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On the Principal Bundles over a Flag Manifold

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Abstract. Let P be a parabolic subgroup of a semisimple simply connected linear algebraic group G over $\mathbb{C}$ and ρ an irreducible homomorphism from P to a complex reductive group H. We show that the associated principal H–bundle over G/P, associated for ρ to the principal P–bundle defined by the quotient map $G \longrightarrow G/P$, is stable. We describe the Harder–Narasimhan reduction of the G–bundle over G/P obtained using the composition $P \longrightarrow L(P) \longrightarrow G$, where $L(P)$ is the Levi factor of P.

1. Introduction

Let G be a semisimple simply connected linear algebraic group over the field of complex numbers and P a proper parabolic subgroup of G. So G/P is an irreducible smooth projection manifold, and the projection of G to G/P defines a principal P–bundle over G/P. Let E denote this principal P–bundle over G/P.

Let H be a complex reductive linear algebraic group and

$$\rho : P \longrightarrow H$$

a homomorphism. The homomorphism ρ will be called irreducible if its image is not contained in a proper parabolic subgroup of H.

Let $E(H)$ be the principal H–bundle over G/P obtained by extending the structure group of the P–bundle E using ρ.

We prove that the principal H–bundle $E(H)$ over G/P is stable with respect to any polarization of G/P provided the homomorphism ρ is irreducible (Theorem 2.6).

We recall that the notion of a stable principal bundle was introduced by A. Ramanathan in [Ra1] generalizing the original notion of a stable vector bundle due to D. Mumford.

Fix $T \subset B \subset P$, where T is a maximal torus and B a Borel subgroup of G. Using the pair (B,T), the Levi quotient $L(P) := P/R_u(P)$, where $R_u(P)$ is the unipotent radical of P, gets identified with a subgroup of P. Let $E(L(P))$ be

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the principal $L(P)$–bundle obtained by extending the structure group of E using the quotient homomorphism of P to $L(P)$. Let $E(L(P))(G)$ be the principal G–bundle over G/P obtained by extending the structure group of $E(L(P))$ using the inclusion of the copy of $L(P)$ in G. This G–bundle $E(L(P))(G)$ s not semistable. In Proposition 3.1 we construct the Harder–Narasimhan reduction of $E(L(P))(G)$ for the polarization on G/P defined by $\bigwedge^{\mathrm{top}} TG/P$.

In Section 2 we also consider the special case of $G = \mathrm{SL}(n, \mathbb{C})$. The stability of vector bundles, associated to some naturally occurring $L(P)$–modules, over a flag variety for $\mathbb{C}^n$ has been investigated (for example, the tangent bundle of a Grassmannian is stable).

2. Extension of structure group and stability

Let G be a semisimple simply connected linear algebraic group over $\mathbb{C}$ and $P \subset G$ a parabolic subgroup. A parabolic subgroup will always be assumed to be a proper subgroup. Let H be a connected reductive linear algebraic group over $\mathbb{C}$.

A homomorphism $\rho : P \longrightarrow H$ is called *irreducible* if there is no parabolic subgroup of H that contains $\rho(P)$.

Let $R_u(P)$ be the unipotent radical of P. So the quotient group $L(P) := P/R_u(P)$, which is called the Levi factor of P, is reductive (see [Bo], [Sp]). If $T \subset B \subset P$, where T is a maximal torus and B a Borel subgroup, then $L(P)$ is identified with the T–invariant maximal reductive subgroup of P. Fix T and B as above. Henceforth $L(P)$ will be considered both as a quotient group of P and a subgroup of P.

Lemma 2.1. *Let $\rho : P \longrightarrow H$ is an irreducible homomorphism. Then $\rho(R_u(P)) = e$.*

Let $ZL(P)$ (respectively, $Z(H)$) be the connected component of the center of $L(P)$ (respectively, H) containing the identity element. The homomorphism $L(P) \longrightarrow H$ induced by the irreducible homomorphism ρ takes $ZL(P)$ into $Z(H)$.

Proof. Assume that $\rho(R_u(P)) \neq e$. Consider the unipotent subgroup $U_1 := \rho(R_u(P))$ of H. The normalizer of U_1 in H will be denoted by N_1. Inductively define U_{i+1}, $i \geq 1$, to be the unipotent radical of N_i, and define N_i, $i \geq 1$, to be the normalizer of U_i in H. So we have

$$N_1 \subset N_2 \subset N_3 \subset \cdots \subset H .$$

Let $Q \subset H$ be the direct limit of the subgroups $\{N_i\}$. Note that Q is a proper subgroup of H (as U_1 is nontrivial and H is reductive). Since Q, by construction, is the normalizer of its own unipotent radical, we conclude that Q is a parabolic subgroup of H.

Since $R_u(P)$ is a normal subgroup of P, it follows immediately that $\rho(P) \subset N_1$. So we have $\rho(P) \subset Q$. This contradicts the assumption that ρ is irreducible, and hence $\rho(R_u(P)) = e$.

To prove the second part, consider the image of the torus $ZL(P)$ in H; we will denote this image by Z'. If $Z' \subsetneq Z(H)$, then the centralizer of the torus

$Z' \subset H$ is a Levi subgroup of some parabolic subgroup $Q \subset H$. In that case we have $\rho(P) \subset Q$ (since $\rho(P)$ is contained in the centralizer). But this contradicts the assumption that ρ is irreducible. Hence $Z' \subset Z(H)$ and the proof of the lemma is complete. ■

Proposition 2.2. *Let $\rho : P \longrightarrow H$ be an irreducible homomorphism, and let $\mathfrak{h}$ denote the Lie algebra of H with $\mathfrak{z} \subset \mathfrak{h}$ its center. The center $\mathfrak{z}$ coincides with the space of all invariants of $\mathfrak{h}$ for the adjoint action of P on it. If H is simple, then $\mathfrak{h}$ is an irreducible P–module.*

Proof. Clearly $\mathfrak{z}$ is contained in $\mathfrak{h}^P$, the space of P–invariants.

Take any $\theta \in \mathfrak{h}^P$. Let
$$\theta = \theta_n + \theta_s$$
be the Jordan decomposition, where θ_n is nilpotent and θ_s is semisimple [Bo, page 83]. From the uniqueness of Jordan decomposition if follows immediately that both θ_n and θ_s are individually preserved by P. If
$$\theta_s \notin \mathfrak{z}$$
then the centralizer (in $\mathfrak{h}$) of θ_s is the Levi subalgebra of a parabolic subalgebra. In that case, $\rho(P) \subset Q$, where Q is the parabolic subgroup corresponding to a parabolic subalgebra of $\mathfrak{h}$ containing the centralizer of θ_s. This contradicts the given condition that ρ is irreducible. Therefore, $\theta_s \in \mathfrak{z}$.

Assume that $\theta_n \neq 0$. Let $U_1 \subset H$ be the unipotent subgroup generated by θ_n. Setting U_1 in the construction described in the proof of Lemma 2.1 we get a parabolic subgroup $Q \subset H$. The normalizer N_1 of U_1 in H (see the proof of Lemma 2.1) contains the subgroup of H that fixes θ_n by the adjoint action (as θ_n generates U_1). Therefore, we conclude that
$$\rho(P) \subset N_1 \subset Q\,.$$
This contradicts the given condition that ρ is irreducible. Therefore, $\theta_n = 0$, and hence $\mathfrak{h}^P = \mathfrak{z}$.

Let the group H be simple. Assume that the P–module $\mathfrak{h}$ is not irreducible. Let $0 \neq V \subsetneq \mathfrak{h}$ be a nonzero proper subspace preserved by the adjoint action of P on $\mathfrak{h}$. Let $Q \subset H$ be the subgroup that preserves V by the adjoint action. This Q is a parabolic subgroup of H. Since $\rho(P) \subset Q$, we conclude that no such V exists. This proves that $\mathfrak{h}$ is an irreducible P–module, and the proof of the proposition is complete. ■

The quotient G/P is a smooth complex projective variety. Fix an ample line bundle ζ on G/P. For any coherent sheaf F on G/P, the *degree* of F is defined as
$$\text{degree}(F) := \int_{G/P} c_1(F)c_1(\zeta)^{d-1}\,,$$
where d is the complex dimension of G/P. For any coherent sheaf F' defined on a nonempty Zariski open subset $U' \subset G/P$ with the codimension of the complement of U' at least two, the direct image ι_*F' is a coherent sheaf on G/P, where ι is the inclusion map of U'. The *degree* of F' is defined to be the degree of ι_*F'.

Let H be a complex connected reductive algebraic group and E_H a holomorphic principal H–bundle over G/P. The H–bundle E_H is called *stable* (respectively, *semistable*) if for any reduction of structure group $\sigma : U' \longrightarrow E_H/Q$ of E_H to any maximal parabolic subgroup $Q \subset H$ over a Zariski open subset $U' \subset G/P$, with $\mathrm{codim}(G/P \setminus U') \geq 2$, the following inequality

$$\mathrm{degree}(\iota_* \sigma^* T_{\mathrm{rel}}) > 0$$

(respectively, $\mathrm{degree}(\iota_* \sigma^* T_{\mathrm{rel}}) \geq 0$) holds; here T_{rel} is the relative tangent bundle for the projection $E_H/Q \longrightarrow G/P$ (see [Ra1]).

Let Z_0 denote the connected component of the center of H containing the identity element. Let $E_Q \subset E_H$ be a reduction of structure group of E_H over G/P to a parabolic subgroup $Q \subset H$. This reduction is called *admissible* if for every character χ of Q trivial on Z_0, the associated line bundle $E_Q(\chi)$ over G/P, associated to E_Q for χ, is of degree zero [Ra2, page 307, Definition 3.3].

A holomorphic principal H–bundle E_H over G/P is called *polystable* if either E_H is stable or there is a parabolic subgroup Q of H and a reduction

$$E_{L(Q)} \subset E_H$$

over G/P of structure group of E_H to the Levi factor $L(Q)$ (the quotient $L(Q)$ can be realized as a subgroup of Q) such that

1. the principal $L(Q)$–bundle $E_{L(Q)}$ is stable;
2. the extension of structure group of $E_{L(Q)}$ to Q, constructed using the inclusion of $L(Q)$ in Q, is an admissible reduction of E_H to Q.

(See [Ra2], [RS] for the details.)

A stable H–bundle is polystable, and a polystable H–bundle is semistable. The following simple proposition gives a criterion for a polystable H–bundle to be stable.

Proposition 2.3. *Let $\mathfrak{z} \subset \mathfrak{h}$ be the center of the Lie algebra of H. A polystable H–bundle E_H over G/P is stable if and only if $H^0(G/P, \mathrm{ad}(E_H)) \cong \mathfrak{z}$, where $\mathrm{ad}(E_H)$ is the adjoint vector bundle.*

Proof. If E_H is stable then $H^0(G/P, \mathrm{ad}(E_H)) \cong \mathfrak{z}$ [Ra1, page 136, Proposition 3.2]. On the other hand, if E_H is only polystable but not stable, then there is a reduction of structure group $E_{L(Q)} \subset E_H$ to a Levi factor $L(Q)$ of some parabolic subgroup $Q \subset H$. The center of $L(Q)$ is contained in the automorphism group of $E_{L(Q)}$, and hence the center is contained in the automorphism group of E_H. But the dimension of the center of a Levi subgroup is more than $\dim \mathfrak{z}$. This completes the proof of the proposition. ∎

Note that the projection $G \longrightarrow G/P$ defines a holomorphic principal P–bundle over G/P; this P–bundle will be denoted by E. Let

$$\beta : P \longrightarrow L(P) := P/R_u(P) \tag{1}$$

be the quotient map. Let $E(L(P))$ denote the principal $L(P)$–bundle obtained by extending the structure group of the principal P–bundle E using β in (1).

Lemma 2.4. *The principal $L(P)$–bundle $E(L(P))$ over G/P is stable with respect to any polarization on G/P.*

Proof. A principal $L(P)$–bundle is polystable if and only if admits an Einstein–Hermitian connection [RS], [AB]. We will prove that $E(L(P))$ is polystable by exhibiting an Einstein–Hermitian connection on it.

Fix an ample line bundle ζ on G/P to define degree of a sheaf. Since G is simply connected, the Picard group of G/P is identified with the group of characters of P. Let χ be the character of P that corresponds to ζ.

Fix a maximal compact subgroup $K \subset G$. Set

$$K(P) := K \cap P .$$

Note that $G/P = K/K(P)$ and $K(P)$ projects isomorphically to a maximal compact subgroup of $L(P)$. The maximal compact subgroup of $L(P)$ defined by $K(P)$ will be denoted by $K(L(P))$.

Consider the action of P on $\mathbb{C}$ defined by the character χ (that gives the polarization ζ). Fix a Hermitian structure H_χ on $\mathbb{C}$ fixed by the action of $K(P)$; since $K(P)$ is compact, such a Hermitian structure exists.

Since $G/P = K/K(P)$ and $\zeta = (K\times\mathbb{C})/K(P)$, the condition that $K(P)$ preserves H_χ implies that H_χ induces a Hermitian structure on the line bundle ζ. The curvature of the corresponding Chern connection on H_χ defines a Kähler structure on G/P. This Kähler form will be denoted by Ω_χ. Note that Ω_χ is K–invariant form on G/P representing $c_1(\zeta)$.

Recall that $E(L(P)) = (G \times L(P))/P$, where the action of any $p \in P$ sends any $(g\,,l) \in G \times L(P)$ to $(gp\,,\beta(p)^{-1}l\beta(p))$, with β defined in (1). Since the submanifold $K \times K(L(P)) \subset G \times L(P)$ is $K(P)$–invariant, we have

$$E_K(L(P)) := (K \times K(L(P)))/K(P) \subset (G \times L(P))/K(P) = E(L(P)) . \tag{2}$$

Note that $E_K(L(P))$ in (2) defines a reduction of structure group of the principal $L(P)$–bundle $E(L(P))$ to the the maximal compact subgroup $K(L(P))$. The action of G/P lifts naturally to $E(L(P))$ preserving the holomorphic structure; the action of G on $G\times L(P)$ defined by $g\circ(z\,,l) = (gz\,,l)$, where $g, z \in G$ and $l \in L(P)$, descends to an action of G on $E(L(P)) = (G \times L(P))/P$. Furthermore, the action of $K \subset G$ on $E(L(P))$ preserves $E_K(L(P))$ in (2).

A reduction of structure group to a maximal compact subgroup of a holomorphic principal bundle with a reductive group as a structure group has a unique connection known as the Chern connection which is compatible with the holomorphic structure as well as with the reduction (see [AB], [RS]). Let

$$\Omega_{L(P)} \in C^\infty(G/P, \Omega^{1,1}_{G/P}(\mathrm{ad}(E(L(P))))) \tag{3}$$

be the curvature of the Chern connection on $E(L(P))$ for the reduction of structure group to $K(L(P))$ in (2), where $\mathrm{ad}(E(L(P)))$ is the adjoint bundle. Since the action of K on $E(L(P))$ preserves the holomorphic structure as well as the reduction of structure group to $K(L(P))$ in (2), it follows that the action of K on $\Omega^{1,1}(\mathrm{ad}(E(L(P))))$ preserves the section $\Omega_{L(P)}$ in (3).

Let $\mathfrak{l}(P)$ be the Lie algebra of $L(P)$ and

$$\mathfrak{z}(L) \subset \mathfrak{l}(P)$$

be the center of $\mathfrak{l}(P)$. Since the adjoint action of $L(P)$ on $\mathfrak{z}(L)$ is trivial, we have

$$\mathfrak{z}(L) \subset H^0(G/P, \operatorname{ad}(E(L(P))))\,.$$

Let

$$\Lambda_\chi : \Omega^{i,j}_{G/P} \longrightarrow \Omega^{i-1,j-1}_{G/P}$$

be the adjoint operator of the multiplication operation by the Kähler form Ω_χ on G/P; recall that Ω_χ is the K–invariant form representing $c_1(\zeta)$. We will show that the Chern connection satisfies the Einstein–Hermitian condition which says that

$$\Lambda_\chi \Omega_{L(P)} \in \mathfrak{z}(L) \tag{4}$$

(we showed earlier that $\mathfrak{z}(L)$ defines a subspace of the space of holomorphic sections of $\operatorname{ad}(E(L(P)))$); it should be clarified that the condition in (4) says that there is a fixed element in $\mathfrak{z}(L)$ independent of the point of G/P such that the section $\Lambda_\chi\Omega_{L(P)}$ takes that value at any point of G/P (see [AB, page 220, Definition 3.2], [RS]).

Since both the Chern connection on $E(L(P))$ and the Kähler form Ω_χ on G/P are preserved by the action of K, it follows immediately that the section $\Lambda_\chi\Omega_{L(P)}$ in (4) is preserved by the action of K on $\operatorname{ad}(E(L(P)))$.

The isotropy subgroup at $eP \in G/P$, for the action of K on G/P, is $K(P)$. Since $K(P)$ is a maximal compact subgroup of $L(P)$, we have

$$(\mathfrak{l}(P))^{K(P)} = \mathfrak{z}(L)$$

for the adjoint action of $K(P)$ the Lie algebra of $L(P)$; here $(\mathfrak{l}(P))^{K(P)}$ is the space of all K–invariants. Since K preserves $\Lambda_\chi\Omega_{L(P)}$ we conclude that the evaluation

$$\Lambda_\chi\Omega_{L(P)}(eP) \in \mathfrak{z}(L)\,.$$

Now, since the action of K on G/P is transitive and $\Lambda_\chi\Omega_{L(P)}$ is K–invariant, it follows that $\Omega_{L(P)}$ satisfies the Einstein–Hermitian condition stated in (4).

Therefore, the principal $L(P)$–bundle $E(L(P))$ over G/P is polystable [RS, page 24, Theorem 1], [AB, page 221, Theorem 3.7]. We will use the criterion in Proposition 2.3 to prove the $E(L(P))$ is stable. For that we need the following proposition.

Proposition 2.5. *Let V be a nontrivial irreducible $L(P)$–module such that $V \cong V^*$. Let $E_V = (E(L(P)) \times V)/L(P)$ be the vector bundle over G/P associated to $E(L(P))$ for V. Then,*

$$H^0(G/P, E_V) = 0\,.$$

Proof. To prove the proposition, first note that the action of G on $E(L(P))$ induces an action of G on E_V lifting the action of G on G/P. Assume that $H^0(G/P, E_V) \neq 0$. Take a nonzero holomorphic section s of E_V. Let $W \subset E_V$ be the coherent subsheaf generated by all the translations of s by the elements of G. Since the action of G on G/P is transitive, W is a subbundle of E_V.

The fiber of E_V over $eP \in G/P$ is naturally identified with V (send any $v \in V$ to the element in $(E_V)_{eP}$ defined by (e, v)). The isotropy subgroup P of eP (for the action of G on G/P) acts on the fiber $(E_V)_{eP}$; the action of P induces an action of $L(P)$ on $(E_V)_{eP}$, and the induced action coincides with the $L(P)$–module structure of V.

Since W is generated by all translates of a section, the subspace

$$W_{eP} \subset (E_V)_{eP} = V$$

is left invariant by the action of $L(P)$. This, in view of the given condition that V is an irreducible $L(P)$–module, implies that $W = E_V$. In particular, E_V is globally generated (generated by its global sections).

Since $V \cong V^*$, we have $E_V \cong E_V^*$. Hence the dual vector bundle E_V^* is also globally generated.

We will now prove that E_V is a trivial vector bundle.

Fix a point $x_0 \in G/P$. Take holomorphic sections

$$v_j \in H^0(G/P, E_V),$$

$j \in [1, \dim V]$, such that $\{v_j(x_0)\}_{j=1}^{\dim V}$ is a basis of the fiber $(E_V)_{x_0}$. Now consider the homomorphism from the trivial vector bundle

$$\psi : (G/P) \times \mathbb{C}^{\dim V} \longrightarrow E_V$$

defined by $(z; c_1, \cdots, c_{\dim V}) \longmapsto \sum_{j=1}^{\dim V} c_j v_j(z)$, where $z \in G/P$ and $v_j \in \mathbb{C}$. This homomorphism ψ of vector bundles is an isomorphism over a Zariski open subset of G/P containing x_0 (as it is an isomorphism over x_0). Now, if ψ is not an isomorphism everywhere, then the dual homomorphism

$$\psi^* : E_V^* \longrightarrow (G/P) \times \mathbb{C}^{\dim V}$$

makes E_V^* a proper subsheaf of the vector bundle $(G/P) \times \mathbb{C}^{\dim \mathfrak{h}_i}$. Since all the global sections of a trivial vector bundle $(G/P) \times \mathbb{C}^{\dim V}$ are constant sections, the dimension of the space of all global sections of any proper subsheaf of the vector bundle $(G/P) \times \mathbb{C}^{\dim V}$ is less than $\dim V$. This means that if ψ is not an isomorphism over G/P, then E_V^* is not globally generated. This contradicts the earlier obtained conclusion that E_V^* is globally generated. Therefore, ψ must be an isomorphism. Hence E_V is a trivial vector bundle.

Since V is a nontrivial $L(P)$–module, the associated vector bundle E_V is not trivial, contradicting the earlier observation. Therefore, we conclude that $H^0(G/P, E_V) = 0$. This completes the proof of the proposition. ∎

Continuing with the proof Lemma 2.4, let

$$\mathfrak{l}(P) \cong \mathfrak{z}(L) \oplus (\bigoplus_{i=1}^{m} V_i)$$

be a decomposition of the $L(P)$–module $\mathfrak{l}(P)$ (module structure is defined by the adjoint action), where each $L(P)$–module V_i is nontrivial and irreducible and $\mathfrak{z}(L)$ is the center. Note that as $L(P)$ is reductive, we have $\mathfrak{l}(P) \cong \mathfrak{l}(P)^*$ as $L(P)$–modules. Therefore, each $L(P)$–module V_i is self–dual.

The above decomposition of $\mathfrak{l}(P)$ induces a decomposition

$$\mathrm{ad}(E(L(P))) \cong (G/P \times \mathfrak{z}(L)) \oplus (\bigoplus_{i=1}^{m} E_{V_i}),$$

where $E_{V_i} = (E(L(P)) \times V_i)/L(P)$ is the vector bundle associated to $E(L(P))$ for the $L(P)$–module V_i, and $G/P \times \mathfrak{z}(L)$ is the trivial vector bundle over G/P with fiber $\mathfrak{z}(L)$.

From Proposition 2.5 it follows that

$$H^0(G/P, E_{V_i}) = 0,$$

and hence we have

$$H^0(G/P, \mathrm{ad}(E(L(P)))) = \mathfrak{z}(L).$$

Now using Proposition 2.3 we conclude that $E(L(P))$ is stable. This completes the proof of the lemma. ■

Fix an irreducible homomorphism

$$\rho : P \longrightarrow H,$$

where H is reductive. Let $E(H) := (G \times H)/P$ be the principal H–bundle over G/P obtained by extending the structure group of the P–bundle E using ρ. (The action of any $p \in P$ sends any $(g, h) \in G \times H$ to $(gp, \rho(p^{-1})h)$.)

The following theorem, which follows from Lemma 2.4 and Proposition 2.2, was proved in [Rm] under the two assumptions that $\mathrm{Pic}(G/P) = \mathbb{Z}$ and $H = \mathrm{GL}(n, \mathbb{C})$ (see [Rm, page 168, Theorem 2]).

Theorem 2.6. *Let $\rho : P \longrightarrow H$ be an irreducible homomorphism. The associated principal H–bundle $E(H)$ over G/P is stable with respect to any polarization on G/P.*

Proof. From Lemma 2.1 it follows that $\rho = \rho' \circ \beta$, where

$$\rho' : L(P) \longrightarrow H$$

is a homomorphism and β is defined in (1). Consequently, $E(H)$ is identified with the principal H–bundle over G/P obtained by extending the structure group of $E(L(P))$ using ρ'.

Consider the connection on $E(H)$ induced by the Einstein–Hermitian connection on E constructed in the proof of Lemma 2.4. If

$$\Omega_H \in C^\infty(G/P, \Omega^{1,1}_{G/P}(\mathrm{ad}(E(H))))$$

is the curvature of the induced connection, then

$$\Lambda_\chi \Omega_H = \mathrm{d}\rho'(\Lambda_\chi \Omega_{L(P)}) \in C^\infty(G/P, \mathrm{ad}(E(H))), \tag{5}$$

where $\Lambda_\chi \Omega_{L(P)}$ is as in (4) and

$$\mathrm{d}\rho' : \mathfrak{l}(P) \longrightarrow \mathfrak{h} \tag{6}$$

is the homomorphism of Lie algebras defined by ρ' (here $\mathfrak{h}$ is the Lie algebra of H).

From the second part of Lemma 2.1 it follows that

$$\mathrm{d}\rho'(\mathfrak{z}(L)) \subset \mathfrak{z},$$

where $\mathfrak{z}$ (respectively, $\mathfrak{z}(L)$) is the center of $\mathfrak{h}$ (respectively, $\mathfrak{l}(P)$), and $\mathrm{d}\rho'$ is defined in (6). This and (5) together immediately imply that the connection on $E(H)$ induced by the Einstein–Hermitian connection is also Einstein–Hermitian. Consequently, the H–bundle $E(H)$ is polystable.

Consider the decomposition of the $L(P)$–module

$$\mathfrak{h} \cong \mathfrak{z} \oplus (\bigoplus_{i=1}^{n} V_i')$$

for the adjoint action, where each V_i' is an irreducible $L(P)$–module. From the first part of Proposition 2.2 it follows that each V_i' is nontrivial.

Since $\mathfrak{h}^* \cong \mathfrak{h}$, we have $(V_i')^* \cong V_i'$ for each $i \in [1, n]$. Now using Proposition 2.5 it follows that $H^0(G/P, E_{V_i'}) = 0$, $i \in [1, n]$, where $E_{V_i'}$ is the vector bundle over G/P associated to the $L(P)$–bundle $E(L(P))$ for the $L(P)$–module V_i'. Consequently, we have

$$H^0(G/P, \mathrm{ad}(E(H))) \cong \mathfrak{z},$$

where $\mathrm{ad}(E(H)) = (E(H) \times \mathfrak{h})/H$ is the adjoint bundle (note that $\mathrm{ad}(E(H)) \cong ((G/P) \times \mathfrak{z}) \oplus (\bigoplus_{i=1}^n E_{V_i'})$. Finally, using Proposition 2.3 it follows that $E(H)$ is polystable. This completes the proof of the theorem. ∎

Note that if $\rho : P \longrightarrow H$ is a homomorphism with the property that the homomorphism $\mathrm{d}\rho'$ (defined in (6)) takes the center $\mathfrak{z}(L)$ to $\mathfrak{z}$, then the principal H–bundle $E(H)$ is polystable; the assumption in Theorem 2.6 that ρ is irreducible was used only to prove that the polystable bundle $E(H)$ is stable.

Let V be a complex vector space of dimension n. Let $P \subset \mathrm{SL}(V)$ be a parabolic subgroup. So there is a filtration

$$0 = V_0 \subset V_1 \subset \cdots \subset V_{l-1} \subset V_l = V$$

of subspaces such that P is the space of all automorphisms T of V with $T \in \mathrm{SL}(V)$ and $T(V_i) = V_i$ for all $i \in [0, l]$. The Levi quotient of P is described as follows:

$$L(P) \subset \prod_{i=1}^{l} \mathrm{GL}(V_i/V_{i-1}) \tag{7}$$

is the subgroup defined by all $\prod_{i=1}^l A_i \in \prod_{i=1}^l \mathrm{GL}(V_i/V_{i-1})$, where $A_i \in \mathrm{GL}(V_i/V_{i-1})$, such that $\prod_{i=1}^l \det(A_i) = 1$.

We use the convention that $W^{\otimes 0} := \mathbb{C}$ and $W^{\otimes -j} = (W^*)^{\otimes j}$ if $j \geq 1$. Consider the $L(P)$–module

$$W_{a_1,\cdots,a_l} := \bigotimes_{i=1}^{l} (V_i/V_{i-1})^{\otimes a_i},$$

where $a_i \in \mathbb{Z}$. The center $ZL(P)$ of $L(P)$ acts as scalar multiplications on $W_{a_1,\cdots,a_l}$; in other words, $ZL(P)$ is mapped to the center of $\mathrm{GL}(W_{a_1,\cdots,a_l})$. Therefore, the vector bundle over G/P associated to the $L(P)$–module $W_{a_1,\cdots,a_l}$ is polystable (with respect to any polarization on G/P).

If we have $-1 \leq a_i \leq 1$ for each $i \in [1,l]$, then the homomorphism

$$L(P) \longrightarrow \mathrm{GL}(W_{a_1,\cdots,a_l})$$

defined using the $L(P)$–module structure is clearly irreducible. Therefore, from Theorem 2.6 we conclude that the vector bundle over G/P associated to the $L(P)$–module $W_{a_1,\cdots,a_l}$, where $a_i \in \{-1,0,1\}$ for each i, is stable with respect to any polarization of G/P.

More generally, let W'_i, $i \in [1,l]$, be an irreducible $\mathrm{GL}(V_i/V_{i-1})$–module. (For example, we can take $W'_i = \mathrm{Sym}^{k_i}(V_i/V_{i-1})$.) So

$$W' := \bigotimes_{i=1}^{l} W'_i$$

is a $\prod_{i=1}^{l} \mathrm{GL}(V_i/V_{i-1})$–module, and using the inclusion in (7) W' is a $L(P)$–module. Since each W'_i is an irreducible $\mathrm{GL}(V_i/V_{i-1})$–module, it follows immediately that W' is an irreducible $L(P)$–module. Therefore, Theorem 2.6 says that the vector bundle over G/P associated to the $L(P)$–module W' is stable.

Consider the special case where $l = 2$. So P is a maximal parabolic subgroup and G/P is a Grassmannian. The vector bundle over G/P corresponding to the $L(P)$–module $W_{-1,1}$ is the tangent bundle of the Grassmannian. Therefore, the tangent bundle of a Grassmannian is stable.

Remark 2.7. The connection on $E(H)$ induced by the Einstein–Hermitian connection (constructed in the proof of Lemma 2.4) on the $L(P)$–bundle $E(L(P))$ can be described as follows. Let $K(H) \subset H$ be a maximal compact subgroup such that $\rho(K(P)) \subset K(H)$, where ρ is as in Theorem 2.6 and $K(P) = K \cap P$ as before. Since $E(H)$ is the extension of structure group of E using ρ, and ρ is defined using a homomorphism from $L(P)$ to H, it follows that the reduction of structure group $E_K(L(P)) \subset E(L(P))$ in (2) gives a reduction of structure group of $E(H)$

$$E(H)_{K(H)} \subset E(H)$$

to $K(H) \subset H$. This reduction is constructed using the natural inclusion

$$E(H)_{K(H)} := (E_K(L(P)) \times K(H))/K(L(P)) \subset (E(L(P)) \times H)/L(P) =: E(H).$$

The connection on $E(H)$ induced by the Einstein–Hermitian connection on $E(L(P))$ is identified with the Chern connection on $E(H)$ corresponding to the above reduction of structure group $E(H)_{K(H)}$ to the maximal compact subgroup. Indeed, this identification follows immediately by comparing this Chern connection with the construction of the Einstein–Hermitian connection on $E(L(P))$. As it was noted in the proof of Theorem 2.6, the induced connection on $E(H)$ is the Einstein–Hermitian connection on it.

3. Harder–Narasimhan reduction

The top exterior power of the tangent bundle TG/P is an ample line bundle over G/P. In this section we fix the polarization on G/P defined by $\bigwedge^{\mathrm{top}} TG/P$.

As before, let E be the principal P–bundle defined by the projection of G to G/P. The principal G–bundle $E(G)$ over G/P obtained by extending the structure group of E using the inclusion $P \hookrightarrow G$ is trivial. Indeed, $E(G)$ has a natural section that sends any point $gP \in G/P$ to the point in $E(G)$ defined by $(g\,, g^{-1})$. Therefore, the G–bundle $E(G)$ is trivial.

Now, let $E_L(G)$ be the principal G–bundle obtained by extending the structure group of the $L(P)$–bundle $E(L(P))$ using the inclusion of $L(P)$ in G. The G–bundle $E_L(G)$ is not trivial, in fact, it is not even semistable (as it will be shown later). We will describe its Harder–Narasimhan reduction.

Let $Q \subset G$ be the opposite parabolic of P. So the roots (with respect to the fixed pair $(B\,, T)$) corresponding to the Lie algebra of Q are dual to the roots corresponding to the Lie algebra of P. We have $P \cap Q = L(P)$, so both P and Q share a common Levi subgroup.

Let $E_L(Q)$ be the principal Q–bundle over G/P obtained by extending the structure group of $E(L(P))$ using the inclusion of $L(P)$ in Q. Since $L(P) \subset Q \subset G$, we have

$$E_L(Q) \subset E_L(G)$$

which defines a reduction of structure group of $E_L(G)$ to Q.

Proposition 3.1. *The reduction $E_L(Q) \subset E_L(G)$ to Q is the Harder–Narasimhan reduction of the G–bundle $E_L(G)$ with respect to the polarization on G/P defined by $\bigwedge^{\mathrm{top}} TG/P$.*

Proof. Let $L(Q) := Q/R_u(Q)$ be the Levi quotient, where $R_u(Q)$ is the unipotent radical of Q. Let $E_L(Q)(L(Q))$ be the principal $L(Q)$–bundle obtained by extending the structure group of $E_L(Q)$ using the quotient map $Q \longrightarrow L(Q)$. The first of the two conditions for a Harder–Narasimhan reduction says that $E_L(Q)(L(Q))$ should be semistable (see [AAB, page 694, Theorem 1]).

Since P and Q share a common Levi subgroup $L(P)$, we have $L(Q) \cong L(P)$, and furthermore, $E_L(Q)(L(Q))$ is identified with $E(L(P))$ using the the isomorphism of $L(Q)$ with $L(P)$. From Theorem 2.6 we know that the principal $L(P)$–bundle $E(L(P))$ is semistable. Therefore, the principal $L(Q)$–bundle $E_L(Q)(L(Q))$ is semistable.

Let $R_n(\mathfrak{q})$ be the Lie algebra of the unipotent radical $R_u(Q)$ of Q. Consider the $L(Q)$–module $R_n(\mathfrak{q})/[R_n(\mathfrak{q})\,, R_n(\mathfrak{q})]$. The second and final condition in [AAB, page 694, Theorem 1] for a Harder–Narasimhan reduction says that for any irreducible $L(Q)$–submodule

$$V \subset \frac{R_n(\mathfrak{q})}{[R_n(\mathfrak{q})\ , R_n(\mathfrak{q})]}\,, \tag{8}$$

the associated vector bundle over G/P

$$E_{L(Q)}(V) \;=\; (E_L(Q)(L(Q)) \times V)/L(Q) \tag{9}$$

(associated to the principal $L(Q)$–bundle $E_L(Q)(L(Q))$ for the $L(Q)$–module V) should be of positive degree.

To prove that the vector bundle $E_{L(Q)}(V)$ in (9) is of positive degree, first note that $R_n(\mathfrak{q})$ is identified with the quotient of the Lie algebra of G by the Lie algebra of P. This identification makes $R_n(\mathfrak{q})$ a P–module. Furthermore, the vector bundle $(E \times R_n(\mathfrak{q}))/P$ (associated to the principal P–bundle E for the P–module $R_n(\mathfrak{q})$) is identified with the (holomorphic) tangent bundle TG/P.

Since V in (8) is a quotient of the $L(P)$–module $R_n(\mathfrak{q})$, we conclude that the vector bundle $E_{L(Q)}(V)$ in (9) is a quotient of the tangent bundle TG/P.

The tangent bundle TG/P is polystable of positive degree. Indeed, G/P admits a Kähler–Einstein metric (see [AzBi] for an explicit construction of a Kähler–Einstein metric on G/P); the existence of a Kähler–Einstein metric on G/P implies that TG/P is polystable with respect to the polarization defined by $\bigwedge^{\text{top}} TG/P$. Therefore, any quotient bundle of TG/P, in particular V, is of positive degree. This completes the proof of the proposition. ■

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