in terms of the Laurent expansion coefficients of the complex powers of the determinant functions $P^{[\vec{a},s]}(x)$ (Theorem 1.8 and Theorem 1.9). These are main results of the paper [22]. In order to apply these constructions of solutions in concrete examples, we have to see the exact pole order of $P^{[\vec{a},s]}(x)$ especially at $s = \lambda \in \frac{1}{2}\mathbb{Z}$.

In this section, we shall give a condition to determine the exact order of pole of $P^{[\vec{a},s]}(x)$ at $s = \lambda \in \frac{1}{2}\mathbb{Z}$ for a given vector $\vec{a} \in \mathbb{C}^{n+1}$. This is a direct application of the author's result in [19].

In order to determine the exact pole of $P^{[\vec{a},s]}(x)$ at $s = s_0$, the author introduced in [19] the coefficient vectors

$$\boldsymbol{d}^{(k)}[s_0] := (d_0^{(k)}[s_0], d_1^{(k)}[s_0], \dots, d_{n-k}^{(k)}[s_0]) \in ((\mathbb{C}^{n+1})^*)^{n-k+1} \tag{30}$$

with $k = 0,1,\dots,n$. Here, $(\mathbb{C}^{n+1})^*$ denotes the dual vector space of $\mathbb{C}^{n+1}$. Each element of $\boldsymbol{d}^{(k)}[s_0]$ is a linear form on $\vec{a} \in \mathbb{C}^{n+1}$ depending on $s_0 \in \mathbb{C}$, i.e., a linear map from $\mathbb{C}^{n+1}$ to $\mathbb{C}$

$$d_i^{(k)}[s_0] : \mathbb{C}^{n+1} \ni \vec{a} \longmapsto \langle d_i^{(k)}[s_0], \vec{a}\rangle \in \mathbb{C}. \tag{31}$$

We set

$$\langle \boldsymbol{d}^{(k)}[s_0], \vec{a}\rangle := (\langle d_0^{(k)}[s_0], \vec{a}\rangle, \langle d_1^{(k)}[s_0], \vec{a}\rangle, \dots, \langle d_{n-k}^{(k)}[s_0], \vec{a}\rangle) \in \mathbb{C}^{n-k+1}. \tag{32}$$

The precise definition of $\boldsymbol{d}^{(k)}[s_0]$ is the following.

Definition 2.1. (Coefficient vectors $\boldsymbol{d}^{(k)}[s_0]$) *Let s_0 be a half-integer, i.e., $s_0 = q/2$ with $q \in \mathbb{Z}$. We define the coefficient vectors $\boldsymbol{d}^{(k)}[s_0]$ $(k = 0,1,\dots,n)$ by induction in the following way.*

1. *First, we set*

$$\boldsymbol{d}^{(0)}[s_0] := (d_0^{(0)}[s_0], d_1^{(0)}[s_0], \dots, d_n^{(0)}[s_0]) \in ((\mathbb{C}^{n+1})^*)^{n+1} \tag{33}$$

where each $d_i^{(0)}[s_0]$ satisfies $\langle d_i^{(0)}[s_0], \vec{a}\rangle = a_i$ for all $\vec{a} \in \mathbb{C}^{n+1}$.

2. *Next, we define $\boldsymbol{d}^{(1)}[s_0]$ and $\boldsymbol{d}^{(2)}[s_0]$ by*

$$\boldsymbol{d}^{(1)}[s_0] := (d_0^{(1)}[s_0], d_1^{(1)}[s_0], \dots, d_{n-1}^{(1)}[s_0]) \in ((\mathbb{C}^{n+1})^*)^{n}, \tag{34}$$

with $d_j^{(1)}[s_0] := d_j^{(0)}[s_0] + \epsilon[s_0] d_{j+1}^{(0)}[s_0]$, *and*

$$\boldsymbol{d}^{(2)}[s_0] := (d_0^{(2)}[s_0], d_1^{(2)}[s_0], \ldots, d_{n-2}^{(2)}[s_0]) \in ((\mathbb{C}^{n+1})^*)^{n-1}, \tag{35}$$

with $d_j^{(2)}[s_0] := d_j^{(0)}[s_0] + d_{j+2}^{(0)}[s_0]$. *Here,*

$$\epsilon[s_0] := \begin{cases} 1 & \text{if } s_0 \text{ is a strict half-integer,} \\ (-1)^{s_0+1} & \text{if } s_0 \text{ is an integer.} \end{cases} \tag{36}$$

A strict half-integer means a rational number given by $q/2$ *with an odd integer* q.

3. *Lastly, by induction on* k, *we define the coefficient vectors* $\boldsymbol{d}^{(k)}[s_0]$ *for* $k = 3, 4, \ldots, n$ *by*

$$\boldsymbol{d}^{(2l+1)}[s_0] := (d_0^{(2l+1)}[s_0], d_1^{(2l+1)}[s_0], \ldots, d_{n-2l-1}^{(2l+1)}[s_0]) \in ((\mathbb{C}^{n+1})^*)^{n-2l}, \tag{37}$$

with $d_j^{(2l+1)}[s_0] := d_j^{(2l-1)}[s_0] - d_{j+2}^{(2l-1)}[s_0]$, *and*

$$\boldsymbol{d}^{(2l)}[s_0] := (d_0^{(2l)}[s_0], d_1^{(2l)}[s_0], \ldots, d_{n-2l}^{(2l)}[s_0]) \in ((\mathbb{C}^{n+1})^*)^{n-2l+1}, \tag{38}$$

with $d_j^{(2l)}[s_0] := d_j^{(2l-2)}[s_0] + d_{j+2}^{(2l-2)}[s_0]$.

By using $\boldsymbol{d}^{(k)}[s_0]$ in Definition 2.1, the author proved in [19] an algorithm to compute the exact order of poles of $P^{[\vec{a},s]}(x)$. Refer to Theorem A1 in Appendix. In this section, we shall characterize the vector space

$$A(\lambda, q) := \{\vec{a} \in \mathbb{C}^{n+1} \mid P^{[\vec{a},s]}(x) \text{ has a pole of order } \leq q \text{ at } s = \lambda\}$$

in terms of the coefficient vectors $\boldsymbol{d}^{(k)}[\lambda]$.

Definition 2.2. *We define the vector subspaces* $D_{half}^{(l)}$, $D_{even}^{(l)}$ *and* $D_{odd}^{(l)}$ *in* $\mathbb{C}^{n+1}$.

1. *Note that* $\boldsymbol{d}^{(2l+2)}[\lambda]$ *does not depend on the choice of* λ *if it is a half-integer. We define*

$$D_{half}^{(l)} := \{\vec{a} \in \mathbb{C}^{n+1} \mid \langle \boldsymbol{d}^{(2l+2)}[\lambda], \vec{a} \rangle = 0 \text{ for any strict half-integer } \lambda\}.$$

2. *Note that* $\boldsymbol{d}^{(2l+1)}[\lambda]$ *does not depend on the choice of* λ *if* λ *is an odd (resp. even) integer. We define*

$$D_{odd}^{(l)} := \{\vec{a} \in \mathbb{C}^{n+1} \mid \langle \boldsymbol{d}^{(2l+1)}[\lambda], \vec{a} \rangle = 0 \text{ for any odd integer } \lambda\}.$$

$$\text{(resp. } D_{even}^{(l)} := \{\vec{a} \in \mathbb{C}^{n+1} \mid \langle \boldsymbol{d}^{(2l+1)}[\lambda], \vec{a} \rangle = 0 \text{ for any even integer } \lambda\}.)$$

Theorem 2.1. *The subspaces $D^{(l)}_{half}$, $D^{(l)}_{even}$ and $D^{(l)}_{odd}$ in $\mathbb{C}^{n+1}$ have the following properties.*

1. *We define*

$$\vec{a}^{\#} = \vec{a}^{\#1} := ((-1)^n a_0, (-1)^{n-1} a_1, \ldots, a_n) \in \mathbb{C}^{n+1}$$

for $\vec{a} = (a_0, a_1, \ldots, a_n) \in \mathbb{C}^{n+1}$. Then we have

$$\vec{a} \in D^{(l)}_{odd} \iff \vec{a}^{\#} \in D^{(l)}_{even}$$

and

$$\vec{a} \in D^{(l)}_{half} \iff \vec{a}^{\#} \in D^{(l)}_{half}.$$

2. *Let l be an integer $0 \le l < PHO(\lambda)$. Then we have*

$$\vec{a} \in A(\lambda, l) \iff \begin{cases} \vec{a} \in D^{(l)}_{half} & \text{if } \lambda \text{ is a strict half-integer,} \\ \vec{a} \in D^{(l)}_{odd} & \text{if } \lambda \text{ is an odd integer,} \\ \vec{a} \in D^{(l)}_{even} & \text{if } \lambda \text{ is an even integer.} \end{cases} \tag{39}$$

In addition, we have $A(\lambda, PHO(\lambda)) = \mathbb{C}^{n+1}$.

Proof. The second statement is nothing but the definition of $D^{(l)}_{half}$, $D^{(l)}_{even}$ and $D^{(l)}_{odd}$ by Theorem A1 in Appendix.

We shall prove the first statement. Let ***odd*** be an odd integer and let ***even*** be an even integer. We have only to prove that

$$\langle \boldsymbol{d}^{(2l+1)}[\boldsymbol{odd}], \vec{a} \rangle = (-1)^n \langle \boldsymbol{d}^{(2l+1)}[\boldsymbol{even}], \vec{a}^{\#} \rangle. \tag{40}$$

for each $l \in \mathbb{Z}_{\ge 0}$. We prove it by induction on l. When $l = 0$, we have

$$\begin{aligned} \langle \boldsymbol{d}^{(1)}[\boldsymbol{odd}], \vec{a} \rangle &= (a_0 + a_1, a_1 + a_2, \ldots, a_{n-1} + a_n) \\ &= (-1)^n (a_0^{\#} - a_1^{\#}, a_1^{\#} - a_2^{\#}, \ldots, a_{n-1}^{\#} - a_n^{\#}) \\ &= (-1)^n \langle \boldsymbol{d}^{(1)}[\boldsymbol{even}], \vec{a}^{\#} \rangle \end{aligned}$$

since $\vec{a}^{\#} = (a_0^{\#}, a_1^{\#}, \ldots, a_n^{\#}) = ((-1)^n a_0, (-1)^{n-1} a_1, \ldots, a_n)$. We see that

$$\langle \boldsymbol{d}^{(2l+1)}[\boldsymbol{odd}], \vec{a} \rangle = (-1)^n \langle \boldsymbol{d}^{(2l+1)}[\boldsymbol{even}], \vec{a}^{\#} \rangle$$

if

$$\langle \boldsymbol{d}^{(2l-1)}[\boldsymbol{odd}], \vec{a} \rangle = (-1)^n \langle \boldsymbol{d}^{(2l-1)}[\boldsymbol{even}], \vec{a}^{\#} \rangle$$

by (37). Thus (40) is valid for all $l \in \mathbb{Z}_{\ge 0}$. By (40), we have

$$\begin{aligned} \vec{a} \in D^{(l)}_{odd} &\iff \langle \boldsymbol{d}^{(2l+1)}[\boldsymbol{odd}], \vec{a} \rangle = 0 \\ &\iff \langle \boldsymbol{d}^{(2l+1)}[\boldsymbol{even}], \vec{a}^{\#} \rangle = 0 \iff \vec{a}^{\#} \in D^{(l)}_{even}. \end{aligned}$$

Next let ***half*** be a strict half-integer. We have only to prove that

$$\langle \boldsymbol{d}^{(2l+2)}[\boldsymbol{half}], \vec{a} \rangle = \langle \boldsymbol{d}^{(2l+2)}[\boldsymbol{half}], \vec{a}^{\#} \rangle^{\#} \tag{41}$$

for each $l \in \mathbb{Z}_{\geq 0}$. We prove it by induction on l. When $l = 0$, we have

$$\begin{aligned}
\langle \boldsymbol{d}^{(2)}[\boldsymbol{half}], \vec{a} \rangle &= (a_0 + a_2, a_1 + a_3, \ldots, a_{n-2} + a_n) \\
&= ((-1)^{n-2}(a_0^{\#} + a_2^{\#}), (-1)^{n-3}(a_1^{\#} + a_3^{\#}), \ldots, (a_{n-2}^{\#} + a_n^{\#})) \\
&= \langle \boldsymbol{d}^{(1)}[\boldsymbol{half}], \vec{a}^{\#} \rangle^{\#}
\end{aligned}$$

since $\vec{a}^{\#} = (a_0^{\#}, a_1^{\#}, \ldots, a_n^{\#}) = ((-1)^n a_0, (-1)^{n-1} a_1, \ldots, a_n)$. We see that

$$\langle \boldsymbol{d}^{(2l+2)}[\boldsymbol{half}], \vec{a} \rangle = \langle \boldsymbol{d}^{(2l+2)}[\boldsymbol{half}], \vec{a}^{\#} \rangle^{\#}$$

if

$$\langle \boldsymbol{d}^{(2l)}[\boldsymbol{half}], \vec{a} \rangle = \langle \boldsymbol{d}^{(2l)}[\boldsymbol{half}], \vec{a}^{\#} \rangle^{\#}$$

by (38). Thus (41) is valid for all $l \in \mathbb{Z}_{\geq 0}$. By (41), we have

$$\begin{aligned}
\vec{a} \in D_{half}^{(l)} &\iff \langle \boldsymbol{d}^{(2l+2)}[\boldsymbol{half}], \vec{a} \rangle = 0 \\
&\iff \langle \boldsymbol{d}^{(2l+2)}[\boldsymbol{half}], \vec{a}^{\#} \rangle^{\#} = 0 \\
&\iff \langle \boldsymbol{d}^{(2l+2)}[\boldsymbol{half}], \vec{a}^{\#} \rangle = 0 \iff \vec{a}^{\#} \in D_{half}^{(l)}.
\end{aligned}$$

■

When $\lambda \notin \frac{1}{2}\mathbb{Z}$, any basis is a standard basis since all $P^{[\vec{a},s]}(x)$ is holomorphic at $s = \lambda$. When $\lambda \in \frac{1}{2}\mathbb{Z}$, we can easily choose one standard basis for a given λ by utilizing Theorem 2.1. However, it is sufficient only to consider the following three kinds of standard basis, SB^{half}, SB^{even} and SB^{odd}.

Definition 2.3. *For $\lambda \in \frac{1}{2}\mathbb{Z}$, we define the bases SB^{half}, SB^{even} and SB^{odd} of $\mathbb{C}^{n+1}$ by*

$$\begin{aligned}
SB^{half} &:= \{\vec{a}_0^{half}, \vec{a}_1^{half}, \ldots, \vec{a}_n^{half}\} \text{ if } \lambda \text{ is a strict half-integer,} \\
SB^{even} &:= \{\vec{a}_0^{even}, \vec{a}_1^{even}, \ldots, \vec{a}_n^{even}\} \text{ if } \lambda \text{ is an even integer,} \\
SB^{odd} &:= \{\vec{a}_0^{odd}, \vec{a}_1^{odd}, \ldots, \vec{a}_n^{odd}\} \text{ if } \lambda \text{ is an odd integer,}
\end{aligned} \tag{42}$$

satisfying that there exists an increasing integer sequence

$$0 < l(0) < l(1) < \cdots < l(p) = n \tag{43}$$

with

$$p := \begin{cases} \lfloor \frac{n}{2} \rfloor & \text{if } i + n \text{ is odd,} \\ \lfloor \frac{n+1}{2} \rfloor & \text{if } i + n \text{ is even,} \end{cases}$$

such that

$$\begin{aligned}
SB_q^{half} &:= \{\vec{a}_0^{half}, \vec{a}_1^{half}, \ldots, \vec{a}_{l(q)}^{half}\} \text{ is a basis of } D_{half}^{(q)}, \\
SB_q^{even} &:= \{\vec{a}_0^{even}, \vec{a}_1^{even}, \ldots, \vec{a}_{l(q)}^{even}\} \text{ is a basis of } D_{even}^{(q)}, \\
SB_q^{odd} &:= \{\vec{a}_0^{odd}, \vec{a}_1^{odd}, \ldots, \vec{a}_{l(q)}^{odd}\} \text{ is a basis of } D_{odd}^{(q)},
\end{aligned}$$

for $q = 0, 1, \ldots, p$, respectively. In particular, we take SB^{even} and SB^{odd} such that

$$\vec{a}_j^{odd} = \vec{a}_j^{even\#} \quad (j = 0, 1, \ldots, n) \tag{44}$$

where $\vec{a}^{\#} := ((-1)^n a_0, (-1)^{n-1} a_1, \ldots, a_n)$ for $\vec{a} := (a_0, a_1, \ldots, a_n) \in \mathbb{C}^{n+1}$. This is possible by Theorem 2.1.

Proposition 2.2. *The bases* (42) *are standard bases for* $\lambda \in \frac{1}{2}\mathbb{Z}$ *in the sense of Definition 1.4. When* $\lambda \notin \frac{1}{2}\mathbb{Z}$, *every basis is a standard basis since every* $P^{[\vec{a},s]}(x)$ *does not have a pole.*

Proof. This is just the definition of the standard basis. ∎

3. Algorithms for constructing solutions — homogeneous equations.

In this section we give algorithms to compute all the hyperfunction solutions to $P(x,\partial)u(x) = 0$ for a homogeneous $\boldsymbol{G}$-invariant differential operator $P(x,\partial)$.

Algorithm 3.1. (The case of homogeneous degree zero) *For a given non-zero* $\boldsymbol{G}$*-invariant differential operator* $P(x,\partial) \in D(\boldsymbol{V})_0^{\boldsymbol{G}}$ *of homogeneous degree* 0 *satisfying the condition* (14) *in Theorem 1.3, an algorithm to compute a basis of the* $\boldsymbol{G}$*-invariant differential equation* $P(x,\partial)u(x) = 0$ *is given in the following.*

Input *A non-zero* $\boldsymbol{G}$*-invariant differential operator* $P(x,\partial) \in D(\boldsymbol{V})_0^{\boldsymbol{G}}$ *satisfying the condition* (14).

Output *A basis of the* $\boldsymbol{G}$*-invariant hyperfunction solutions to the differential equation* $P(x,\partial)u(x) = 0$.

Procedure

1. *Compute the* b_P*-function for* $P(x,\partial)$. *It is denoted by* $b_P(s) = (s-\lambda_1)^{p_1}\cdots(s-\lambda_l)^{p_l}$.
2. *For each* λ_i $(i = 1,\ldots,l)$, *take one standard basis at* $s = \lambda_i$ $SB^{\lambda_i} = \{\vec{a}_0(\lambda_i),\cdots,\vec{a}_n(\lambda_i)\}$, *which is defined in Definition 1.4.*
3. *Compute the Laurent expansion coefficients* $P_k^{[\vec{a}_j(\lambda_i),\lambda_i]}(x)$ *for each* $\vec{a}_j(\lambda_i)$ $(i = 1,\ldots,l;\ j = 0,\ldots,n)$ *and* $k \in \mathbb{Z}$ *in* $-o_{ij} \le k \le -o_{ij}+p_i-1$ *with* $o_{ij} := o(\vec{a}_j(\lambda_i),\lambda_i)$. *Then we have the generators of the vector space.*
$$L_{ij} := \text{the vector space generated by} \quad \{P_k^{[\vec{a}_j(\lambda_i),\lambda_i]}(x) \mid k = -o_{ij},\ldots,-o_{ij}+p_i-1\}. \tag{45}$$
4. *Then*
$$\bigoplus_{\substack{i=1,\ldots,l\\ j=0,\ldots,n}} L_{ij} \tag{46}$$
forms a basis of the $\boldsymbol{G}$*-invariant hyperfunction solution space to the equation* $P(x,\partial)u(x) = 0$.

Proof. Note that, by Theorem 1.3 and Corollary 1.7, every $\boldsymbol{G}$-invariant hyperfunction solution to $P(x,\partial)u(x) = 0$ is written as a finite combination of Laurent expansion coefficients of $|P(x)|_i^s$ $(i = 0,\ldots,n)$. Suppose that $u(x)$ is written as $u(x) = u_1(x)+\cdots+u_l(x)$ where each $u_i(x)$ is quasi-homogeneous of degree s_i and $s_1,\ldots,s_l$ are pairwise different. If $P(x,\partial)u(x) = 0$, then $P(x,\partial)u_i(x) = 0$ for all $i = 1,\ldots,l$ since the homogeneous degrees of $P(x,\partial)u_i(x)$ $(i = 1,\ldots,l)$ are pairwise different and hence linearly independent. Then, for each complex number $\lambda \in \mathbb{C}$, we have only to see which $u(x)$ is annihilated by $P(x,\partial)$ when $u(x)$ is a finite combination of Laurent expansion coefficients of $|P(x)|_i^s$ $(i = 0, \ldots, n)$ at $s = \lambda$.

Let $SB := \{\vec{a}_0, \vec{a}_1, \ldots, \vec{a}_n\}$ be a standard basis of $\mathbb{C}^{n+1}$ at $s = \lambda$ with an increasing sequence

$$0 < k(0) < k(1) < \cdots < k(PHO(\lambda)) = n \tag{47}$$

such that $SB_q := \{\vec{a}_0, \vec{a}_1, \ldots, \vec{a}_{k(q)}\}$ is a basis of $A(\lambda, q)$ for each q in $0 \le q \le PHO(\lambda)$. Then every $u(x)$ given as a finite combination of Laurent expansion coefficients of $|P(x)|_i^s$ $(i = 0, \ldots, n)$ at $s = \lambda$ is expressed as

$$u(x) = \sum_{\substack{f,g \in \mathbb{Z} \\ 0 \le g \le n}} c_{f,g} P_f^{[\vec{a}_g, \lambda]}(x) \tag{48}$$

with $c_{f,g} \in \mathbb{C}$. Let $b_P(s) = \sum_{i=0}^q b_i (s-\lambda)^{p+i}$ be an expansion of $b_P(s)$ with respect to $(s - \lambda)$. The number p is the multiplicity of $b_P(s)$ at $s = \lambda$. Then what we have to prove is that

$$\begin{aligned} &c_{f,g} = 0 \text{ except for } -o(\vec{a}_g, \lambda) \le f \le -o(\vec{a}_g, \lambda) + p - 1 \\ &\text{if and only if } P(x,\partial)u(x) = 0 \end{aligned} \tag{49}$$

since $P_f^{[\vec{a}_g, \lambda]}(x) = 0$ if $f < -o(\vec{a}_g, \lambda)$ by the definition. Indeed, the basis of L_{ij} in (45) is the basis of the remainder terms in the expression (48) with the condition (49) when $\lambda = \lambda_i$ and $p = k_i$. In particular, if λ is not a root of $b_P(s) = 0$, i.e., $p = 0$, then there is no $\boldsymbol{G}$-invariant solution to $P(x,\partial)u(x) = 0$.

The rest of the proof is devoted to showing (49). Let the Laurent expansion of $P^{[\vec{a},s]}(x)$ at $s = \lambda$ be denoted by $P^{[\vec{a},s]}(x) = \sum_{w \in \mathbb{Z}} P_w^{[\vec{a},\lambda]}(x)(s-\lambda)^w$. Then we have

$$\begin{aligned} P(x,\partial)P^{[\vec{a},s]}(x) &= \sum_{w \in \mathbb{Z}} P(x,\partial) P_w^{[\vec{a},\lambda]}(x)(s-\lambda)^w = b_P(s)P^{[\vec{a},s]}(x) \\ &= \left(\sum_{i=0}^q b_i(s-\lambda)^{p+i}\right)\left(\sum_{j \in \mathbb{Z}} P_j^{[\vec{a},\lambda]}(x)(s-\lambda)^j\right) \\ &= \sum_{w \in \mathbb{Z}} \sum_{i+j+p=w} b_i P_j^{[\vec{a},\lambda]}(x)(s-\lambda)^w \end{aligned}$$

and hence

$$P(x,\partial)P_w^{[\vec{a},\lambda]}(x) = \sum_{i+j+p=w} b_i P_j^{[\vec{a},\lambda]}(x). \tag{50}$$

Here $b_i = 0$ except for i in $0 \le i \le q$ and $P_j^{[\vec{a},\lambda]}(x) = 0$ for sufficiently small j.

Then since $u(x)=\sum_{\substack{f,g\in\mathbb{Z}\\ 0\le g\le n}} c_{f,g}P_f^{[\vec{a}_g,\lambda]}(x)$, we have

$$\begin{aligned}
P(x,\partial)u(x) &= \sum_{f,g\in\mathbb{Z}} c_{f,g}P(x,\partial)P_f^{[\vec{a}_g,\lambda]}(x) = \sum_{f,g\in\mathbb{Z}} c_{f,g}\sum_{i=0}^{q} b_i P_{f-p-i}^{[\vec{a}_g,\lambda]}(x)\\
&= \sum_{f,g\in\mathbb{Z}} c_{f,g}\sum_{j\in\mathbb{Z}} b_{f-p-j}P_j^{[\vec{a}_g,\lambda]}(x) = \sum_{j\in\mathbb{Z}}\sum_{f,g\in\mathbb{Z}} c_{f,g}b_{f-p-j}P_j^{[\vec{a}_g,\lambda]}(x)\\
&= \sum_{j\in\mathbb{Z}} P_j^{[\sum_{f,g\in\mathbb{Z}} c_{f,g}b_{f-p-j}\vec{a}_g,\lambda]}(x) = 0
\end{aligned}$$

where g runs in $0\le g\le n$. Therefore

$$\sum_{f,g\in\mathbb{Z}} c_{f,g}b_{f-p-j}\vec{a}_g = \sum_{g=0}^{n}\sum_{f\in\mathbb{Z}} c_{f,g}b_{f-p-j}\vec{a}_g \in A(\lambda,-j-1)$$

for all $j\in\mathbb{Z}$ by Theorem 1.6. This means that,

$$\begin{aligned}
&\text{for each } g=0,1,\ldots,n,\\
&\sum_{f\in\mathbb{Z}} c_{f,g}b_{f-p-j}=0 \ \text{ for all } j\in\mathbb{Z} \text{ satisfying } g\ge k(-j)
\end{aligned}\tag{51}$$

since $\vec{a}_g\notin A(\lambda,-j-1)$ if $g\ge k(-j)$ by definition. Here $k(-j)$ is the number defined by (47) if $0\le -j\le PHO(\lambda)$ and $k(-j)=0$ (resp. $k(-j)=n+1$) if $-j<0$ (resp. $-j>PHO(\lambda)$). Since $g\ge k(-j)$ is equivalent to $o(\vec{a}_g,\lambda)\ge -j$ by definition and $\sum_{f\in\mathbb{Z}} c_{f,g}b_{f-p-j}=\sum_{f=p+j}^{p+j+q} c_{f,g}b_{f-p-j}=\sum_{s=0}^{q} c_{p+j+s,g}b_s=0$, the condition (51) is rewritten as the condition:

$$\begin{aligned}
&\text{for each } g=0,1,\ldots,n,\\
&\sum_{s=0}^{q} c_{p+j+s,g}b_s=0 \text{ for all } j\in\mathbb{Z} \text{ satisfying } j\ge -o(\vec{a}_g,\lambda).
\end{aligned}\tag{52}$$

Note that coefficients b_0 and b_q are not zero. Then the condition (52) is equivalent to the condition:

$$\begin{aligned}
&\text{for each } g=0,1,\ldots,n,\\
&c_{p+j,g}=0 \text{ for all } j\in\mathbb{Z} \text{ satisfying } j\ge -o(\vec{a}_g,\lambda).
\end{aligned}\tag{53}$$

This is just equivalent to the condition (49), which we have to prove. ∎

Next we consider $P(x,\partial)$ of non-zero homogeneous degree.

Algorithm 3.2. (The case of non-zero homogeneous degree) *For a given non-zero $\boldsymbol{G}$-invariant differential operator $P(x,\partial)\in D(\boldsymbol{V})^{\boldsymbol{G}}$ of homogeneous degree $q_1 n\ne 0$ satisfying the condition (14) in Theorem 1.3, an algorithm to compute a basis of the $\boldsymbol{G}$-invariant differential equation $P(x,\partial)u(x)=0$ is given in the following.*

1. *The case of the homogeneous degree $q_1<0$*

Input *A non-zero $\boldsymbol{G}$-invariant differential operator $P(x,\partial) \in D(\boldsymbol{V})^{\boldsymbol{G}}$ of homogeneous degree $q_1 n < 0$ satisfying the condition (14).*

Output *A basis of the $\boldsymbol{G}$-invariant hyperfunction solutions to the differential equation $P(x,\partial)u(x) = 0$.*

Procedure

(a) Compute the b_P-function for $P(x,\partial)$. It is denoted by $b_P(s) = (s-\lambda_1)^{p_1}\cdots(s-\lambda_l)^{p_l}$.

(b) For each λ_i $(i = 1,\ldots,l)$, take one standard basis

$$SB^{\lambda_i} = \{\vec{a}_0(\lambda_i), \cdots, \vec{a}_n(\lambda_i)\}$$

at $s = \lambda_i$, which is the standard basis defined by (42) when $\lambda_i \in \frac{1}{2}\mathbb{Z}$ and the one defined in Definition 1.4 otherwise.

(c) Compute the Laurent expansion coefficients $P_k^{[\vec{a}_j(\lambda_i),\lambda_i]}(x)$ for each $\vec{a}_j(\lambda_i)$ $(i = 1,\ldots,l, j = 0,\ldots,n)$ and $k \in \mathbb{Z}$ in $-o_{ij} \le k \le -o_{ij}^{\#q_1} + p_i - 1$ with $o_{ij} := o(\vec{a}_j(\lambda_i),\lambda_i)$ and $o_{ij}^{\#q_1} := o(\vec{a}_j(\lambda_i)^{\#q_1}, \lambda_i + q_1)$. Then we have the generators of the vector space L_{ij} by (54).

$$\begin{aligned} L_{ij} &:= \text{the vector space generated by} \\ &\quad \{P_k^{[\vec{a}_j(\lambda_i),\lambda_i]}(x) \mid k = -o_{ij},\ldots,-o_{ij}^{\#q_1} + p_i - 1\}. \end{aligned} \tag{54}$$

We set $L_{ij} := \{0\}$ if $-o_{ij} > -o_{ij}^{\#q_1} + p_i - 1$.

(d) Then

$$\bigoplus_{\substack{i=1,\ldots,l \\ j=0,\ldots,n}} L_{ij} \tag{55}$$

forms a basis of the solution space.

2. The case of the homogeneous degree $q_1 > 0$.

Input *A non-zero $\boldsymbol{G}$-invariant differential operator $P(x,\partial) \in D(\boldsymbol{V})^{\boldsymbol{G}}$ of homogeneous degree $q_1 n > 0$ satisfying the condition (14).*

Output *A basis of the $\boldsymbol{G}$-invariant hyperfunction solutions to the differential equation $P(x,\partial)u(x) = 0$.*

Procedure

(a) Compute the b_P-function $b_P(s)$ and consider the set $R := R_1 \cup R_2$ with

$$\begin{aligned} R_1 &:= \{\lambda_i := -\frac{i+1}{2} \mid i = 1,2,\ldots,n+2q_1-2\}, \\ R_2 &:= \{\lambda \in \mathbb{C} \mid b_P(\lambda) = 0\}. \end{aligned}$$

Let q_2 be the number of elements of the set $R_2 - R_1$. We denote by

$$\lambda_{n+2q_1-1}, \lambda_{n+2q_1}, \ldots, \lambda_{n+2q_1+q_2-2}$$

the elements of $R_2 - R_1$. Then we can write the elements of R by

$$R = \{\lambda_1, \lambda_2, \ldots, \lambda_{n+2q_1+q_2-2}\}.$$

(b) *We define the multiplicity* k_i *of* λ_i *by*

$$p_i := \begin{cases} \text{multiplicity of } s-\lambda_i \text{ in } b_P(s) & \text{if } b_P(\lambda_i)=0, \\ 0 & \text{if } b_P(\lambda_i)\neq 0. \end{cases} \tag{56}$$

(c) *For each* λ_i $(i=1,\dots,n+2q_1+q_2-2)$, *take one standard basis* $SB^{\lambda_i}=\{\vec{a}_0(\lambda_i),\cdots,\vec{a}_n(\lambda_i)\}$ *at* $s=\lambda_i$, *which is the standard basis defined by* (42) *when* $\lambda_i \in \frac{1}{2}\mathbb{Z}$ *and the one defined in Definition 1.4 otherwise.*

(d) *Compute the Laurent expansion coefficients* $P_k^{[\vec{a}_j(\lambda_i),\lambda_i]}(x)$ *for each* $\vec{a}_j(\lambda_i)$ $(i=1,\dots,n+2q_1+q_2-2, j=0,\dots,n)$ *and* $k\in\mathbb{Z}$ *in* $-o_{ij}\le k\le -o_{ij}^{\#q_1}+p_i-1$ *with* $o_{ij}:=o(\vec{a}_j(\lambda_i),\lambda_i)$ *and* $o_{ij}^{\#q_1}:=o(\vec{a}_j(\lambda_i),\lambda_i+q_1)$. *Then we have the generators of the vector space* L_{ij} *by* (57).

$$\begin{aligned} L_{ij} := &\text{ the vector space generated by} \\ &\{P_k^{[\vec{a}_j(\lambda_i),\lambda_i]}(x) \mid k=-o_{ij},\dots,-o_{ij}^{\#q_1}+p_i-1\}. \end{aligned} \tag{57}$$

We set $L_{ij}:=\{0\}$ *if* $-o_{ij} > -o_{ij}^{\#q_1}+p_i-1$.

(e) *Then*

$$\bigoplus_{\substack{i=1,\dots,n+2q_1+q_2-2\\ j=0,\dots,n}} L_{ij} \tag{58}$$

forms a basis of the solution space.

Proof. Note that, for each complex number $\lambda\in\mathbb{C}$, we have only to decide which $u(x)$ is annihilated by $P(x,\partial)$ when $u(x)$ is given as a finite combination of Laurent expansion coefficients of $|P(x)|_i^s$ $(i=0,\dots,n)$ at $s=\lambda$. This is because of the same reason as in the proof of Algorithm 3.1.

Let $SB:=\{\vec{a}_0,\vec{a}_1,\dots,\vec{a}_n\}$ be a standard basis of $\mathbb{C}^{n+1}$ at $s=\lambda$ with an increasing sequence

$$0<k(0)<k(1)<\cdots<k(PHO(\lambda))=n \tag{59}$$

such that $SB_q:=\{\vec{a}_0,\vec{a}_1,\dots,\vec{a}_{k(q)}\}$ is a basis of $A(\lambda,q)$ for each q in $0\le q\le PHO(\lambda)$. In particular, we suppose that it is the standard basis defined by (42) when $\lambda_i\in\frac{1}{2}\mathbb{Z}$ or the one from Definition 1.4 otherwise. Then, by the property (44), we see easily that $SB^{\#q_1}:=\{\vec{a}_0^{\#q_1},\vec{a}_1^{\#q_1},\dots,\vec{a}_n^{\#q_1}\}$ is a standard basis of $\mathbb{C}^{n+1}$ at $s=\lambda+q_1$ with an increasing sequence

$$0<k^{\#q_1}(0)<k^{\#q_1}(1)<\cdots<k^{\#q_1}(PHO(\lambda+q_1))=n \tag{60}$$

such that $SB_q^{\#q_1}:=\{\vec{a}_0^{\#q_1},\vec{a}_1^{\#q_1},\dots,\vec{a}_{k^{\#q_1}(q)}^{\#q_1}\}$ is a basis of $A(\lambda+q_1,q)$ for each q in $0\le q\le PHO(\lambda+q_1)$. Here, we see from the definition that

$$\begin{aligned} PHO(\lambda+q_1)\ge PHO(\lambda) \quad &\text{if} \quad q_1<0, \\ PHO(\lambda+q_1)\le PHO(\lambda) \quad &\text{if} \quad q_1>0, \end{aligned}$$

and that $k(q) = k^{\#q_1}(q)$ for $q < PHO(\lambda)$ if $q_1 < 0$ or for $q < PHO(\lambda + q_1)$ if $q_1 > 0$.

Every $u(x)$ given as a finite combination of Laurent expansion coefficients of $|P(x)|_i^s$ $(i = 0, \ldots, n)$ at $s = \lambda$ is expressed as

$$u(x) = \sum_{\substack{f,g \in \mathbb{Z} \\ 0 \le g \le n}} c_{f,g} P_f^{[\vec{a}_g, \lambda]}(x) \tag{61}$$

with $c_{f,g} \in \mathbb{C}$. Let $b_P(s) = \sum_{i=0}^{q} b_i (s-\lambda)^{p+i}$ be an expansion of $b_P(s)$ with respect to $(s - \lambda)$. The number p is the multiplicity of $b_P(s)$ at $s = \lambda$. Now our goal is to prove the statement:

$$\begin{aligned} &c_{f,g} = 0 \textit{ except for } -o(\vec{a}_g, \lambda) \le f \le -o(\vec{a}_g^{\#q_1}, \lambda + q_1) + p - 1 \\ &\textit{if and only if } P(x, \partial)u(x) = 0 \end{aligned} \tag{62}$$

since $P_f^{[\vec{a}_g, \lambda]}(x) = 0$ if $f < -o(\vec{a}_g, \lambda)$ from the definition.

First we consider the situation that λ is not a root of $b_P(s) = 0$, i.e., $p = 0$. When $q_1 < 0$ (Algorithm 3.2-1), there is no non-zero $\boldsymbol{G}$-invariant homogeneous solutions of homogeneous degree $n\lambda$ to $P(x, \partial)u(x) = 0$. When $q_1 > 0$ (Algorithm 3.2-2), there is no non-zero $\boldsymbol{G}$-invariant homogeneous solutions of homogeneous degree $n\lambda$ to $P(x, \partial)u(x) = 0$ *except when* $\lambda \in R_1$. Then we have only to consider the cases that λ is a root of $b_P(\lambda) = 0$ when $q_1 < 0$ (Algorithm 3.2-1), and the cases λ is a root of $b_P(\lambda) = 0$ or $\lambda \in R_1$ when $q_1 > 0$ (Algorithm 3.2-2). This is the reason why we restrict the λ's to the finite sets of numbers in the first step of the procedures in the algorithms. We can easily see that the basis of L_{ij} in (57) is just the basis of the terms in the expression (61) with the condition (62) when $\lambda = \lambda_i$ and $p = k_i$.

The rest of the proof is devoted to showing (62). The Laurent expansion of $P^{[\vec{a},s]}(x)$ at $s = \lambda$ is denoted by $P^{[\vec{a},s]}(x) = \sum_{w \in \mathbb{Z}} P_w^{[\vec{a},\lambda]}(x)(s-\lambda)^w$. Then we have $P(x,\partial)P^{[\vec{a},s]}(x) = \sum_{w \in \mathbb{Z}} \sum_{i+j+p=w} b_i P_j^{[\vec{a}^{\#q_1}, \lambda+q_1]}(x)(s-\lambda)^w$ and hence we have

$$P(x,\partial)P_w^{[\vec{a},\lambda]}(x) = \sum_{i+j+p=w} b_i P_j^{[\vec{a}^{\#q_1}, \lambda+q_1]}(x). \tag{63}$$

Here $b_i = 0$ except for i in $0 \le i \le q$ and $P_j^{[\vec{a}^{\#q_1}, \lambda+q_1]}(x) = 0$ for sufficiently small j. Then, for the function $u(x) = \sum_{\substack{f,g \in \mathbb{Z} \\ 0 \le g \le n}} c_{f,g} P_f^{[\vec{a}_g, \lambda]}(x)$, we see that $P(x,\partial)u(x) = \sum_{j \in \mathbb{Z}} P_j^{[\sum_{f,g \in \mathbb{Z}} c_{f,g} b_{f-p-j} \vec{a}_g^{\#q_1}, \lambda+q_1]}(x) = 0$ where g runs in $0 \le g \le n$, i.e., $\vec{a}_g = 0$ except for $0 \le g \le n$. Thus we have $\sum_{f,g \in \mathbb{Z}} c_{f,g} b_{f-p-j} \vec{a}_g = \sum_{g=0}^{n} \sum_{f \in \mathbb{Z}} c_{f,g} b_{f-p-j} \vec{a}_g^{\#q_1} \in A(\lambda + q_1, -j-1)$ for all $j \in \mathbb{Z}$ by Theorem 1.6. Therefore we see that

$$\begin{aligned} &\text{for each } g = 0, 1, \ldots, n, \\ &\sum_{f \in \mathbb{Z}} c_{f,g} b_{f-p-j} = 0 \text{ for all } j \in \mathbb{Z} \text{ satisfying } g \ge k^{\#q_1}(-j) \end{aligned} \tag{64}$$

since $\vec{a}_g^{\#q_1} \notin A(\lambda + q_1, -j-1)$ if $g \ge k^{\#q_1}(-j)$ by definition. Here $k^{\#q_1}(-j)$ is the number defined by (60) if $0 \le -j \le PHO(\lambda + q_1)$ and $k^{\#q_1}(-j) = 0$

(resp. $k^{\#q_1}(-j) = n+1$) if $-j < 0$ (resp. $-j > PHO(\lambda + q_1)$). Since $g \geq k^{\#q_1}(-j)$ is equivalent to $o(\vec{a}_g^{\#q_1}, \lambda + q_1) \geq -j$ by definition and $\sum_{f \in \mathbb{Z}} c_{f,g} b_{f-p-j} = \sum_{f=p+j}^{p+j+q} c_{f,g} b_{f-p-j} = \sum_{s=0}^{q} c_{p+j+s,g} b_s = 0$, the condition (64) is rewritten as the condition:

$$\begin{aligned}&\text{for each } g = 0, 1, \ldots, n,\\ &\sum_{s=0}^{q} c_{p+j+s,g} b_s = 0 \text{ for all } j \in \mathbb{Z} \text{ satisfying } j \geq -o(\vec{a}_g^{\#q_1}, \lambda + q_1).\end{aligned} \tag{65}$$

Note that coefficients b_0 and b_q are not zero. Then the condition (65) is equivalent to the condition:

$$\begin{aligned}&\text{for each } g = 0, 1, \ldots, n,\\ &c_{p+j,g} = 0 \text{ for all } j \in \mathbb{Z} \text{ satisfying } j \geq -o(\vec{a}_g^{\#q_1}, \lambda + q_1).\end{aligned} \tag{66}$$

This is equivalent to the condition (62), which we wanted to prove. ∎

4. Algorithms for constructing solutions — inhomogeneous equations.

For the inhomogeneous differential equation $P(x, \partial)u(x) = v(x)$, the algorithm for the computation of invariant hyperfunction solutions is carried out under rather a weak condition like (29).

Algorithm 4.1. (the case of inhomogeneous equation) *For a given non-zero $\boldsymbol{G}$-invariant homogeneous differential operator $P(x, \partial) \in D(\boldsymbol{V})^{\boldsymbol{G}}$ of homogeneous degree $kn \in \mathbb{Z}$ satisfying the condition (29) in Theorem 1.8, an algorithm to compute a $\boldsymbol{G}$-invariant hyperfunction solution of the $\boldsymbol{G}$-invariant differential equation $P(x, \partial)u(x) = v(x)$ is given in the following.*

Input *A non-zero $\boldsymbol{G}$-invariant homogeneous differential operator $P(x, \partial) \in D(\boldsymbol{V})^{\boldsymbol{G}}$ of homogeneous degree $kn \in \mathbb{Z}$ satisfying the condition (29) and a non-zero quasi-homogeneous $\boldsymbol{G}$-invariant hyperfunction $v(x)$.*

Output *A non-zero $\boldsymbol{G}$-invariant hyperfunction solution $u(x)$ to the differential equation $P(x, \partial)u(x) = v(x)$.*

Procedure

1. *Write $v(x)$ as a sum of Laurent expansion coefficients. Namely, $v(x)$ is given by $v(x) = \sum_{l=1}^{q} P_{w_l}^{[\vec{a}_l, \lambda]}(x)$ with $\vec{a}_1, \ldots, \vec{a}_k \in \mathbb{C}^{n+1}$ and $w_1, \ldots, w_k \in \mathbb{Z}$.*
2. *Compute the b_P-function $b_P(s)$ and factorize it as $b_P(s) = (s - \lambda + k)^p \widetilde{b}(s)$ with $\widetilde{b}(\lambda - k) \neq 0$ and expand $\widetilde{b}(s)^{-1}$ into a Taylor series $\widetilde{b}(s)^{-1} = \sum_{i=0}^{\infty} b_i (s - \lambda + k)^i$ at $s = \lambda - k$.*
3. *Then*
$$u(x) = \sum_{l=1}^{q} \sum_{i+j=w_l+p} b_i P_j^{[\vec{a}_l^{\#-k}, \lambda - k]}(x) \tag{67}$$
is an $\boldsymbol{G}$-invariant hyperfunction solution to $P(x, \partial)u(x) = v(x)$.

Proof. The proof can be found in the proof of Theorem 1.8. See [22, Theorem 6.1]. ∎

5. Explicit computations of examples

In this section we shall give some examples. Some homogeneous differential equations generated by $\det(x)$ and $\det(\partial^*)$ are treated here.

5.1. The equations $\det(\partial^*)\det(x)u(x) = 0$ **and** $\det(x)\det(\partial^*)u(x) = 0$**.** We consider two examples of differential equation of homogeneous degree 0. Let us consider the case of $P(x, \partial) = \det(\partial^*)\det(x)$ and $P(x, \partial) = \det(x)\det(\partial^*)$. The homogeneous degrees of $P(x, \partial)$ are 0 and the b_P-functions are $b_P(s) = (s+1)(s+\frac{3}{2})\cdots(s+\frac{n+1}{2})$ and $b_P(s) = (s)(s+\frac{1}{2})\cdots(s+\frac{n-1}{2})$, respectively.

Proposition 5.1. *The invariant hyperfunction solution to the differential equations* $\det(\partial^*)\det(x)u(x) = 0$ *and* $\det(x)\det(\partial^*)u(x) = 0$ *are calculated in the following.*

1. *The* $\boldsymbol{G}$*-invariant hyperfunction solution space to the differential equation* $\det(\partial^*)\det(x)u(x) = 0$ *is generated by*

$$\bigcup_{i=1}^{n} \bigcup_{q=0}^{\lfloor \frac{i+1}{2} \rfloor} \left\{ P_{-q}^{[\vec{a}, -\frac{i+1}{2}]}(x) \;\middle|\; \vec{a} \in A(-\tfrac{i+1}{2}, q) \right\}. \tag{68}$$

Here, $A(-\frac{i+1}{2}, q)$ *is the vector subspace of* $\mathbb{C}^{n+1}$ *defined by* (19) *in Definition 1.2 and explicitly computed in* (39) *of Theorem 2.1. Similarly, the* $\boldsymbol{G}$*-invariant hyperfunction solution space to the equation* $\det(x)\det(\partial^*)u(x) = 0$ *is generated by*

$$\bigcup_{i=-1}^{n-2} \bigcup_{q=0}^{\lfloor \frac{i+1}{2} \rfloor} \left\{ P_{-q}^{[\vec{a}, -\frac{i+1}{2}]}(x) \;\middle|\; \vec{a} \in A(-\tfrac{i+1}{2}, q) \right\}. \tag{69}$$

2. *In particular, for* $i = -1, 0, 1, 2, \ldots, n$,

$$\bigcup_{q=0}^{\lfloor \frac{i+1}{2} \rfloor} \left\{ P_{-q}^{[\vec{a}, -\frac{i+1}{2}]}(x) \;\middle|\; \vec{a} \in A(-\tfrac{i+1}{2}, q) \right\} \tag{70}$$

spans an $(n+1)$*-dimensional vector space generated by all the relatively invariant hyperfunctions corresponding to the character* $\det(g)^{-i-1}$*. The dimensions of* $\boldsymbol{G}$*-invariant hyperfunction solutions to* $\det(\partial^*)\det(x)u(x) = 0$ *and* $\det(x)\det(\partial^*)u(x) = 0$ *are* $n(n+1)$.

Proof. 1. We compute the solution space following Algorithm 3.1. For the differential operator $P(x, \partial) = \det(\partial^*)\det(x)$, the b_P-function is $b_P(s) = (s+1)(s+\frac{3}{2})\cdots(s+\frac{n+1}{2})$. In the first step of the procedure of Algorithm 3.1, we have $l = n$ and the roots of $b_P(s) = 0$ are $\lambda_i = \frac{i+1}{2}$ with multiplicity $p_i = 1$ $(i = 1, \ldots, n)$. Since they are all half-integers, we can take a standard basis at $s = \lambda_i$ $SB^{\lambda_i} = \{\vec{a}_0(\lambda_i), \ldots, \vec{a}_n(\lambda_i)\}$ as defined in Definition 2.3. Let $SB^{\lambda_i}_q$ be a subset of SB^{λ_i} such that $SB^{\lambda_i}_q$ forms a basis of $A(\lambda_i, q)$ for each q in $0 \le q \le PHO(\lambda_i) = \lfloor \frac{i+1}{2} \rfloor$. Then we have $SB^{\lambda_i} = \bigsqcup_{q=0}^{\lfloor \frac{i+1}{2} \rfloor} SB^{\lambda_i}_q - SB^{\lambda_i}_{q-1}$.

and the set $SB_q^{\lambda_i} - SB_{q-1}^{\lambda_i}$ forms a basis of $\overline{A(\lambda_i, q)} := A(\lambda_i, q)/A(\lambda_i, q-1)$, where $A(\lambda_i, -1) = \{0\}$ and $SB_{-1}^{\lambda_i} := \emptyset$. For each $\vec{a}_j(\lambda_i) \in SB_q^{\lambda_i} - SB_{q-1}^{\lambda_i}$, we have $o_{ij} := o(\vec{a}_j(\lambda_i), \lambda_i) = q$ and hence

$$-o_{ij} \leq k \leq -o_{ij} + p_i - 1 \ \Rightarrow\ -q \leq k \leq -q + p_i - 1 \ \Rightarrow\ k = -q$$

for each q in $0 \leq q \leq \lfloor \frac{i+1}{2} \rfloor$. Since $P_{-q}^{[\vec{a},\lambda_i]}(x) = 0$ if $\vec{a} \in A(\lambda_i, q-1)$, we have

$$\begin{aligned} &\text{vector space generated by } \{P_{-q}^{[\vec{a},\lambda_i]}(x) \mid \vec{a} \in A(\lambda_i, q)\} \\ = &\text{ vector space generated by } \{P_{-q}^{[\vec{a},\lambda_i]}(x) \mid \vec{a} \in SB_q^{\lambda_i} - SB_{q-1}^{\lambda_i}\} \end{aligned}$$

for each q in $0 \leq q \leq \lfloor \frac{i+1}{2} \rfloor$. Then the vector spaces generated by $\bigcup_{q=0}^{\lfloor \frac{i+1}{2} \rfloor} \{P_{-q}^{[\vec{a},\lambda_i]}(x) \mid \vec{a} \in A(\lambda_i, q)\}$ and $\bigcup_{q=0}^{\lfloor \frac{i+1}{2} \rfloor} \bigcup_{\vec{a} \in SB_q^{\lambda_i} - SB_{q-1}^{\lambda_i}} \{P_{-q}^{[\vec{a},\lambda_i]}(x)\} = \bigoplus_{j=0}^{n} L_{ij}$ coincide, and so do the vector spaces generated by

$$\bigcup_{i=1}^{n} \bigcup_{q=0}^{\lfloor \frac{i+1}{2} \rfloor} \{P_{-q}^{[\vec{a},\lambda_i]}(x) \mid \vec{a} \in A(\lambda_i, q)\} \tag{71}$$

and the vector space generated by

$$\bigoplus_{i=1}^{n} \bigoplus_{j=0}^{n} L_{ij} = \bigcup_{i=1}^{n} \bigcup_{q=0}^{\lfloor \frac{i+1}{2} \rfloor} \bigcup_{\vec{a} \in SB_q^{\lambda_i} - SB_{q-1}^{\lambda_i}} \{P_{-q}^{[\vec{a},\lambda_i]}(x)\}.$$

Thus we have proved that (68), which is (71), forms a basis of the vector space of $\boldsymbol{G}$-invariant hyperfunction solution space to the differential equation $\det(\partial^*) \det(x) u(x) = 0$.

For the differential equation $\det(x) \det(\partial^*) u(x) = 0$, we can prove in the same way that the vector space of $\boldsymbol{G}$-invariant hyperfunction solution space to the differential equation $\det(x) \det(\partial^*) u(x) = 0$ is generated by (69) since the b_P-function of $\det(x) \det(\partial^*)$ is $b_P(s) = (s)(s + \frac{1}{2}) \cdots (s + \frac{n-1}{2})$.

2. By Proposition 1.5, the elements in

$$\bigcup_{q=0}^{\lfloor \frac{i+1}{2} \rfloor} \bigcup_{\vec{a} \in SB_q^{\lambda_i} - SB_{q-1}^{\lambda_i}} \{P_{-q}^{[\vec{a},\lambda_i]}(x)\} \tag{72}$$

are linearly independent and spans an $(n+1)$-dimensional vector space. On the other hand, by Theorem 1.6, each $P_{-q}^{[\vec{a},\lambda_i]}(x)$ for $\vec{a} \in SB_q^{\lambda_i} - SB_{q-1}^{\lambda_i}$ is a homogeneous $\boldsymbol{G}$-invariant hyperfunction of homogeneous degree $n\lambda_i = -n(i+1)/2$. This means that it is relatively invariant under the action of $\mathbf{GL}_n(\mathbb{R})$ corresponding to the character $\det(g)^{-i-1}$. By the main result of [18], the space of relatively invariant hyperfunctions for a fixed character $\det(g)^{2s}$ ($s \in \mathbb{C}$) is $n+1$. Then (72) forms a basis of all relatively invariant hyperfunctions corresponding to the character $\det(g)^{-i-1}$. Then we see that the dimensions of $\boldsymbol{G}$-invariant hyperfunction solutions to $\det(\partial^*) \det(x) u(x) = 0$ and $\det(x) \det(\partial^*) u(x) = 0$ are $n(n+1)$. ∎

5.2. The equations $\det(x)u(x) = 0$**.**

Let us consider the case of $P(x,\partial) = \det(x)$. The total homogeneous degree of $\det(x)$ is n and $b_P(s) = 1$. We can prove by our algorithm that the $\boldsymbol{G}$-invariant solution space of the differential equation $\det(x)u(x) = 0$ is generated by the $\boldsymbol{G}$-invariant measures on all the singular orbits, i.e., $\boldsymbol{G}$-orbits contained in $\det(x) = 0$, and hence it is $\frac{n(n+1)}{2}$-dimensional. The dimension coincides with the number of singular orbits. Here the $\boldsymbol{G}$-invariant measure on each singular orbit is a relatively invariant hyperfunction. Namely we have the following proposition.

Proposition 5.2. *Consider the differential equation* $\det(x)u(x) = 0$.

1. *The* $\boldsymbol{G}$*-invariant hyperfunction solution space to the differential equation* $\det(x)u(x) = 0$ *is generated by*

$$\bigcup_{i=1}^{n} \left\{ P^{[\vec{a}, -\frac{i+1}{2}]}_{-\lfloor \frac{i+1}{2} \rfloor}(x) \;\middle|\; \vec{a} \in \mathbb{C}^{n+1} \right\}. \tag{73}$$

2. *In particular, for* $i = 1, 2, \ldots, n$*, the set*

$$\left\{ P^{[\vec{a}, -\frac{i+1}{2}]}_{-\lfloor \frac{i+1}{2} \rfloor}(x) \;\middle|\; \vec{a} \in \mathbb{C}^{n+1} \right\} \tag{74}$$

spans an $(n+1-i)$*-dimensional vector space generated by the tempered distributions*

$$f(x) \longmapsto \int f(x) d\nu_i^j \quad (f(x) \in \mathcal{S}(\boldsymbol{V}))$$

($j = 0, 1, \ldots, n-i$) *where* $d\nu_i^j$ *is the* $\boldsymbol{G}$*-invariant measure on*

$$\boldsymbol{S}_i^j := \{x \in \mathrm{Sym}_n(\mathbb{R}) \mid \mathrm{sgn}(x) = (j, n-i-j)\}.$$

Proof. 1. We shall prove the first part of the proposition by carrying out Algorithm 3.2-2.

The b_P-function is $b_P(s) \equiv 1$ and we have $R = R_1 = \{\lambda_i = -\frac{i+1}{2} \mid i = 1, \ldots, n\}$ and $q_1 = 1$ and $q_2 = 0$ in the first step of the procedure. In the second step, we have $p_i = 0$ for all $i = 1, \ldots, n$. Since all $\lambda_i \in R$ are half-integers, we take the standard basis $SB^{\lambda_i} = \{\vec{a}_0(\lambda_i), \ldots, \vec{a}_n(\lambda_i)\}$ of Definition 1.4 as a standard basis at $s = \lambda_i$. For each $\vec{a}_j(\lambda_i)$ ($i = 1, \ldots, n$ and $j = 0, \ldots, n$), we have defined $o_{ij} := o(\vec{a}_j(\lambda_i), \lambda_i)$ and $o_{ij}^{\#q_1} := o_{ij}^{\#} = o(\vec{a}_j(\lambda_i)^{\#}, \lambda_i + 1)$. Let $SB^{\lambda_i}_{q-1}$ be the subset of SB^{λ_i} consisting of the vectors $\vec{a}_j(\lambda_i)$ such that $P^{[\vec{a}_j(\lambda_i), s]}(x)$ has a pole of order less than $q-1$ at $s = \lambda_i$. If $\vec{a}_j(\lambda_i) \in SB^{\lambda_i} - SB^{\lambda_i}_{q-1}$, then $P^{[\vec{a}_j(\lambda_i), s]}(x)$ has a pole of order q at $s = \lambda_i$ since the possible highest order of $P^{[\vec{a}, s]}(x)$ at $s = \lambda_i$ is $q = \lfloor \frac{i+1}{2} \rfloor$. Then we have, by putting $q = \lfloor \frac{i+1}{2} \rfloor$,

$$\begin{aligned} o_{ij} &= o_{ij}^{\#} + 1 = q \quad \text{if} \quad \vec{a}_j(\lambda_i) \in SB^{\lambda_i} - SB^{\lambda_i}_{q-1}, \\ o_{ij} &= o_{ij}^{\#} \leq q - 1 \quad \text{if} \quad \vec{a}_j(\lambda_i) \in SB^{\lambda_i}_{q-1}. \end{aligned} \tag{75}$$

Indeed, by Theorem 2.1, Definition 2.3 and the property (44), we see that

$$\begin{aligned}&(\text{the order of } P^{[\vec{a}_j(\lambda_i)^{\#},s]}(x) \text{ at } s=\lambda_i+1)\\ =&(\text{the order of } P^{[\vec{a}_j(\lambda_i),s]}(x) \text{ at } s=\lambda_i)\end{aligned}$$

for all $\vec{a}_j(\lambda_i)\in SB^{\lambda_i}_{q-1}$ and that the order of $P^{[\vec{a}_j(\lambda_i)^{\#},s]}(x)$ at $s=\lambda_i+1$ is equal to $q-1$ for all $\vec{a}_j(\lambda_i)\in SB^{\lambda_i}-SB^{\lambda_i}_{q-1}$. Then we have (75). Thus, for $\vec{a}_j(\lambda_i)\in SB^{\lambda_i}-SB^{\lambda_i}_{q-1}$, we have

$$-o_{ij}=-(o^{\#}_{ij}+1)=-o^{\#}_{ij}+p_i-1=-q=-\lfloor\frac{i+1}{2}\rfloor,$$

and L_{ij} in (57) is generated by $P^{[\vec{a}_j(\lambda_i),\lambda_i]}_{-\lfloor\frac{i+1}{2}\rfloor}(x)$. For $\vec{a}_j(\lambda_i)\in SB^{\lambda_i}_{q-1}$, we have $-o_{ij}=-o^{\#}_{ij}>-o_{ij}-1=-o_{ij}+p_i-1$, and hence L_{ij} in (57) is $\{0\}$. Therefore we have

$$\begin{aligned}\bigoplus_{\substack{i=1,\dots,n\\ j=0,\dots,n}} L_{ij}&=\bigoplus_{i=1,\dots,n}\left\{\begin{array}{l}\text{the vector space generated by}\\ \{P^{[\vec{a}_j(\lambda_i),\lambda_i]}_{-\lfloor\frac{i+1}{2}\rfloor}(x)\mid \vec{a}_j(\lambda_i)\in SB^{\lambda_i}-SB^{\lambda_i}_{q-1}\}\end{array}\right\}\\ &=\left\{\begin{array}{l}\text{the vector space generated by}\\ \displaystyle\bigcup_{i=1}^{n}\{P^{[\vec{a},-\frac{i+1}{2}]}_{-\lfloor\frac{i+1}{2}\rfloor}(x)\mid \vec{a}\in\mathbb{C}^{n+1}\}\end{array}\right\},\end{aligned}$$

since $P^{[\vec{a}_j(\lambda_i),-\frac{i+1}{2}]}_{-\lfloor\frac{i+1}{2}\rfloor}(x)=0$ if $\vec{a}_j(\lambda_i)\in SB^{\lambda_i}_{q-1}$. This is what we wanted to prove.

2. For $i=1,\dots,n$, each element of $\{P^{[\vec{a},-\frac{i+1}{2}]}_{-\lfloor\frac{i+1}{2}\rfloor}(x)\mid\vec{a}\in\mathbb{C}^{n+1}\}$ is a homogeneous $\boldsymbol{G}$-invariant hyperfunction of homogeneous degree $-n(\frac{i+1}{2})$ and its support is contained in $\boldsymbol{S}_i$. See Theorem A2. It is proved that such hyperfunctions are given as a linear sum of $\boldsymbol{G}$-invariant measures on the $(n-i+1)$ open orbits $\boldsymbol{S}^j_i$ $(j=1,\dots,n-i+1)$ in $\boldsymbol{S}_i$. See, for example, §4 in [16].

■

5.3. The equations $\det(\partial^*)u(x)=0$.

Similar argument is possible in the case of $P(x,\partial)=\det(\partial)$. In this case, the total homogeneous degree of $P(x,\partial)$ is $-n$ and we see that $b_P(s)=\prod_{i=1}^n(s+\frac{i-1}{2})$. The solution space of $\det(\partial)u(x)=0$ is the Fourier transform of that of $\det(x)u(x)=0$, and hence it is $\frac{n(n+1)}{2}$-dimensional and generated by relatively invariant hyperfunctions.

Proposition 5.3. *Consider the differential equation* $\det(\partial^*)u(x)=0$.

1. *The $\boldsymbol{G}$-invariant hyperfunction solution space to the differential equation* $\det(\partial^*)u(x)=0$ *is generated by*

$$\bigcup_{i=-1}^{n-2}\bigcup_{q=0}^{\lfloor\frac{i+1}{2}\rfloor}\left\{P^{[\vec{a},-\frac{i+1}{2}]}_{-q}(x)\;\middle|\;\vec{a}\in D^{(q)}_{*}\right\}. \tag{76}$$

Here, $D_*^{(j)}$ is a vector subspace of $\mathbb{C}^{n+1}$ defined by Definition 2.2. The $*$ in $D_*^{(j)}$ is substituted the word $half$, even or *odd according as $-\frac{i+1}{2}$ is* a strictly half integer, an even integer or an odd integer , *respectively.*

2. *In particular, for $i=-1,0,1,\ldots,n-2$,*

$$\bigcup_{q=0}^{\lfloor\frac{i+1}{2}\rfloor}\left\{P_{-q}^{[\vec{a},-\frac{i+1}{2}]}(x)\;\middle|\;\vec{a}\in D_*^{(q)}\right\} \tag{77}$$

spans an $(i+2)$-dimensional vector space generated by the Fourier transforms of the tempered distributions in (74).

Proof. We follow Algorithm 3.2-1. The first step and the second step of the procedure of Algorithm 3.2-1 are the same as those of Algorithm 3.1. The roots of the b_P-function are $\lambda_i=-\frac{i+1}{2}$ $(i=-1,0,\ldots,n-2)$ and their multiplicity p_i is 1. We can determine the generators of the solution space in the same way as the proof of Proposition 5.1. Since they are all half-integers, we can take a standard basis at $s=\lambda_i$: $SB^{\lambda_i}=\{\vec{a}_0(\lambda_i),\ldots,\vec{a}_n(\lambda_i)\}$ as defined in Definition 2.3. For each $\vec{a}_j(\lambda_i)\in SB^{\lambda_i}$, we define $o_{ij}:=o(\vec{a}_j(\lambda_i),\lambda_i)$ and $o_{ij}^{\#q_1}:=o_{ij}^{\#-1}=o(\vec{a}_j(\lambda_i)^{\#-1},\lambda_i-1)$. We have only to pick up the vector $\vec{a}_j(\lambda_i)$ satisfying

$$-o_{ij}\leq -o_{ij}^{\#-1}+p_i-1=-o_{ij}^{\#-1}. \tag{78}$$

Since, for $i=-1,0,\ldots,n-2$, we see $-o_{ij}\geq -o_{ij}^{\#-1}$ by Theorem 2.1, Definition 2.3 and Theorem A1, (78) means $-o_{ij}=-o_{ij}^{\#-1}$. Namely, we have to choose $\vec{a}_j(\lambda_i)$ satisfying

$$\begin{aligned}&(\text{ order of } P^{[\vec{a}_j(\lambda_i)^{\#-1},s]}(x) \text{ at } s=\lambda_i-1)\\ =&(\text{ order of } P^{[\vec{a}_j(\lambda_i),s]}(x) \text{ at } s=\lambda_i).\end{aligned}$$

By Theorem 2.1, Definition 2.3 and Theorem A1, we see that this condition is equivalent to that $\vec{a}_j(\lambda_i)\in D_*^{(q)}$ with some $q=0,1,\ldots,\lfloor\frac{i+1}{2}\rfloor$. Here, the $*$ in $D_*^{(j)}$ is substituted $half$, $even$ or odd according as $-\frac{i+1}{2}$ is a strictly half integer, an even integer or an odd integer, respectively. Thus we have the first result.

The second result is easily verified and we omit the proof. ∎

5.4. The equations $P(x,\partial)u(x)=\delta(x)$.

We shall find a $\boldsymbol{G}$-invariant fundamental solution to the homogeneous $\boldsymbol{G}$-invariant differential operator $P(x,\partial)$. First note that the delta function $\delta(x)$ on $\mathrm{Sym}_n(\mathbb{R})$ is given as $\delta(x)=(const.)\times P_{-\lfloor\frac{n+1}{2}\rfloor}^{[\vec{a},-\frac{n+1}{2}]}(x)$ with a vector $\vec{a}\in A(-\frac{n+1}{2},\lfloor\frac{n+1}{2}\rfloor)$ which is non-zero in $\overline{A(-\frac{n+1}{2},\lfloor\frac{n+1}{2}\rfloor)}$. Henceforth, we fix $\vec{a}=\vec{a}_0$ satisfying that

$$\delta(x)=P_{-\lfloor\frac{n+1}{2}\rfloor}^{[\vec{a}_0,-\frac{n+1}{2}]}(x). \tag{79}$$

We are using the procedure of Algorithm 4.1 in the following examples.

Example 5.1. The differential operator $P(x,\partial)=\det(\partial^*)\det(x)$ is an operator of homogeneous degree 0 and hence $k=0$ in Algorithm 4.1. The b_P-function is $b_P(s)=(s+1)(s+\frac{3}{2})\dots(s+\frac{n+1}{2})$. The function $v(x)$ in Algorithm 4.1 is given by (79) and hence $\lambda=-\frac{n+1}{2}$ in Algorithm 4.1. Since $(s-\lambda+k)=(s+\frac{n+1}{2})$, we have $b_P(s)=(s+\frac{n+1}{2})\widetilde{b}(s)$ where $\widetilde{b}(s)=(s+1)(s+\frac{3}{2})\dots(s+\frac{n}{2})$. Then $p=1$ in Algorithm 4.1. We have the Taylor expansion $\widetilde{b}(s)^{-1}=\sum_{i=0}^{\infty}b_i(s+\frac{n-1}{2})^i$ with $b_0=\left((-\frac{n-1}{2})(-\frac{n-2}{2})\cdots(-\frac{1}{2})\right)^{-1}$. Then, by the third step of the procedure in Algorithm 4.1, we have

$$u(x)=\sum_{i+j=-\lfloor\frac{n+1}{2}\rfloor+1}b_iP_j^{[\vec{a}_0,-\frac{n+1}{2}]}(x)=\sum_{i+j=-\lfloor\frac{n-1}{2}\rfloor}b_iP_j^{[\vec{a}_0,-\frac{n+1}{2}]}(x).$$

Since $i\geq 0$ and $P_j^{[\vec{a}_0,-\frac{n+1}{2}]}(x)=0$ except for $j\geq-\lfloor\frac{n+1}{2}\rfloor$, we have

$$u(x)=b_0\times P_{-\lfloor\frac{n-1}{2}\rfloor}^{[\vec{a}_0,-\frac{n+1}{2}]}(x)+b_1\times P_{-\lfloor\frac{n+1}{2}\rfloor}^{[\vec{a}_0,-\frac{n+1}{2}]}(x).$$

This is a $\boldsymbol{G}$-invariant fundamental solution to $P(x,\partial)=\det(\partial^*)\det(x)$. In this case, $b_1\times P_{-\lfloor\frac{n+1}{2}\rfloor}^{[\vec{a}_0,-\frac{n+1}{2}]}(x)$ is not necessary since it is annihilated by $P(x,\partial)$.

Next we consider the differential operator $P(x,\partial)=\det(x)\det(\partial^*)$. It is an operator of homogeneous degree 0 and hence $k=0$. The b_P-function is $b_P(s)=s(s+\frac{1}{2})\dots(s+\frac{n-1}{2})$. The function $v(x)$ is given by (79) and hence $\lambda=-\frac{n+1}{2}$. Since $(s-\lambda+k)=(s+\frac{n+1}{2})$, we have $b_P(s)=\widetilde{b}(s)=s(s+\frac{1}{2})\dots(s+\frac{n-1}{2})$. Then $p=0$. We have the Taylor expansion $\widetilde{b}(s)^{-1}=\sum_{i=0}^{\infty}b_i(s+\frac{n-1}{2})^i$ with $b_0=\left((-\frac{n+1}{2})(-\frac{n}{2})\cdots(-1)\right)^{-1}$. Then, by the third step of the procedure, we have $u(x)=\sum_{i+j=-\lfloor\frac{n+1}{2}\rfloor}b_iP_j^{[\vec{a}_0,-\frac{n+1}{2}]}(x)$. Since $i\geq 0$ and $P_j^{[\vec{a}_0,-\frac{n+1}{2}]}(x)=0$ except for $j\geq-\lfloor\frac{n+1}{2}\rfloor$, we have

$$u(x)=\left((-\frac{n+1}{2})(-\frac{n}{2})\cdots(-1)\right)^{-1}\times P_{-\lfloor\frac{n+1}{2}\rfloor}^{[\vec{a}_0,-\frac{n+1}{2}]}(x).$$

This is a $\boldsymbol{G}$-invariant fundamental solution of $P(x,\partial)=\det(x)\det(\partial^*)$.

Example 5.2. We consider the differential operator $P(x,\partial)=\det(x)$. It is an operator of homogeneous degree n and hence $k=1$. The b_P-function is $b_P(s)=1$. The function $v(x)$ is given by (79) and hence $\lambda=-\frac{n+1}{2}$. Since $(s-\lambda+k)=(s+\frac{n+3}{2})$, we have $p=0$ and $\widetilde{b}(s)^{-1}=1$, and hence $b_0=1$ and $b_i=0$ for $i>1$. Then, by the third step of the procedure, we have

$$u(x)=\sum_{i+j=-\lfloor\frac{n+1}{2}\rfloor+0}b_iP_j^{[\vec{a}_0^{\#-1},-\frac{n+1}{2}-1]}(x)=\sum_{i+j=-\lfloor\frac{n+1}{2}\rfloor}b_iP_j^{[\vec{a}_0^{\#-1},-\frac{n+3}{2}]}(x).$$

Since $i\geq 0$ and $P_j^{[\vec{a}_0^{\#1},-\frac{n+3}{2}]}(x)=0$ except for $j\geq-\lfloor\frac{n+1}{2}\rfloor$, we have $u(x)=P_{-\lfloor\frac{n+1}{2}\rfloor}^{[\vec{a}_0^{\#1},-\frac{n+3}{2}]}(x)$. This is a $\boldsymbol{G}$-invariant fundamental solution of $P(x,\partial)=\det(x)$.

Example 5.3. The differential operator $P(x,\partial) = \det(\partial^*)$ is an operator of homogeneous degree $-n$ and hence $k=-1$. The b_P-function is $b_P(s) = s(s+\frac{1}{2})\dots(s+\frac{n-1}{2})$. The function $v(x)$ is given by (79) and hence $\lambda = -\frac{n+1}{2}$. Since $(s-\lambda+k) = (s+\frac{n-1}{2})$, we have $b_P(s) = (s+\frac{n-1}{2})\widetilde{b}(s)$ where $\widetilde{b}(s) = s(s+\frac{1}{2})\dots(s+\frac{n-2}{2})$. Then $p=1$. We have the Taylor expansion $\widetilde{b}(s)^{-1} = \sum_{i=0}^{\infty} b_i(s+\frac{n-1}{2})^i$ with $b_0 = \left((-\frac{n-1}{2})(-\frac{n-2}{2})\cdots(-\frac{1}{2})\right)^{-1}$. Then, by the third step of the procedure, we have

$$u(x) = \sum_{i+j=-\lfloor\frac{n+1}{2}\rfloor+1} b_i P_j^{[\vec{a}_0^{\#1}, -\frac{n+1}{2}+1]}(x) = \sum_{i+j=-\lfloor\frac{n-1}{2}\rfloor} b_i P_j^{[\vec{a}_0^{\#1}, -\frac{n-1}{2}]}(x).$$

Since $i \geq 0$ and $P_j^{[\vec{a}_0^{\#1}, -\frac{n-1}{2}]}(x) = 0$ except for $j \geq -\lfloor\frac{n-1}{2}\rfloor$, we have

$$u(x) = \left((-\frac{n-1}{2})(-\frac{n-2}{2})\cdots(-\frac{1}{2})\right)^{-1} \times P_{-\lfloor\frac{n-1}{2}\rfloor}^{[\vec{a}_0^{\#1}, -\frac{n-1}{2}]}(x).$$

This is a $\boldsymbol{G}$-invariant fundamental solution to $P(x,\partial) = \det(\partial^*)$.

A Some preliminary results.

In this section we state the main results of the author's paper [19], which play crucial roles in this paper. We shall give the statements of the theorems used in this paper for the reader's convenience without proof.

A1. The exact order of complex power functions.

Using the vectors $\boldsymbol{d}^{(k)}[s_0]$ defined in (30), we can determine the exact orders of poles of $P^{[\vec{a},s]}(x)$.

Theorem A1. (Exact orders of poles) *The exact orders of poles of $P^{[\vec{a},s]}(x)$ are computed by the following algorithm.*

1. *At $s = -\frac{2m+1}{2}(m=1,2,\dots)$, let the coefficient vectors $\boldsymbol{d}^{(k)}[-\frac{2m+1}{2}]$ be as defined in Definition 2.1. The exact pole order $P^{[\vec{a},s]}(x)$ at $s=-\frac{2m+1}{2}(m=1,2,\dots)$ is given in terms of the coefficient vector $\boldsymbol{d}^{(2k)}[-\frac{2m+1}{2}]$.*
   - (a) *If $1 \leq m \leq \frac{n}{2}$, then $P^{[\vec{a},s]}(x)$ has a possible pole of order not larger than m.*
     - *If $\langle \boldsymbol{d}^{(2)}[-\frac{2m+1}{2}], \vec{a}\rangle = 0$, then $P^{[\vec{a},s]}(x)$ is holomorphic, and the converse is true.*
     - *Suppose integers p is in $1 \leq p < m$. If $\langle \boldsymbol{d}^{(2p+2)}[-\frac{2m+1}{2}], \vec{a}\rangle = 0$ and $\langle \boldsymbol{d}^{(2p)}[-\frac{2m+1}{2}], \vec{a}\rangle \neq 0$, then $P^{[\vec{a},s]}(x)$ has a pole of order p, and the converse is true.*
     - *Lastly, if $\langle \boldsymbol{d}^{(2m)}[-\frac{2m+1}{2}], \vec{a}\rangle \neq 0$, then $P^{[\vec{a},s]}(x)$ has a pole of order m, and the converse is true.*
   - (b) *If $m > \frac{n}{2}$, then $P^{[\vec{a},s]}(x)$ has a possible pole of order not larger than $n' := \lfloor\frac{n}{2}\rfloor$*
     - *If $\langle \boldsymbol{d}^{(2)}[-\frac{2m+1}{2}], \vec{a}\rangle = 0$, then $P^{[\vec{a},s]}(x)$ is holomorphic, and the converse is true.*

- *Suppose integers p is in $1 \leq p < n'$. If $\langle \boldsymbol{d}^{(2p+2)}[-\frac{2m+1}{2}], \vec{a} \rangle = 0$ and $\langle \boldsymbol{d}^{(2p)}[-\frac{2m+1}{2}], \vec{a} \rangle \neq 0$, then $P^{[\vec{a},s]}(x)$ has a pole of order p, and the converse is true.*
- *Lastly, $P^{[\vec{a},s]}(x)$ has a pole of order n' if $\langle \boldsymbol{d}^{(n-1)}[-\frac{2m+1}{2}], \vec{a} \rangle \neq 0$ (when n is odd) or $\langle \boldsymbol{d}^{(n)}[-\frac{2m+1}{2}], \vec{a} \rangle \neq 0$ (when n is even), and the converse is true.*

2. *At $s = -m (m = 1, 2, \dots)$, the coefficient vectors $\boldsymbol{d}^{(k)}[-m]$ are defined in Definition 2.1 with $\epsilon[-m] = (-1)^{-m+1}$. We obtain the exact order at $s = -m (m = 1, 2, \dots)$ in terms of the coefficient vectors $\boldsymbol{d}^{(2k+1)}[-m]$.*

   (a) *If $1 \leq m \leq \frac{n}{2}$, then $P^{[\vec{a},s]}(x)$ has a possible pole of order not larger than m.*
   - *If $\langle \boldsymbol{d}^{(1)}[-m], \vec{a} \rangle = 0$, then $P^{[\vec{a},s]}(x)$ is holomorphic, and the converse is true.*
   - *Suppose integers p is in $1 \leq p < m$. If $\langle \boldsymbol{d}^{(2p+1)}[-m], \vec{a} \rangle = 0$ and $\langle \boldsymbol{d}^{(2p-1)}[-m], \vec{a} \rangle \neq 0$, then $P^{[\vec{a},s]}(x)$ has a pole of order p, and the converse is true.*
   - *Lastly, if $\langle \boldsymbol{d}^{(2m-1)}[-m], \vec{a} \rangle \neq 0$, then $P^{[\vec{a},s]}(x)$ has a pole of order m, and the converse is true.*

   (b) *If $m > \frac{n}{2}$, then $P^{[\vec{a},s]}(x)$ has a possible pole of order not larger than $n' := \lfloor \frac{n+1}{2} \rfloor$*
   - *If $\langle \boldsymbol{d}^{(1)}[-m], \vec{a} \rangle = 0$, then $P^{[\vec{a},s]}(x)$ is holomorphic, and the converse is true.*
   - *Suppose integers p is in $1 \leq p < n'$. If $\langle \boldsymbol{d}^{(2p+1)}[-m], \vec{a} \rangle = 0$ and $\langle \boldsymbol{d}^{(2p-1)}[-m], \vec{a} \rangle \neq 0$, then $P^{[\vec{a},s]}(x)$ has a pole of order p, and the converse is true.*
   - *Lastly, $P^{[\vec{a},s]}(x)$ has a pole of order n' if $\langle \boldsymbol{d}^{(n)}[-m], \vec{a} \rangle \neq 0$ (when n is odd) or $\langle \boldsymbol{d}^{(n-1)}[-m], \vec{a} \rangle \neq 0$ (when n is even), and the converse is true.*

A2. The exact support of complex power functions.

The exact support of $P^{[\vec{a},s]}(x)$ is given by the following theorem.

Theorem A2. (Support of the singular invariant hyperfunctions) *Let q be a positive integer. Suppose that $P^{[\vec{a},s]}(x)$ has a pole of order p at $s = -\frac{q+1}{2}$ Let*

$$P^{[\vec{a},s]}(x) = \sum_{w=-p}^{\infty} P_w^{[\vec{a},-\frac{q+1}{2}]}(x)(s + \frac{q+1}{2})^w \tag{80}$$

be the Laurent expansion of $P^{[\vec{a},s]}(x)$ at $s = -\frac{q+1}{2}$. The support of the Laurent expansion coefficients $P_w^{[\vec{a},-\frac{q+1}{2}]}(x)$ is contained in $\boldsymbol{S}$ if $w < 0$.

1. *Let q be an even positive integer. Then the support of $P_w^{[\vec{a},-\frac{q+1}{2}]}(x)$ for $w = -1, -2, \dots, -p$ is contained in the closure $\overline{S_{-2w}}$. More precisely, it is given by*

$$\mathbf{Supp}(P_w^{[\vec{a},-\frac{q+1}{2}]}(x)) = \overline{(\bigcup_{j \in \{0 \leq j \leq n+2w \ | \ \langle d_j^{(-2w)}[-\frac{q+1}{2}], \vec{a} \rangle \neq 0\}} \boldsymbol{S}_{-2w}^{j}}). \tag{81}$$

2. *Let q be an odd positive integer. Then the support of $P_w^{[\vec{a},-\frac{q+1}{2}]}(x)$ for $w = -1,-2,\dots,-p$ is contained in the closure $\overline{\boldsymbol{S}_{-2w-1}}$. More precisely, it is given by*

$$\mathbf{Supp}(P_w^{[\vec{a},-\frac{q+1}{2}]}(x)) = \overline{\Big(\bigcup_{j\in\{0\le j\le n+2w+1 \ | \ \langle d_j^{(-2w-1)}[-\frac{q+1}{2}],\vec{a}\rangle\neq 0\}} \boldsymbol{S}^j_{-2w-1}\Big)}. \qquad (82)$$

Here, $\mathbf{Supp}(-)$ *denotes the support of the hyperfunction in* $(-)$*.*

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Received October 1, 2002
and in final form April 23, 2003