Journal of Lie Theory
Volume **14** (2004) 1–9

Weil Representations of SL∗(2,A) for a Locally Profinite Ring A with Involution

Roberto Johnson and José Pantoja *

Communicated by D. Poguntke

Abstract. We construct, via a complex $G-$bundle space, a Weil representation for the group $G = SL_*(2, \mathbf{A})$, where $(\mathbf{A}, *)$ is a locally profinite ring with involution. The construction is obtained using maximal isotropic lattices and Heisenberg groups.

1. Preliminaries.

Let $(\mathbf{A}, *)$ be a locally profinite ring with involution , i.e. a unitary locally compact and totally disconnected ring with an involutive anti-automorphism $a \longrightarrow a^*,\ a \in \mathbf{A}$. Let $Z_s(\mathbf{A})$ be the subring of central symmetric elements of $\mathbf{A}$.

We define the group $GL_*(2, \mathbf{A})$ of matrices $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a,\ b,\ c,\ d \in \mathbf{A}$, such that:

1. $ab^* = ba^*, \qquad cd^* = dc^*$
2. $a^*c = c^*a, \qquad b^*d = d^*b$
3. $ad^* - bc^* = a^*d - c^*b$ is an invertible central symmetric element of $\mathbf{A}$, i.e. an element of $Z_s(\mathbf{A})^\times$.

We set $\det_*(g) = ad^* - bc^* = a^*d - c^*b$; then

$$g^{-1} = [\det_*(g)]^{-1} \begin{pmatrix} d^* & -b^* \\ -c^* & a^* \end{pmatrix}$$

We observe that the function $\det_* : GL_*(2, \mathbf{A}) \longrightarrow Z_s(\mathbf{A})^\times$ is an epimorphism so that $G = SL_*(2, \mathbf{A}) = \operatorname{Ker} \det_*$ is a normal subgroup of $GL_*(2, \mathbf{A})$.

* Both authors have been partially supported by FONDECYT grant 1990029, PICS (CNRS-CONICYT) and Universidad Católica de Valparaíso

ISSN 0949–5932 / $2.50

In what follows we will assume that $Z_s(\mathbf{A}) = \mathbf{F}$ is a p-adic field. We denote by $O_{\mathbf{F}}$ the ring of integers of $\mathbf{F}$, $P_{\mathbf{F}}$ is the maximal ideal of $O_{\mathbf{F}}$, ϖ is a generator of $P_{\mathbf{F}}$ and $k_{\mathbf{F}}$ is the residual field of $\mathbf{F}$ which has q elements.

Some such rings are: $\mathbf{A} = M_n(\mathbf{F})$, $\mathbf{F}$ a p-adic field, with $*$ the transposition; $\mathbf{A} = \mathbf{K}$ a separable quadratic extension of $\mathbf{F}$, $\mathbf{F}$ as above with $*$ the non trivial Galois element; $\mathbf{A} = \bigwedge^0 V \oplus \bigwedge^1 V \oplus \bigwedge^2 V$ where V is a two dimensional vector space over a p-adic field $\mathbf{F}$ with basis (e_1, e_2) and $*$ is given by the basis transposition (e_1, e_2) to (e_2, e_1).

2. General Setting

Let H be a locally profinite group and Γ a subgroup of $\mathrm{Aut}(H)$. Let (π, V) be an irreducible smooth (complex) representation of H such that $\pi^\gamma \simeq \pi$ ($\pi^\gamma = \pi \circ \gamma$) for every γ in Γ.

If $\gamma \in \Gamma$ then there exists $T_\gamma \in \mathrm{Aut}_{\mathbf{C}}(V)$ such that $T_\gamma \pi(x) = \pi\gamma(x) T_\gamma$ for every $x \in H$.

Set G be the semidirect product of Γ and H. For (γ, h) in G we define $\widetilde{\pi}(\gamma, h)$ in $\mathrm{Aut}_{\mathbf{C}}(V)$ by

$$\widetilde{\pi}(\gamma, h) = T_\gamma \pi(h).$$

Proposition 2.1. *The endomorphism $\widetilde{\pi}$, defined above, is a projective extension of π to G.*

Proof. We want to prove that $T_{\gamma\delta}^{-1} T_\gamma T_\delta$ is a scalar.

Since $T_\gamma T_\delta \pi(x) = T_\gamma \pi(\delta(x)) T_\delta = \pi(\gamma\delta(x)) T_\gamma T_\delta$ and $T_{\gamma\delta}\pi(x) = \pi(\gamma\delta(x)) T_{\gamma\delta}$ then

$$T_{\gamma\delta}^{-1} T_\gamma T_\delta \pi(x) = \pi(x) T_{\gamma\delta}^{-1} T_\gamma T_\delta$$

It follows, by Schur's Lemma, that $T_{\gamma\delta}^{-1} T_\gamma T_\delta = \sigma(\gamma, \delta) id_V$, for a cocycle σ.

We compute now $\widetilde{\pi}(\gamma, h)\widetilde{\pi}(\delta, k)$. We have

$$\widetilde{\pi}(\gamma, h)\widetilde{\pi}(\delta, k) = \sigma(\gamma, \delta) T_{\gamma\delta} \pi(\delta^{-1}(h))\pi(k).$$

Since $\widetilde{\pi}((\gamma, h)(\delta, k)) = \widetilde{\pi}(\gamma\delta, \delta^{-1}(h)k) = T_{\gamma\delta}\pi(\delta^{-1}(h)k)$ we get

$$\widetilde{\pi}(\gamma, h)\widetilde{\pi}(\delta, k) = \sigma(\gamma, \delta)\widetilde{\pi}((\gamma, h)(\delta, k)).$$

Therefore $\widetilde{\pi}$ is a projective representation of G with cocycle σ.

We recall now the definition of compact induction, c-Ind, as we will use it: Let L be a an open subgroup of H, compact modulo the centre of H, and let (ρ, W) be a smooth representation of L. Let V denote the space of compactly supported modulo the centre of H functions $f : H \to W$ with the property $f(lh) = \rho(l) f(h), l \in L, h \in H$. The group acts on this space by right translation of functions; the implied representation is smooth. We will assume now that $(\pi, V) = c - \mathrm{Ind}_L^H \rho$, where L is an open, compact modulo the centre, subgroup of H and ρ is a one dimensional representation of L.

We assume also that $\rho^\gamma = \rho$ on $L^\gamma \cap L$, where $L^\gamma = \gamma(L)$ and $\rho^\gamma(y) = \rho(\gamma^{-1}(y))$ with $y \in L^\gamma$. We can define, similarly,

$$(\pi_\gamma, V_\gamma) = c - \mathrm{Ind}_{L^\gamma}^H \rho^\gamma.$$

Let S_γ be a non zero intertwining operator from (π, V) to $(\pi_{\gamma^{-1}}, V_{\gamma^{-1}})$. So S_γ is an isomorphism between π and $\pi_{\gamma^{-1}}$ when π (and then $\pi_{\gamma^{-1}}$) is irreducible. Then $S_\gamma \pi(x) = \pi_{\gamma^{-1}}(x) S_\gamma$.

We define now $I_\gamma : V_{\gamma^{-1}} \longrightarrow V$ by $(I_\gamma(f))(x) = f(\gamma^{-1}(x))$. The operator I_γ is well defined and intertwining, in fact, $I_\gamma(f(lx)) = \rho(l)f(x)$ and $I_\gamma\pi_{\gamma^{-1}}(x) = \pi(\gamma(x))I_\gamma$. On the other hand, we have that $I_\gamma S_\gamma : V \longrightarrow V$ is an intertwining operator since $I_\gamma S_\gamma \pi(x) = \pi(\gamma(x))I_\gamma S_\gamma$. Let us define $T_\gamma = I_\gamma S_\gamma$. We want to compute the cocycle σ. In order to do this we look first at I_γ on V_δ, Since $\gamma^{-1}(h) \in \delta(L)$ implies that $h \in \gamma\delta(L)$, we have $(I_\gamma f)(hx) = f(\gamma^{-1}(h)\gamma^{-1}(x))$.
We can define $I_{\gamma,\delta} : V_{\gamma^{-1}\delta} \longrightarrow V_\delta$ by $(I_{\gamma,\delta}f)(x) = f(\gamma^{-1}x)$, and $S_{\delta,\gamma} : V_{\gamma^{-1}} \longrightarrow V_{\gamma^{-1}\delta^{-1}}$ by $S_{\delta,\gamma} = I^{-1}_{\gamma,\delta^{-1}} S_\delta I_{\gamma,1}$ a computation shows that $S_{\delta,\gamma}$ is an intertwining map.

Since the operators $S_{\delta,\gamma} \circ S_\gamma : V \longrightarrow V_{\gamma^{-1}\delta^{-1}}$ and $S_{\delta\gamma} : V \longrightarrow V_{\gamma^{-1}\delta^{-1}}$ are both intertwining, the irreductibility of V implies that they differ on a scalar i.e. $S_{\delta,\gamma} \circ S_\gamma = kS_{\delta\gamma}$.

Lemma 2.2. *The intertwining operators defined above satisfy the equation* $I_\delta \circ I_{\gamma,\delta^{-1}} = I_{\delta\gamma}$.
Proof. Straightforward. ■

We finally show that $k = \sigma(\delta, \gamma)$: Since $S_{\delta,\gamma} \circ S_\gamma = kS_{\delta\gamma}$ we have $I^{-1}_{\gamma,\delta^{-1}} S_\delta I_\gamma S_\gamma = kS_{\delta\gamma}$. So $S_\delta I_\gamma S_\gamma = kI_{\gamma,\delta^{-1}} S_{\delta\gamma}$ and then $I_\delta S_\delta I_\gamma S_\gamma = kI_\delta I_{\gamma\delta^{-1}} S_{\delta\gamma}$. Using Lemma 2.2 we get $I_\delta S_\delta I_\gamma S_\gamma = kI_{\delta\gamma} S_{\delta\gamma}$ i.e. $T_\delta T_\gamma = kT_{\delta\gamma}$.

3. Heisenberg Construction

Given a $\mathbf{F}-$vector space W we can define $H = \mathbf{F} \oplus W$ which has a structure of group with respect to

$$(a, w) \cdot (a', w') = (a + a' + B(w, w'), w + w')$$

where $B : W \times W \longrightarrow \mathbf{F}$ is a non-degenerate alternating form.
If M is any subgroup of W we write $\widetilde{M} = \mathbf{F} \oplus M$, which is a subgroup of H.
Definition 3.1. Let M be an any subset of W. We define $M^* = \{w \in W \mid B(m, w) \in O_{\mathbf{F}}\, \forall m \in M\}$ and $M^\perp = \{w \in W \mid B(m, w) = 0\, \forall m \in M\}$.

Observation 3.2.
a) If M is a $\mathbf{F}-$subspace of W, then $M^* = M^\perp$. In fact, the inclusion $M^\perp \subset M^*$ is obvious. On the other hand, since $\alpha B(m, w) = B(\alpha m, w)$ we have that $w \in M^*$ implies that $\alpha B(m, w) \in O_{\mathbf{F}}$ $\forall m \in M$ $\forall \alpha \in \mathbf{F}$, so $B(m, w) = 0$.
b) Another fact that we will use later, is the following

$$[(a, w), (a', w')] = (2B(w, w'), 0).$$

Let $\mathfrak{L}$ be a maximal isotropic lattice i.e. $\mathfrak{L}$ is compact and open and $\mathfrak{L}^* = \mathfrak{L}$. Set $\widetilde{\mathfrak{L}} = \mathbf{F} \oplus \mathfrak{L}$ and let ψ be a character of $\mathbf{F}$ of conductor $O_{\mathbf{F}}$. Define $\psi_{\mathfrak{L}}$ on $\widetilde{\mathfrak{L}}$ by $\psi_{\mathfrak{L}}(a, l) = \psi(a)$ for $a \in \mathbf{F}$.

Proposition 3.3. *With the above notation and assuming that* $2 \in O^\times_{\mathbf{F}}$ *we have:*
a) $\psi_{\mathfrak{L}}$ *is a character of* $\widetilde{\mathfrak{L}}$.
b) *If we define* $\mathrm{Int}_H(\psi_{\mathfrak{L}}) = \{h \in H \mid \mathrm{Hom}_{\widetilde{\mathfrak{L}} \cap \widetilde{\mathfrak{L}}^h} \left(\psi_{\mathfrak{L}}, \psi^h_{\mathfrak{L}}\right) \neq 0\}$, *where* $\widetilde{\mathfrak{L}}^h = h\widetilde{\mathfrak{L}}h^{-1}$

and $\psi_{\mathfrak{L}}^h(x)=\psi_{\mathfrak{L}}(h^{-1}xh)$ *for any* $x\in\widetilde{\mathfrak{L}}^h$*, then* $\mathrm{Int}_H(\psi_{\mathfrak{L}})$ *is equal to* $\widetilde{\mathfrak{L}}$.

Proof. a) $\psi_{\mathfrak{L}}((a,w)(a',w'))=\psi_{\mathfrak{L}}(a+a'+B(w,w'),w+w')$, since $\mathfrak{L}$ is a maximal isotropic lattice, $B(w,w')\in O_{\mathbf{F}}$. Then $\psi_{\mathfrak{L}}((a,w)(a',w'))=\psi(a)\psi(a')=\psi_{\mathfrak{L}}(a,w)\psi_{\mathfrak{L}}(a',w')$.

b) If $(a,w)\in H$ Since $(-a,-w)(\alpha,y)(a,w)=(\alpha+2B(y,w),y)$ and $\widetilde{\mathfrak{L}}\lhd H$, we have $\psi_{\mathfrak{L}}^{(a,w)}=\psi_{\mathfrak{L}}$ on $\widetilde{\mathfrak{L}}\cap(-a,-w)\widetilde{\mathfrak{L}}(a,w)=\widetilde{\mathfrak{L}}$ if and only if $2B(y,w)\in O_{\mathbf{F}}$ $\forall y\in\mathfrak{L}$ if and only if $B(y,w)\in O_{\mathbf{F}}$ $\forall y\in\mathfrak{L}$ (given that $2\in O_{\mathbf{F}}^{\times}$) and this is the case if and only if $w\in\mathfrak{L}$. ■

Now let $\Pi_{\mathfrak{L}}=c-\mathrm{Ind}_{\widetilde{\mathfrak{L}}}^{H}\psi_{\mathfrak{L}}$ be the compact induction of the character $\psi_{\mathfrak{L}}$ from $\widetilde{\mathfrak{L}}$ to H as defined in Section 2.

Proposition 3.4. *The representation* $\Pi_{\mathfrak{L}}$ *defined above is an irreducible admissible supercuspidal representation of* H*.*

Proof. The representation $\Pi_{\mathfrak{L}}$ is the Heisenberg representation realized in the lattice model (see [5], Chapter 2). Stone-von Neumann theorem implies that $\Pi_{\mathfrak{L}}$ is a smooth irreducible (thus admissible) representation. Then, using theorem 1 of [2], we get that it is supercuspidal.

Now let Γ be the subgroup of $\mathrm{Aut}(H)$ of all automorphism $\gamma:H\longrightarrow H$ such that $\gamma_{|\mathbf{F}}=id_{\mathbf{F}}$ and $\gamma_{|W}$ is a symplectic linear automorphism. The subgroup Γ acts transitively over the set Θ of all maximal isotropic lattices in W, by $\mathfrak{L}^{\gamma}=\gamma(\mathfrak{L})$ ($\gamma\in\Gamma$ and $\mathfrak{L}\in\Theta$). Furthermore $\psi_{\mathfrak{L}}^{\gamma}=\psi_{\mathfrak{L}}$ on $\mathfrak{L}^{\gamma}\cap\mathfrak{L}$ where $\psi_{\mathfrak{L}}^{\gamma}(y)=\psi_{\mathfrak{L}}(\gamma^{-1}(y))$, $\forall y\in\mathfrak{L}^{\gamma}$.

On the other hand, by Proposition 3.4 , $(\Pi_{\mathfrak{L}},V_{\mathfrak{L}})=c-\mathrm{Ind}_{\widetilde{\mathfrak{L}}}^{H}\psi_{\mathfrak{L}}$ is an irreducible admissible supercuspidal representation of H, where $V_{\mathfrak{L}}=\{f:H\to\mathbf{C}\,|f(lx)=\psi_{\mathfrak{L}}(l)f(x),\ \forall l\in\mathfrak{L},\forall x\in H,\ f$ compactly supported modulo the centre of $H\}$. So, we can define $(\Pi_{\mathfrak{L}^{\gamma}},V_{\mathfrak{L}^{\gamma}})=c-\mathrm{Ind}_{\widetilde{\mathfrak{L}^{\gamma}}}^{H}\psi_{\mathfrak{L}}^{\gamma}$, where $V_{\mathfrak{L}^{\gamma}}=\{f:H\to\mathbf{C}\,|f(lx)=\psi_{\mathfrak{L}}^{\gamma}(l)f(x),\ \forall l\in\mathfrak{L}^{\gamma}\}$ and now the general set-up of Section 2 applies.

Define the function $\tau_{\gamma}:H\to\mathbf{C}$ by

$$\tau_{\gamma}(xy)=\begin{cases}\psi_{\mathfrak{L}}^{\gamma}(x)\psi_{\mathfrak{L}}(y) & \text{if}\quad x\in\widetilde{\mathfrak{L}^{\gamma}},y\in\widetilde{\mathfrak{L}}\\[2ex] 0 & \text{otherwise}\end{cases}$$

Note that τ_{γ} is well defined since $\psi_{\mathfrak{L}}^{\gamma}=\psi_{\mathfrak{L}}$ on $\widetilde{\mathfrak{L}^{\gamma}}\cap\widetilde{\mathfrak{L}}$. For any f in the space of $\Pi_{\mathfrak{L}}$ we can define $\Upsilon_{\gamma}(f):H\longrightarrow\mathbf{C}$ by

$$\Upsilon_{\gamma}(f)(x)=\int\limits_{H/\mathbf{F}}\tau_{\gamma}(y)f(y^{-1}x)dy$$

for an appropriate Haar measure on $W=H/\mathbf{F}$. We can observe that $\Upsilon_{\gamma}:V_{\mathfrak{L}}\longrightarrow V_{\mathfrak{L}^{\gamma^{-1}}}$ is a non zero intertwining operator and since $\Pi_{\mathfrak{L}}$ is irreducible (and also $\Pi_{\mathfrak{L}^{\gamma^{-1}}}$), we have that Υ_{γ} is an isomorphism.

We define now $I_{\gamma}:V_{\mathfrak{L}^{\gamma^{-1}}}\longrightarrow V_{\mathfrak{L}}$ by $(I_{\gamma}f)(x)=f(\gamma^{-1}(x))$ and so we have, as in section 2, that $T_{\gamma}=I_{\gamma}\Upsilon_{\gamma}$ is an intertwining of $V_{\mathfrak{L}}$ which verify

$$T_{\delta}\circ T_{\gamma}=\sigma(\delta,\gamma)T_{\delta\gamma}.$$

4. Lagrangians

Let S be a left $\mathbf{A}$- module whose $\mathbf{F}$-dimension is n. We note that S is a right $\mathbf{A}$- module with $sa = a^* s$, $a \in \mathbf{A}$, $s \in S$.

Let $b : S \times S \longrightarrow \mathbf{F}$ be a non degenerate bilinear symmetric form such that

$$b(x_1 a, x_2) = b(x_1, a x_2) \qquad (a \in \mathbf{A};\, x_1, x_2 \in S).$$

We set now $W = S \oplus S$ and define $B : W \times W \longrightarrow \mathbf{F}$ by $B(x,y) = b(x_{1,} y_2) - b(y_1, x_2)$ for $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in W. Observe that B is a non degenerate alternating form and we can define $M^{\perp} = \{w \in W \mid B(w,m) = 0,\ \forall m \in M\}$ for any $O_{\mathbf{F}}-$submodule M of W. The following properties are straightforward.

If either M is an $\mathbf{F}-$subspace or if M is a compact open $O_{\mathbf{F}}-$submodule of W (an $O_{\mathbf{F}}-$lattice in W) then $(M^{\perp})^{\perp} = M$.

If M, N are any $O_{\mathbf{F}}-$submodules which satisfy $(M^{\perp})^{\perp} = M$ and $(N^{\perp})^{\perp} = N$ then $(M \cap N)^{\perp} = M^{\perp} + N^{\perp}$.

We call an $O_{\mathbf{F}}-$submodule M of W isotropic if M is an $O_{\mathbf{F}}-$submodule of $M^{\perp}$. We say that M is maximal isotropic if $M = M^{\perp}$

Fixing an additive (continuous) character ψ of $\mathbf{F}$ of conductor $O_{\mathbf{F}}$, we can define the function $\chi : W \times W \longrightarrow \mathbf{T}$, where $\mathbf{T}$ is the group of complex numbers of module one, by

$$\chi(x,y) = (\psi \circ B)(x,y) \qquad ((x,y) \in W \times W).$$

which is a symplectic bicharacter.

Definition 4.1. Let M be a subset of W. The orthogonal component M^* of M is the set of $y \in W$ such that $\chi(x,y) = 1$, for every $x \in M$.

Observation 4.2. In the case where M is a $\mathbf{F}-$subspace of W we have that M^* is also a $\mathbf{F}-$subspace of W and $M^{\perp} = M^*$.

Definition 4.3. Let L be a $\mathbf{F}-$subspace of W such that $L^{\perp} = L$. L is called a Lagrangian subspace of W.

Observation 4.4. If M is an $\mathbf{F}-$subspace of W then M is maximal isotropic if and only if M is Lagrangian.

Lemma 4.5. *Let W and χ be as above. Let L and L' be two Lagrangian subspaces of W. Then there exists a symplectic basis $\{w_1, w_2, \ldots, w_n, w'_1, w'_2, \ldots, w'_n\}$ of W, i.e.*

1. $\chi(w_j, w'_j) \neq 1, \quad j = 1, \ldots, n$
2. $\chi(w_i, w_j) = \chi(w'_i, w'_j) = 1$ *for every* i, j.
3. $\chi(w_i, w'_j) = 1$ *for every* $i \neq j$

such that:

$$L = \mathbf{F}w_1 \oplus \mathbf{F}w_2 \oplus \cdots \oplus \mathbf{F}w_k \oplus \mathbf{F}w_{k+1} \oplus \cdots \oplus \mathbf{F}w_n$$
$$L' = \mathbf{F}w'_1 \oplus \mathbf{F}w'_2 \oplus \cdots \oplus \mathbf{F}w'_k \oplus \mathbf{F}w_{k+1} \oplus \cdots \oplus \mathbf{F}w_n$$

Proof. See Lemma 1.4.6. in [4]. ■

Corollary 4.6. *Given a Lagrangian L, there exists a Lagrangian L' such that $W = L \oplus L'$.*

Proof. If $L = \langle w_1, w_2, \ldots, w_n \rangle$, then L is a proper subspace of $\langle w_2, \ldots, w_n \rangle^{\perp}$. We consider an element $v_1 \in \langle w_2, \ldots, w_n \rangle^{\perp} - L$. Then $\chi(w_1, v_1) \neq 1$. Now we can pick an element $v_2 \in \langle w_1, w_3, w_{4,} \ldots, w_n, v_1 \rangle^{\perp} - \langle w_1, w_2, w_3, \ldots, w_n, v_1 \rangle^{\perp}$, and so $\chi(w_2, v_2) \neq 1$. By induction we have $\{w_1, v_1\}$, $\{w_2, v_2\}, \ldots, \{w_n, v_n\}$ such that $\chi(w_i, v_i) \neq 1$, $i = 1, \ldots, n$; $\chi(w_i, w_j) = \chi(v_i, v_j) = 1$ for every i, j and $\chi(w_i, v_j) = 1$ for every $i \neq j$. Hence $L' = \langle v_1, v_2, \ldots, v_n \rangle$ is such that $W = L \oplus L'$. ■

Corollary 4.7. *There exists a maximal isotropic $O_{\mathbf{F}}-$lattice $\mathfrak{L}$ in W.* ■

Let L be a Lagrangian in W and define ψ_L on $\widetilde{L} = \mathbf{F} \oplus L$ as above. Let $\Pi_L = c - \mathrm{Ind}_{\widetilde{L}}^{H} \psi_L$ and consider the group $H = \mathbf{F} \oplus W$. Let $\widetilde{\mathfrak{L}} = \mathbf{F} \oplus \mathfrak{L}$, $\mathfrak{L}$ a maximal isotropic $O_{\mathbf{F}}-$lattice in W.

Now we can define the function $\rho : H \longrightarrow \mathbf{C}$ by

$$\rho(z) = \begin{cases} \psi_L(x)\psi_{\mathfrak{L}}(y) & \text{if} \quad z = x \cdot y,\ x \in \widetilde{L}, y \in \widetilde{\mathfrak{L}} \\ \\ 0 & \text{if} \quad z \notin \widetilde{L}\widetilde{\mathfrak{L}} = \mathbf{F} \oplus (L + \mathfrak{L}) \end{cases}$$

Note that ρ is well defined since $\psi_L = \psi_{\mathfrak{L}}$ on $\widetilde{L} \cap \widetilde{\mathfrak{L}}$ and $\widetilde{L} \cap \widetilde{\mathfrak{L}} = \mathbf{F} \oplus (L \cap \mathfrak{L})$.

For any f in the space of $\Pi_{\mathfrak{L}}$ we can define $S(f) : H \longrightarrow \mathbf{C}$ by

$$S(f)(x) = \int_{H/\mathbf{F}} \rho(y) f(y^{-1}x) dy.$$

Given an $O_{\mathbf{F}}-$lattice $\mathfrak{M}$ submodule of $\mathfrak{L}$, we define the function

$$\rho_{\mathfrak{M}}(z) = \begin{cases} \psi_L(x)\psi_{\mathfrak{L}}(y) & \text{if} \quad z = x \cdot y,\ x \in \widetilde{L}, y \in \widetilde{\mathfrak{M}} \\ \\ 0 & \text{if} \quad z \notin \widetilde{L}\widetilde{\mathfrak{M}} = \mathbf{F} \oplus (L + \mathfrak{M}). \end{cases}$$

Proposition 4.8. *The map S defined above is an $H-$isomorphism from $\Pi_{\mathfrak{L}}$ to Π_L*

Proof. Let f_0 be the function, in the space of $\Pi_{\mathfrak{L}}$, defined by

$$f_0(z) = \begin{cases} \psi_{\mathfrak{L}}(z) & \text{if} \quad z \in \widetilde{\mathfrak{L}} \\ \\ 0 & \text{otherwise} \end{cases}$$

and

$$f_{\mathfrak{M}}(z) = \begin{cases} \psi_{\mathfrak{M}}(z) & \text{if} \quad z \in \widetilde{\mathfrak{M}} \\ \\ 0 & \text{otherwise} \end{cases}$$

A computation shows $S(f_0) = \rho$ and $S(f_{\mathfrak{M}}) = \rho_{\mathfrak{M}}$.

Since S is different from 0 and $\Pi_{\mathfrak{L}}$ is irreducible, S is injective.

We will prove now that S is onto. To this end we prove that the space of Π_L is equal to $\langle\{\rho_{\mathfrak{M}} \mid \mathfrak{M} \subset \mathfrak{L}\}\rangle$. First, $S(f_{\mathfrak{M}}) = \rho_{\mathfrak{M}}$ so $\langle\{\rho_{\mathfrak{M}} \mid \mathfrak{M} \subset \mathfrak{L}\}\rangle \subset \Pi_L$. On the other hand any f in Π_L has support compact modulo $\widetilde{\mathfrak{L}}$ and it is locally constant. From this, it can be seen that any function f is a linear combination of $\rho_{\mathfrak{M}}$ for different lattices $\mathfrak{M} \subset \mathfrak{L}$. Hence we can conclude that S is an isomorphism.

Define now $T : \Pi_L \longrightarrow \Pi_{\mathfrak{L}}$ by

$$T(f)(x) = \int_{H/\mathbf{F}} \theta(y) f(y^{-1}x)dy$$

where θ is given by

$$\theta(z) = \begin{cases} \psi_{\mathfrak{L}}(x)\psi_L(y) & \text{if} \quad z = x \cdot y, \; x \in \mathfrak{L}, y \in L \\ \\ 0 & \text{if} \quad z \notin \mathfrak{L}L \end{cases}$$

We have that $T \neq 0$ and by Schur´s Lemma [1] [3], $TS = cI$, so $TS(f_0) = cf_0$ which implies $c = 1$, and finally

$$TS = I_{\Pi_{\mathfrak{L}}}$$

5. Connections over $SL_*(2, A)$.

The group $G = SL_*(2, \mathbf{A})$ acts naturally by matrix multiplication on W by fixing the bicharacter χ,

$$\chi(gx, gy) = \chi(x, y) \qquad (x, y \in W)$$

We define a complex $G-$bundle space $\mathfrak{F} = (\mathfrak{E}, p, \Gamma, \tau)$ by:

1. $\Gamma = \{L \mid L \text{ a Lagrangian of } W\}$
2. Fix a Haar measure dw on W and dw_L on a Lagrangian L such that $d\overline{w_L}$ is the unique Haar measure on W/L which verify that $dw = d\overline{w_L}dw_L$.

   For each Lagrangian L we consider the set $\mathfrak{E}_L$ of all functions $f : W \longrightarrow \mathbf{C}$ which are locally constant, compactly supported modulo L, and such that $f(w + l) = \chi(w, l)f(w)$ for every $w \in W$ and $l \in L$.

   We set

$$\mathfrak{E} = \dot{\bigcup_{L \in \mathfrak{d}}} \mathfrak{E}_L$$

and we define an inner product on each $\mathfrak{E}_L$ by

$$\langle f, h\rangle = \int_{W/L} f(w)\overline{h(w)}dw_L \qquad (f, h \in \mathfrak{E}_L)$$

3. Let $p : \mathfrak{E} \longrightarrow \Gamma$ be the canonical projection which sends each f of $\mathfrak{E}_L$ to L.

4. The group G acts on $\mathfrak{E}$ and Γ by

$$[\tau_g(f)](w) = f(g^{-1}w) \qquad\qquad (f \in \mathfrak{E},\ g \in G,\ w \in W)$$

and by

$$\tau_g(L) = gL \qquad\qquad (L \in \mathfrak{d},\ g \in G)$$

respectively.

Lemma 5.1. *Let L be a Lagrangian subspace of W. Let M be an $O_{\mathbf{F}}-$lattice of W. We set*

$$g_M(w) = \begin{cases} \overline{\chi(x,c)} & \textit{if} \quad w = x + c \in L + M \\ \\ 0 & \qquad\qquad \textit{otherwise.} \end{cases}$$

Then, the set $\{g_M \mid M$ be an $O_{\mathbf{F}}-$lattice of $L\}$ span $\mathfrak{E}_L$ as a $\mathbf{C}$-vector space.

Proof. For each f in $\mathfrak{E}_L$ we can pick an $O_{\mathbf{F}}-$lattice M such that $\text{Supp}(f) = L + M$. We use that f is locally constant and M is compact, to write f as linear combination of $g_{M'}$ as above.

Let L and L' be Lagrangians included in a fixed maximal $O_{\mathbf{F}}-$lattice $\mathfrak{L}$ in W. As we have seen, there are two isomorphisms, namely $S_L : \Pi_{\mathfrak{L}} \longrightarrow \Pi_L$ and $S_{L'} : \Pi_{\mathfrak{L}} \longrightarrow \Pi_{L'}$ with $T_L : \Pi_L \longrightarrow \Pi_{\mathfrak{L}}$ and $T_{L'} : \Pi_{L'} \longrightarrow \Pi_{\mathfrak{L}}$ as the respective inverses.

We now define isomorphisms $\widetilde{\gamma}_{L',L} : \Pi_L \longrightarrow \Pi_{L'}$, by

$$\widetilde{\gamma}_{L',L} = S_{L'} \circ T_L$$

Let $\Lambda^L : \Pi_L \longrightarrow \mathfrak{E}_L$ be defined by $\Lambda^L(f)(w) = f(0,w)$, for $f \in \Pi_L$ and $w \in W$, and let, $\Omega^L : \mathfrak{E}_L \longrightarrow \Pi_L$ be defined by $\Omega^L(f)(a,w) = \psi(a)f(w)$, for $f \in \mathfrak{E}_L$ and $(a,w) \in \widetilde{L}$. A computation shows that Λ^L and Ω^L are inverse to each other and both are intertwining operators.

We can define now isomorphisms (which we will call connections) $\gamma_{L,L'} : \mathfrak{E}_L \to \mathfrak{E}_{L'}$ by $\gamma_{L,L'} = \Lambda^{L'} \circ \widetilde{\gamma}_{L,L'} \circ \Omega^L$.

Then the diagram

$$\begin{array}{ccc} \Pi_L & \xrightarrow{\widetilde{\gamma}_{L,L'}} & \Pi_{L'} \\ \Omega^L \uparrow & & \downarrow \Lambda^{L'} \\ \mathfrak{E}_L & \xrightarrow{\gamma_{L,L'}} & \mathfrak{E}_{L'} \end{array}$$

is commutative.

We obtain

Theorem 5.2. *The set $\Gamma = \{\gamma_{L',L} \mid L', L \in \mathfrak{d}\}$ is a family of G-equivariant connections over the fiber bundle $\mathfrak{F}$ which verifies, for $L, L', L'' \in \mathfrak{d}$; $f, f' \in \mathfrak{E}_L$; $h \in \mathfrak{E}_{L'}$ $g \in G$ the following properties:*

1. $\gamma_{L,L'} \circ \gamma_{L',L} = \gamma_{L,L} = id_{\mathfrak{E}_L}$

1. $\langle \gamma_{L',L}(f), h\rangle = \langle f, \gamma_{L,L'}(h)\rangle$
2. $\langle \gamma_{L',L}(f), \gamma_{L',L}(f')\rangle = \langle f, f'\rangle$
3. $\gamma_{L,L''} \circ \gamma_{L'',L'} \circ \gamma_{L',L} = S_W(L; L', L'') id_{\mathfrak{E}_L}$
4. *where* $S_W(L; L', L'')$ *is a constant.*
5. $\tau_g \circ \gamma_{L',L} = \gamma_{gL',gL} \circ \tau_g$

∎

Note that $S_W(L; L', L'')$ is the analogous of the Maslov index in [4] and this theorem is comparable with theorem 1.4 in [6].

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Roberto Johnson
Universidad Católica de Valparaíso
Chile
rjohnson@ucv.cl

José Pantoja
Universidad Católica de Valparaíso
Chile
jpantoja@ucv.cl

Received January 25, 2001
and in final form July 16, 2003