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H^4(BK,Z) and Operator Algebras

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Abstract. There is a well-known interpretation of group cohomology in terms of (generalized) group extensions. For a connected semisimple compact Lie group K, we prove that the extensions corresponding to classes in $H^4(BK,\mathbb{Z})$ can be interpreted in terms of automorphisms of a pair consisting of a type II_1 von Neumann algebra and a Cartan subalgebra.

0. Introduction

Throughout this paper K will denote a compact Lie group, BK will denote a classifying space for K, and if K acts continuously on an abelian separable metrizable topological group A, then $H_K^*(A)$ will denote the corresponding Borel group cohomology as in [8] (see also [9], [11]).

There are natural isomorphisms

$$H^4(BK,\mathbb{Z}) \cong H_K^4(\mathbb{Z}) \cong H_K^3(\mathbb{T}) \cong H_K^2(U(K,\mathbb{T})/\mathbb{T}), \tag{0.1}$$

where $U(K,\mathbb{T}) = U(L^\infty K)$, the unitary group of the abelian von Neumann algebra $L^\infty K$, equipped with the strong operator topology, and the action of K on $U(K,\mathbb{T})$ is induced by left translation. The first isomorphism depends only upon the fact that the coefficient group $\mathbb{Z}$ is discrete (see [9] or [11]). The second isomorphism follows from dimension shifting, using the short exact sequence $\mathbb{Z} \to \mathbb{R} \to \mathbb{T}$, and depends upon the vanishing of $H_K^*(\mathbb{R})$ (see Theorem 3 of [11], [4], and [1]). The last isomorphism is a consequence of dimension shifting, using the short exact sequence $\mathbb{T} \to U(K,\mathbb{T}) \to U(K,\mathbb{T})/\mathbb{T}$, and depends upon the vanishing of $H_K^*(U(K,\mathbb{T}))$ (see [8]).

The isomorphisms (0.1) imply that given a level $l \in H^4(BK,\mathbb{Z})$, we can identify l with (the isomorphism class of) an abelian noncentral extension E_l,

$$0 \to \mathbb{T} \to U(K,\mathbb{T}) \to E_l \to K \to 0, \tag{0.2}$$

where K acts on $U(K,\mathbb{T})$ via its left action on K (see (4.2) below for an explicit realization of E_l in the case $K = SU(2)$).

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For various reasons the extension (0.2) is reminiscent of

$$0 \to \mathbb{T} \to U(\mathcal{M}) \to Aut(\mathcal{M}) \to Out(\mathcal{M}) \to 0, \tag{0.3}$$

where $\mathcal{M}$ is a von Neumann algebra factor, and $Aut(\mathcal{M})$ acts naturally on $U(\mathcal{M})$, the unitary group of $\mathcal{M}$. This suggests the following

Problem 0.4. Given a level l, does there exist a factor $\mathcal{M}_l$ (preferably finite) such that K is faithfully represented as outer automorphisms of $\mathcal{M}_l$ and the induced extension

$$0 \to \mathbb{T} \to U(\mathcal{M}_l) \to \mathcal{E} \to K \to 0 \tag{0.5}$$

(where $\mathcal{E} \subset Aut(\mathcal{M}_l)$ is the inverse image of K) is equivalent to (0.2) (in particular, K lifts to a group of automorphisms of $\mathcal{M}_l$ if and only if l is trivial)?

The answer is affirmative if $K = \mathbb{Z}_n$, where $\mathcal{M}_l = \mathcal{R}$, the hyperfinite II_1 factor (see $V.6.\epsilon$ of [3]).

In this paper we will prove a weaker result. Recall that a Cartan subalgebra $\mathcal{A}$ of a type II_1 von Neumann algebra $\mathcal{M}$ is a maximal abelian self-adjoint subalgebra such that $N_{U(\mathcal{M})}(\mathcal{A})$, the normalizer of $\mathcal{A}$ in $U(\mathcal{M})$, generates $\mathcal{M}$ (see §3 of [6]). We will use the Feldman-Moore structure theory of pairs $(\mathcal{M}, \mathcal{A})$ ([6]) to prove the following theorem.

Theorem 0.6. *If K is connected and semisimple, then given $l \in H^4(BK, \mathbb{Z})$, there exists a faithful representation of the extension E_l of the form*

$$\begin{array}{ccccccccccc}
0 & \to & \mathbb{T} & \to & U(K,\mathbb{T}) & \to & E_l & \to & K & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathbb{T} & \to & N_{U(\mathcal{M})}(\mathcal{A}) & \to & Aut(\mathcal{M},\mathcal{A}) & \to & Out(\mathcal{M},\mathcal{A}) & \to & 0
\end{array}$$

where $\mathcal{M}$ is a type II_1 factor and $\mathcal{A}$ is a Cartan subalgebra naturally identified with $L^\infty K$.

One motivation for pursuing these questions can be explained in terms of an analogy. We have $H^3(BK, \mathbb{Z}) \cong H^2_K(\mathbb{T})$, the group of central $\mathbb{T}$-extensions of K. In particular there is the $spin^c$ extension

$$0 \to \mathbb{T} \to Spin^c \to SO \to 0, \tag{0.7}$$

which is related to orientation in K-theory and spin geometry (see [7], especially Appendix D).

For connected K, $H^4(BK, \mathbb{Z})$ can be identified (using Chern-Weil theory) with the set of AdK-invariant symmetric forms $\langle \cdot, \cdot \rangle$ satisfying $\langle x, x \rangle \in 2\mathbb{Z}$ whenever $e^x = 1$, $x \in \mathfrak{k}$. It is the parameter space for central $\mathbb{T}$-extensions of LK, the loop group, which have a reciprocity property introduced by Segal. The spin extension in this loop space picture is related to orientation in elliptic cohomology. However, from this point of view what currently constitutes "elliptic geometry" is Witten's loop space Dirac operator, which lacks a proper mathematical definition (see [10] and references there).

The realization of $H^4(BK,\mathbb{Z})$ above suggests the possibility that an appropriate $\mathcal{M}$ plays a role in "elliptic geometry".

This paper is organized as follows. In §1 we recall the analysis in §4 of [6] of the group $Aut(\mathcal{M},\mathcal{A})$, and in terms of this analysis, we show that the questions above have natural cohomological interpretations in terms of a double complex. In §2 we define the double complex and apply spectral sequence techniques to relate the relevant cohomology groups. In §3 we prove (0.6), and add a few comments. In §4 we discuss issues relevant to the more fundamental question (0.4). Finally in §5 we discuss our proposed application to "elliptic geometry".

(0.8) Acknowledgement and Notation. This paper depends heavily upon [5] and [6], and our own contributions are relatively minor. Throughout this paper we will use the terminology and notation established in [5] and [6] (which we will regularly explain in the text).

As in [5] and [6], R will denote an equivalence relation (in our case R will always be defined by the right action of a countable dense subgroup π on K); $H^*(R,\mathbb{T})$ will denote cohomology of the relation R with coefficients in $\mathbb{T}$ (in our case this is isomorphic to $H^*_\pi(U(K,\mathbb{T}))$, the Eilenberg-Maclane cohomology of π, where π acts on the coefficients through its right action on K); C^*, Z^* and B^* will denote groups of cochains, cocycles and coboundaries, respectively; $x\sim y$ denotes a pair $(x,y)\in R$, $x\sim y\sim z$, or $(x,y,z)\in R^2$, denotes a triple such that $x\sim y$ and $y\sim z$, and so on; and δ denotes the differential (see §6 of [5]).

1. Feldman-Moore Theory

Lemma 1.1. *If R is an equivalence relation on a group G which is normalized by the left action of G (i.e. $x\sim y$ if and only if $gx\sim gy$, $\forall g\in G$), then the equivalence class of 1 is a subgroup π, and R is defined by the right action of π on G.*

The proof is trivial.

Throughout the remainder of the paper we suppose that R is an ergodic countable equivalence relation on K, so that the corresponding subgroup π in the Lemma is a countable dense subgroup of K. The left action of K on itself induces a natural map $K\to N(R)$, the normalizer of R. Here $N(R)$ is the group of (classes of) Borel automorphisms θ of K (regarded as a Borel space, not as a group) such that $\theta\times\theta(R)\subset R$; $N(R)$ is much larger than K, which equals the continuous automorphisms in $N(R)$.

Given the relation R and $\sigma\in H^2(R,\mathbb{T})$, Feldman and Moore construct a type II_1 factor $\mathcal{M}(R,\sigma)$ with Cartan subalgebra $\mathcal{A}=L^\infty K$ (see §2 of [6]). The details of this construction are not important for the limited purposes of this paper; we only need to note that because Haar measure is finite, π-invariant and ergodic, $\mathcal{M}(R,\sigma)$ is a type II_1 factor.

According to Theorem 3 of §4 of [6], there is a short exact sequence

$$0\to Z^1(R,\mathbb{T})\to Aut(\mathcal{M}(R,\sigma),\mathcal{A})\to N(R,\sigma)\to 0, \tag{1.2}$$

where $N(R,\sigma)$ denotes the subgroup of $N(R)$ fixing σ.

If we assume that the cohomology class σ is fixed by K, i.e. $K \subset N(R,\sigma)$, then (1.2) induces an extension of K by $Z^1(R,\mathbb{T})$. This induced extension is close in form to (0.2).

Because π is dense in K, we have

$$0 \to \mathbb{T} \to U(K,\mathbb{T}) = C^0(R,\mathbb{T}) \xrightarrow{\delta} B^1(R,\mathbb{T}) \to 0 \tag{1.4}$$

where the differential δ is given by

$$\delta f(x \sim y) = \frac{f(y)}{f(x)} \tag{1.5}$$

(and $U(K,\mathbb{T}) = U(L^\infty(K))$, as before). Thus the heart of the matter is to cut down the kernel of the extension (1.2), restricted to K, from $Z^1(R,\mathbb{T})$ to $B^1(R,\mathbb{T})$. To analyze this question, we need to recall some details concerning (1.2).

We fix a function $s : R^2 \to \mathbb{T}$ representing σ. By Prop 7.7 of [5], we can suppose that s is normalized, meaning that $s(x \sim y \sim z) = 1$ whenever two of the variables are the same. Because σ is fixed by K, there exists a function

$$d : K \to U(R,\mathbb{T}) = C^1(R,\mathbb{T}) : g \to d_g \tag{1.6}$$

such that $\delta d_g = \frac{s}{s^g}$ and $d_1 = 1$; here s^g denotes the transform of s by g, where g acts naturally (from the left) on the domain of s, R^2. We can choose d to be a Borel function, for d is a lift of a well-defined map $\underline{d}$,

$$\begin{array}{ccc} & & C^1(R,\mathbb{T}) \\ & d \nearrow & \downarrow \\ K & \xrightarrow{\underline{d}} & C^1(R,\mathbb{T})/Z^1(R,\mathbb{T}) \end{array}, \tag{1.7}$$

and there exist Borel cross-sections for the projection. Because s is normalized, $d_g(x \sim y)$ is automatically skew-symmetric in x, y (to see this, in the equation $\delta d_g = s s_g^{-1}$, evaluated on the triple $x \sim y \sim z$, take $y = z$). Note that s has values in $\mathbb{T}$, so that we cannot average to obtain an s which is K-invariant.

According to Theorem 2 of [6] there is an isomorphism of $Aut(\mathcal{M},\mathcal{A})$ with a group $G(s)$, where $G(s)$ consists of pairs $(c,\theta) \in Z^1(R,\mathbb{T}) \times N(R,\sigma)$ with multiplication

$$(c_1,\theta_1) * (c_2,\theta_2) = (c_1 c_2^{\theta_1} \frac{d_{\theta_1\theta_2}}{d_{\theta_1} d_{\theta_2}^{\theta_1}}, \theta_1\theta_2) \tag{1.8}$$

(where we have momentarily extended d to $N(R,\sigma)$, as in §4 of [6]). Recall from (1.6) that d has values in $C^1(R,\mathbb{T})$, not necessarily $Z^1(R,\mathbb{T})$, so that the extension (1.8) is not necessarily equivalent to a semidirect product.

We are interested in the restriction of the multiplication (1.8) to $Z^1(R,\mathbb{T}) \times K$, so that θ_1, θ_2 are replaced by $g, h \in K$. From this we see that we can cut down the extension (1.2), restricted to K, to $B^1(R,\mathbb{T})$, i.e. $B^1(K,\mathbb{T})\times K$ is closed under the multiplication in (1.8), if and only if

$$(\delta^K d)_{g,h} = \frac{d_h^g d_g}{d_{gh}} \in B^1(R,\mathbb{T}); \tag{1.9}$$

here δ^K is the differential for the group cohomology of K corresponding to its action on $U(R,\mathbb{T}) = C^1(R,\mathbb{T})$, where K is acting from the left. Assuming that (1.9) holds, we obtain a cocycle having values in $B^1(R,\mathbb{T})$ (again, we remark that this cocycle is not necessarily trivial, because d may not have values in $B^1(R,\mathbb{T})$). Recalling (1.4), the class $E \in H_K^2(U(K,\mathbb{T})/\mathbb{T})$ is represented by a cocycle $C: K\times K \to U(K,\mathbb{T})/\mathbb{T} : g,h \to C_{g,h}$ satisfying

$$\delta C_{g,h}(x\sim y) = \frac{d_{gh}}{d_g d_h^g}(x\sim y) \tag{1.10}$$

(g,h are treated as constants in this equation).

The compatibility conditions satisfied by the triple (C,d,s) are summarized in the following diagram:

$$\begin{array}{cccccc}
 & 2 & C_{g,h}(x) & \stackrel{\delta}{\to} & \frac{d_h^g d_h}{d_{gh}}(x\sim y) & & \\
 & & & & \uparrow \delta^K & & \\
(1.11) & 1 & & & d_g(x\sim y) & \stackrel{\delta}{\to} & \frac{s}{s^g}(x\sim y\sim z) \\
 & & & & & & \uparrow -\delta^K \\
 & 0 & & & & & s(x\sim y\sim z) \\
 & & 0 & & 1 & & 2
\end{array}$$

2. Cohomological Formulation.

In this section we will formalize the discussion in §1 in terms of a double complex. In particular we will see that a triple (C,d,s), satisfying the compatibility conditions (1.11), is a cocycle for the total differential of the complex. Standard spectral sequence techniques will then be used to relate the cohomology for the double complex to the group cohomology for K (the cohomology of the first column), and the cohomology for the relation R (the cohomology of the first row); see [2], especially chapter 3).

We consider the following double complex, $\Omega^{p,q}$:

$$\begin{array}{cccccccc}
 & \uparrow & & \uparrow & & \uparrow & & \\
2 & U(K^2, U(R^0,\mathbb{T})/\mathbb{T}) & \stackrel{\delta}{\to} & U(R\times K^2,\mathbb{T}) & \stackrel{\delta}{\to} & .. & & \\
 & \uparrow \delta^K & & \uparrow \delta^K & & & & \\
1 & U(K, U(R^0,\mathbb{T})/\mathbb{T}) & \stackrel{\delta}{\to} & U(R\times K,\mathbb{T}) & \stackrel{\delta}{\to} & U(R^2\times K,\mathbb{T}) & & \\
 & \uparrow \delta^K & & \uparrow \delta^K & & \uparrow \delta^K & & \\
q=0 & U(K^0, U(R^0,\mathbb{T})/\mathbb{T}) & \stackrel{\delta}{\to} & U(R\times K^0,\mathbb{T}) & \stackrel{\delta}{\to} & U(R^2,\mathbb{T}) & \stackrel{\delta}{\to} & \\
 & p=0 & & 1 & & 2 & &
\end{array}$$

Here

$$(2.2)\qquad \Omega^{p,q}=\left\{\begin{array}{cc} U(K^q,U(R^0,\mathbb{T})/\mathbb{T})=U(R^0,U(K^q,\mathbb{T}))/U(K^q,\mathbb{T}), & p=0\\ U(R^p\times K^q,\mathbb{T}) & p>0\end{array}\right),$$

where $R^0=K$, $R^1=R$, $R^2=\{x\sim y\sim z\in K^3\}$, and so on, as in [5]. The group K acts on $R^p\times K^q$ from the left, and this induces an action on $\Omega^{p,q}$. For fixed q we obtain a horizontal differential $\delta=\delta^R$, and for fixed p we have a vertical differential δ^K, where the coefficient K-module is $U(R^0,\mathbb{T})/\mathbb{T}$ for $p=0$ and $U(R^p,\mathbb{T})$ for $p>0$. Since δ^R originates from the right action of π on K, whereas δ^K originates from the left action of K on K, these differentials commute.

The group $\Omega^{p,q}$ is defined in terms of functions on $R^p\times K^q$. As a mnemonic device, we will use the letters g,h when referring to K degrees of freedom, and letters x,y,z when referring to the relation R.

(2.3) Remarks. (a) A key point is that the pth column is exact for $p>0$. For we have a K-equivariant isomorphism

$$(2.4)\qquad U(R^p,\mathbb{T})=U(K,U(\pi^p,\mathbb{T}))$$

induced by the correspondence

$$(2.5)\qquad x_0\sim x_1\sim ...\sim x_p\leftrightarrow(x_0,x_0^{-1}x_1,..,x_{p-1}^{-1}x_p).$$

This is the basis of the natural identification $H^*(R,\mathbb{T})=H^*_\pi(U(K,\mathbb{T}))$; see Theorem 5 of [5]. Hence the coefficient module is an induced module, and the cohomology vanishes in positive degree.

(b) There is an easily forgotten assymetry between the roles of K and π in the above complex. On the one hand, in defining the differential δ^K, K is acting on all the variables, $g,h,x..$ (because in the pth column, $U(R^p,\mathbb{T})$ is regarded as a K-module). On the other hand, in defining the differential δ, the variables g,h are treated as constant and only the variables $x,y,..$ are taken into account.

Let $D=\delta+(-1)^p\delta^K$ (in additive notation) denote the total differential for the double complex Ω. Then we have natural maps

$$(2.6)\qquad H^*_D(\Omega)\overset{pr_K}{\to}H^*_K(U(K,\mathbb{T})/\mathbb{T}),\quad H^*_D(\Omega)\overset{pr_R}{\to}H^*(R,\mathbb{T}).$$

For example in degree 2,

$$(2.7)\qquad [C_{g,h}(x)]\leftarrow[C_{g,h}(x),d_g(x\sim y),s(s\sim y\sim z)]\to[s(x\sim y\sim z)],$$

where $[\cdot]$ denotes a cohomology class, the left arrow is pr_K, and the right arrow is pr_R. The compatibility conditions (1.11) are equivalent to saying that the triple (C,d,s) is D-closed. Thus proving that pr_K is surjective in degree 2 is equivalent to realizing E_l as a group of automorphisms of a pair $(\mathcal{M}(R,\sigma).L^\infty K)$, by what we said in §1.

To analyze the map pr_K, we consider the spectral sequence corresponding to the filtration obtained by deleting successive rows, $\Omega_n = \{\Omega^{p,q} : n \leq q\}$. We have $(E_0, d_0) = (\Omega, \delta^K)$, hence by (a) of (2.3), E_1 is given by

$$\begin{array}{cccccc}
2 & H^2_K(U(K,\mathbb{T})/\mathbb{T}) & & & & \\
1 & H^1_K(U(K,\mathbb{T})/\mathbb{T}) & & & & \\
0 & & U(R,\mathbb{T})^K & U(R^2,\mathbb{T})^K & U(R^3,\mathbb{T})^K & \ldots \\
 & 0 & 1 & 2 & 3 &
\end{array} \tag{2.8}$$

The differential $d_1 = \delta$, and using (2.4) to calculate $U(R^p, \mathbb{T})^K$, we can also compute E_2:

$$\begin{array}{cccccc}
2 & H^2_K(U(K,\mathbb{T})/\mathbb{T}) & & & & \\
1 & H^1_K(U(K,\mathbb{T})/\mathbb{T}) & & & & \\
0 & & H^1_\pi(\mathbb{T}) & H^2_\pi(\mathbb{T}) & H^3_\pi(\mathbb{T}) & \ldots \\
 & 0 & 1 & 2 & 3 &
\end{array} \tag{2.9}$$

(In calculating the cohomology of the first row of (2.8), we used (2.4) to calculate the K-invariants, and the identification of the differential for R, δ, and the differential for the group cohomology of π; see (2.5) and Theorem 5 of [5]).

It is straightforward to check that the nontrivial higher differentials d_r are identified with the natural maps $H^{r-1}_K(U(K,\mathbb{T})/\mathbb{T}) \to H^r_K(\mathbb{T}) \to H^r_\pi(\mathbb{T})$.

We now focus on degree 2. We can summarize what we have learned to this point, as follows.

Proposition 2.10. *In degree 2, we have*

$$Im(pr_K) = E^{0,2}_\infty = ker(H^2_K(U(K,\mathbb{T})/\mathbb{T}) \stackrel{d_3}{\to} H^3_\pi(\mathbb{T}))$$

$E^{1,1}_\infty = 0$, *and*

$$ker(pr_K) = E^{2,0}_\infty = coker(H^1_K(U(K,\mathbb{T})/\mathbb{T}) \stackrel{d_2}{\to} H^2_\pi(\mathbb{T})).$$

To analyze the map pr_R, we consider the spectral sequence corresponding to the filtration obtained by deleting successive columns, $\Omega'_n = \{\Omega^{p,q} : n \leq p\}$. We have $(E'_0, d'_0) = (\Omega, \delta)$, hence E'_1 is given by

$$\begin{array}{cccccc}
2 & & H^1(R, U(K^2,\mathbb{T})) & .. & & \\
1 & & H^1(R, U(K,\mathbb{T})) & H^2(R, U(K,\mathbb{T})) & .. & \\
0 & & H^1(R,\mathbb{T}) & H^2(R,\mathbb{T}) & H^3(R,\mathbb{T}) & \ldots \\
 & p=0 & 1 & 2 & 3 &
\end{array} \tag{2.11}$$

The differential $d_1' = \delta^K$. Now π does not act on the K variables when we calculate δ. This implies that

$$H^1(R, U(K, \mathbb{T})) = U(K, H^1(R, \mathbb{T})), \tag{2.12}$$

by Theorem 1 of part IV of [8]. The point is that we again obtain an induced module, and we can compute E_2':

$$\begin{array}{ccccccc}
 & 2 & & H_K^2(H^1(R,\mathbb{T})) & H_K^2(H^2(R,\mathbb{T})) & & \\
(2.13) & 1 & & H_K^1(H^1(R,\mathbb{T})) & H_K^1(H^2(R,\mathbb{T})) & & \\
 & 0 & & H^1(R,\mathbb{T})^K & H^2(R,\mathbb{T})^K & H^3(R,\mathbb{T})^K & \dots \\
 & & p=0 & 1 & 2 & 3 &
\end{array}$$

Proposition 2.14. *We have*

$$Im(pr_R) = E_\infty'^{2,0} = ker(H^2(R,\mathbb{T})^K \overset{d_3'}{\to} H_K^2(H^1(R,\mathbb{T})))$$

and

$$ker(pr_R) = E_\infty'^{0,2} = H_K^1(H^1(R,\mathbb{T})).$$

To summarize, we have two exact sequences:

$$H_K^1(U(K,\mathbb{T})/\mathbb{T}) \overset{d_3}{\to} H_\pi^2(\mathbb{T}) \to H_D^2(\Omega) \overset{pr_K}{\to} H_K^2(U(K,\mathbb{T})/\mathbb{T}) \overset{d_4}{\to} H_\pi^3(\mathbb{T}) \tag{2.15}$$

and

$$H_K^1(H^1(R,\mathbb{T})) \to H_D^2(\Omega) \overset{pr_R}{\to} H^2(R,\mathbb{T})^K \overset{d_3'}{\to} H_K^2(H^1(R,\mathbb{T})). \tag{2.16}$$

(2.17) Example. Suppose that $K = SO(3)$ and π is isomorphic to F_2, the free group on 2 generators. In this case we are asserting that

$$H_K^2(U(K,\mathbb{T})/\mathbb{T}) = H_D^2(\Omega) = H_K^1(H^1(R,\mathbb{T})). \tag{2.18}$$

To check this directly, note that

$$H^1(R,\mathbb{T}) = H_{F_2}^1(U(K,\mathbb{T})), \tag{2.19}$$

the moduli space of crossed-homomorphisms. Because F_2 is free, this group fits into a short exact sequence

$$0 \to U(K,\mathbb{T})/\mathbb{T} \to U(K,\mathbb{T}) \times U(K,\mathbb{T}) \to H^1(R,\mathbb{T}) \to 0. \tag{2.20}$$

The corresponding long exact sequence in cohomology implies (2.18).

3. Consequences.

Theorem 3.1. *If K is connected and semisimple, then given $l \in H^4(BK, \mathbb{Z})$, there exists a faithful representation of E_l of the form*

$$\begin{array}{ccccccccc} 0 & \rightarrow & \mathbb{T} & \rightarrow & U(K, \mathbb{T}) & \rightarrow & E & \rightarrow & K & \rightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & \mathbb{T} & \rightarrow & N_{U(\mathcal{M})}(\mathcal{A}) & \rightarrow & Aut(\mathcal{M}, \mathcal{A}) & \rightarrow & Out(\mathcal{M}, \mathcal{A}) & \rightarrow & 0 \end{array}$$

where $\mathcal{M} = \mathcal{M}(R, \sigma)$, for some $\sigma \in H^2(R, \mathbb{T})$, and R corresponds to any π with $H^3_\pi(\mathbb{T}) = 0$ and $\pi \cap C(K) = 0$, e.g. π isomorphic to a free group, or $\pi_1\Sigma$, the fundamental group of a closed surface of genus > 1.

Proof. For a countable dense π satisfying $H^3_\pi(\mathbb{T}) = 0$, Proposition (2.10) implies that pr_K in (2.7) (the left map) is surjective. This implies the existence of a triple (C, d, s) satisfying the conditions in (1.11). Thus by the discussion in §1, with $\sigma = [s] \in H^2(R, \mathbb{T})$, we obtain a representation of the extension E_l (with l corresponding to [3] via the isomorphisms in (0.1)).

The only thing that remains is to show that the map $K \to N(R)$ induces an injection $K \to Out(\mathcal{M}, \mathcal{A})$. This is equivalent to proving that $K \cap G(R) = \{1\}$, where $G(R)$ is the group of (equivalence classes of) Borel automorphisms θ of K (as a Borel space, not as a group), such that $x \sim \theta(x)$, $\forall x \in K$ (see §4 of [6]).

Suppose $g \in K$ and L_g, left translation by g, equals $\theta \in G(R)$. There exists a Borel function $\alpha : K \to \pi$ such that $\theta(x) = x\alpha(x)$, $\forall x \in K$. We enumerate the elements of π by α_1, α_2,.., and we let E_i denote the inverse image of α_i. Then we have $gx = x\alpha_i$, or $g = x\alpha_i x^{-1}$ for all $x \in \bar{E}_i$, the closure of E_i. One of the sets $\bar{E}_i$ must have nonempty interior. Since $\bar{E}_i$ has nonempty interior, α_i is necessarily central and $g = \alpha_i \in \pi \cap C(K)$, hence $g = 1$. ■

(3.2) Remarks. (a) In the case of a torus, given $l \in H^4(BK, \mathbb{Z})$, there exists a representation $E \to Aut(\mathcal{R}, \mathcal{A})$, where $\mathcal{R}$ is the unique hyperfinite type II_1 factor (and $\mathcal{A}$ is unique up to automorphism). Here we can take $\pi = \mathbb{Z}^n$ i.e. n independent irrational rotations. In this case, because the left and right actions are the same, the intersection of K (acting from the left) and $G(R)$ contains π, so that we do not have a faithful map of T into $Out(\mathcal{M}, \mathcal{A})$.

(b) In the case of a torus, π is abelian, hence $\mathcal{M}$ is necessarily hyperfinite (see Proposition 4.4 of [5]). This is amazing (but well-known to experts in von Neumann algebra theory), given the variety of possible π's (for example for $\mathbb{T}$, $\pi = \mathbb{Z}^n$, $Tor(\mathbb{T}) = \mathbb{Q}/\mathbb{Z}$, $SO(2, \mathbb{Q})$ (which has torsion subgroup $\{\pm 1, \pm i\}$ and quotient $\mathbb{Z}^\infty$), and so on.

(c) In the semisimple case, one suspects that $\mathcal{M}$ cannot be hyperfinite. For by a result of Sakai (see (4.5) of [5]), $\mathcal{M}$ hyperfinite implies that π is amenable. It would be surprising if a countable dense subgroup of a simple compact Lie group could be amenable (it obviously cannot be solvable). Beyond this result of Sakai, apparently not much is known about how the isomorphism type of $\mathcal{M}(R,\sigma)$ depends on (R,σ) (but for a conjecture in a related context, see Problem 1 of $V.B.\epsilon$ of [3]).

4. On $H^2(R,\mathbb{T})$ for Surface Groups $\pi=\pi_1\Sigma$.

At this point it is natural to turn to question 0.4 in the introduction. A cohomological variation of the question is, given K, can we find π such that there is a correspondence

$$H^4(BK,\mathbb{Z})\leftrightarrow H^2(R,\mathbb{T})^K : l\leftrightarrow \sigma_l=[s], \tag{4.1}$$

with $H^2_D(\Omega)$ as intermediary (see (2.15) and (2.16)), where K acts as a group of automorphisms of $\mathcal{M}(R,\sigma_l)$ if and only if σ_l is trivial? We are not able to answer this question for lack of feeling for $H^2(R,\mathbb{T})$ (which might vanish in some generality, for all we know). The point of this section is to at least lay out the issues in a relatively explicit way.

Suppose that K is connected and $\mathfrak{k}$ is simple. In this case $H^4(BK,\mathbb{Z})=\mathbb{Z}\langle\cdot,\cdot\rangle$, where $\langle\cdot,\cdot\rangle$ is normalized so that $\langle\cdot,[\cdot,\cdot]\rangle\in(\Lambda^3\mathfrak{k})^{AdK}$ corresponds to a generator for $H^3(K,\mathbb{Z})$. We write $[c_3]$ for the corresponding class in $H^3_K(\mathbb{T})$.

(4.2) Example. If $K=SU(2,\mathbb{C})=S^3$, then $\langle\cdot,[\cdot,\cdot]\rangle$ is the normalized volume element of S^3. The corresponding generator of $H^3_K(\mathbb{T})$ is represented by the Cheeger-Simons Borel cocycle

$$c_3(a,b,c,d)=exp2\pi iVol(a,b,c,d) \tag{4.3}$$

(in the homogeneous picture of group cohomology), where $Vol(a,b,c,d)$ denotes the oriented volume of the oriented geodesic simplex of S^3 with vertices a,b,c,d (see §8 of [4]). Thus the multiplication of the extension E in (0.3), in terms of a Borel identification $E=U(K,\mathbb{T})/\mathbb{T}\times K$, is given by

$$(\lambda,g)\cdot(\eta,h)=(\lambda(\cdot)\eta^g(\cdot)e^{2\pi iVol(1,g,h,\cdot)},gh). \tag{4.4}$$

The general strategy for obtaining a class in $H^2(R,\mathbb{T})^K$ corresponding to $[c_3]$ is the following. Let K_r denote a copy of K acting on $U=U(K,\mathbb{T})$ (or $U/\mathbb{T}$) via its right action on K. The results of §2, imply that we have a diagram

$$\begin{tikzcd}
H^2_\pi(\mathbb{T}) \arrow[dr] & & H^1_K(H^1(R,\mathbb{T})) \arrow[dl] & \\
& H^2_D(\Omega) \arrow[dl] \arrow[dr] & & \\
H^2(R,\mathbb{T}) \arrow[d, equal] & H^2_{K_r}(U/\mathbb{T}) \arrow[r, phantom, "\cong"] \arrow[d] & H^2_K(U/\mathbb{T}) \arrow[r, phantom, "\cong"] \arrow[dr] & H^3_K(\mathbb{T}) \arrow[d] \\
H^2_\pi(U) \arrow[r] & H^2_\pi(U/\mathbb{T}) \arrow[rr] & & H^3_\pi(\mathbb{T})
\end{tikzcd} \tag{4.5}$$

where the southeast (respectively, southwest) diagonal maps are derived from (2.15) (respectively, (2.16)), the bottom row is part of the long exact sequence corresponding to the short exact sequence of coefficients $\mathbb{T} \to U \to U/\mathbb{T}$, $U = U(K, \mathbb{T})$, and the vertical arrows are induced by restriction (for the vertical isomorphism, see Theorem 5 of [5]). We want to know that the restriction of c_3 to π will be zero. This will imply that the corresponding extension of π by $U/\mathbb{T}$ comes from an extension by U, and this opens the possibility of obtaining a nontrivial deformation class $\sigma \in H^2(R, \mathbb{T}) \cong H^2_\pi(U)$.

For π isomorphic to F_n or $\pi_1\Sigma$, $H^3_\pi(\mathbb{T})$ is automatically zero. Note that for our example (4.2), this translates into a strange statement about the possibility of computing volumes (modulo $\mathbb{Z}$) of simplices with vertices in π as an alternating sum of a function applied to three vertices at a time.

For $\pi = F_n$, $H^2(R, \mathbb{T}) = H^2_\pi(U(K, \mathbb{T}))$ also vanishes, because BF_n is one dimensional. Hence we do not obtain a nontrivial deformation class in this case.

We consider the next simplest case. Fix a compact connected oriented surface Σ of genus $g > 1$, and a faithful homomorphism $\pi_1\Sigma \hookrightarrow K$ with dense image. There is an entire moduli space of such homomorphisms (which depends upon the semisimplicity of K). We employ the abbreviations $\pi = image(\pi_1\Sigma \to K)$, $U = U(K, \mathbb{T})$, and $PU = U(K, \mathbb{T})/\mathbb{T}$.

To calculate $H^2(R, \mathbb{T}) = H^2_\pi(U)$, we use the standard presentation

$$0 \to N = \langle r = [\alpha_1, \beta_1]..[\alpha_g, \beta_g]\rangle \to F_{2g} \to \pi \to 0, \tag{4.6}$$

where the free group F_{2g} will act on anything that π acts on.

Suppose that π acts on A (an abelian group), and we have an extension

$$0 \to A \to E \to \pi \to 0. \tag{4.7}$$

This will be the quotient of a trivial extension

$$\begin{array}{ccccc} A & \to & F \propto A & \to & F \\ \| & & \downarrow & & \downarrow \\ A & \to & E & & \pi \end{array}, \tag{4.8}$$

because we can always find a cross-section for the pullback of E to an extension of the free group F. The restriction of the cross-section, a map $N \to A$, is F-equivariant and hence determined by its value, say a, on r. We will write $E = F \propto_{r \sim a} A$ to denote this realization of E.

Now suppose that we have two such realizations of extensions

$$E_i = F \propto_{r \sim a_i} A \tag{4.9}$$

These will be equivalent if and only if we can find a crossed homomorphism $\phi : F \to A$ such that $\phi(r)a_1 = a_2$, where the equivalence is given by the map

$$E_1 \to E_2 : [f, a] \to [f, \phi(f)a]. \tag{4.10}$$

The value $\phi(r)$ is 1 if ϕ is exact. This proves the following lemma.

Lemma 4.11. *We have*

$$H^1_F(A) \to A \to H^2_\pi(A) \to 0,$$

where

$$\phi \in H^1_F(A) \to \phi(r) \in A$$

and

$$a \in A \to E_a = F \propto_{r \sim a} A.$$

In general, given a presentation $N \to F \to \pi$*,*

$$H^1_F(A) \to Hom_F(N, A) \to H^2_\pi(A) \to 0$$

where $\theta \in Hom_F(N, A)$ *maps to the extension*

$$F \propto_\theta A$$

and $[\phi] \in H^1_F(A)$ *maps to* $\phi|_N$*.*

(4.12) Remark. If π acts trivially on A, then $\phi \in H^1_F(A)$ is an ordinary homomorphism, hence vanishes on r, so that in this case $H^2_\pi(A) = A$, as it should.

Consider the linear case $A = U(K, \mathbb{R})$. A $\phi \in Z^1_F(U)$ is completely determined by its values $\phi_{\alpha_1}(k), .., \phi_{\beta_g}(k)$ (functions of $k \in K$), and conversely given arbitrary functions $\phi_{\alpha_1}, ..\phi_{\beta_g} \in U$, there is a corresponding ϕ. Using the crossed-homomorphism property (in particular $\phi_{\gamma^{-1}}(k)\phi_\gamma(k)^{\gamma^{-1}} = 1$), we see

$$\phi_r(k) = \phi_{\alpha_1}(k) - \phi_{\alpha_1}(k\alpha_1\beta_1\alpha_1^{-1}) + \phi_{\beta_1}(k\alpha_1) - \phi_{\beta_1}(k[\alpha_1, \beta_1]) + ..$$

$$+\phi_{\beta_g}(k\alpha_1..\alpha_g) - \phi_{\beta_g}(kr) \tag{4.13}$$

Here we have abused notation, in that in the arguments of these functions we have written α_1 and so on for the elements of K corresponding to these symbols; in particular the r in the last factor $= 1$.

Note that it is tempting to assert that $\int \phi_r = 0$, with respect to Haar measure. But this does not make sense because functions in $U(K, \mathbb{R})$ are generally nonintegrable. It seems more likely that the following is true.

Conjecture 4.14. The operator $L : U(K, \mathbb{R})^{2g} \to U(K, \mathbb{R})$ defined by (4.13) is surjective. Consequently for a relation defined by $\pi_1\Sigma$, $H^2(R, \mathbb{R})$ and $H^2(R, \mathbb{T})$ vanish.

Although we are assuming $g > 1$ and π is dense in K, we note that (4.14) is true for $g = 1$ and any faithful embedding $\pi_1\Sigma \to K$. The first statement is Theorem 7 of [5], and together with (4.11) this implies the second statement, in the case $g = 1$.

5. Globalization.

Suppose that X is a compact oriented Riemannian manifold. We can consider the oriented orthonormal frame bundle

$$\begin{array}{ccc} P_{SO} & \leftarrow & SO \\ \downarrow & & \\ X & & \end{array} \tag{5.1}$$

In spin geometry the basic topological question is whether P_{SO} is derived from a spin bundle $P_{Spin} \to X$. This is analyzed in the following way (see Appendix A of [7]). Corresponding to the spin extension there is an exact sequence of nonabelian sheaves on X,

$$\underline{\mathbb{Z}_2} \to \underline{Spin} \to \underline{SO}. \tag{5.2}$$

There is a corresponding exact sequence of pointed cohomology spaces,

$$.. \to H^1(X, \mathbb{Z}_2) \to H^1(X, \underline{Spin}) \to H^1(X, \underline{SO}) \overset{w_2}{\to} H^2(X, \mathbb{Z}_2). \tag{5.3}$$

Thus there exists a spin structure iff $w_2(P_{SO}) = 0$, and the possibilities are acted upon simply and transitively by $H^1(X, \mathbb{Z}_2)$.

Let $K = Spin(n)$, and for simplicity of exposition we suppose that $n \neq 4$. Suppose that $w_2(X) = 0$ and fix a particular spin structure. We fix a level $l \in H^4(BK, \mathbb{Z}) = \mathbb{Z}\frac{1}{2}p_1$, we let $E = E_l$ denote the corresponding group extension, as in (0.2), and we ask whether P_{Spin} is derived from a bundle with structure group E. In this context $\mathbb{Z}_2$ is replaced by $U(K, \mathbb{T})/\mathbb{T}$. This is not a Lie group, so that we must think of sheaves in this context in a purely topological way. As in the spin context, we have an exact sequence

(5.4)

$$.. \to H^1(X, \underline{U(K, \mathbb{T})/\mathbb{T}}) \to H^1(X, \underline{E}) \to H^1(X, \underline{K}) \overset{\Delta}{\to} H^2(X, \underline{U(K, \mathbb{T})/\mathbb{T}})$$

and we have to understand the meaning of the connecting map Δ.

We first note that $U(K, \mathbb{T})$ is contractible for K connected. To see this we can replace K by I, the unit interval. A contraction is given by the map

$$U(I, \mathbb{T}) \times I \to U(I, \mathbb{T}) : g, s \to g_s, \quad g_s(x) = \left\{ \begin{array}{cc} 1, & x \leq s \\ g(x), & s < x \end{array} \right\}. \tag{5.5}$$

It follows that

$$H^*(X, \underline{U(K, \mathbb{T})}) = 0 \tag{5.6}$$

in positive degrees. Thus the connecting map in (6.4) can be identified with a map to $H^3(X, \underline{\mathbb{T}})$. Since $H^*(X, \underline{\mathbb{R}}) = 0$ in positive degrees, (5.4) is equivalent to an exact sequence

$$.. \to H^3(X, \mathbb{Z}) \to H^1(X, \underline{E}) \to H^1(X, \underline{K}) \overset{\Delta}{\to} H^4(X, \mathbb{Z}) \tag{5.7}$$

Proposition 5.8. *The connecting map satisfies* $\Delta(P_{Spin}) = \frac{l}{2}p_1(X)$. *Thus there exists a covering of bundles*

$$P_E \to P_{Spin} \to X$$

if and only if $\frac{l}{2}p_1(X) = 0$ *in* $H^4(X, \mathbb{Z})$, *and the possible coverings are acted upon simply and transitively by* $H^3(X, \mathbb{Z})$.

Proof. We formulate the question in slightly different terms. Because $U(K, \mathbb{T})$ is contractible, $U(K, \mathbb{T})/\mathbb{T}$ has the homotopy type $K(\mathbb{Z}, 2)$, and $B[U(K, \mathbb{T})/\mathbb{T}]$ has the homotopy type $K(\mathbb{Z}, 3)$. Also the cohomology class l is identified with a classifying map $BK \to K(\mathbb{Z}, 4)$. Hence we are given a diagram

$$(5.9) \qquad \begin{array}{ccccccc} K(\mathbb{Z},3) & \to & BE & \to & BK & \xrightarrow{l} & K(\mathbb{Z},4) \\ & & & & \uparrow P & \nearrow \Delta & \\ & & & & X & & \end{array}$$

where $P = P_{Spin}$, and the question is whether there is a lift $P_E : X \to BE$. The first statement follows immediately from this, and the second statement follows from either (5.7) or (5.9). ∎

In this result the specific form of the extension E, especially the specific model for the kernel, $U = U(K, \mathbb{T})$, is unimportant. The important point is simply that the quotient $U/\mathbb{T}$ has homotopy type $B\mathbb{T} = K(\mathbb{Z}, 2)$ (see the development of group cohomology in [9], which emphasizes this flexibility). However, the specific model for E involving $U(K, \mathbb{T})$ seems quite natural, and it is clearly related to von Neumann algebras in an interesting way. Although von Neumann algebras are not normally associated with topology, (5.8), together with 3.1, suggests that it might be interesting to consider a Grothendieck type group of bundles of von Neumann algebras of type II_1 (or the associated algebras of unbounded operators, in a given representation), possibly in relation to elliptic cohomology.

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