Journal of Lie Theory
Volume **14** (2004) 427–441

Separation of Unitary Representations of $\mathbb{R} \times \mathbb{R}^d$

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Communicated by J. Ludwig

Abstract. We consider a type I, solvable Lie group G, of the form $\mathbb{R} \times \mathbb{R}^d$. We show that every irreducible unitary representation π of G is characterized by its generalized moment set.

1. Introduction

Let G be a real Lie group with Lie algebra $\mathfrak{g}$, $(\pi, \mathcal{H}_\pi)$ a unitary irreducible representation of G and $\mathcal{H}_\pi^\infty$ the space of $\mathcal{C}^\infty$ vectors for π. Let $\mathfrak{g}^*$ be the dual space of $\mathfrak{g}$. In [7], Wildberger defined the moment map Ψ_π of π. For all ξ in $\mathcal{H}_\pi^\infty \setminus \{0\}$, the element $\Psi_\pi(\xi)$ in $\mathfrak{g}^*$ is defined by:

$$\Psi_\pi(\xi)(X) := \frac{1}{i}\frac{\langle d\pi(X)\xi, \xi\rangle}{\langle \xi, \xi\rangle}, \qquad X \in \mathfrak{g}.$$

The moment set I_π of the representation π is by definition the closure in $\mathfrak{g}^*$ of the image of the moment map:

$$\Psi_\pi : \mathcal{H}_\pi^\infty \setminus \{0\} \longrightarrow \mathfrak{g}^*.$$

Wildberger gave an explicit description of the moment set I_π when G is a connected, simply connected nilpotent Lie group. More precisely, he showed (Theorem 4.2 in [7]) that I_π is the closure of the convex hull of the coadjoint orbit $\mathcal{O}_\pi$ associated to π via the Kirillov theory, *i.e.*

$$I_\pi = \overline{\mathrm{Conv}(\Omega_\pi)}.$$

This result has been generalized by Arnal and Ludwig [2] for solvable Lie groups. Nevertheless, as shown in [7], the moment set does not characterize the representation even for nilpotent Lie groups.

A. Baklouti, J. Ludwig and M. Selmi extended the moment map to the dual of the complex universal envelopping algebra $\mathcal{U}(\mathfrak{g})$ of the complexification $\mathfrak{g}_\mathbb{C}$ of $\mathfrak{g}$, as follows:

ISSN 0949–5932 / \$2.50

For all A in $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ and ξ in $\mathcal{H}_\pi^\infty \setminus \{0\}$,

$$\tilde{\Psi}_\pi(\xi)(A) := \mathfrak{Re}\left(\frac{1}{i}\frac{\langle d\pi(A)\xi,\xi\rangle}{\langle \xi,\xi\rangle}\right),$$

and considered the convex hull $J(\pi)$ of the image of this generalized moment map $\tilde{\Psi}_\pi$:

$$J(\pi) := \mathrm{Conv}(\tilde{\Psi}_\pi(\mathcal{H}_\pi^\infty \setminus \{0\})).$$

Let $\mathcal{U}_n$ be the subspace of $\mathcal{U}(\mathfrak{g})$ consisting of elements of degree less or equal to n. By restriction to $\mathcal{U}_n$, we define :

$$J^n(\pi) = J(\pi)|_{\mathcal{U}_n} = \Big\{F|_{\mathcal{U}_n}, \quad F \in J(\pi)\Big\}.$$

In [5], A. Baklouti, J. Ludwig and M. Selmi shown that for all nilpotent Lie group, there exists an integer n such that, for any unitary irreducible representations π and ρ of G, we have

$$J^n(\pi) = J^n(\rho) \quad \text{if and only if} \quad \pi \simeq \rho.$$

Later on, they generalized this result for an exponential Lie group and, in [4], they shown with D. Arnal, the following result:

Theorem 0. (Separation of unitary representations of exponential Lie groups) *Let $G = \exp(\mathfrak{g})$ be an exponential Lie group. Let π and ρ be two unitary irreducible representations of G. Then:*

$$\pi \simeq \rho \quad \textit{if and only if} \quad J(\pi) = J(\rho).$$

For exponential groups, there is a bijection between coadjoint orbits and classes of unitary irreducible representations. Thus this result means it is possible to find the coadjoint orbit of a representation π, just by looking at the image of the generalized moment map of π.

Since quantizing a symplectic manifold like a coadjoint orbit is building a Hilbert space and a representation of the algebra of observables, theorem 0 can be viewed as a dequantization of the representation π: we refind directly the coadjoint orbit from the representation and its moment map.

In this paper, we study the simplest solvable, non exponential example: the case of a type I solvable connected and simply connected Lie group of the form $G = \mathbb{R} \times \mathbb{R}^d$. We show that the set $J(\pi)$, characterizes the irreducible unitary representations of G and we prove the following:

Theorem 1. *Let G be a semi-direct product of $\mathbb{R}$ by $\mathbb{R}^d$, $G = \mathbb{R} \times \mathbb{R}^d$. Suppose G is type I, let π and ρ be two unitary irreducible representations of G. Then:*

$$\pi \simeq \rho \quad \textit{if and only if} \quad J(\pi) = J(\rho).$$

For type I groups, even for our simple example, we have not only to find the coadjoint orbit associated to a representation π, but also the extension of the inducing character to the non simply connected inducing "small group".

Thus, we can say that, for non exponential Lie groups, the dequantization procedure is not injective, there is many different “quantum” models for the coadjoint orbit and we have to separate them.

In this article, we get this separation thanks to a case-by-case analysis which is not directly generalizable to a general type I, solvable Lie group. However, we conjecture that theorem 1 holds for any type I, solvable Lie group and we hope that our present proof is the first step in proving a more general result.

2. Preliminaries

The unitary representations of our group are very easy to describe, by using unitary induction.

Let $\mathfrak{g}$ be the Lie algebra of G, $\mathfrak{n}$ its abelian ideal $\mathbb{R}^d$ and H an element in $\mathfrak{g}$ not belonging to $\mathfrak{n}$. Each irreducible unitary representation π is associated to a coadjoint orbit in $\mathfrak{g}^*$, the dual of $\mathfrak{g}$.

Let f be in $\mathfrak{g}^*$. Its coadjoint orbit is either $\{f\}$ and the associated representation is the character $\chi_f : \exp\ X \mapsto e^{if(X)}$, or a two dimensional manifold: denote f_0 the restriction of f to $\mathfrak{n}$, the coadjoint action of $\exp(\mathbb{R}H)$ on f_0 is a one-dimensional submanifold of $\mathfrak{n}^* \simeq \mathbb{R}^d$, which is diffeomorphic to a circle or to a line.

The coadjoint orbit of f is thus the cylinder:

$$G.f = \{\ell \in \mathfrak{g}^* \ \text{ such that } \ \ell|_{\mathfrak{n}} \in \exp(\mathbb{R}H).f_0\}.$$

Note $G(f)$ the stabilizer of f in G, $\mathfrak{h} = \mathfrak{n}$ is a real polarization in the point f, stable under the $G(f)$-adjoint action. Let us put $D = G(f)\exp(\mathfrak{h})$, this is a closed, normal subgroup of G and the representations associated to f are:

$$\pi = \pi(f, \chi_f, \mathfrak{h}, G),$$

where χ_f is any character of D, with differential if.

Let us remark that the space $\mathcal{H}_\pi$ of the representation π is isomorphic to $L^2(G/D)$, which is either $L^2(\mathbb{R})$ or $L^2(\mathbb{R}/\mathbb{Z})$.

Auslander and Kostant shown in [1] that the representation π does not depend on the choice of the polarization and each unitary representation of G can be obtained in this way.

Since H belongs to $\mathfrak{g} \setminus \mathfrak{n}$, each element g of G can be written in an unique way as:

$$g = \exp\ tH\ n \qquad (t \in \mathbb{R},\ n \in N).$$

We complexify the space $\mathbb{R}^d = \mathfrak{n}$ and we put the matrix of ad_H restricted to $\mathfrak{n}_{\mathbb{C}}$ in its Jordan form. There is four possible types of Jordan blocks:

those associated to the eigenvalue 0, the corresponding subspaces $E_j(0)$ are real;

those associated to a real, non zero eigenvalue a, the corresponding subspaces $E_j(a)$ are real;

those associated to a non real, non purely imaginary eigenvalue $a+ib$, the corresponding subspaces $E_j(a+ib)$ are not real, but $a-ib$ is another eigenvalue and we choose $E_j(a-ib)=\overline{E_j(a+ib)}$;

those associated to an eigenvalue $ib\neq 0$, the corresponding subspaces $E_j(ib)$ are not real, but $-ib$ is another eigenvalue and we choose $E_j(-ib)=\overline{E_j(ib)}$. In this last case, the real number b will called a period of the group G.

Note $p_j(\lambda)$ the dimensionality of $E_j(\lambda)$ and fix a Jordan basis for ad_H:

$$\begin{array}{ll}
Z_{jk}(0)=X_{jk}(0) & (1\leq k\leq p_j(0)),\\
Z_{jk}(a)=X_{jk}(a) & (1\leq k\leq p_j(a)),\\
Z_{jk}(a+ib)=X_{jk}(a+ib)+iY_{jk}(a+ib), & (1\leq k\leq p_j(a+ib),\quad b>0),\\
Z_{jk}(a-ib)=\overline{Z_{jk}(a+ib)} & \\
Z_{jk}(ib)=X_{jk}(ib)+iY_{jk}(ib), & (1\leq k\leq p_j(ib),\quad b>0).\\
Z_{jk}(-ib)=\overline{Z_{jk}(ib)} &
\end{array}$$

Then:

$$ad_H\left(Z_{jk}(\lambda)\right)=\lambda Z_{jk}(\lambda)+Z_{j(k+1)}(\lambda),\quad (k<p_j(\lambda))$$

and

$$ad_H\left(Z_{jp_j(\lambda)}(\lambda)\right)=\lambda Z_{jp_j(\lambda)}(\lambda).$$

Now, for the dual basis $Z_{jk}(\lambda)^*$, we have:

$$ad^*_{-H}\left(Z_{jk}(\lambda)^*\right)=\lambda Z_{jk}(\lambda)^*+Z_{j(k-1)}(\lambda)^*,\quad (k>1),$$

and

$$ad^*_{-H}\left(Z_{j1}(\lambda)^*\right)=\lambda Z_{j1}(\lambda)^*.$$

Let f_0 be a point in the dual $\mathfrak{n}^*$ of $\mathfrak{n}$,

$$f_0=\sum_{\lambda,\ j,\ k}\zeta_{jk}(\lambda)Z_{jk}(\lambda)^*\qquad (\overline{f_0}=f_0),$$

then:

$$\mathrm{Coad}\left(\exp(-tH)\right)f_0=\sum_{\lambda,\ j,\ k}z_{jk}(\lambda)(t)Z_{jk}(\lambda)^*,$$

with

$$z_{jk}(\lambda)(t)=e^{t\lambda}\left(\zeta_{jk}(\lambda)+t\zeta_{j(k+1)}(\lambda)+\ldots+\frac{t^{p_j(\lambda)-k}}{(p_j(\lambda)-k)!}\zeta_{jp_j(\lambda)}\right).$$

Now G is supposed to be of type I. Let us call period group of G the subgroup of $\mathbb{R}$ generated by the periods b of G. The following is well known.

Proposition 2.1. (Characterization of G-type [6]) *The type of G is I if and only if G has a discrete period group.*

3. The coadjoints orbits

We have already seen that the coadjoints orbits are:

either points, if and only if H is in the Lie algebra $\mathfrak{g}(f)$, then, since H is arbitrary, if and only if $\mathfrak{g}(f) \not\subset \mathfrak{n}$, or if and only if:

$$f_0 = \sum_j \zeta_{1j}(0) Z_{1j}(0)^*, \; f = hH^* + \sum_j \zeta_{1j}(0) Z_{1j}(0)^*,$$

or diffeomorphic to the cylinder $\mathbb{R}H^* \times O$, if O is the orbit of f_0, O is diffeomorphic to $\mathbb{R}$ or to a circle S^1.

In order to give a parametrization of the orbits, we choose an ordering on the subspaces $E_{jk}(\lambda)$. Let us be more precise, we class these spaces as follow: first the spaces $E_j(0)$, then the spaces $E_j(a)$ (a real, $a \neq 0$), then the spaces $E_j(a+ib)$ (a and b reals, $a \neq 0$ and $b \neq 0$), finally the spaces $E_j(ib)$. This induced a natural ordering on the basis $(Z_{kj}(\lambda)^*)$ of $\mathfrak{n}^*_{\mathbb{C}}$.

Each non trivial orbit meets the orthogonal of H. We choose a point f of the orbit in this orthogonal.

First case: there exists a non null component $\zeta_{jk}(0)$, $k > 1$. Let j_0 be the smallest j, k_0 the largest k possible such that $\zeta_{jk}(0) \neq 0$. Then:

$$z_{j_0k_0}(0)(t) = \zeta_{j_0k_0}(0), \quad z_{j_0(k_0-1)}(0)(t) = \zeta_{j_0(k_0-1)}(0) + t\zeta_{j_0k_0}(0).$$

We choose, as a base point of the orbit, the only point ℓ orthogonal to H and to $Z_{j_0(k_0-1)}(0)$. We deduce that the invariant functions which characterize all the orbits of this category having the same j_0 and k_0 are the coordinates of ℓ, *i.e.* the functions:

$$f \mapsto z_{jk}(\lambda)\left(-\frac{f(Z_{j_0(k_0-1)}(0))}{f(Z_{j_0k_0}(0))}\right) \qquad \big((j,k,\lambda) > (j_0,k_0,0)\big).$$

For these orbits, O is diffeomorphic to $\mathbb{R}$ and $G(\ell)$ is a subgroup of N.

Second case: all the components $\zeta_{jk}(0)$, $k > 1$ vanish, but there exists a non vanishing component $\zeta_{jk}(a)$ ($a \neq 0$). Let a_0 be the first a, j_0 the smallest j, k_0 the largest k such that $\zeta_{jk}(a)$ is non null. Then:

$$z_{j_0k_0}(a_0)(t) = e^{a_0t}\zeta_{j_0k_0}(a_0).$$

We choose, as a base point of the orbit, the only point ℓ orthogonal to H and such that $|\ell(Z_{j_0k_0}(a_0))| = 1$. Then the invariant functions which characterize all the orbits of this category, having the same a_0, j_0 and k_0 are the coordinates of ℓ, *i.e.* the functions:

$$f \mapsto z_{jk}(\lambda)\left(-\frac{1}{a_0}\operatorname{Log}|f(Z_{j_0k_0}(a_0)|\right) \qquad \big((j,k,\lambda) > (j_0,k_0,a_0)\big).$$

For these orbits, O is diffeomorphic to $\mathbb{R}$ and $G(\ell)$ is a subgroup of N.

Third case: all the components $\zeta_{jk}(0)$, $k>1$, are vanishing, as well as all the components $\zeta_{jk}(a)$ $(a\neq 0)$, but there exists a non vanishing component $\zeta_{jk}(a+ib)$ (a and b real and non zero). Let a_0+ib_0 the first $a+ib$ (by convention, we take $b_0>0$), j_0 the smallest j and k_0 the largest k such that $\zeta_{jk}(a+ib)\neq 0$. Then:

$$z_{j_0k_0}(a_0+ib_0)(t)=e^{a_0t}e^{ib_0t}\zeta_{j_0k_0}(a_0).$$

We choose, as a base point of the orbit, the only point ℓ orthogonal to H and such that $|\ell\left(Z_{j_0k_0}(a_0)\right)|=1$. Then the invariant functions which characterize all the orbits of this category, having the same a_0+ib_0, j_0 and k_0 are the coordinates of ℓ, *i.e.* the functions:

$$f\mapsto z_{jk}(\lambda)\left(-\frac{1}{2a_0}\,\mathrm{Log}\,|f(Z_{j_0k_0}(a_0+ib_0)|^2\right)\qquad\big((j,k,\lambda)>(j_0,k_0,a_0+ib_0)\big).$$

For these orbits, O is diffeomorphic to $\mathbb{R}$ and $G(\ell)$ is a subgroup of N.

Fourth case: all the components $\zeta_{jk}(0)$, with $k>1$, are vanishing, as well as all the components $\zeta_{jk}(a)$ $(a\neq 0)$ and all the components $\zeta_{jk}(a+ib)$ (a and b real and non zero), but there exists a non vanishing component $\zeta_{jk}(ib)$ with $k>1$. Let ib_0 be the the first ib (by convention, we take $b_0>0$), j_0 the smallest j, k_0 the largest $k>1$ such that $\zeta_{jk}(ib)\neq 0$. Then:

$$z_{j_0k_0}(ib_0)(t)=e^{ib_0t}\zeta_{j_0k_0}(ib_0),$$

$$z_{j_0(k_0-1)}(ib_0)(t)=e^{ib_0t}\left(\zeta_{j_0(k_0-1)}(ib_0)+t\zeta_{j_0k_0}(ib_0)\right).$$

We choose, as a base point of the orbit, the only point ℓ orthogonal to H and such the quantity $t\mapsto|z_{j_0(k_0-1)}(ib_0)(t)|^2$ is minimal. Then if we replace f by ℓ, this quantity becomes:

$$t^2|\zeta_{j_0k_0}(ib_0)|^2+|\zeta_{j_0(k_0-1)}(ib_0)|^2$$

else, the coordinates of ℓ verify:

$$\zeta_{j_0k_0}(ib_0)\overline{\zeta_{j_0(k_0-1)}(ib_0)}+\overline{\zeta_{j_0k_0}(ib_0)}\zeta_{j_0(k_0-1)}(ib_0)=0.$$

The invariants functions which characterize the orbits of this category having the same b_0, j_0 and k_0 are the coordinats of ℓ, i.e. the functions

$$f\mapsto -z_{jk}(\lambda)\frac{f(Z_{j_0k_0}(ib_0))\overline{f(Z_{j_0(k_0-1)}(ib_0))}+f(Z_{j_0(k_0-1)}(ib_0))\overline{f(Z_{j_0k_0}(ib_0))}}{2\,|f(Z_{j_0k_0}(ib_0))|^2}$$

for $(j,k,\lambda)>(j_0,k_0,ib_0)$. For these orbits, O is diffeomorphic to $\mathbb{R}$ and $G(\ell)$ is a subgroup of N.

Fifth case: all the components $\zeta_{jk}(0)$, with $k>1$, are vanishing, as well as all the components $\zeta_{jk}(a)$ $(a\neq 0)$ and all the components $\zeta_{jk}(a+ib)$ ($a\neq 0$ and

$b \neq 0$, a and b real) and all the components $\zeta_{jk}(ib)$ $(k > 1)$, but there exists a non vanishing component $\zeta_{j1}(ib)$. Then, we have:

$$f = \sum_j \zeta_{j1}(0) Z_{j1}(0)^* + \sum_{ib} \sum_j \zeta_{j1}(ib) Z_{j1}(ib)^*.$$

We call period group of f, the subgroup of $\mathbb{R}$ generated by the numbers b such that there exists j for which $\zeta_{j1}(ib) \neq 0$. For each b, let j_0 be the first j such that $\zeta_{j1}(ib) \neq 0$, by construction, we have:

$$\zeta_{j_0 1}(-ib) = \overline{\zeta_{j_0 1}(ib)}.$$

The period group of f can be written as $\beta\mathbb{Z}$ with $\beta > 0$. This implies that $\mathrm{Coad}(\exp(tH))f = f$ if and only if $t = \frac{2n\pi}{\beta}$ for some integral number n. We choose integral numbers $q(b)$ such that $\beta = \sum_b q(b)b$. Putting $q(-b) = -q(b)$ and $q(b)b = q(-b)(-b)$, we can suppose $q(b) \geq 0$ for any b. Then:

$$\prod_b \left(z_{j_0 1}(b)(t)\right)^{q(b)} = e^{i\beta t} \prod_b \left(\zeta_{j_0 1}(b)\right)^{q(b)}.$$

As a base point in the orbit, we choose the only point ℓ of O for which this quantity is real positive. For orbits in this category, with the same family of b and j_0, we identify each orbit with the coordinates of ℓ, i.e. the functions:

$$f \mapsto z_{jk}(\lambda)\left(-\frac{1}{\beta}\operatorname{Arg}\left(\prod_b \left(f(Z_{j_0 1}(b))\right)^{q(b)}\right)\right).$$

For these orbits, O is diffeomorphic to S^1 and $G(\ell)$ is the non connected subgroup $\exp\left(\frac{2\pi}{\beta}\mathbb{Z}H\right)\exp\left(\mathfrak{g}(\ell)\right)$.

Proposition 3.1. (The orbit description) *There exist six classes of coadjoints orbits in $\mathfrak{g}^*$: the family of trivial orbits and the five classes of non trivial orbits. Each class is the union of the subclasses $\mathcal{C}(\lambda, j_0, k_0)$ constituted by the orbits with the same λ, j_0 and k_0.*

Each subclass is characterized by the value of polynomial functions, constant on the orbits.

The class of trivial orbits is characterized by the nullity of all the functions $f \mapsto f(Z_{jk}(\lambda))$ with $\lambda \neq 0$ or $k > 1$,

a subclass $\mathcal{C}(0, j_0, k_0)$ of the first type is characterized by the vanishing of the functions $f \mapsto f(Z_{jk}(0))$ if $j < j_0$ or $j = j_0$ and $k > k_0$ and the non vanishing of the function $f \mapsto f(Z_{j_0 k_0}(0))$,

a subclass $\mathcal{C}(a_0, j_0, k_0)$ of the second type is characterized by the vanishing of the functions $f \mapsto f(Z_{jk}(0))$ and $f \mapsto f(Z_{jk}(a))$ for $a < a_0$ or $a = a_0$ and $j < j_0$ or $a = a_0$, $j = j_0$ and $k > k_0$ and the non vanishing of the function $f \mapsto f(Z_{j_0 k_0}(a_0))$, *a subclass* $\mathcal{C}(a_0+ib_0,j_0,k_0)$ *of the third type is characterized by the vanishing of the functions* $f\mapsto f(Z_{jk}(0))$, $f\mapsto f(Z_{jk}(a))$ *and* $f\mapsto f(Z_{jk}(a+ib))$ *for* $a+ib<a_0+ib_0$ *or* $a+ib=a_0+ib_0$ *and* $j<j_0$ *or* $a+ib=a_0+ib_0$, $j=j_0$ *and* $k>k_0$ *and the non vanishing of the function* $f\mapsto f(Z_{j_0k_0}(a_0+ib_0))f(Z_{j_0k_0}(a_0-ib_0))$,

a subclass $\mathcal{C}(ib_0,j_0,k_0)$ *of the fourth type is characterized by the vanishing of the functions* $f\mapsto f(Z_{jk}(0))$, $f\mapsto f(Z_{jk}(a))$, $f\mapsto f(Z_{jk}(a+ib))$ *and* $f\mapsto f(Z_{jk}(ib))$ *for* $ib<ib_0$ *and* $k>1$ *or* $ib=ib_0$ *and* $j<j_0$ *and* $k>1$ *or* $ib=ib_0$, $j=j_0$ *and* $k>k_0$ *and the non vanishing of the function* $f\mapsto f(Z_{j_0k_0}(ib_0))f(Z_{j_0k_0}(-ib_0))$,

a subclass $\mathcal{C}(ib_0,j_0,1)$ *of the fifth type is characterized by the vanishing of the functions* $f\mapsto f(Z_{jk}(0))$, $f\mapsto f(Z_{jk}(a))$, $f\mapsto f(Z_{jk}(a+ib))$ *and* $f\mapsto f(Z_{jk}(ib))$ *for* $k>1$ *or* $ib<ib_0$ *or* $ib=ib_0$ *and* $j<j_0$ *or* $ib=ib_0$, $j=j_0$ *and the non vanishing of the function* $f\mapsto f(Z_{j_01}(ib_0))f(Z_{j_01}(-ib_0))$.

4. Separation of generic representations

We built in paragraph 2 the unitary irreducible representations π of G. Let ℓ be in $\mathfrak{g}$ and π be the representation associated to ℓ. The space of this representation is $L^2(O)$, where the restriction of the orbit O of ℓ to $\mathfrak{n}^*$ is parametrized by the map $t\mapsto f(t)=\mathrm{Coad}(\exp\ tH)\ell$.

Thus π is a character if the orbit of ℓ is trivial, or realized in the space $L^2(\mathbb{R})$, if the orbit is in the one of the first four type of subclasses, or realized in the space $L^2\left({}^{\mathbb{R}}/_{\frac{2\pi}{\beta}\mathbb{Z}}\right)$, if the orbit is in a fifth type subclass.

By construction, if Z is an element of $\mathfrak{n}$, the operator $d\pi(Z)$ is the multiplication operator by $if(t)(Z)$.

Especially, if π_1 and π_2 are two unitary irreducible representations with the same moment set and if for example, for π_1, the function

$$t\mapsto f_1(t)(Z_{jk}(\lambda))=z_{jk}^{(1)}(\lambda)(t)$$

is constant, then the same holds for

$$t\mapsto f_2(t)(Z_{jk}(\lambda))=z_{jk}^{(2)}(\lambda)(t)$$

and these two functions coincide. As a consequence, the subclass of the orbit associated to π_1 coincides with the subclass of the orbit associated to π_2.

Moreover the representations π_i $(i=1,\ 2)$ are induced from characters defined by the form ℓ_i such that, for each Jordan bloc $E_j(\lambda)$ of $ad(H)$,

$$k_0=\sup\big\{k\ \text{ such that }\ \ell_1(Z_{jk}(\lambda))\neq 0\big\}=\sup\big\{k\ \text{ such that }\ \ell_2(Z_{jk}(\lambda))\neq 0\big\}.$$

For the separation of two representations which are both in one of the four first type of subclasses, we use the method of [4]. In fact, if π is not in the fifth class, we have $\pi=\mathrm{Ind}_N^G\chi_\ell$, where:

$$\chi_\ell:N\longrightarrow\mathbb{C},\quad \chi_\ell(\exp\ X)=e^{i\ell(X)},\quad \forall X\in\mathfrak{n}.$$

Lemma 4.1. *Let π_1 and π_2 be two unitary irreducible representations of G which are both in one of the first four classes. Suppose $J(\pi_1) = J(\pi_2)$. Then we can choose the point ℓ_i such that there exists a strictly increasing C^∞ function h, such that $h(0) > 0$ and an element u in $\mathcal{U}(\mathfrak{n})$ such that, for each C^∞-vector ϕ_i for π_i ($i = 1, 2$),*

$$(d\pi_1(u)\phi_1)(t) = h(t)\phi_1(t), \quad (d\pi_2(u)\phi_2)(t) = h(t)\phi_2(t), \quad \forall t.$$

Proof. We prove the lemma case by case.

If the representations $\pi_i = \mathrm{Ind}_N^G \chi_{\ell_i}$ are both in a first type subclass, with our notations, there exists j_0 and $k_0 > 1$ such that:

$$\zeta^{(1)}_{j_0k_0}(0) = \ell_1(Z_{j_0k_0}(0)) = \zeta^{(2)}_{j_0k_0}(0) = \ell_2(Z_{j_0k_0}(0)) \neq 0.$$

Then

$$\frac{1}{i}\left(d\pi_i(Z_{j_0k_0}(0))\phi_i\right)(t) = \zeta^{(i)}_{j_0k_0}(0)\phi_i(t),$$

$$\frac{1}{i}\left(d\pi_i(Z_{j_0(k_0-1)}(0))\phi_i\right)(t) = \left(\zeta^{(i)}_{j_0(k_0-1)}(0) + t\zeta^{(i)}_{j_0k_0}(0)\right)\phi_i(t).$$

Replacing ℓ_i by

$$\ell_i' = \mathrm{Coad}\left(\exp\ -\frac{\zeta^{(i)}_{j_0(k_0-1)}(0)}{\zeta^{(i)}_{j_0k_0}(0)}H\right)\ell_i,$$

this relation becomes:

$$\frac{1}{i}\left(d\pi_i(Z_{j_0(k_0-1)}(0))\phi_i\right)(t) = t\zeta^{(i)}_{j_0k_0}(0)\phi_i(t).$$

The lemma holds for

$$h(t) = t\left|\zeta^{(i)}_{j_0k_0}(0)\right| + 1 \quad \text{and} \quad u = \frac{\zeta^{(i)}_{j_0k_0}(0)}{i\left|\zeta^{(i)}_{j_0k_0}(0)\right|}Z_{j(k_0-1)}(0) + 1.$$

If the representations $\pi_i = \mathrm{Ind}_N^G \chi_{\ell_i}$ are both in a second type class, with our notations, there exists a_0, j_0 and k_0 such that:

$$\zeta^{(1)}_{j_0k_0}(a_0) = \ell_1(Z_{j_0k_0}(a_0)) \neq 0, \quad \zeta^{(2)}_{j_0k_0}(a_0) = \ell_2(Z_{j_0k_0}(a_0)) \neq 0.$$

Then

$$\frac{1}{i}\left(d\pi_i(Z_{j_0k_0}(a_0))\phi_i\right)(t) = \zeta^{(i)}_{j_0k_0}(a_0)e^{a_0t}\phi_i(t).$$

Comparing the signs of $\langle d\pi_i(Z_{j_0k_0}(a_0))\phi_i, \phi_i\rangle$, we see that the numbers $\zeta^{(i)}_{j_0k_0}(a_0)$ have the same sign $\varepsilon = \pm 1$. Replacing ℓ_i by

$$\ell_i' = \mathrm{Coad}\left(\exp\ -\frac{1}{a_0}\,\mathrm{Log}\,|\zeta^{(i)}_{j_0(k_0-1)}(a_0)|H\right)\ell_i,$$

we get:

$$\frac{1}{i}\left(d\pi_i(Z_{j_0k_0}(a_0))\phi_i\right)(t) = \varepsilon e^{a_0 t}\phi_i(t).$$

The lemma holds for $h(t) = \frac{a_0}{|a_0|}e^{a_0 t}$ and $u = \frac{\varepsilon a_0}{i|a_0|}Z_{j_0k_0}(a_0)$.

If the representations $\pi_i = \mathrm{Ind}_N^G \chi_{\ell_i}$ are both in a third type class, there exists $a_0 + ib_0$, j_0 and k_0 such that:

$$\zeta^{(1)}_{j_0k_0}(a_0+ib_0) = \ell_1(Z_{j_0k_0}(a_0+ib_0)) \neq 0, \quad \zeta^{(2)}_{j_0k_0}(a_0+ib_0) = \ell_2(Z_{j_0k_0}(a_0+ib_0)) \neq 0.$$

Put

$$Z_{j_0k_0}(a_0+ib_0) = X_{j_0k_0}(a_0+ib_0) + iY_{j_0k_0}(a_0+ib_0)$$

and

$$\zeta^{(i)}_{j_0k_0}(a_0+ib_0) = \xi^{(i)}_{j_0k_0} + i\eta^{(i)}_{j_0k_0},$$

we get:

$$\frac{1}{i}\left(d\pi_i(Z_{j_0k_0}(a_0+ib_0))\phi_i\right)(t) = \zeta^{(i)}_{j_0k_0}(a_0+ib_0)e^{(a_0+ib_0)t}\phi_i(t).$$

Or:

$$\begin{aligned}\frac{1}{i}\left(d\pi_i(X_{j_0k_0}(a_0+ib_0))\phi_i\right)(t) &= e^{a_0t}\left(\xi^{(i)}_{j_0k_0}\cos\ b_0t - \eta^{(i)}_{j_0k_0}\sin\ b_0t\right)\phi_i(t)\\ \frac{1}{i}\left(d\pi_i(Y_{j_0k_0}(a_0+ib_0))\phi_i\right)(t) &= e^{a_0t}\left(\xi^{(i)}_{j_0k_0}\sin\ b_0t + \eta^{(i)}_{j_0k_0}\cos\ b_0t\right)\phi_i(t).\end{aligned}$$

Replacing ℓ_i by

$$\ell_i' = \mathrm{Coad}\left(\exp\ -\frac{1}{a_0}\mathrm{Log}\left(\xi^{(i)\ 2}_{j_0(k_0-1)} + \eta^{(i)\ 2}_{j_0(k_0-1)}\right)H\right)\ell_i$$

and putting $\zeta^{(i)}_{j_0k_0}(a_0+ib_0) = e^{i\theta^{(i)}}$, these relations become:

$$\begin{aligned}\frac{1}{i}\left(d\pi_i(X_{j_0k_0}(a_0+ib_0))\phi_i\right)(t) &= e^{a_0t}\cos(b_0t+\theta^{(i)})\phi_i(t)\\ \frac{1}{i}\left(d\pi_i(Y_{j_0k_0}(a_0+ib_0))\phi_i\right)(t) &= e^{a_0t}\sin(b_0t+\theta^{(i)})\phi_i(t).\end{aligned}$$

Thus:

$$-\left(d\pi_i(X_{j_0k_0}(a_0+ib_0)^2 + Y_{j_0k_0}(a_0+ib_0)^2)\phi_i\right)(t) = e^{2a_0t}\phi_i(t).$$

The lemma holds for $h(t) = \frac{a_0}{|a_0|}e^{2a_0t}$ and

$$u = -\frac{a_0}{|a_0|}\left(X_{j_0k_0}(a_0+ib_0)^2 - Y_{j_0k_0}(a_0+ib_0)^2\right).$$

If the representations $\pi_i = \mathrm{Ind}_N^G \chi_{\ell_i}$ are both in a fourth type subclass, there exists ib_0, j_0 and $k_0 > 1$ such that:

$$\zeta^{(1)}_{j_0k_0}(ib_0) = \ell_1(Z_{j_0k_0}(ib_0)) = \zeta^{(2)}_{j_0k_0}(ib_0) = \ell_2(Z_{j_0k_0}(ib)) \neq 0.$$

Then

$$\frac{1}{i}\left(d\pi_i(Z_{j_0k_0}(ib_0))\phi_i\right)(t) = \zeta^{(i)}_{j_0k_0}(ib_0)e^{ib_0t}\phi_i(t)$$

and

$$\frac{1}{i}\left(d\pi_i(Z_{j_0(k_0-1)}(ib_0))\phi_i\right)(t) = e^{ib_0t}\left(\zeta^{(i)}_{j_0(k_0-1)}(ib_0) + t\zeta^{(i)}_{j_0k_0}(ib_0)\right)\phi_i(t).$$

From these relations, we see:

$$-\langle d\pi_i(Z_{j_0k_0}(ib_0)\overline{Z_{j_0k_0}(ib_0)})\phi_i,\phi_i\rangle = |\zeta^{(i)}_{j_0k_0}|^2\langle\phi_i,\phi_i\rangle$$

so $|\zeta^{(1)}_{j_0k_0}|^2 = |\zeta^{(2)}_{j_0k_0}|^2$.

On the other hand, we have

$$\begin{aligned}-&\left(d\pi_i(Z_{j_0(k_0-1)}(ib_0)\overline{Z_{j_0k_0}(ib_0)} + Z_{j_0k_0}(ib_0)\overline{Z_{j_0(k_0-1)}(ib_0)})\phi_i\right)(t) = \\ &= \left(2t|\zeta^{(i)}_{j_0k_0}(ib_0)|^2 + \zeta^{(i)}_{j(k_0-1)}(ib_0)\overline{\zeta^{(i)}_{j_0k_0}(ib_0)} + \zeta^{(i)}_{j_0k_0}(ib_0)\overline{\zeta^{(i)}_{j_0(k_0-1)}(ib_0)}\right)\phi_i(t).\end{aligned}$$

Replace ℓ_i by

$$\ell_i' = \mathrm{Coad}\left(\exp\ -\frac{\zeta^{(i)}_{j_0(k_0-1)}(ib_0)\overline{\zeta^{(i)}_{j_0k_0}(ib_0)} + \zeta^{(i)}_{j_0k_0}(ib_0)\overline{\zeta^{(i)}_{j_0(k_0-1)}(ib_0)}}{2|\zeta^{(i)}_{j_0k_0}(ib_0)|^2}H\right)\ell_i,$$

this relation becomes:

$$\begin{aligned}-&\left(d\pi_i(Z_{j_0(k_0-1)}(ib_0)\overline{Z_{j_0k_0}(ib_0)} + Z_{j_0k_0}(ib_0)\overline{Z_{j_0(k_0-1)}(ib_0)})\phi_i\right)(t) = \\ &= 2t|\zeta^{(i)}_{j_0k_0}(ib_0)|^2\phi_i(t).\end{aligned}$$

The lemma holds for $h(t) = 2t|\zeta^{(i)}_{j_0k_0}(ib_0)|^2 + 1$ and

$$u = -Z_{j_0(k_0-1)}(ib_0)\overline{Z_{j_0k_0}(ib_0)} - Z_{j_0k_0}(ib_0)\overline{Z_{j_0(k_0-1)}(ib_0)} + 1.$$

Corollary 4.2. (Case of the fourth first type of classes) *Let π_1 and π_2 be two unitary irreducible representations of G such that $J(\pi_1) = J(\pi_2)$. Suppose π_1 is not in a fifth type class, then π_2 is in the same subclass as π_1 and:*

$$\pi_1 \simeq \pi_2.$$

Proof. Indeed, we can now apply lemma 3.1 of [4].

5. The fifth type classes

In the only remaining case, the two representations π_1 and π_2 are in a fifth type class, the only non vanishing quantities $\langle d\pi_i(Z_{jk}(\lambda))\phi_i, \phi_i\rangle$ with $\lambda \neq 0$ are such that $\lambda = ib$ and $k = 1$. For them:

$$\begin{aligned}|\zeta_{j1}^{(1)}(ib)|^2 &= -\langle d\pi_1(Z_{j1}(ib)\overline{Z_{j1}(ib)}\phi_1, \phi_1\rangle \\ &= -\langle d\pi_2(Z_{j1}(ib)\overline{Z_{j1}(ib)}\phi_2, \phi_2\rangle = |\zeta_{j1}^{(2)}(ib)|^2.\end{aligned}$$

The coadjoint orbits associated to π_i are both diffeomorphic to a cylinder with a circular basis.

Moreover, the period groups of ℓ_1 and ℓ_2 coincide since:

$$\text{period group}(\ell_1) = \sum_{b,\zeta_{j1}^{(1)}(ib)\neq 0} b\mathbb{Z} = \sum_{b,\zeta_{j1}^{(2)}(ib)\neq 0} b\mathbb{Z} = \text{period group}(\ell_2).$$

Let $\beta > 0$ be the generator of this period group, the stabilizer $G(\ell_i)$ contains $\exp(\frac{2\pi}{\beta}\mathbb{Z}H)$, there is two characters of $G(\ell_1)N = G(\ell_2)N$ defined by:

$$\chi_{\ell_i,\mu_i}\left(\exp(\frac{2\pi}{\beta}nH)\exp(X)\right) = e^{2i\pi\mu_i n}e^{i\ell_i(X)},$$

where μ_i belongs to $[0, 1[$, such that:

$$\pi_i = \operatorname{Ind}_{G(\ell_i)N}^{G}\chi_{\ell_i,\mu_i}.$$

Lemma 5.1. (Separation of the μ_i) *With our notations, if* $J(\pi_1) = J(\pi_2)$, *then* $\mu_1 = \mu_2$.

Proof. First, we identify the space $\mathcal{H}_{\pi_i}$ to $L^2(S^1)$ by identifying each function $\Phi : G \longrightarrow \mathbb{C}$ of $\mathcal{H}_{\pi_i}$, with the function $\phi : S^1 \longrightarrow \mathbb{C}$ defined by:

$$\phi(t) = e^{i\mu_i t}\Phi(\exp\ \frac{t}{\beta}H).$$

Then we have:

$$\begin{aligned}\left(d\pi_i(\frac{H}{\beta})\phi\right)(t) &= -\left(\frac{d}{dt}\phi(t) + i\mu_i\phi(t)\right) \\ \left(d\pi_i(Z_{j1}(ib))\phi\right)(t) &= e^{-i\frac{b}{\beta}t}\zeta_{j1}^{(i)}(ib)\phi(t).\end{aligned}$$

Now, let us suppose for instance that $0 \leq \mu_2 < \mu_1 < 1$. We can find $\delta > 0$ such that:

$$\delta < \mu_1 - \mu_2 < 1 - \delta.$$

Let ϕ_2 be a C^∞ vector for π_2. In the space $L^2(S^1)$, ϕ_2 is the sum of its Fourier series:

$$\phi_2 = \sum_{n\in\mathbb{Z}} c_n\theta_n.$$

where $\theta_n(t) = e^{int}$.

Let A be the element $-i(\frac{H}{\beta} + i\mu_1)^2$ in $\mathcal{U}(\mathfrak{g})$. We get:

$$\begin{aligned}\frac{1}{i}\langle d\pi_2(A)\varphi_2, \varphi_2\rangle &= \langle -\sum_n c_n \left(i(-\mu_2 - n) + i\mu_1\right)^2 \theta_n, \sum_m c_m\theta_m\rangle \\ &= \sum_n |c_n|^2(\mu_1 - \mu_2 - n)^2.\end{aligned}$$

But: $|\mu_1 - \mu_2 - n| > \delta$ thus:

$$\frac{1}{i}\langle d\pi_2(A)\varphi_2, \varphi_2 >\geq \delta^2\langle\phi_2, \phi_2\rangle,$$

for all ϕ_2, in $\mathcal{H}^\infty_{\pi_2}$. The same inequality holds for any convex combination $\sum_r \nu_r\Psi_{\pi_2}(\phi_2^{(r)})$ of points in the moment set of π_2, *i.e.*

$$\left(\sum_r \nu_r\Psi_{\pi_2}(\phi_2^{(r)})\right)(A) = \sum_r \nu_r\frac{1}{i}\langle d\pi_2(A)\varphi_2^{(r)}, \varphi_2^{(r)}\rangle \geq \delta^2\sum_r \nu_r = \delta^2.$$

On the other hand, θ_0 is a C^∞ vector of the representation π_1 and we have:

$$\left(\Psi_{\pi_1}(\theta_0)\right)(A) = \frac{1}{i}\langle d\pi_1(A)\theta_0, \theta_0\rangle = 0.$$

Then, our hypothesis $\mu_1 \neq \mu_2$ does not hold.

Moreover, we remark that:

$$\langle d\pi_1(u)\theta_0, \theta_0\rangle = \langle d\pi_2(u)\theta_0, \theta_0\rangle$$

for all u in $\mathcal{U}(\mathfrak{g})$. Indeeed, since the moment sets of π_1 and π_2 coincide, then $\Psi_{\pi_1}(\theta_0)$ is a convex combination:

$$\Psi_{\pi_1}(\theta_0) = \sum_r \nu_r\Psi_{\pi_2}(\phi_2^{(r)})$$

of points in $J(\pi_2)$, the only possibility is $\phi_2^{(r)} = c_{r0}\theta_0$ for all r, or

$$\Psi_{\pi_1}(\theta_0) = \Psi_{\pi_2}(\theta_0).$$

Lemma 5.2. (The fifth type classes) *Let π_1 and π_2 be two unitary irreducible representations of G in a fifth type class. Suppose that $J(\pi_1) = J(\pi_2)$, then: $\pi_1 \simeq \pi_2$.*

Proof. As in section three, choose the positif integers $q(b)$ such that:

$$\beta = \sum_{b,\zeta_{j1}^{(i)}(ib)\neq 0} q(b)b.$$

Define now:

$$B = \frac{1}{\prod_b |\zeta_{j1}^{(i)}(ib)|^{q(b)}} \prod_b Z_{j1}(b)^{q(b)}.$$

If $\zeta_{j1}^{(i)}(b) = |\zeta_{j1}^{(i)}(b)|e^{i\alpha_j^{(i)}(b)}$ we get:

$$\left(d\pi_i(Z_{j1}(b)^{q(b)})\phi_i\right)(t) = |\zeta_{j1}^{(i)}(b)|^{q(b)} e^{iq(b)(\alpha_j^{(i)}(b)-\frac{1}{\beta}tb)}\phi_i(t).$$

Thus:

$$\left(d\pi_i(B)\phi_i\right)(t) = e^{i\sum_b q(b)\alpha_j^{(i)}(b)} e^{-it}\phi_i(t)$$

and we get two orthogonal basis of $L^2(S^1)$:

$$\omega_0^{(i)} = \theta_0, \quad \omega_n^{(i)} = d\pi_i(B^n)\theta_0 \quad (n > 0), \quad \omega_n^{(i)} = d\pi_i(\overline{B^{-n}})\theta_0 \quad (n < 0).$$

This defines an intertwining operator for π_1 and π_2, for instance, if n is positive, then, for all u in $\mathcal{U}(\mathfrak{g})$,

$$\begin{aligned}\langle d\pi_1(u)\omega_n^{(1)}, \omega_n^{(1)}\rangle &= \langle d\pi_1(u)d\pi_1(B^n)\theta_0, d\pi_1(B^n)\theta_0\rangle \\ &= \langle d\pi_1(\overline{B^n}uB^n)\theta_0, \theta_0\rangle \\ &= \langle d\pi_2(\overline{B^n}uB^n)\theta_0, \theta_0\rangle \\ &= \langle d\pi_2(u)\omega_n^{(2)}, \omega_n^{(2)}\rangle.\end{aligned}$$

Replacing B by $\overline{B}$, we get the same result for negative n. By polarization, for all n, m,

$$\langle d\pi_1(u)\omega_n^{(1)}, \omega_m^{(1)}\rangle = \langle d\pi_2(u)\omega_n^{(2)}, \omega_m^{(2)}\rangle.$$

Now, let U be the unitary operator of $L^2(S^1)$, defined by $U\omega_n^{(1)} = \omega_n^{(2)}$, we have

$$\begin{aligned}\langle d\pi_2(u)\omega_n^{(2)}, \omega_m^{(2)}\rangle &= \langle d\pi_2(u)U\omega_n^{(1)}, U\omega_m^{(1)}\rangle \\ &= \langle (U^{-1}\circ d\pi_2(u)\circ U)\omega_n^{(1)}, \omega_m^{(1)}\rangle \\ &= \langle d\pi_1(u)\omega_n^{(1)}, \omega_m^{(1)}\rangle\end{aligned}$$

or $U^{-1}\circ d\pi_2(u)\circ U = d\pi_1(u)$, this shows that π_1 and π_2 are unitary equivalent.

The corollary 4.2 and the lemma 5.2 prove our theorem 1.

Acknowledgments. This work was supported by the CMCU contract 00/F 1502. L. Abdelmoula and M. Selmi thank the Institut de Mathématiques de l'Université de Bourgogne and D. Arnal the Faculté des Sciences de Sfax for their kind hospitality during their stay.

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Received April 10, 2003
and in final form October 4, 2003