Journal of Lie Theory
Volume **14** (2004) 509–522

Injectivity of the Double Fibration Transform for Cycle Spaces of Flag Domains

Alan T. Huckleberry[1] **and Joseph A. Wolf** [2]

Communicated by K.-H. Neeb

Abstract. The basic setup consists of a complex flag manifold $Z = G/Q$ where G is a complex semisimple Lie group and Q is a parabolic subgroup, an open orbit $D = G_0(z) \subset Z$ where G_0 is a real form of G, and a G_0–homogeneous holomorphic vector bundle $\mathbb{E} \to D$. The topic here is the double fibration transform $\mathcal{P} : H^q(D; \mathcal{O}(\mathbb{E})) \to H^0(\mathcal{M}_D; \mathcal{O}(\mathbb{E}'))$ where q is given by the geometry of D, $\mathcal{M}_D$ is the cycle space of D, and $\mathbb{E}' \to \mathcal{M}_D$ is a certain naturally derived holomorphic vector bundle. Schubert intersection theory is used to show that $\mathcal{P}$ is injective whenever $\mathbb{E}$ is sufficiently negative.

Contents

1. Introduction

Let G_0 be a non–compact real form of a complex semisimple Lie group G and consider its action on a compact G–homogeneous projective algebraic manifold $Z = G/Q$. It is of interest to understand the G_0–representation theory associated

[1]Research partially supported by the DFG Schwerpunkt “Global Methods in Complex Geometry”.

[2]Research partially supported by DFG Schwerpunkt “Global Methods in Complex Geometry” and by NSF Grant DMS 99–88643.

ISSN 0949–5932 / \$2.50

to each of its (finitely many) orbits. In this note we restrict to the case of an open orbit $D = G_0(z_0)$, and we often refer to such open orbits as flag domains.

The simplest example of this situation, where $G_0 = SU(1,1)$, $G = SL_2(\mathbb{C})$ and $Z = \mathbb{P}_1(\mathbb{C})$, is at first sight perhaps somewhat misleading. In this case D is either the set of of negative or positive lines and therefore is biholomorphic to the unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. The naturally associated G_0–representations can be regarded as being in L^2–spaces of holomorphic functions on $\mathcal{O}(\Delta)$, or their complex conjugates.

Unless G_0 is of Hermitian type and D is biholomorphically equivalent to the associated bounded symmetric domain as in the above example, the holomorphic functions do not separate the points of D; in fact in most cases $\mathcal{O}(D) \cong \mathbb{C}$. One explanation for this is that D generally contains positive–dimensional compact analytic subsets which are in fact closely related to the representation theory at hand. These arise initially as orbits of maximal compact subgroups K_0 of G_0: *There is a unique complex K_0–orbit C_0 in D* [20].

Of course C_0 can be just a point, which is exactly the case when D is a bounded Hermitian symmetric domain, but in general it has positive dimension.

For example, if $G_0 = SL_3(\mathbb{R})$, $G = SL_3(\mathbb{C})$ and $Z = \mathbb{P}_2(\mathbb{C})$, then the closed G_0–orbit in Z is the set of real points $Z_{\mathbb{R}} = \mathbb{P}_2(\mathbb{R})$ and its complement $D = Z \setminus Z_{\mathbb{R}}$ is the only other orbit. If K_0 is chosen to be the real orthogonal group $K_0 = SO_3(\mathbb{R})$, then

$$C_0 = \{[z_0 : z_1 : z_2] : z_0^2 + z_1^2 + z_2^2 = 0\}$$

is the standard quadric. Here one easily checks that $\mathcal{O}(D) \cong \mathbb{C}$ and, in view of basic results of Andreotti and Grauert [2], looks for G_0–representations in Dolbeault cohomology $H^1(D, \mathbb{E})$, where $\mathbb{E} \to D$ is a sufficiently negative holomorphic vector bundle.

Cohomology classes, e.g., classes of bundle valued differential forms, are technically and psychologically more difficult to handle than holomorphic functions or sections of vector bundles. Thus, for $q := \dim_{\mathbb{C}} C_0$ one is led to consider the space $\mathcal{C}^q(D)$ of q–dimensional cycles in D, where C_0 can be considered as a point.

An element $C \in \mathcal{C}^q(D)$ is a linear combination, $C = n_1 C_1 + \dots n_m C_m$, with $n_j \in \mathbb{N}^{>0}$ and where C_j is a q–dimensional irreducible compact analytic subset of D. It is also necessary to consider $\mathcal{C}^q(Z)$, where the C_j are not required to be contained in D.

The cycle space $\mathcal{C}^q(X)$ of a complex space X has a canonical structure of (locally) finite–dimensional complex space (see [4] and, e.g., [14], for this and other basic properties). In the case at hand, where Z is projective algebraic, $\mathcal{C}^q(Z)$ is often referred to as the Chow variety; in particular, its irreducible components are projective algebraic varieties.

To simplify the notation we redefine $\mathcal{C}^q(Z)$ to be the topological component containing the base cycle C_0. It contains $\mathcal{C}^q(D)$ as an open semialgebraic subset. This is well defined independent of the choice of the maximal compact subgroup K_0.

It is known that the induced action of G on $\mathcal{C}^q(Z)$ is algebraic (see, e.g., [15] for a detailed proof) and therefore the orbit $\mathcal{M}_Z := G.C_0$ is Zariski open in its closure X in $\mathcal{C}^q(Z)$. It should be noted that $\mathcal{M}_Z$ is a spherical homogeneous space of the reductive group G and therefore the Luna–Vust theory of compactifications [5] applies. It would be extremely interesting to determine the G–varieties X which occur in this way.

In general it may be difficult to understand the full cycle space $\mathcal{C}^q(D)$ and therefore one cuts down to a simpler space which is more closely related to the group actions at hand. That simpler space is

$$\mathcal{M}_D: \text{ topological component of } C_0 \text{ in } \mathcal{M}_Z \cap \mathcal{C}^q(D).$$

We have the incidence space

$$\mathfrak{X}_D = \{(z, C) \in D \times \mathcal{M}_D : z \in C\}.$$

and the projections $\mu : \mathfrak{X}_D \to D$ defined by $(z, C) \mapsto z$, and $\nu : \mathfrak{X}_D \to \mathcal{M}_D$ defined by $(z, C) \mapsto C$.

If $\mathbb{E} \to D$ is a holomorphic vector bundle, we may lift it to $\mu^*\mathbb{E} \to \mathfrak{X}_D$ consider the associated ν–direct image sheaves on $\mathcal{M}_D$. In this way Dolbeault cohomology spaces $H^q(D, \mathbb{E})$ are transformed to the level of sections of holomorphic vector bundles $\mathbb{E}' \to \mathcal{M}_D$. This *double fibration transform* is explained in detail in the next section. Since $\mathcal{M}_D$ is now known to be a Stein domain ([21], [17]), our somewhat technical initial setting is transformed to one that is more tractable.

In recent work we developed complex geometric methods aimed at describing the cycle spaces $\mathcal{M}_D$. For example, for a fixed real form, the space $\mathcal{M}_D$ is essentially always biholomorphically equivalent to a universal domain $\mathcal{U}$ which is defined independent of D and Z ([17], [10]).

On the other hand, here we prove that $\mathcal{M}_D$ possesses canonically defined holomorphic fibrations which do indeed depend on D and Z and which give it interesting refined structure. As a consequence we in particular show that the double fibration transform is injective, a fact that should be useful for representation theoretic considerations.

The present paper is organized as follows. In §2 we recall the basics of the double fibration transformation. The recently proved results of [17] and [10] are summarized in §3 and in §4 we give details on the method of Schubert slices. In the final section these slices are used to define the fibrations of $\mathcal{M}_D$ which were mentioned above. The injectivity of the double fibration transform is a consequence of the topological triviality of these bundles.

2. Basics of the Double Fibration Transform

Let D be a complex manifold (later it will be an open orbit of a real reductive group G_0 on a complex flag manifold $Z = G/Q$ of its complexification). We suppose that D fits into what we loosely call a *holomorphic double fibration*. This means that there are complex manifolds $\mathcal{M}$ and $\mathfrak{X}$ with maps

$$\begin{array}{ccc} & \mathfrak{X} & \\ {}^{\mu}\swarrow & & \searrow^{\nu} \\ D & & \mathcal{M} \end{array} \tag{2.1}$$

where μ is a holomorphic submersion and ν is a proper holomorphic map which is a locally trivial bundle. Given a locally free coherent analytic sheaf $\mathcal{E} \to D$ we construct a locally free coherent analytic sheaf $\mathcal{E}' \to \mathcal{M}$ and a transform

$$\mathcal{P} : H^q(D;\mathcal{E}) \to H^0(\mathcal{M};\mathcal{E}') \tag{2.2}$$

under mild conditions on (2.1). This construction is fairly standard, but we need several results specific to the case of flag domains.

Pull–back.

The first step is to pull cohomology back from D to $\mathfrak{X}$. Let $\mu^{-1}(\mathcal{E}) \to \mathfrak{X}$ denote the inverse image sheaf. For every integer $r \geqq 0$ there is a natural map

$$\mu^{(r)} : H^r(D;\mathcal{E}) \to H^r(\mathfrak{X};\mu^{-1}(\mathcal{E})) \tag{2.3}$$

given on the Čech cocycle level by $\mu^{(r)}(c)(\sigma) = c(\mu(\sigma))$ where $c \in Z^r(D;\mathcal{E})$ and where $\sigma = (w_0, \dots, w_r)$ is a simplex. For $q \geqq 0$ we consider the Buchdahl q–condition on the fiber F of $\mu : \mathfrak{X} \to D$:

$$F \text{ is connected and } H^r(F;\mathbb{C}) = 0 \text{ for } 1 \leqq r \leqq q-1. \tag{2.4}$$

Proposition 2.5. (See [6].) *Fix $q \geqq 0$. If* (2.4) *holds, then* (2.3) *is an isomorphism for $r \leqq q-1$ and is injective for $r = q$. If the fibers of μ are cohomologically acyclic then* (2.3) *is an isomorphism for all r.*

As usual, $\mathcal{O}_X \to X$ denotes the structure sheaf of a complex manifold X and $\mathcal{O}(\mathbb{E}) \to X$ denotes the sheaf of germs of holomorphic sections of a holomorphic vector bundle $\mathbb{E} \to X$. Let $\mu^*(\mathcal{E}) := \mu^{-1}(\mathcal{E})\widehat{\otimes}_{\mu^{-1}(\mathcal{O}_D)}\mathcal{O}_{\mathfrak{X}} \to \mathfrak{X}$ denote the pull–back sheaf. It is a coherent analytic sheaf of $\mathcal{O}_{\mathfrak{X}}$–modules. Now $[\sigma] \mapsto [\sigma] \otimes 1$ defines a map $i : \mu^{-1}(\mathcal{E}) \to \mu^*(\mathcal{E})$ which in turn specifies maps in cohomology, the coefficient morphisms

$$i_p : H^p(\mathfrak{X};\mu^{-1}(\mathcal{E})) \to H^p(\mathfrak{X};\mu^*(\mathcal{E})) \quad \text{for } p \geqq 0. \tag{2.6}$$

Our natural pull–back maps are the compositions $j^{(p)} = i_p \cdot \mu^{(p)}$ of (2.3) and (2.6):

$$j^{(p)} : H^p(D;\mathcal{E}) \to H^p(\mathfrak{X};\mu^*(\mathcal{E})) \quad \text{for } p \geqq 0. \tag{2.7}$$

We have $\mathcal{E} = \mathcal{O}(\mathbb{E})$ for some holomorphic vector bundle $\mathbb{E} \to D$, because we assumed $\mathcal{E} \to D$ locally free. Thus $\mu^*(\mathcal{E}) = \mathcal{O}(\mu^*(\mathbb{E}))$ and we realize these sheaf cohomologies as Dolbeault cohomologies. In the context of Dolbeault cohomology, the pull–back maps (2.7) are given by pulling back $[\omega] \mapsto [\mu^*(\omega)]$ on the level of differential forms.

Push–down.

In order to push the $H^q(\mathfrak{X};\mu^*(\mathcal{E}))$ down to $\mathcal{M}$ we assume that

$$\mathcal{M} \text{ is a Stein manifold.} \tag{2.8}$$

Since $\nu : \mathfrak{X} \to \mathcal{M}$ was assumed proper, we have the Leray direct image sheaves $\mathcal{R}^p(\mu^*(\mathcal{E})) \to \mathcal{M}$. Those sheaves are coherent [13]. As $\mathcal{M}$ is Stein

$$H^q(\mathcal{M};\mathcal{R}^p(\mathcal{E})) = 0 \quad \text{for} \quad p \geqq 0 \text{ and } q > 0. \tag{2.9}$$

Thus the Leray spectral sequence of $\nu : \mathfrak{X} \to \mathcal{M}$ collapses and gives

$$H^p(\mathfrak{X}; \mu^*(\mathcal{E})) \cong H^0(\mathcal{M}; \mathcal{R}^p(\mu^*(\mathcal{E}))). \tag{2.10}$$

Definition. The *double fibration transform* for the double fibration (2.1) is the composition

$$\mathcal{P} : H^p(D; \mathcal{E}) \to H^0(\mathcal{M}; \mathcal{R}^p(\mu^*(\mathcal{E}))) \tag{2.11}$$

of the maps (2.7) and (2.10). ∎

In order that the double fibration transform (2.11) be useful, one wants two conditions to be satisfied. They are

$$\mathcal{P} : H^p(D; \mathcal{E}) \to H^0(\mathcal{M}; \mathcal{R}^p(\mu^*(\mathcal{E}))) \text{ should be injective, and} \tag{2.12}$$

$$\text{there should be an explicit description of the image of } \mathcal{P}. \tag{2.13}$$

Assuming (2.8), injectivity of $\mathcal{P}$ is equivalent to injectivity of $j^{(p)}$ in (2.7). The most general way to approach this is the combination of vanishing and negativity in Theorem 2.14 below, taking the Buchdahl conditions (2.4) into consideration.

Given the setting of (2.8) our attack on the injectivity question uses a spectral sequence argument for the relative de Rham complex of the holomorphic submersion $\mu : \mathfrak{X} \to D$. See [23] for the details. The end result is

Theorem 2.14. *Let $\mathcal{E} = \mathcal{O}(\mathbb{E})$ for some holomorphic vector bundle $\mathbb{E} \to D$. Fix $q \geqq 0$. Suppose that the fiber F of $\mu : \mathfrak{X} \to D$ is connected and satisfies* (2.4). *Assume* (2.8) *that $\mathcal{M}$ is Stein. Suppose that $H^p(C; \Omega^r_\mu(\mathbb{E})|_C) = 0$ for $p < q$, and $r \geqq 1$ for every fiber C of $\nu : \mathfrak{X} \to \mathcal{M}$ where $\Omega^r_\mu(\mathbb{E}) \to \mathfrak{X}$ denotes the sheaf of relative $\mu^*\mathbb{E}$–valued holomorphic r–forms on $\mathfrak{X}$ with respect to $\mu : \mathfrak{X} \to D$. Then $\mathcal{P} : H^q(D; \mathcal{E}) \to H^0(\mathcal{M}; \mathcal{R}^q(\mu^*\mathcal{E}))$ is injective.*

Flag domain case.

In the cases of interest to us, D will be a flag domain, we will have $\mathcal{E} = \mathcal{O}(\mathbb{E})$ as in Theorem 2.14, and the transform $\mathcal{P}$ will have an explicit formula. The Leray derived sheaf will be given by

$$\begin{aligned} &\mathcal{R}^q(\mu^*(\mathcal{O}(\mathbb{E}))) = \mathcal{O}(\mathbb{E}') \text{ where} \\ &\quad \mathbb{E}' \to \mathcal{M} \text{ has fiber } H^q(\nu^{-1}(C); \mathcal{O}(\mu^*(\mathbb{E})|_{\nu^{-1}(C)})) \text{ at } C. \end{aligned} \tag{2.15}$$

Then $\mathcal{P}$ will be given on the level of Dolbeault cohomology, as follows. Let ω be an $\mathbb{E}$–valued $(0,q)$–form on D and $[\omega] \in H^q_{\bar\partial}(D, \mathbb{E})$ its Dolbeault class. Then

$$\begin{aligned} &\mathcal{P}([\omega]) \text{ is the holomorphic section of } \mathbb{E}' \to \mathcal{M} \\ &\qquad \text{whose value } \mathcal{P}([\omega])(C) \text{ at } C \in \mathcal{M} \text{ is } [\mu^*(\omega)|_{\nu^{-1}(C)}]. \end{aligned}$$

In other words,

$$\mathcal{P}([\omega])(C) = [\mu^*(\omega)|_{\nu^{-1}(C)}] \in H^0_{\bar\partial}(\mathcal{M}; \mathbb{E}') \tag{2.16}$$

This is most conveniently interpreted by viewing $\mathcal{P}([\omega])(C)$ as the Dolbeault class of $\omega|_C$, and by viewing $C \mapsto [\omega|_C]$ as a holomorphic section of the holomorphic vector bundle $\mathbb{E}' \to \mathcal{M}$.

Now let $D = G_0(z_0)$ be an open orbit in the complex flag manifold $Z = G/Q$, and $\mathcal{M}$ is replaced by the cycle space $\mathcal{M}_D$. Our double fibration (2.1) is replaced by

$$\begin{tikzcd} & \mathfrak{X}_D \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & \\ D & & \mathcal{M}_D \end{tikzcd} \tag{2.17}$$

where $\mathfrak{X}_D := \{(z, C) \in D \times \mathcal{M}_D \mid z \in C\}$ is the incidence space. Given a G_0–homogeneous holomorphic vector bundle $\mathbb{E} \to D$, and the number $q = \dim_{\mathbb{C}} C_0$, we will see in Section 4. that the Leray derived sheaf involved in the double fibration transform satisfies (2.15). Here (2.15) will become a little bit more explicit and take the form

$$\begin{aligned} &\mathcal{R}^q(\mu^*(\mathcal{O}(\mathbb{E}))) = \mathcal{O}(\mathbb{E}') \text{ where} \\ &\qquad\qquad \mathbb{E}' \to \mathcal{M}_D \text{ has fiber } H^q(C; \mathcal{O}(\mathbb{E}|_C)) \text{ at } C \in \mathcal{M}_D\ . \end{aligned} \tag{2.18}$$

Evidently, $\mathbb{E}' \to \mathcal{M}_D$ will be globally G_0–homogeneous. It cannot be G–homogeneous unless $\mathcal{M}_D$ is G–invariant, and that only happens in the degenerate case where G_0 is transitive on Z. However, in Section 4. we will see that $\mathbb{E}' \to \mathcal{M}_D$ is the restriction of a G–homogenous holomorphic vector bundle $\widetilde{\mathbb{E}'} \to \mathcal{M}_{\mathcal{Z}}$, and in particular (2.15) is satisfied. In any case, $H^q(C; \mathcal{O}(\mathbb{E}|_C))$ can be calculated from the Bott–Borel–Weil Theorem. Thus $\mathcal{R}^q(\mu^*(\mathcal{O}(\mathbb{E})))$ will be given explicitly by (2.18) in the flag domain case.

Using methods of complex geometry as described in Section 3. below it was shown ([17] plus [10]) that $\mathcal{M}_D$ is biholomorphically equivalent to a certain universal domain $\mathcal{U}$ or to a bounded symmetric domain. It follows in general that $\mathcal{M}_D$ is a contractible Stein manifold, so $\mathbb{E}' \to \mathcal{M}_D$ is holomorphically trivial. We will use those same methods in Section 5. to construct certain holomorphic fibrations of the $\mathcal{M}_D$ and use those "Schubert fibrations" to show that F satisfies (2.4) for all q. That is how we will prove that the double fibration transforms are injective. Thus, in the flag domain case, we will have a complete answer to (2.12) and some progress toward (2.13).

3. A Computable Description of $\mathcal{M}_D$

In order to understand the structure of Z, D and $\mathcal{M}_D$ we may assume that G_0 is simple, because G_0 is local direct product of simple groups, and Z, D and $\mathcal{M}_D$ break up as global direct products along the local direct product decomposition of G_0. From this point on G_0 is simple unless we say otherwise. We also assume that G_0 is not compact, and we avoid the two trivial noncompact cases (see [22]), where G_0 acts transitively on Z so that $\mathcal{M}_D$ is reduced to a single point.

As above let $\mathcal{M}_Z := G.C_0$. Since C_0 is a complex manifold, the isotropy group G_{C_0} contains the complexification K of K_0, which is a closed complex subgroup of G. If G_0 is not Hermitian, then $\mathfrak{k}$ is a maximal subalgebra of $\mathfrak{g}$, so G_{C_0}/K is finite. In this nonhermitian case, without further discussion, we replace $\mathcal{M}_Z$ by the finite cover G/K. There is no loss of generality in doing this because, as we will see later, $\mathcal{M}_D$ pulls back biholomorphically to a domain in G/K under the finite covering $G/K \to G.C_0$.

If G_0 is Hermitian, then the symmetric space G_0/K_0 possesses two invariant complex structures, $\mathcal{B}$ and $\overline{\mathcal{B}}$, as bounded Hermitian symmetric domains. These are realized as open orbits D in their compact duals G/P and $G/\bar{P}$ respectively. In these cases the base cycles are just the K_0 fixed points and clearly G_{C_0} is either P or $\overline{P}$ in such a situation; in particular, as opposed to being the affine homogeneous space G/K, the space $\mathcal{M}_Z$ is a compact homogeneous manifold.

The above mentioned phenomenon is characterized by G_{C_0} being either P or $\bar{P}$. It occurs in more interesting situations than just that where D itself is a bounded domain, but it should be regarded as an exceptional case which is completely understood [21]. In particular, in any such example the cycle space $\mathcal{M}_D$ is either $\mathcal{B}$ or $\overline{\mathcal{B}}$.

In all other cases it was recently shown that $\mathcal{M}_D$ is naturally biholomorphic to a universal domain $\mathcal{U} \subset \mathcal{M}_Z = G/K$ ([10], [17]). It should be emphasized that $\mathcal{U}$, which we now define, depends *only* on the real form G_0, and not on D.

The domain $\mathcal{U}$ can be defined in a number of different ways. We choose the historical starting point which is of a differential geometric nature.

Let M denote the Riemannian symmetric space G_0/K_0 of negative curvature and consider its tangent bundle TM. As usual let θ be a Cartan involution of $\mathfrak{g}$ that commutes with complex conjugation over its real form $\mathfrak{g}_0$. θ defines the Cartan decompositions $\mathfrak{g} = \mathfrak{g}_u \oplus i\mathfrak{g}_u$ and $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{s}_0$, where $\mathfrak{g}_u$ is the compact real form of $\mathfrak{g}$ with $\mathfrak{k}_0 = \mathfrak{g}_u \cap \mathfrak{g}_0$.

We identify $\mathfrak{s}_0$ with TM_{x_0}, where x_0 is the base point with G_0–isotropy K_0, and regard TM as the homogeneous bundle $G_0 \times_{K_0} \mathfrak{s}_0$. One defines the *polar coordinates* mapping by

$$\Pi : TM \to G/K, \;\; ([g_0, \xi]) \mapsto g_0 \exp(i\xi).x_0.$$

Clearly Π is a diffeomorphism in a neighborhood of the 0–section and thus it is of interest to consider the canonically defined domain

$$\Omega_{max} := \{v \in TM : \mathrm{rank}(\Pi_*(v)) = \dim TM\}^0.$$

Here the connected component is that which contains the 0–section.

It turns out that Ω_{max} is determined by differential geometric properties of the compact dual $N := G_u.x_0 = G_u/K_0$, where G_u is the maximal compact subgroup of G defined by $\mathfrak{g}_u$. For this it is essential that the geodesics emanating from x_0 in N are just orbits $\exp(i\xi).x_0$, $\xi \in \mathfrak{s}_0$, of 1–parameter groups.

Let $\frac{1}{2}N$ denote the set of points in N which are at most halfway from x_0 to the cut point locus and define $\Omega_C = G_0.\frac{1}{2}N$.

Theorem 3.1. (Crittenden [8]) *The polar coordinates mapping* Π *restricts to a diffeomorphism from* Ω_{max} *to* Ω_C.

The domain Ω_C can be computed in an elementary way. For this regard K_0 as acting on $\mathfrak{s}_0$ by the adjoint representation, let $\mathfrak{a}_0$ be a maximal Abelian subalgebra in $\mathfrak{s}_0$ and recall that $K_0.\mathfrak{a}_0 = \mathfrak{s}_0$. Thus Ω_C is determined by a domain in $\mathfrak{a}_0$. This is computed as follows.

Since $\mathfrak{a}_0$ acts on $\mathfrak{g}_0$ as a commutative algebra of self–adjoint transformations, the associated joint eigenvalues $\alpha \in \mathfrak{a}_0^*$ are real valued. (The nonzero ones are the $(\mathfrak{g}_0, \mathfrak{a}_0)$–roots or *restricted* roots.) Consider the convex polygon

$$V = \{\xi \in \mathfrak{a}_0 \mid |\alpha(\xi)| < \tfrac{\pi}{2} \text{ for all restricted roots } \alpha\}$$

and define

$$\mathcal{U} := G_0. \exp(iV).x_0.$$

Theorem 3.2. ([8], [1]) $\Omega_C = \mathcal{U}$.

Remark 3.3. The domain $\mathcal{U}$, which is indeed explicitly computable, was first brought to our attention by the work in [1]. As a consequence we originally denoted it by Ω_{AG}. It turns out that it is naturally equivalent to a number of other domains, including the cycle spaces, which are defined from a variety of viewpoints. So now, unless we have a particular construction in mind, we denote it by $\mathcal{U}$ to underline its universal character.

Proposition 3.4. *The domain $\mathcal{U}$ is contractible.*

Proof. For this we regard $\mathcal{U}$ as being contained in TM. Since V is a (polyhedral) domain in $\mathfrak{s}_0$ star–shaped from 0, scalar multiplication $\varphi_t : TM \times TM$, $v \mapsto tv$, stabilizes V, and for $0 \leq t \leq 1$ defines a strong deformation retraction of $\mathcal{U}$ to the 0–section M. That 0–section is also contractible. ∎

Complex geometric properties of $\mathcal{U}$ are also of importance. These include the fact that $\mathcal{U}$ is a Stein domain in $\mathcal{M}_{\mathcal{Z}}$ ([7]; also see [3], [16], [17], [11], [12]).

For later reference let us state the main points in the context of cycle spaces.

Theorem 3.5. *If D is not one of the above exceptions where $\mathcal{M}_D$ is either a single point or $\mathcal{B}$ or $\bar{\mathcal{B}}$, then $\mathcal{M}_D = \mathcal{U}$. Thus in all cases $\mathcal{M}_D$ is contractible and Stein.*

Corollary 3.6. *Let $P \subset Q$ be parabolic subgroups of G. Let π denote the natural projection $1P \mapsto 1Q$ of $W = G/P$ onto $Z = G/Q$. Suppose that the flag domain $D \subset Z$ is not one of the above exceptions where $\mathcal{M}_D$ is either a single point or $\mathcal{B}$ or $\bar{\mathcal{B}}$. Let $\widetilde{D} \subset W$ be a flag domain such that $\pi(\widetilde{D}) = D$. Then π induces a holomorphic diffeomorphism of $\mathcal{M}_{\widetilde{D}}$ onto $\mathcal{M}_D$.*

4. Globalization of the Bundles

We will show that various bundles are restrictions of bundles homogeneous under the complex group G, and use that to carry bundles over D to bundles over $\mathcal{M}_D$. For that we need an old result on homogeneous holomorphic vector bundles from [19, Section 3].

$M = A_0/B_0$ be a homogeneous complex manifold. Let $p : A_0 \to M$ denote the natural projection. View the Lie algebra $\mathfrak{a}_0$, and thus its complexification $\mathfrak{a}$, as Lie algebras of holomorphic vector fields on M. Let $m \in M$ denote the base point $1B_0$ and define $\mathfrak{p} = \{\xi \in \mathfrak{a} \mid \xi_m = 0\}$. Then $\mathfrak{p}$ is an $\mathrm{Ad}(B_0)$–stable complex subalgebra of $\mathfrak{a}$ such that $\mathfrak{a} = \mathfrak{p} + \overline{\mathfrak{p}}$ and $\mathfrak{b} = \mathfrak{p} \cap \overline{\mathfrak{p}}$. Here $\mathfrak{b}$ is the complexification of the Lie algebra $\mathfrak{b}_0$ of B_0.

Let χ be a continuous representation of B_0 on a finite dimensional complex vector space $E = E_\chi$. By *extension of* χ to $\mathfrak{p}$ we mean a Lie algebra representation λ of $\mathfrak{p}$ on E such that $\lambda|_{\mathfrak{b}_0} = d\chi$ and $\lambda(\mathrm{Ad}(b)\xi) = \chi(b)\lambda(\xi)\chi(b)^{-1}$ for all $b \in B_0$ and $\xi \in \mathfrak{p}$. Thus an extension of χ to $\mathfrak{p}$ is a $(\mathfrak{p}, B_0)$–module structure on E_χ.

The representation χ defines a real analytic, A_0–homogeneous, complex vector bundle $\mathbb{E}_\chi \to M = A_0/B_0$, by $\mathbb{E}_\chi = A_0 \times_{B_0} E_\chi$. We identify local sections $s : U \to \mathbb{E}_\chi$ with functions $f_s : p^{-1}(U) \to E_\chi$ such that $f_s(gb) = \chi(b)^{-1} f_s(b)$.

The holomorphic vector bundle structures on $\mathbb{E}_\chi \to M$ are given as follows.

Theorem 4.1. [19, Theorem 3.6] *The structures of A_0–homogeneous holomorphic vector bundle on $\mathbb{E}_\chi \to M$ are on one to one correspondence with the extensions λ of χ from B_0 to $\mathfrak{p}$. The structure corresponding to λ is the one for which the holomorphic sections s over any open set $U \subset M$ are characterized by $\xi \cdot f_s + \lambda(\xi) f_s = 0$ on $p^{-1}(U)$ for all $\xi \in \mathfrak{p}$.*

Now we return to the flag domain setting and extend the double fibration (2.17) to

$$\begin{array}{ccc} & \mathfrak{X}_Z & \\ \widetilde{\mu} \swarrow & & \searrow \widetilde{\nu} \\ Z & & \mathcal{M}_Z \end{array} \tag{4.2}$$

where $\mathcal{M}_Z = \{gC_0 \mid g \in G\}$ and $\mathfrak{X}_Z := \{(z, C) \in Z \times \mathcal{M}_Z \mid z \in C\}$ is the incidence space. Evidently $\mathcal{M}_D$ is an open submanifold of $\mathcal{M}_Z$, and μ and ν are the respective restrictions of $\widetilde{\mu}$ and $\widetilde{\nu}$.

Theorem 4.3. *Let D be an open G_0–orbit on Z, let $\mathbb{E} \to D$ be a G_0–homogeneous holomorphic vector bundle, and let $q \geqq 0$. Suppose $G_0 \subset G$. Then (1) $\mathbb{E} \to D$ is the restriction of a G–homogeneous holomorphic vector bundle $\widetilde{\mathbb{E}} \to Z$, (2) the Leray derived sheaf for $\widetilde{\nu}$ is given by $\mathcal{R}^q(\mathcal{O}(\widetilde{\mu}^*\widetilde{\mathbb{E}})) = \mathcal{O}(\widetilde{\mathbb{E}}')$ where $\widetilde{\mathbb{E}}' \to \mathcal{M}_Z$ is the G–homogeneous, holomorphic vector bundle with fiber $H^q(C; \mathcal{O}(\widetilde{\mathbb{E}}|_C))$ over $C \in \mathcal{M}_Z$, and (3) the Leray derived sheaf for ν is given by $\mathcal{R}^q(\mathcal{O}(\mu^*\mathbb{E})) = \mathcal{O}(\mathbb{E}')$ where $\mathbb{E}'$ is the restriction of $\widetilde{\mathbb{E}}'$ to $\mathcal{M}_D$.*

Proof. We translate the result of Theorem 4.1 to our situation of the flag domain $D = G_0(z) \cong G_0/L_0$ where $L_0 = G_0 \cap Q_z$ is the isotropy subgroup of G_0 at z. Here G_0 replaces A_0, L_0 replaces B_0, $E = E_\chi$ is the fiber of $\mathbb{E} \to D$ over z, the representation χ is the action of L_0 on E, and $\mathfrak{q}_z$ replaces $\mathfrak{p}$. The homogeneous holomorphic vector bundle structure on $\mathbb{E} \to D$ comes from an extension λ of χ from L_0 to $\mathfrak{q}_z$. Here λ integrates to L_0 by construction and then to Q_z because $G_0 \subset G$. Thus we have a holomorphic representation $\widetilde{\chi}$ of Q_z on E that extends χ. That defines the G–homogeneous holomorphic vector bundle $\widetilde{\mathbb{E}} \to G/Q_z = Z$, and $\mathbb{E} = \widetilde{\mathbb{E}}|_D$ by construction. Statement (1) is proved.

By G–homogeneity of $\widetilde{\mathbb{E}} \to Z$ all the $\widetilde{\mathbb{E}}|_C \to C$ are holomorphically equivalent. Now (2) follows from the construction of Leray derived sheaves. The same considerations prove (3). ∎

5. The Method of Schubert Slices

Schubert slices play a major role in the classification part of the above theorem (see [10] and [17]). Here we present the mini–version of this theory which is all that is needed for our applications in §6. (see [16]).

Let us begin by recalling that a Borel subgroup B of G has only finitely many orbits in Z. Such an orbit $\mathcal{O}$ is called a Schubert cell, $\mathcal{O} \cong \mathbb{C}^{m(\mathcal{O})}$, and its closure $S = \bar{\mathcal{O}} = \mathcal{O}\dot{\cup}Y$ is referred to as the associated Schubert variety. The set $\mathcal{S} = \{S\}$ of all B–Schubert varieties freely generates the integral homology $H_*(Z)$.

If $G_0 = K_0A_0N_0$ is an Iwasawa decomposition and $B \supset A_0N_0$, then we refer to B as an Iwasawa–Borel subgroup. Recalling that a given Borel subgroup has a unique fixed point in $Z = G/Q$, it follows that the Iwasawa–Borel groups are exactly those which fix a point of the closed G_0–orbit in Z.

We now prove several elementary propositions.

Proposition 5.1. *Let D be an open G_0–orbit in Z, C_0 the base cycle in D and $z \in D$. Then $A_0N_0.z \cap C_0 \neq \emptyset$. In particular, if B is an Iwasawa–Borel subgroup, then every B–orbit $\mathcal{O}$ which has non–empty intersection with D satisfies $\mathcal{O} \cap C_0 \neq \emptyset$.*

Proof. Since $G_0 = A_0N_0K_0$ and C_0 is a K_0–orbit in D, it is immediate that $D = A_0N_0.C_0$. ■

The following holds for similar reasons.

Lemma 5.2. *If $z \in D$ then $T_z(K_0.z) + T_z(A_0N_0.z) = T_zD$.*

Proposition 5.3. *If S is a Schubert variety of an Iwasawa–Borel subgroup B with* $\mathrm{codim}_{\mathbb{C}}S > q$, *then $S \cap D = \emptyset$.*

Proof. Let $\mathcal{O}$ be the open dense B–orbit in S. If $S \cap D \neq \emptyset$, then $\mathcal{O} \cap D \neq \emptyset$ as well. Thus, by Proposition 5.1, $\mathcal{O} \cap C_0 \neq \emptyset$. But if $z \in \mathcal{O} \cap C_0$ then $\dim_{\mathbb{R}} A_0N_.z <$ $\mathrm{codim}_{\mathbb{R}}K_0.z$ would contradict Lemma 5.2. ■

We now come to a basic fact.

Proposition 5.4. *If* $\mathrm{codim}_{\mathbb{C}}S = q$ *and $S \cap D \neq \emptyset$, then $S \cap C_0 = \{z_1, \ldots z_d\}$ is non–empty, finite and contained in the B–orbit $\mathcal{O}$. This intersection is transversal in the sense that*

$$T_{z_i}C_0 \oplus T_{z_i}\mathcal{O} = T_{z_i}Z$$

for all i. Furthermore, $\Sigma_i := A_0N_0.z_i$ is open in $\mathcal{O}$ and closed in D.

Proof. Proposition 5.1 tells us that $S \cap C_0$ is non–empty. If $S \cap C_0$ is infinite, then it has positive dimension at one or more of its points, contrary to Lemma 5.2. The same argument proves the transversality and the fact that $A_0N_0.z_i$ is open in $\mathcal{O}$. If Σ_i were not closed in $\mathcal{O}$, then we would find an A_0N_0–orbit of smaller dimension on its boundary. By Proposition 5.1 this A_0N_0–orbit would meet C_0, contrary to Lemma 5.2. ■

An A_0N_0–orbit Σ as above is called a *Schubert slice.* Since $[C_0]$ is non–zero in $H_*(Z;\mathbb{Z})$ and $\mathcal{S} = \{S\}$ generates this homology, every Iwasawa–Borel subgroup B gives us q–codimensional Schubert varieties with non–empty intersection $S \cap D$. In particular, there exist such Schubert slices.

It is known that D is retractable to C_0 ([20], see also [17]) and therefore $\pi_1(D) = 1$. Let us translate this into a statement on the isotropy groups along the base cycle.

Lemma 5.5. *If Σ is a Schubert slice and $z \in \Sigma \cap C_0$, then*

$$\alpha : (K_0)_z \times (A_0N_0)_z \to (G_0)_z,\ \ (k_0, a_0n_0) \mapsto k_0a_0n_0,$$

is bijective.

Proof. Since K_0 is compact and A_0N_0 is closed in G_0, it follows that α is a diffeomorphism onto a closed submanifold of G_0. A dimension count shows that it is open in G_z. But G_z is connected, because $\pi_1(D) = 1$. Thus the image of α is all of G_z. ∎

We now come to the main result of this section.

Theorem 5.6. *If Σ is a Schubert slice, then for every $C \in \mathcal{M}_D$ the intersection $\Sigma \cap C$ is transversal and consists of exactly one point.*

Proof. First, we prove this for $C = C_0$. Let $z_0 \in C_0 \cap \Sigma$. If $z_1 = a_0n_0.z_0$ were any other point in this intersection, then $z_1 = k_0^{-1}.z_0$ for some $k_0 \in K_0$. But then $g_0 = k_0a_0n_0 \in (G_0)_{z_0}$ and consequently by Lemma 5.5 both k_0 and a_0n_0 fix z_0, i.e., $z_1 = z_0$ is the unique point in $C_0 \cap \Sigma$.

Now let $C \in \mathcal{M}_D$ be an arbitrary element of the cycle space. Denote by $\mathcal{O}$ the open B-orbit in the Schubert variety $S = \mathcal{O} \dot{\cup} Y$ which contains Σ. Since $\mathcal{O} \cong \mathbb{C}^{m(\mathcal{O})}$ is Stein, if $C \cap \Sigma$ were positive–dimensional then $C \cap Y \neq \emptyset$. But $\mathrm{codim}_{\mathbb{C}} Y > q$ and therefore $Y \cap D = \emptyset$. Thus this would be contrary to $C \subset D$. Now $C \cap D$ is finite.

If $C \cap \Sigma$ is empty we obtain a contradiction as follows. Let C_t be a curve from C_0 to C in $\mathcal{M}_D$. Define

$$t_0 := \sup\{t : C_s \cap \Sigma \neq \emptyset \text{ for all } s < t\}.$$

Then there is a sequence $\{z_n\} \subset D$ with $z_n \in C_{t_n}$ corresponding to $\{t_n\}$ such that $\{t_n\} \to t_0$ and $\{z_n\} \to z_0 \in \mathrm{bd}(\Sigma) \subset \mathrm{bd}(D)$. Therefore $C_{t_0} \cap \mathrm{bd}(D) \neq \emptyset$, contrary to $C_{t_0} \in \mathcal{M}_D$.

This argument holds for each of the Σ_i. In particular, $C \cap S$ contains at least d distinct (isolated) points. Since $[C].[S] = d$, it follows that $|C \cap S| = d$ and that the intersection at each of these points is transversal. ∎

6. Canonical Fibrations and DFT–Injectivity

Let Σ be a Schubert slice defined by an Iwasawa–Borel subgroup B, denote $\{z_0\} = \Sigma \cap C_0$, In the context of the double fibration (2.17), the projection ν

carries the fiber $F = \mu^{-1}(z_0) = \{(z_0, C) : z_0 \in C\}$ biholomorphically onto the analytic subset $\{C \in \mathcal{M}_D : z_0 \in C\}$ of $\mathcal{M}_D$.

Now consider an arbitrary element $C \in \mathcal{M}_Z$ with $z_0 \in C$. By definition $C = g(C_0)$ for some $g \in G$. Since $z_0 \in C$, by adjusting g by an appropriate element of K_0 we may assume that $g \in Q = G_{z_0}$. Thus $\widetilde{F} \cong \{C \in \mathcal{M}_Z : z_0 \in C\} = Q.C_0$. $\widetilde{F}$ is closed in $\mathcal{M}_Z$, for if a net $\{C_i\}$ in $\widetilde{F}$ converges to $C \in \mathcal{M}_Z$ then $z_0 \in C$ because $z_0 \in C_i$ for each i, so $C \in \widetilde{F}$.

The Z–analog of the double fibration (2.17) is given by (4.2). There the Q–orbit $\widetilde{F} \subset \mathcal{M}_Z$ is identified with the fiber $\widetilde{\mu}^{-1}(z_0)$. In particular, its open subset $F = \widetilde{F} \cap \mathcal{M}_D$ is a closed complex submanifold on $\mathcal{M}_D$.

By Theorem 5.6, an A_0N_0–equivariant map $\varphi : \mathcal{M}_D \to \Sigma$ is defined by mapping C to its point of intersection with Σ. The fiber over $z_0 \in \Sigma$ is of course F.

Let $J_0 := (A_0N_0)_{z_0}$ be the A_0N_0–isotropy at the base point and note that

$$A_0N_0 \times_{J_0} F \to \mathcal{M}_D, \quad \text{defined by } [(a_0n_0, C)] \mapsto a_0n_0(C),$$

is well–defined, smooth and bijective. Thus $\varphi : \mathcal{M}_D \to \Sigma$ is naturally identified with the smooth A_0N_0–equivariant bundle

$$\pi_\Sigma : A_0N_0 \times_{J_0} F \to A_0N_0/J_0 = \Sigma.$$

In this sense, every Schubert slice defines a *Schubert fibration* of the cycle space $\mathcal{M}_D$.

Theorem 6.1. *Let B be an Iwasawa–Borel subgroup of G and Σ an associated Schubert slice for the open orbit D. Then the fibration $\pi_\Sigma : \mathcal{M}_D \to \Sigma$ is a holomorphic map on to a contractible base Σ and diffeomorphically realizes $\mathcal{M}_D$ as the product $\Sigma \times F$.*

Proof. Let $J := B_{z_0}$. The inclusions $A_0N_0 \hookrightarrow B$ and $F \hookrightarrow \widetilde{F}$ together define a map $A_0N_0 \times_{J_0} F \hookrightarrow B \times_J F$. That map realizes $A_0N_0 \times_{J_0} F \cong \mathcal{M}_D$ as an open subset of $B \times_J \widetilde{F}$. The latter is fibered over the open A_0N_0–orbit Σ in $\mathcal{O} = B.z_0$ by the natural holomorphic projection $\pi : B \times_J \widetilde{F} \to B/J$. Since π_Σ is the restriction $\pi|_{\mathcal{M}_D}$, it follows that π_Σ is holomorphic as well.

The fact that Σ is a cell follows form the simple connectivity of the solvable group A_0N_0 and the fact that it is acting algebraically. ∎

Recall the notation: $\Omega^r_\mu(\mathbb{E}) \to \mathfrak{X}_D$ is the sheaf of relative $\mu^*\mathbb{E}$–valued holomorphic r–forms on $\mathfrak{X}$ with respect to $\mu : \mathfrak{X}_D \to D$.

Corollary 6.2. *Suppose that $\mathbb{E} \to D$ a holomorphic G_0–homogeneous vector bundle which is sufficiently negative so that $H^p(C; \Omega^r_\mu(\mathbb{E})|_C) = 0$ for $p < q$, and $r \geqq 1$. Then the double fibration transform*

$$\mathcal{P} : H^q(D, \mathcal{O}(\mathbb{E})) \to H^0(M, \mathcal{O}(\mathbb{E}'))$$

is injective.

Proof. $\mathcal{M}_D$ is contractible by Proposition 3.4. Since Σ is likewise contractible and $\mathcal{M}_D$ is diffeomorphic to $\Sigma \times F$, it follows that F is cohomologically trivial. The Buchdahl conditions (2.4) follow. Proposition 2.5 now says that (2.3) is an isomorphism for all r. Composing with coefficient morphisms, the maps (2.6) also are isomorphisms. By Theorem 3.5 we know that the conditions (2.8) are satisfied. The assertion now follows from Theorem 2.14. ∎

With a bit more work one can see that the fiber F of $\mathcal{M}_D \to \Sigma$ is contractible, not just cohomologically trivial. We thank Peter Michor for showing us the following result for the C^∞ category, from which contractibility of F is immediate. His argument is based on the existence of a complete Ehresmann connections for smooth fiber bundles.

Proposition 6.3. *Let $p : M \to S$ be a smooth fiber bundle with fiber $F = p^{-1}(s_0)$. If both M and S are contractible then F is contractible.*

Proof. Since S is contractible and smooth, approximation gives us a smooth contraction $h : [0,1] \times S \to S$; here $h(0,s) = s$ and $h(1,s) = s_0$. Following [18, §9.9] the bundle $p : M \to S$ has a complete Ehresmann connection. Completeness means that every smooth curve in S has horizontal lifts to M. If $m \in M$ let $t \mapsto H(t,m)$ denote the horizontal lift of $t \mapsto h(t,p(m))$ such that $H(0,m) = m$. Note $H(1,m) \in F$. Fix a base point $m_0 \in M$ and a smooth contraction $I : [0,1] \times M \to M$ of M to m_0; if $m \in M$ then $I(0,m) = m$ and $I(1,m) = m_0$. Denote $f_0 = H(1,m_0) \in F$. Define $J : [0,1] \times F$ by $J(t,f) = H(t, I(t,f))$, so $J(0,f) = f$ and $J(1,f) = f_0$. Thus J is a contraction of F to f_0, and so F is contractible. ∎

References

[1] Akhiezer, D. N., and S. Gindikin, *On the Stein extensions of real symmetric spaces*, Math. Annalen **286** (1990), 1–12.

[2] Andreotti, A., and H. Grauert, *Théorèmes de finitude pour la cohomologie des espaces complexes*, Bull. Soc. Math France **90** (1962), 193–259.

[3] Barchini, L., *Stein extensions of real symmetric spaces and the geometry of the flag manifold*, Math. Ann. **326** (2003), 331–346.

[4] Barlet, D., *Familles analytiques de cycles et classes fondamentales relatives*, Springer Lecture Notes Math. **807** (1980), 1–158.

[5] Brion, M., D. Luna, and T. Vust, Espaces homogènes sphériques, Invent. Math. **84** (1986), 617–632.

[6] Buchdahl, N., *On the relative DeRham sequence*, Proc. Amer. Math. Soc. **87** (1983), 363–366.

[7] Burns, D., S. Halverscheid, and R. Hind, *The geometry of Grauert tubes and complexification of symmetric spaces*, Duke. J. Math. **118** (2003), 465–491.

[8] Crittenden, R. J., *Minimum and conjugate points in symmetric spaces*, Canad. J. Math. **14** (1962), 320–328.

[9] Fels, G., Habilitationsarbeit, in progress.

[10] Fels, G., and A. T. Huckleberry, *Characterization of cycle domains via Kobayashi hyperbolicity*, to appear in the Bull. Soc. Math. de France, preprint AG/0204341.

[11] Gindikin, S., and B. Krötz, *Complex crowns of Riemannian symmetric spaces and non-compactly causal symmetric spaces*, Int. Math. Res. Notes (2002), 959–971.

[12] Gindikin, S., and T. Matsuki, *Stein extensions of Riemannian symmetric spaces and dualities of orbits on flag manifolds*, preprint.

[13] Grauert, H., and R. Remmert, "Coherent Analytic Sheaves", Springer–Verlag, 1984.

[14] Grauert, H., T. Peternell, and R. Remmert, *Sheaf–theoretical methods in complex analysis*, Several Complex Variables, VII, Encyclopaedia of Mathematical Sciences **74**, Springer–Verlag, 1994.

[15] Heier, G., *Die komplexe Geometrie des Periodengebietes der K3–Flächen*, Diplomarbeit, Ruhr–Universität Bochum, 1999.

[16] Huckleberry, A. T., *On certain domains in cycle spaces of flag manifolds*, Math. Annalen **323** (2002), 797–810.

[17] Huckleberry, A. T., and J. A. Wolf, *Schubert varieties and cycle spaces*, Duke Math. J., to appear.

[18] Kolář, I, P. W. Michor, and J. Slovák, "Natural Operations in Differential Geometry," Springer–Verlag, 1993, `http://www.emis.de/monographs/KSM/index.html`.

[19] Tirao, J. A., and J. A. Wolf, *Homogeneous holomorphic vector bundles*, Journal of Mathematics and Mechanics (= Indiana University Mathematics Journal) **20** (1970), 15–31.

[20] Wolf, J. A., *The action of a real semisimple Lie group on a complex manifold,* I*: Orbit structure and holomorphic arc components*, Bull. Amer. Math. Soc. **75** (1969), 1121–1237.

[21] Wolf, J. A., *The Stein condition for cycle spaces of open orbits on complex flag manifolds*, Annals of Math. **136** (1992), 541–555.

[22] Wolf, J. A., *Real groups transitive on complex flag manifolds*, Proc. American Math. Soc. **129** (2001), 2483–2487.

[23] Wolf, J. A., and R. Zierau, *Holomorphic double fibration transforms*, In "The Mathematical Legacy of Harish–Chandra", PSPM **68**, AMS, 2000, 527–551.

Alan T. Huckleberry
Fakultät für Mathematik
Ruhr–Universität Bochum
D–44780 Bochum, Germany
ahuck@cplx.ruhr-uni-bochum.de

Joseph A. Wolf
Department of Mathematics
University of California
Berkeley, CA 94720–3840, USA
jawolf@math.berkeley.edu

Received June 2, 2003
and in final form August 22, 2003