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Koszul Duality of Translation—and Zuckerman Functors

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Abstract. We review Koszul duality in representation theory of category $\mathcal{O}$, especially we give a new presentation of the Koszul duality functor. Combining this with work of Backelin, we show that the translation and Zuckerman functors are Koszul dual to each other, thus verifying a conjecture of Bernstein, Frenkel and Khovanov. Finally we use Koszul duality to give a short proof of the Enright-Shelton equivalence.

1. Introduction.

The definite exposition of Koszul duality in representation theory is the paper of Beilinson, Ginzburg and Soergel [5]. The main theme of that paper is that the category $\mathcal{O}$ of Bernstein, Gelfand and Gelfand is the module category of a Koszul ring, from which many properties of $\mathcal{O}$ can be deduced very elegantly. They also consider singular as well as parabolic versions of $\mathcal{O}$ and show that they are are equivalent to each other under the Koszul duality functor. Two of the main tools to obtain these results were the Zuckerman functors and the translation functors.

In a recent paper by Bernstein, Frenkel and Khovanov [6] these functors are studied on certain simultaneously singular and parabolic subcategories of $\mathcal{O}$, in type A. They show, partially conjecture, that each one of them – in combination with an equivalence of certain categories due to Enright-Shelton – can be used to categorify the Temperley-Lieb algebra and also conjecture that these two pictures should be Koszul dual to each other.

The purpose of this note is to show that the translation– and Zuckerman functors are indeed Koszul dual to each other. By this we mean that both admit graded versions and that these correspond under the Koszul duality functor. Although this of course is one of the main philosophical points of [5], a rigorous proof was never given.

We furthermore use the Koszul duality theory to give a simple proof of the Enright-Shelton equivalence.

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Added: Recently C. Stroppel [11] has obtained a proof of the full Bernstein-Frenkel-Khovanov conjectures, in part using results from the present paper.

2. Preliminaries

In this section we will mostly recall some of the basic definitions and concepts from the theory of Koszul duality, see also [5].

Let $\mathfrak{g}$ be a complex semisimple Lie algebra containing a Borel algebra $\mathfrak{b}$ with Cartan part $\mathfrak{h}$ and let $\mathcal{O}$ be the category of $\mathfrak{g}$-modules associated to it by Bernstein, Gelfand and Gelfand. For $\lambda \in \mathfrak{h}^*$ integral but possibly singular, denote by $\mathcal{O}_\lambda$ the subcategory of $\mathcal{O}$ consisting of modules of generalized central character χ_λ – we refer to it as the singular category $\mathcal{O}$. Let S be the set of simple reflections corresponding to our data, and let S_λ be the subset consisting of those reflections that fix λ under the dot operation. Then S_λ defines a parabolic subalgebra $\mathfrak{q}(\lambda)$. We define the parabolic category $\mathcal{O}^\lambda$ to consist of the $\mathfrak{q}(\lambda)$-finite objects in $\mathcal{O}$. For $\lambda = 0$ we omit the index, i.e. we write $\mathcal{O}_0 = \mathcal{O}$, this should not cause confusion.

Let P denote the sum of all indecomposable projectives in $\mathcal{O}$, hence P is a projective generator of $\mathcal{O}$. Analogously, we can construct projective generators P_λ (resp. P^λ) of $\mathcal{O}_\lambda$ (resp. $\mathcal{O}^\lambda$). Let $A = \mathrm{End}_\mathcal{O}\, P$ (resp. $A_\lambda = \mathrm{End}_{\mathcal{O}_\lambda}\, P_\lambda$, $A^\lambda = \mathrm{End}_{\mathcal{O}^\lambda}\, P^\lambda$). By general principles, we can then identify $\mathcal{O}$ with $\mathrm{Mod}{-}A$, the category of finitely generated A-right modules (and analogously for A_λ, A^λ).

Let $T_0^\lambda : \mathcal{O} \to \mathcal{O}_\lambda$, $T_\lambda^0 : \mathcal{O}_\lambda \to \mathcal{O}$ be the Jantzen translation functors onto and out of the wall. Passing to $\mathrm{Mod}{-}A$, T_0^λ corresponds to the functor

$$\mathrm{Mod}{-}A \to \mathrm{Mod}{-}A_\lambda : \; M \mapsto M \otimes_A X$$

where X is the (A, A_λ) bimodule $X = \mathrm{Hom}_\mathcal{O}(P_\lambda, T_0^\lambda P)$. There is a similar description of $T_\lambda^0 : \mathcal{O}_\lambda \to \mathcal{O}$.

Let $\tau_\lambda : \mathcal{O} \to \mathcal{O}^\lambda$ be the parabolic truncation functor, by definition it takes $M \in \mathcal{O}$ to its largest $\mathfrak{q}(\lambda)$-finite quotient. It is right exact and left adjoint to the inclusion functor $\iota_\lambda : \mathcal{O}^\lambda \to \mathcal{O}$, which is exact, and it thus takes projectives to projectives. It even takes indecomposable projectives to indecomposable projectives. Its top degree left derived functor is the Zuckerman functor that takes a module $M \in \mathcal{O}$ to its largest $\mathfrak{q}(\lambda)$-finite submodule.

On the $\mathrm{Mod}{-}A$ level, τ_λ is given by the tensor product with the (A, A^λ)-bimodule $Y = \mathrm{Hom}_{\mathcal{O}^\lambda}(P^\lambda, \tau_\lambda P)$. However, from the above considerations we deduce that $Y = A^\lambda$, the left A-structure coming from τ_λ and the right A^λ-structure coming from the multiplication in A^λ.

Let us finally recall the definition of a Koszul ring.

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Definition 2.1. Let $R = \bigoplus_{i\geq 0} R_i$ be a positively graded ring with R_0 semisimple and put $R_+ = \bigoplus_{i>0} R_i$. R is called Koszul if the right module $R_0 \cong R/R_+$ admits a graded projective resolution $P^\bullet \twoheadrightarrow R_0$ such that P^i is generated by its component P^i_i in degree i. The Koszul dual ring of R is defined as $R^! := \mathrm{Ext}^\bullet_R(R_0, R_0)$.

The main point of [5] is now that all the rings appearing above can be given a Koszul grading.

3. Koszul duality.

The main purpose of this section is to give a new construction of the Koszul duality functor.

Koszul duality is an equivalence at the level of derived categories between the graded module categories of a Koszul ring and its dual. It exists under some mild finiteness conditions. The concept appeared for the first time in the paper [3] where it is shown that for any finite dimensional vector space V the derived graded module categories of the symmetric algebra SV and the exterior algebra $\bigwedge V^*$ are equivalent. The argument used there carries over to general Koszul rings and is the one used in [5].

We shall here present another approach to the Koszul duality functor, using the language of differential graded algebras (DG-algebras). Although also the papers [4] and [9] link DG-algebras with Koszul duality, we were rather inspired by the construction of the localization functor in the book of Bernstein and Lunts [7].

Let us start out by repeating the basic definitions.

Definition 3.1. A DG-algebra $\mathcal{A} = (A, d)$ is a graded associative algebra $A = \bigoplus_{i=-\infty}^{\infty} A^i$ with a unit $1_A \in A^0$ and an additive endomorphism d of degree 1, s.t.

$$d^2 = 0$$

$$d(a \cdot b) = da \cdot b + (-1)^{deg(a)} a \cdot db$$

$$\textit{and} \;\; d(1_A) = 0.$$

Definition 3.2. A left module $\mathcal{M} = (M, d_M)$ over a DG-algebra $\mathcal{A} = (A, d)$ is a graded unitary right A-module $M = \bigoplus_{i=-\infty}^{\infty} M^i$ with an additive endomorphism $d_M : \mathcal{M} \to \mathcal{M}$ of degree 1, s.t. $d_M^2 = 0$ and

$$d_M(am) = da \cdot m + (-1)^{deg(a)} a \cdot d_M m$$

Definition 3.3. A right module $\mathcal{M} = (M, d_M)$ over a DG-algebra $\mathcal{A} = (A, d)$ is a graded unitary right A-module $M = \bigoplus_{i=-\infty}^{\infty} M^i$ with an additive endomorphism $d_M : \mathcal{M} \to \mathcal{M}$ of degree 1, s.t. $d_M^2 = 0$ and

$$d_M(ma) = d_M m \cdot a + (-1)^{deg(m)} m \cdot da$$

Unless otherwise stated, A will from now on be the ring $\mathrm{End}_{\mathcal{O}}(P)$ from the former section.

By the selfduality theorem of Soergel [10] we know that $A = \mathrm{Ext}^{\bullet}_{\mathcal{O}}(L, L)$, where L is the sum of all simples in $\mathcal{O}$. This provides A with a grading which by [10] or [5] is a Koszul grading.

We regard A as a DG-algebra $\mathcal{A}$ with $A = A^0$. At this stage we neglect the grading on A, hence $\mathcal{A}$ just has one grading, the DG-algebra grading.

Let $K^{\bullet}$ be the Koszul complex of A, see e.g. [5]. It is a projective resolution of k (which corresponds to the sum of all simples in $\mathcal{O}$). In [5] $K^{\bullet}$ is a resolution of left A-modules of k; we however prefer to modify it to a resolution of right A-modules. Furthermore we change the sign of the indices so that the differential has degree $+1$.

Let k_w denote the simple A-module corresponding to $L_w \in \mathcal{O}$ and let $K^{\bullet}_w$ be a projective resolution of k_w. The Koszul complex $K^{\bullet}$ is then quasiisomorphic to $\bigoplus_{\lambda} K^{\bullet}_w$.

We can consider $K^{\bullet}$ as a DG right module of the DG-algebra $\mathcal{A}$, and as such it is $\mathcal{K}$-projective in the sense of [7]: the homotopy category of DG-modules of A is simply the homotopy category of complexes of A-modules, hence any quasiisomorphism from $K^{\bullet}$ to an DG-module $\mathcal{M}$ is a homotopy isomorphism.

Given two DG-modules $\mathcal{M}, \mathcal{N}$ of the (arbitrary) DG-algebra $\mathcal{A}$, recall the construction of the complex $Hom^{\bullet}_{\mathcal{A}}(M, N)$:

$$Hom^{n}_{\mathcal{A}}(M, N) := \mathrm{Hom}_A\left(M, N[n]\right)$$

$$df := d_N f - (-1)^n f d_M, \ \ f \in Hom^{n}_{\mathcal{A}}(M, N).$$

If we perform this construction on $K^{\bullet}$, we obtain a DG-algebra

$$End^{\bullet}_{\mathcal{A}}(K^{\bullet}) = Hom^{\bullet}_{\mathcal{A}}(K^{\bullet}, K^{\bullet}),$$

the multiplication being given by composition of maps (notice that this construction does not involve the Koszul grading on A).

The signs match up to give $K^{\bullet}$ the structure of an $End^{\bullet}_{\mathcal{A}}(K^{\bullet})$ left DG-module.

Let for any DG-algebra $\mathcal{A}$, $\mathcal{D}^b(\mathrm{Mod}{-}\mathcal{A})$ denote the bounded derived category of right $\mathcal{A}$-modules, see for example Bernstein-Lunts [7] for an exposition of this theory.

Let $\mathcal{D}^b(\mathrm{Mof}-A)$ denote the bounded derived category of finitely generated A-modules, and let $\mathcal{D}^b(\mathrm{Mof}-\mathcal{A})$ be the corresponding category of $\mathcal{A}$-modules. Let $\mathcal{D}^b(\mathrm{Mof} - End^{\bullet}_{\mathcal{A}}(K^{\bullet}))$ be the subcategory of $\mathcal{D}^b(\mathrm{Mof} - End^{\bullet}_{\mathcal{A}}(K^{\bullet}))$ consisting of finitely generated $End^{\bullet}_{\mathcal{A}}(K^{\bullet})$-modules in the algebra sense. Then we have

Lemma 3.4. *$\mathcal{D}^b(\mathrm{Mof} - End^{\bullet}_{\mathcal{A}}(K^{\bullet}))$ is generated, in the sense of triangulated categories, by the $Hom^{\bullet}_{\mathcal{A}}(K^{\bullet}, K^{\bullet}_w)$'s.*

Proof. Since our ring $A$is Koszul, the DG-algebras

$$End^\bullet_{\mathcal{A}}(K^\bullet) \text{ and } (Ext^\bullet_{\mathcal{A}}(k,k), d=0)$$

are quasiisomorphic [5, Theorem 2.10.1] and hence their bounded derived categories are equivalent. Under this equivalence the $End^\bullet_{\mathcal{A}}(K^\bullet)$-module $Hom^\bullet_{\mathcal{A}}(K^\bullet, K^\bullet_w)$ becomes the $(Ext^\bullet_{\mathcal{A}}(k,k), d=0)$-module $(Ext^\bullet_{\mathcal{A}}(k,k_w), d=0)$.

Now a certain polynomial DG-algebra $\mathcal{B}$ with $d=0$ is studied in chapter 11 of [7]. Corollary 11.1.5 there is the statement that $\mathcal{D}^b(\mathrm{Mof} - \mathcal{B})$ is generated by $\mathcal{B}$ itself. The proof of this Corollary 11.1.5 relies on the differential being zero for $\mathcal{B}$ and of the fact that finitely generated projective modules over a polynomial algebra are free. We do not have this last property in our situation and should therefore adjust the statement of the Lemma accordingly. Let us give a few details on the modifications:

Let thus $(\mathcal{M}, d_M)$ be an object of $\mathcal{D}^b(\mathrm{Mof} - (Ext^\bullet_{\mathcal{A}}(k,k), d=0))$. Then we have a triangle in that category of the form

$$\operatorname{Ker} d_M \to M \to M/\operatorname{Ker} d_M \to$$

But the first and the third terms have zero differential so we may assume that $d_M = 0$.

Let now

$$0 \to P^{-n} \to P^{-n+1} \to \ldots \xrightarrow{\epsilon} M \to 0$$

be a projective graded resolution of M considered as $Ext^\bullet_{\mathcal{A}}(k,k)$-module. Defining $\mathcal{P} \in \mathcal{D}^b(\mathrm{Mof} - (Ext^\bullet_{\mathcal{A}}(k,k), d=0))$ as having k'th DG-part $\oplus_i P^i_{k-i}$ and differential from the resolution, we get a quasiisomorphism $\mathcal{P} \cong \mathcal{M}$ induced by ϵ.

On the other hand the finitely generated, graded projectives over $Ext^\bullet_{\mathcal{A}}(k,k)$ are the direct sums of the $Ext^\bullet_{\mathcal{A}}(k,k_w)$'s and we are done. ∎

Now since $K^\bullet$ is $\mathcal{K}$-projective, it can be used in itself to calculate $RHom_{\mathcal{A}}(K^\bullet, M)$ for an $\mathcal{A}$-module $\mathcal{M}$. But $K^\bullet$ is an $End^\bullet_{\mathcal{A}}(K^\bullet)$- module, so we get a functor:

$$RHom_{\mathcal{A}}(K^\bullet, -) : \mathcal{D}^b(\mathrm{Mod}-\mathcal{A}) \to \mathcal{D}^b(\mathrm{Mod}- End^\bullet_{\mathcal{A}}(K^\bullet))$$

This might require a little consideration: one should check that $RHom_{\mathcal{A}}(K^\bullet, -)$ takes homotopies to homotopies, even when considered as a functor to the category of right $End^\bullet_{\mathcal{A}}(K^\bullet)$-modules, and likewise for quasiisomorphisms. For homotopies, one checks this by hand (after having consulted e.g. Bernstein-Lunts [7] for the definition of the homotopy category of DG-modules). For quasiisomorphisms one uses the standard argument: if $f : M \to N$ is a quasi-isomorphism, then $Hom^\bullet_{\mathcal{A}}(K^\bullet, C(f))$ is acyclic ($C(f)$ denoting the cone of f) and so on: the action of $End^\bullet_{\mathcal{A}}(K^\bullet)$ is irrelevant for this argument.

Now we also have a functor in the other direction:

$$- \overset{L}{\otimes}_{End^\bullet_{\mathcal{A}}(K^\bullet)} K^\bullet : \ \mathcal{D}^b(\mathrm{Mod}-\mathcal{A}) \leftarrow \mathcal{D}^b(\mathrm{Mod}- End^\bullet_{\mathcal{A}}(K^\bullet))$$

Recall that this construction involves the replacement of

$$\mathcal{N} \in \mathcal{D}^b(\text{Mod}-\, End^\bullet_{\mathcal{A}}(K^\bullet))$$

by a $\mathcal{K}$-flat object (actually a $\mathcal{K}$-projective object will do), to which it is quasi-isomorphic. The action of A on $K^\bullet$ then provides $N \overset{L}{\otimes}_{End^\bullet_{\mathcal{A}}(K^\bullet)} K^\bullet$ with the structure of DG-module over $\mathcal{A}$:

$$\begin{aligned} d(n \otimes ka) &= d(n) \otimes ka + (-1)^{deg(n)} n \otimes d(ka) \\ &= d(n) \otimes ka + (-1)^{deg(n)} n \otimes d(k)a \\ &= d(n \otimes k)a \end{aligned}$$

since the differential of $\mathcal{A}$ is zero. Once again, one should here check that the functor makes sense as a functor into the category of $\mathcal{A}$-modules.

We can then prove the following Theorem:

Theorem 3.5. *The functor $RHom_{\mathcal{A}}(K^\bullet, -)$ establishes an equivalence of the triangulated categories $\mathcal{D}^b(\text{Mof}\,-\mathcal{A})$ and $\mathcal{D}^b(\text{Mof}\,-\,End^\bullet_{\mathcal{A}}(K^\bullet))$. The inverse functor is $-\overset{L}{\otimes}_{End^\bullet_{\mathcal{A}}(K^\bullet)} K^\bullet$.*

Proof. Note first that $\mathcal{D}^b(\text{Mof}\,-A)$ is generated by the $K^\bullet_w$, since they are resolutions of the simple A-modules and short exact sequences $\mathcal{O}$-modules induce triangles in $\mathcal{D}^b(\text{Mod}-A)$.

But then $RHom_{\mathcal{A}}(K^\bullet, M)$ belongs to $\mathcal{D}^b(\text{Mof}\,-\,End^\bullet_{\mathcal{A}}(K^\bullet))$ for any $M \in \mathcal{D}^b(\text{Mof}\,-\,End^\bullet_{\mathcal{A}}(K^\bullet))$, since it clearly does so for each of the generators $K^\bullet_w$.

We furthermore see that $RHom_{\mathcal{A}}(K^\bullet, M)$ is a $\mathcal{K}$-projective $End^\bullet_{\mathcal{A}}(K^\bullet)$-module for any M in $\mathcal{D}^b(\text{Mof}\,-\mathcal{A})$ since it clearly holds for $K^\bullet_w$ and the $\mathcal{K}$-projectives form a triangulated subcategory of $\mathcal{K}(\text{Mod}-\, End^\bullet_{\mathcal{A}}(K^\bullet))$, the homotopy category of $End^\bullet_{\mathcal{A}}(K^\bullet)$-modules.

We then obtain a morphism ψ_M in $\mathcal{D}^b(\text{Mod}\,-\,\mathcal{A})$ in the following way.

$$\psi_M :\; RHom_{\mathcal{A}}(K^\bullet, M) \overset{L}{\otimes}_{End^\bullet_{\mathcal{A}}(K^\bullet)} K^\bullet \to M :\;\; f \otimes k \mapsto f(k)$$

One checks that ψ_M defines a natural transformation from the functor $RHom_{\mathcal{A}}(K^\bullet, -) \overset{L}{\otimes}_{End^\bullet_{\mathcal{A}}(K^\bullet)} K^\bullet$ to the identity functor Id. It is a quasiisomorphism for $M = K^\bullet_w$, and thus for all M.

On the other hand we have for a $\mathcal{K}$-projective $End^\bullet_{\mathcal{A}}(K^\bullet)$-module N the canonical morphism ϕ_N given as follows:

$$\begin{aligned} \phi_N :\; N &\;\to\; RHom_{\mathcal{A}}\left(K^\bullet, N \overset{L}{\otimes}_{End^\bullet_{\mathcal{A}}(K^\bullet)} K^\bullet\right) \\ n_i \in N^i &\;\mapsto\; (\, k_j \in K^j \mapsto n_i \otimes k_j \,) \end{aligned}$$

One readily sees that ϕ_N is a quasiisomorphism for $N = Hom^\bullet_{\mathcal{A}}(K^\bullet, K^\bullet_w)$. But these objects generate $\mathcal{D}^b(\text{Mof}\,-\,End^\bullet_{\mathcal{A}}(K^\bullet))$ and we can argue as above.

Let us finally comment on the shifts [1]. It should here be noticed that since we are working with right DG-modules, the appropriate definition of $\mathcal{M}[1]$ for a DG-module $\mathcal{M}$ over a DG-algebra $\mathcal{A}$ is the following:

$$(M[1])^i = M^{i+1},\ d_{M[1]} = -d_M \text{ and } m \circ a = ma$$

where $m \circ a$ is the multiplication in $M[1]$ while ma is the multiplication in M. So unlike the left module situation, the A structure on M is here left unchanged. One now checks that the functors commute with the shifts [1]. ■

Now A has a Koszul grading so we may consider the category $\mathrm{mof}-A$ of finitely generated, graded A-modules. Let us use a lower index to denote the graded parts with respect to this $\mathbb{Z}$-grading. This passes to the derived category of graded DG-modules which we denote by $\mathcal{D}^b(\mathrm{mof}-\mathcal{A})$. It may be identified with the derived category of $\mathrm{mof}-A$ and carries a twist $\langle 1\rangle$ which we arrange the following way:

$$M\langle 1\rangle_i = M_{i-1}$$

Now the grading on $K^\bullet$ induces a grading on $End^\bullet_{\mathcal{A}}(K^\bullet)$ as well; it satisfies the rule

$$End^\bullet_{\mathcal{A}}(K^\bullet)_i := end^\bullet_{\mathcal{A}}(K^\bullet, K^\bullet\langle -i\rangle)$$

With this grading on $End^\bullet_{\mathcal{A}}(K^\bullet)$, we may consider its graded module category, which we denote $\mathcal{D}^b(\mathrm{mof}-End^\bullet_{\mathcal{A}}(K^\bullet))$. Let us moreover denote the subcategory of this generated, with twists, by the graded summands of the DG-module $End^\bullet_{\mathcal{A}}(K^\bullet)$ itself by

$$\mathcal{D}^b(\langle End^\bullet_{\mathcal{A}}(K^\bullet)\rangle)$$

(It would have been more consistent with the earlier notation to write $\mathcal{D}^b(\mathrm{mof}-End^\bullet_{\mathcal{A}}(K^\bullet))$ for this category; we however prefer the above notation in order to save space).

It is now straightforward to prove the following strengthening of the former theorem:

Theorem 3.6. *The functor $RHom_{\mathcal{A}}(K^\bullet, -)$ establishes an equivalence of the triangulated categories $\mathcal{D}^b(\mathrm{mof}-\mathcal{A})$ and $\mathcal{D}^b(\langle End^\bullet_{\mathcal{A}}(K^\bullet)\rangle)$. It commutes with the twists $[1]$ and $\langle 1\rangle$.*

Our next task will be to study the DG-algebra $(Ext^\bullet_{\mathcal{A}}(K^\bullet), d=0)$. More precisely, we are going to compare $\mathcal{D}^b(\langle Ext^\bullet_{\mathcal{A}}(K^\bullet), d=0\rangle)$ with the standard derived category of the category of finitely generated, graded modules over $Ext^\bullet_A(k,k)$, i.e. $\mathcal{D}^b(\mathrm{mof}-Ext^\bullet_A(k,k))$. Here, we define the $\mathbb{Z}$-grading on $(Ext^\bullet_{\mathcal{A}}(K^\bullet), d=0)$ to be the negative of the DG-grading.

An object of $\mathcal{D}^b\langle(\mathrm{mof}-Ext^\bullet_A(k,k))$ is a bounded complex

$$0 \to P^{-n} \to P^{-n+1} \to \ldots \to P_0 \to \ldots \to P^{m-1} \to P^m \to 0$$

of projective graded $Ext^\bullet_A(k,k)$-modules (so the differential has degree 0). Using this, one constructs a DG-module $\mathcal{P}$ of the DG-algebra $(Ext^\bullet_A(k,k), d=0)$ in the following way:

$$0 \to P^{-n}\langle -n\rangle \to P^{-n+1}\langle -n+1\rangle \to \ldots \to P^0\langle 0\rangle \to \cdots P^m\langle m\rangle \to 0.$$

In other words, $\mathcal{P}$ is the module $\mathcal{P} = \bigoplus_n P^n$ whose k'th DG-piece is $\mathcal{P}^k = \bigoplus_i P^i_{k-i}$. We make $\mathcal{P}$ into a graded DG-module by the rule $\mathcal{P}_j = \bigoplus_i P^i_{-j}$. This construction defines a functor

$$F:\ \mathcal{D}^b\left(\mathrm{mof} - \mathit{Ext}^\bullet_A(k,k)\right) \to \mathcal{D}^b\left(\mathrm{mod}{-}(\mathit{Ext}^\bullet_A(k,k), d=0)\right)$$

since a graded homotopy between two morphisms $f, g : P^\bullet \to Q^\bullet$ of modules will be mapped to a homotopy in the DG-category.

I will now argue that actually F is a functor into the category $\mathcal{D}^b\left(\langle \mathit{Ext}^\bullet_A(k,k), d=0\rangle\right)$. To see this note first that the $P_i\langle i\rangle$ all lie in this category since they are summands of (shifts of) $(\mathit{Ext}^\bullet_A(k,k), d=0)$. On the other hand F can be viewed as an iterated graded mapping cone construction in the DG-sense, starting with the morphism $P^{m-1}\langle m\rangle \to P^m\langle m\rangle$, which is already a DG-morphism, and the claim follows. The graded mapping cone of a graded DG-module morphism $\mathcal{M} \stackrel{u}{\to} \mathcal{N}$ is given by $C(u) = N \oplus M[1]$ as DG-module, and reversing the degrees with respect to the $\mathbb{Z}$-grading.

It is now clear that F commutes with the twists $[1]$. On the other hand F takes a sequence of $\mathit{Ext}^\bullet_A(k,k)$-modules isomorphic to a standard triangle

$$M \stackrel{u}{\to} N \to C(u) \to M[1]$$

to a sequence of graded DG-modules isomorphic to

$$F(M) \stackrel{F(u)}{\to} F(N) \to F(C(u)) \to F(M[1]).$$

But this is easily seen to be a standard triangle of DG-modules; in other words F is a triangulated functor.

One furthermore checks that F is full and faithful and it is thus an equivalence of triangulated categories once we have shown that the generators $(\mathit{Ext}^\bullet_A(k,k_w), d=0)$ lie in the image of F. But this is clear.

Since

$$M_1 \to M_2 \to M_3 \to$$

is a triangle if and only if

$$F(M_1) \to F(M_2) \to F(M_3) \to$$

is a triangle we deduce that the inverse functor G is triangulated as well.

Our next task will be to analyze the behavior of the twists $\langle 1\rangle$ with respect to F. Now apart from the change of the sign of the differentials, we obtain the relation:

$$F(M\langle 1\rangle) = F(M)[-1]\langle -1\rangle.$$

On the other hand, let for a complex $P^\bullet$ of graded $\mathit{Ext}^\bullet_A(k,k)$-modules $P = \bigoplus P^k_n$ be the total module. We may then consider the map σ on P defined as follows:

$$p \in P^k_n \mapsto (-1)^n p.$$

One now checks that σ defines an isomorphism between the functors $F(-\langle 1\rangle)$ and $F(-)[-1]\langle -1\rangle$.

Recall once again that since $A$is Koszul, the DG-algebras

$$End^\bullet_{\mathcal{A}}(K^\bullet) \text{ and } (Ext^\bullet_A(k,k), d=0)$$

are quasiisomorphic so there is an equivalence

$$\phi: \mathcal{D}^b(\mathrm{Mod}-End^\bullet_{\mathcal{A}}(K^\bullet)) \to \mathcal{D}^b(\mathrm{Mod}-(Ext^\bullet_A(k,k), d=0)).$$

A closer look at the above quoted proof reveals that the quasiisomorphism is even a graded one so we obtain an equivalence

$$\phi: \mathcal{D}^b(\langle End^\bullet_{\mathcal{A}}(K^\bullet)\rangle) \to \mathcal{D}^b(\langle Ext^\bullet_A(k,k), d=0\rangle)$$

Let us gather the results in one theorem.

Theorem 3.7. *The composition of the above functors*

$$G\circ\phi\circ RHom_{\mathcal{A}}(K^\bullet,-): \mathcal{D}^b(\mathrm{mof}-A) \to \mathcal{D}^b(\mathrm{mof}-Ext^\bullet_A(k,k))$$

is an equivalence of triangulated categories: the Koszul duality functor. It commutes with [1] *and satisfies the rule*

$$F(M\langle 1\rangle) = F(M)[-1]\langle -1\rangle.$$

4. Translation and Zuckerman functors.

Let us return to the translation functors to and from the wall $T_0^\lambda: \mathcal{O}\to\mathcal{O}_\lambda$, $T_\lambda^0: \mathcal{O}\to\mathcal{O}_\lambda$. We saw in the first section, that they can be viewed as functors between the categories $\mathrm{Mod}-A$ and $\mathrm{Mod}-A_\lambda$. Now, in [2] it is shown that they can be lifted to graded functors

$$T_0^\lambda: \mathrm{mod}-A \to \mathrm{mod}-A_\lambda$$

$$T_\lambda^0: \mathrm{mod}-A_\lambda \to \mathrm{mod}-A$$

that still are adjoint and exact and such that T_0^λ takes pure objects of weight n to pure objects of the same weight; this construction is based on Soergel's theory of modules over the coinvariants of the Weyl group. See also [1], where graded translation functors are constructed in the setting of modular representation theory.

The graded translation functor T_0^λ induces a homomorphism of graded DG-algebras $End^\bullet_{\mathcal{A}}(K^\bullet)\to End^\bullet_{\mathcal{A}_\lambda}(T_0^\lambda K^\bullet)$. But T_0^λ and T_λ^0 are exact and $K^\bullet$ is a projective graded resolution of the pure module k, so $T_0^\lambda K^\bullet$ is a projective graded resolution of k_λ. Hence $T_0^\lambda K^\bullet$ is graded homotopic to the Koszul complex $K_\lambda^\bullet$ of A_λ and this implies that $End^\bullet_{\mathcal{A}_\lambda}(T_0^\lambda K^\bullet)$ and $End^\bullet_{\mathcal{A}_\lambda}(K_\lambda^\bullet)$ are quasiisomorphic graded DG-algebras.

We already saw that A being a Koszul ring implies that

$$End^\bullet_{\mathcal{A}}(K^\bullet)\cong(Ext^\bullet_{\mathcal{A}}(k,k), d=0)$$

and by the above and since A_λ is Koszul as well we have that

$$End^\bullet_{\mathcal{A}_\lambda}(T_0^\lambda K^\bullet)\cong(Ext^\bullet_{\mathcal{A}_\lambda}(T_0^\lambda k, T_0^\lambda k), d=0)\cong(Ext^\bullet_{\mathcal{A}_\lambda}(k_\lambda,k_\lambda), d=0)$$

We now obtain the following commutative diagram of functors:

$$\begin{array}{ccccc}
\mathcal{D}^b(\mathrm{mod}{-}A) & \overset{-\otimes K^\bullet}{\longleftarrow} & \mathcal{D}^b(\langle End^\bullet_{\mathcal{A}}(K^\bullet)\rangle) & \leftarrow & \mathcal{D}^b(\langle Ext^\bullet_{\mathcal{A}}(k,k), d=0\rangle) \\
\downarrow T_0^\lambda & & \downarrow & & \downarrow T_0^\lambda \\
\mathcal{D}^b(\mathrm{mod}{-}A_\lambda) & \overset{-\otimes T_0^\lambda K^\bullet}{\longleftarrow} & \mathcal{D}^b(\langle End^\bullet_{\mathcal{A}_\lambda}(T_0^\lambda K^\bullet)\rangle) & \leftarrow & \mathcal{D}^b(\langle Ext^\bullet_{\mathcal{A}_\lambda}(T_0^\lambda k, T_0^\lambda k), d=0\rangle)
\end{array}$$

where the middle vertical arrow is the graded tensor product functor

$$-\otimes_{End^\bullet_{\mathcal{A}}(K^\bullet)} End^\bullet_{\mathcal{A}}(T_0^\lambda K^\bullet)$$

while the first and third vertical arrows are the graded translation functors. The commutativity of the first square is here obvious, whereas the quasiisomorphism of $(Ext^\bullet_{\mathcal{A}}(k,k), d=0)) \times End^\bullet_{\mathcal{A}_\lambda}(T_0^\lambda K^\bullet)$-bimodules

$$End^\bullet_{\mathcal{A}}(T_0^\lambda K^\bullet) \cong Ext^\bullet_{\mathcal{A}_\lambda}(T_0^\lambda k, T_0^\lambda k)$$

gives the natural transformation that makes the second square commutative.

By the same reasoning as in the last section, the two lower arrows define equivalences of triangulated categories.

Now Backelin [2] shows that one can choose the isomorphisms $Ext^\bullet_{\mathcal{A}}(k,k) \simeq A$ and $Ext^\bullet_{\mathcal{A}_\lambda}(T_0^\lambda k, T_0^\lambda k) \simeq A^\lambda$ to obtain the following commutative diagram:

$$\begin{array}{ccc}
Ext^\bullet_{\mathcal{A}}(k,k) & \overset{\sim}{\longrightarrow} & A \\
T_0^\lambda \downarrow & & \downarrow \tau \\
Ext^\bullet_{\mathcal{A}_\lambda}(T_0^\lambda k, T_0^\lambda k) & \overset{\sim}{\longrightarrow} & A^\lambda
\end{array}$$

The key point is here that both vertical maps are surjections: for τ this is clear, while for T_0^λ an argument involving the Koszul property of $A$is required (one can here give a simple alternate argument along the lines of the Cline, Parshall, Scott approach to Kashdan-Luzstig theory [8]). It then follows that the kernels of the two vertical maps are the ideals generated by corresponding idempotents (thus in the degree 0 part).

Although the diagram involves non-graded maps, it can be used to give τ a grading – and then of course it is a commutative diagram of graded homomorphisms.

This diagram, on the other hand, gives rise to the following commutative diagram of functors:

$$\begin{array}{ccc}
\mathcal{D}^b(\langle Ext^\bullet_{\mathcal{A}}(k,k), d=0\rangle) & \overset{F}{\leftarrow} & \mathcal{D}^b(\mathrm{mod}{-}A) \\
T_0^\lambda \downarrow & & \downarrow \tau \\
\mathcal{D}^b(\langle Ext^\bullet_{\mathcal{A}_\lambda}(T_0^\lambda k, T_0^\lambda k), d=0\rangle) & \overset{F}{\leftarrow} & \mathcal{D}^b(\mathrm{mod}{-}A^\lambda)
\end{array}$$

We now join the two diagrams of functors to obtain a diagram, in which the upper arrows compose to the Koszul duality functor of $\mathcal{O}$ while the composition of the lower arrows is isomorphic to the Koszul duality functor of $\mathcal{O}_\lambda$. Let us formulate this as a Theorem

Theorem 4.1. *The translation– and Zuckerman functors are Kozsul to each other, in other words there is a commutative diagram of functors:*

$$\begin{array}{ccc} \mathcal{D}^b(\text{mod}-A) & \overset{D}{\leftarrow} & \mathcal{D}^b(\text{mod}-A) \\ {\scriptstyle T_0^\lambda}\downarrow & & \downarrow{\scriptstyle \tau} \\ \mathcal{D}^b(\text{mod}-A_\lambda) & \overset{D_\lambda}{\leftarrow} & \mathcal{D}^b(\text{mod}-A^\lambda) \end{array}$$

where D (resp. D_λ) is the Koszul duality functor as described above.

This is the Theorem announced in the introduction of the paper.

5. The Enright-Shelton Equivalence

We now consider the categorification of the Temperley-Lieb algebra. Let thus $\mathfrak{g} := gl_n$ and $\mathcal{O} := \mathcal{O}(\mathfrak{gl}_n)$. Let $\epsilon_k, k \in \{1, 2, \ldots, n\}$ be the standard basis of the weight lattice and define for $k \in \{1, 2, \ldots, n\}, \lambda_k := \epsilon_1 + \epsilon_2 + \ldots \epsilon_k$. We then let $\mathcal{O}_{k,n-k}$ be the singular block of $\mathcal{O}$ consisting of modules with central character $\theta(\lambda_k)$. So the Verma module with highest weight $\lambda_k - \rho$ lies in $\mathcal{O}_{k,n-k}$. Let $\mathfrak{g}_i, 1 \le i \le i-1$ be the subalgebra of $\mathfrak{g}$ consisting of the matrices whose entries are nonzero only on the intersection of the i-th and $(i+1)$-th rows and columns. We then denote by $\mathcal{O}^i_{k,n-k}$ the parabolic subcategory of $\mathcal{O}_{k,n-k}$ whose modules are the locally $\mathfrak{g}_i$-finite ones in $\mathcal{O}_{k,n-k}$.

We shall also consider the following dual picture: let $\mathfrak{p}_k$ be the parabolic subalgebra of $\mathfrak{g}$, whose Levi part is $\mathfrak{g}_k \oplus \mathfrak{g}_{n-k}$ and such that $\mathfrak{n}_+ \subseteq \mathfrak{p}_k$ and let $\mathcal{O}^{k,n-k}$ be the the full subcategory of $\mathcal{O}$ consisting of the locally $\mathfrak{p}_k$-finite modules. Choose an integral dominant regular weight μ and integral dominant subregular weights μ_i on the i-th wall, $i = 1, 2, \ldots, n-1$ (so the coordinates of μ_i in the ϵ_i basis are strictly decreasing, except for the i-th and the $(i+1)$-th that are equal). Let finally $\mathcal{O}^{k,n-k}_i$ be the subcategory of $\mathcal{O}^{k,n-k}$ with central character $\theta(\mu_i)$.

All of this is the setup of [6]

Let $R^i_{k,n-k}$(resp. $R^{k,n-k}_i$) be the endomorphism ring of the minimal projective generator of $\mathcal{O}^i_{k,n-k}$ (resp. $\mathcal{O}^{k,n-k}_i$). The following theorem is a direct consequence of Backelin's work [2]:

Theorem 5.1. $(R^i_{k,n-k})^! = R^{k,n-k}_i$

Proof. The main result of [2] is that

$$(R^\psi_\phi)^! = R^\phi_{-w_0\psi}$$

with the notation as in [5].

Now one should first observe that $c\,Id \in \mathfrak{gl}_n$ acts on $\mathcal{O}_\lambda$ through multiplication with $c \sum \lambda_i$; hence $\mathcal{O}_\lambda$ is equivalent to the category $\mathcal{O}_{\overline{\lambda}}$ of $\mathfrak{sl}_n$-modules, where $\overline{\lambda}$ denotes the image of λ under the projection of the weight lattice with kernel $\mathbb{Z} \sum \epsilon_i$. We can thus restrict ourselves to the semisimple situation and may indeed use the results quoted.

Now μ_i and $\mathfrak{sl}_i \subseteq \mathfrak{sl}_n$ are both given by the simple root $\alpha_i = \epsilon_i - \epsilon_{i+1}$ and also $\mathfrak{p}_{k,n-k}$ and λ_k are given by the same simple roots $(= \Delta \setminus \{\alpha_i\})$. Finally, we have in type A that $w_0\lambda = -\lambda$ and the theorem follows. ■

As a corollary we obtain a simple proof of the following equivalence of categories first proven by Enright and Shelton.

Corollary 5.2. $\mathcal{O}_1^{k,n-k} \cong \mathcal{O}^{k-1,n-k-1}$

Proof. There is first of all a standard equivalence of categories:

$$\mathcal{O}^1_{k,n-k} \cong \mathcal{O}_{k-1,n-k-1}$$

The functor ν_n from left to right takes the sum of all weight spaces of weight $\epsilon_1 + x_3\epsilon_3 + \dots x_n\epsilon_n - \rho_n$ with $x_i \in \mathbb{Z}$. This is a $\mathfrak{gl}_{n-2}$-module and ν_n is then the tensor product of it with the module defined by $\epsilon_1 + \epsilon_2 + \dots \epsilon_{n-2}$. The inverse functor comes from an induction procedure, see [6] for the details. Now this equivalence gives us a ring isomorphism

$$(R^1_{k,n-k})^! \cong (R_{k-1,n-k-1})^!$$

which combined with the theorem yields a ring isomorphism

$$\xi : R_1^{k,n-k} \cong R^{k-1,n-k-1}$$

But then the module categories of $R_1^{k,n-k}$ and $R^{k-1,n-k-1}$ are equivalent, by the restriction and extension of scalars along ξ. The corollary is proved. ∎

This might be useful in proving the full conjectures of [6].

References

[1] Andersen, H. H. , J. C. Jantzen, and W. Soergel, *Representations of Quantum Groups at a p-th root of unity and of semisimple Groups in Characteristic p: Independence of p*, Astérisque **220**, 1994, 1–320.

[2] Backelin, E., *Koszul Duality for Parabolic and Singular Category $\mathcal{O}$, Representation Theory*, **3**, (1999), 139–152.

[3] Bernstein, J., I. Gelfand, and S. Gelfand, *Algebraic bundles over $\mathbb{P}^n$ and problems of linear algebra*, Funcional Analysis and its Appl. **12** (1978), 66–67.

[4] Beilinson, A., J. Ginzburg, and V. Schechtmann, *Koszul Duality*, J. of Geometry and Physics **5** (1988), 317–350.

[5] Beilinson, A., V. Ginzburg, and W. Soergel, *Koszul duality patterns in representation theory*, J. Amer. Math. Soc. **9**, (1996), 473–527.

[6] Bernstein, J., I. Frenkel, and M. Khovanov, *A categorification of the Temperley-Lieb Algebra and Schur quotients of $U(\mathfrak{sl}_2)$ via projective and Zuckerman functors*, Selecta Math. (N.S.), **5** (1999), 199–241.

[7] Bernstein, J., and V. Lunts, "Equivariant Sheaves and Functors," Lecture Notes in Mathematics (Springer-Verlag) **1578**, (1994).

[8] Cline, E., B. Parshall, and L. Scott, *Abstract Kashdan-Luzstig Theories*, Tôhoko Math. J. **45** (1993), 511–534.

[9] B. Keller, E., *Deriving DG-categories*, Ann. Sci. École Norm. Sup. **27** (1994), 63–102.

[10] Soergel, W., *Kategorie $\mathcal{O}$, Perverse Garben und Moduln über den Koinvarianten zur Weylgruppe*, J. Amer. Math. Soc. **3** (1990), 421–445.

[11] Stroppel, C., *Categorifications of the Temperley 2-category, tangles and cobordisms via projective functors* , Preprint, C.Stroppel@mcs.le.ac.uk.

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