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Classification of Spherical Nilpotent Orbits in Complex Symmetric Space

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Abstract. Let G be the adjoint group of the simple real Lie algebra $\mathfrak{g}$, and let $K_{\mathbf{C}} \to \mathrm{Aut}(\mathfrak{p}_{\mathbf{C}})$ be the complexified isotropy representation at the identity coset of the corresponding symmetric space. We classify the spherical nilpotent $K_{\mathbf{C}}$ orbits in $\mathfrak{p}_{\mathbf{C}}$.

1. Introduction

When L is a complex simple Lie group, the spherical nilpotent orbits for the adjoint action of L on its Lie algebra have been determined by Panyushev [22] and McGovern [15]. These orbits are significant in the study of the completely prime primitive ideals in the enveloping algebra of L. For example, McGovern has shown how to associate Dixmier algebras to spherical nilpotent orbits (and their covers) [15]. The Dixmier algebra associated to a spherical orbit has a nice structure owing to the fact that the co-ordinate ring of the orbit is multiplicity free as an L module, i.e., each irreducible finite dimensional representation of L occurs with multiplicity 0 or 1.

The goal of this paper is to classify completely the spherical nilpotent $K_{\mathbf{C}}$-orbits in $\mathfrak{p}_{\mathbf{C}}$, the complexified tangent space at the identity coset of the symmetric space formed by a simple group G (of adjoint type) and its maximal compact subgroup K. Here $K_{\mathbf{C}}$ is the complexification of K. This classification is contained in Theorems 6.1 (section 6.) and 9.1 (section 9.). Panyushev presented a partial classification of spherical nilpotent $K_{\mathbf{C}}$-orbits in [21]. The classification presented here is especially significant since each spherical nilpotent $K_{\mathbf{C}}$-orbit is diffeomorphic to a nilpotent G-orbit in the Lie algebra of G that is multiplicity free as a Hamiltonian K-space [11].

Many specific spherical nilpotent orbits have been investigated. If G is simple, the number of non-zero minimal nilpotent $K_{\mathbf{C}}$-orbits in $\mathfrak{p}_{\mathbf{C}}$ is either 1 or 2. These orbits are spherical and have been studied extensively by representation theorists, notably [25]. Other spherical nilpotent $K_{\mathbf{C}}$-orbits are studied in [1] and [16]. One expects spherical nilpotent orbits to play an increasingly prominent role in the representation theory of real simple Lie groups.

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2. Basic Notation

Throughout this article, we assume that $\mathfrak{g}$ is a real simple Lie algebra with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. θ is the associated Cartan involution. Let $\mathfrak{g}_{\mathbf{C}}$, $\mathfrak{k}_{\mathbf{C}}$ and $\mathfrak{p}_{\mathbf{C}}$ denote the complexifications of $\mathfrak{g}$, $\mathfrak{k}$ and $\mathfrak{p}$ respectively. θ extends to a complex linear involution on $\mathfrak{g}_{\mathbf{C}}$. Let σ denote conjugation on $\mathfrak{g}_{\mathbf{C}}$ relative to the real form $\mathfrak{g}$. $G_{\mathbf{C}}$ is the adjoint group of $\mathfrak{g}_{\mathbf{C}}$. G, K, and $K_{\mathbf{C}}$ are the connected subgroups of $G_{\mathbf{C}}$ corresponding to the Lie algebras $\mathfrak{g}$, $\mathfrak{k}$, and $\mathfrak{k}_{\mathbf{C}}$, respectively. $G_{\mathbf{C}}^{\theta}$ is the subgroup of $G_{\mathbf{C}}$ which is fixed by θ.

3. Kostant-Sekiguchi correspondence

In order to define the Kostant Sekiguchi correspondence, we consider the adjoint actions of $G_{\mathbf{C}}$ on $\mathfrak{g}_{\mathbf{C}}$, G on $\mathfrak{g}$ and $K_{\mathbf{C}}$ on $\mathfrak{p}_{\mathbf{C}}$.

Definition 3.1. Let $\mathcal{N}[\mathfrak{g}]$, $\mathcal{N}[\mathfrak{g}_{\mathbf{C}}]$, and $\mathcal{N}[\mathfrak{p}_{\mathbf{C}}]$ denote the set of nilpotent elements of $\mathfrak{g}$, $\mathfrak{g}_{\mathbf{C}}$ and $\mathfrak{p}_{\mathbf{C}}$ respectively. $\mathcal{N}[\mathfrak{g}]/G$, $\mathcal{N}[\mathfrak{g}_{\mathbf{C}}]/G_{\mathbf{C}}$, and $\mathcal{N}[\mathfrak{p}_{\mathbf{C}}]/K_{\mathbf{C}}$ will denote the orbits (conjugacy classes) in $\mathcal{N}[\mathfrak{g}]$, $\mathcal{N}[\mathfrak{g}_{\mathbf{C}}]$, and $\mathcal{N}[\mathfrak{p}_{\mathbf{C}}]$ under G, $G_{\mathbf{C}}$ and $K_{\mathbf{C}}$ respectively.

The Kostant-Sekiguchi correspondence is a special bijection between $\mathcal{N}[\mathfrak{g}]/G$ and $\mathcal{N}[\mathfrak{p}_{\mathbf{C}}]/K_{\mathbf{C}}$. It is defined by means of $sl(2)$-triples.

Definition 3.2. An ordered triple $\{Z_1, Z_2, Z_3\}$ of elements in $\mathfrak{g}_{\mathbf{C}}$ is said to be an $sl(2)$-triple if the following commutation relations are satisfied:

$$[Z_1, Z_2] = 2Z_2,\ [Z_1, Z_3] = -2Z_3,\ \text{and } [Z_2, Z_3] = Z_1.$$

Two $sl(2)$-triples $\{Z_1, Z_2, Z_3\}$ and $\{Z_1', Z_2', Z_3'\}$ are said to be conjugate under a subgroup $\mathbf{W}$ of $G_{\mathbf{C}}$ if there exists an element $w \in \mathbf{W}$ such that $Z_i = w \cdot Z_i'$ for $i = 1, 2, 3$. ("$\cdot$" denotes the adjoint action.)

Using the Jacobson-Morosov Theorem, one can prove the following characterization of $\mathcal{N}[\mathfrak{g}]/G$.

Theorem 3.3. *[3] There is a bijection between each of the following sets.*

(1) *G conjugacy classes of $sl(2)$-triples of $\mathfrak{g}$*

(2) *$\mathcal{N}[\mathfrak{g}]/G$.*

The $sl(2)$-triple $\{Z_1, Z_2, Z_3\}$ is said to be normal if $Z_1 \in \mathfrak{k}_{\mathbf{C}}$, and $Z_2,\ Z_3 \in \mathfrak{p}_{\mathbf{C}}$. Normal $sl(2)$-triples are at the heart of the Kostant-Rallis description of $\mathcal{N}[\mathfrak{p}_{\mathbf{C}}]/K_{\mathbf{C}}$.

Theorem 3.4. *[13] There is a bijection between each of the following sets.*

(1) *$K_{\mathbf{C}}$ conjugacy classes of normal $sl(2)$-triples of $\mathfrak{g}_{\mathbf{C}}$*

(2) *$\mathcal{N}[\mathfrak{p}_{\mathbf{C}}]/K_{\mathbf{C}}$.*

We need to define two notable classes of $sl(2)$-triples.

Definition 3.5. (Kostant-Sekiguchi triples) An $sl(2)$-triple $\{H,\ E,\ F\}$ in $\mathfrak{g}$ is said to be a KS-triple in $\mathfrak{g}$ if $\theta(E) = -F$, and hence $\theta(H) = -H$. A normal $sl(2)$-triple $\{x,\ e,\ f\}$ in $\mathfrak{g}_{\mathbf{C}}$ is said to be a KS-triple in $\mathfrak{g}_{\mathbf{C}}$ if $f = \sigma(e)$.

Sekiguchi established the following facts about KS-triples.

Theorem 3.6. *[23] (1) Every $sl(2)$-triple $\{H',\ E',\ F'\}$ in $\mathfrak{g}$ is conjugate under G to a KS-triple in $\mathfrak{g}$. Two KS-triples in $\mathfrak{g}$ are conjugate under G to the same $sl(2)$-triple in $\mathfrak{g}$ if and only if the KS-triples are conjugate under K.*

(2) Every normal $sl(2)$-triple $\{x',\ e',\ f'\}$ in $\mathfrak{g}_{\mathbf{C}}$ is conjugate under $K_{\mathbf{C}}$ to a KS-triple in $\mathfrak{g}_{\mathbf{C}}$. Two KS-triples in $\mathfrak{g}_{\mathbf{C}}$ are conjugate under $K_{\mathbf{C}}$ to the same normal $sl(2)$-triple in $\mathfrak{g}_{\mathbf{C}}$ if and only if the KS-triples are conjugate under K.

Definition 3.7. Let $KS(\mathfrak{g})$ denote the set of KS-triples in $\mathfrak{g}$ and $KS(\mathfrak{g}_{\mathbf{C}})$ denote the set of KS-triples in $\mathfrak{g}_{\mathbf{C}}$. $KS(\mathfrak{g})/K$ and $KS(\mathfrak{g}_{\mathbf{C}})/K$ will denote the set of K conjugacy classes in $KS(\mathfrak{g})$ and $KS(\mathfrak{g}_{\mathbf{C}})$ respectively.

Combining Theorems 3.4, 3.3 and 3.6 we have:

Theorem 3.8. *There are bijections:*

$$\mathcal{N}[\mathfrak{g}]/G \longleftrightarrow KS(\mathfrak{g})/K \text{ and } \mathcal{N}[\mathfrak{p}_{\mathbf{C}}]/K_{\mathbf{C}} \longleftrightarrow KS(\mathfrak{g}_{\mathbf{C}})/K.$$

The Kostant-Sekiguchi correspondence is a consequence of Theorem 3.8 and the following observation.

Proposition 3.9. *There is a bijection: $KS(\mathfrak{g})/K \longleftrightarrow KS(\mathfrak{g}_{\mathbf{C}})/K$ defined as follows: $\{H,\ E,\ F\} \in KS(\mathfrak{g})$ is mapped to the $\{x,\ e,\ f\} \in KS(\mathfrak{g}_{\mathbf{C}})$, where*

$$x = i(E - F), \quad e = \frac{E + F + iH}{2}, \quad f = \frac{E + F - iH}{2}. \tag{1}$$

The map just defined is K-equivariant.

Let Ω be a conjugacy class in $\mathcal{N}[\mathfrak{g}]/G$. Let $\{H,\ E,\ F\} = \{H_\Omega,\ E_\Omega,\ F_\Omega\}$ be a representative of the conjugacy class in $KS(\mathfrak{g})$ that is associated to Ω by Theorem 3.8. Then set

$$\mathcal{S}(\Omega) \stackrel{\text{def}}{=} K_{\mathbf{C}} \cdot \frac{E + F + iH}{2}. \tag{2}$$

We obtain our main result.

Theorem 3.10. (The Kostant-Sekiguchi Correspondence [23]) *The mapping $\mathcal{S} : \mathcal{N}[\mathfrak{g}]/G \longrightarrow \mathcal{N}[\mathfrak{p}_{\mathbf{C}}]/K_{\mathbf{C}}$, given by $\Omega \mapsto \mathcal{S}(\Omega) \stackrel{\text{def}}{=} \mathcal{O}_\Omega$ (see formula (2)) is a bijection.*

Proof. Combine Theorem 3.8 and Proposition 3.9. ∎

4. Results about spherical nilpotents in $\mathfrak{p}_{\mathbf{C}}$

We need several results (mostly due to Panyushev) in order to state necessary and sufficient conditions for $\mathcal{O}$ to be $K_{\mathbf{C}}$ spherical. Fix a KS-triple $\{x,\ e,\ f\}$ in $\mathfrak{g}_{\mathbf{C}}$ with $e \in \mathcal{O}$. Thus $x \in i\mathfrak{k}$, $\sigma(e) = f$. It follows that the complex subalgebra $\mathfrak{a}_{\mathbf{C}} = \mathbf{C}x + \mathbf{C}e + \mathbf{C}f$ has a θ-stable real form $\mathfrak{a} \subset \mathfrak{g}$. Let $\mathfrak{g}_{\mathbf{C}}(j)$, $\mathfrak{k}_{\mathbf{C}}(j)$, and $\mathfrak{p}_{\mathbf{C}}(j)$ denote the j-eigenspace of $ad(x)$ on $\mathfrak{g}_{\mathbf{C}}$, $\mathfrak{k}_{\mathbf{C}}$, and $\mathfrak{p}_{\mathbf{C}}$ respectively.

Since $\sigma(x) = -x$, and $\mathfrak{k}_{\mathbf{C}}$ and $\mathfrak{p}_{\mathbf{C}}$ are preserved by σ, for all j we have

$$\dim_{\mathbf{C}} \mathfrak{k}_{\mathbf{C}}(j) = \dim_{\mathbf{C}} \mathfrak{k}_{\mathbf{C}}(-j); \quad \dim_{\mathbf{C}} \mathfrak{p}_{\mathbf{C}}(j) = \dim_{\mathbf{C}} \mathfrak{p}_{\mathbf{C}}(-j).$$

Definition 4.1. We define $\mathfrak{k}_{\mathbf{C}}$-height$(e)$ (resp., $\mathfrak{p}_{\mathbf{C}}$-height$(e)$) to be the largest non-negative integer j such that $\mathfrak{k}_{\mathbf{C}}(j) \neq (0)$ (resp., $\mathfrak{p}_{\mathbf{C}}(j) \neq (0)$). height$(e)$ is the largest non-negative integer j such that $\mathfrak{g}_{\mathbf{C}}(j) \neq (0)$.

Definition 4.2. Let $\mathfrak{u}$ denote the sum of the positive eigenspaces of $ad(x)$ on $\mathfrak{g}_{\mathbf{C}}$. Set $Z = \mathfrak{u} \cap \mathfrak{k}_{\mathbf{C}}/(\mathfrak{u} \cap \mathfrak{k}_{\mathbf{C}})^e$.

Lemma 4.3. (1) *For each $i \geq 0$, as a $K_{\mathbf{C}}^{\{x,\ e,\ f\}}$ module, $\mathfrak{k}_{\mathbf{C}}(i)/(\mathfrak{k}_{\mathbf{C}}(i))^e$ is isomorphic to $[f,\ \mathfrak{p}_{\mathbf{C}}(i+2)]$; and*

(2) *As $K_{\mathbf{C}}^{\{x,\ e,\ f\}}$ modules, Z and $\sum_{i\geq 1} \mathfrak{p}_{\mathbf{C}}(i+2) = \sum_{i\geq 3} \mathfrak{p}_{\mathbf{C}}(i)$ are isomorphic.*

Proof. For (1) apply the representation theory of $sl(2,\ \mathbf{C})$. (2) follows from (1). ∎

Let $\mathbf{u} = \mathfrak{k} + i\mathfrak{p}$. Then $\mathbf{u}$ is a compact real form of $\mathfrak{g}_{\mathbf{C}}$. $\tau = \sigma \circ \theta$ is the conjugation on $\mathfrak{g}_{\mathbf{C}}$ with respect to $\mathbf{u}$. Let $\mathbf{U}$ be the connected subgroup of $G_{\mathbf{C}}$ with Lie algebra $\mathbf{u}$.

Definition 4.4. Let B denote the Killing form of $\mathfrak{g}_{\mathbf{C}}$ and $B_{\mathfrak{g}}$ denote the restriction of B to $\mathfrak{g}$. Set $\langle z,\ w\rangle = -B_{\mathfrak{g}_{\mathbf{C}}}(z,\ \tau(w))$. Then $\langle\cdot,\ \cdot\rangle$ is a $\mathbf{U}$ invariant, positive definite Hermitian inner product on $\mathfrak{g}_{\mathbf{C}}$. If $z \in \mathfrak{g}_{\mathbf{C}}$, set $\|z\|^2 = \langle z,\ z\rangle$. Note that if $z, w \in \mathfrak{g}$, then $\langle z,\ w\rangle = -B_{\mathfrak{g}}(z,\ \theta(w))$.

Let $\mathfrak{m}$ be the orthogonal complement of $\mathfrak{k}_{\mathbf{C}}^{\{x,\ e,\ f\}}$ (relative to $\langle\cdot,\ \cdot\rangle$) inside $\mathfrak{k}_{\mathbf{C}}^x$. $\mathfrak{m}$ is a $K_{\mathbf{C}}^{\{x,\ e,\ f\}}$ module. Note that $\mathfrak{m}$ and $\mathfrak{p}_{\mathbf{C}}(2)$ are isomorphic as $K_{\mathbf{C}}^{\{x,\ e,\ f\}}$ modules.

Recall from [20] the notion of a stabilizer in general position (*s.g.p.*) for the action of an algebraic group on an irreducible variety.

Definition 4.5. We fix $\mathbf{S}$ to be an *s.g.p.* for the representation of $K_{\mathbf{C}}^{\{x,\ e,\ f\}}$ on $\mathfrak{m}$.

That is $\mathbf{S}$ is the stabilizer in $K_{\mathbf{C}}^{\{x,\ e,\ f\}}$ of a point whose orbit under $K_{\mathbf{C}}^{\{x,\ e,\ f\}}$ has maximal dimension. Such a point lies in an open subset of $\mathfrak{m}$ such that the stabilizers of any two points in this subset are conjugate under $K_{\mathbf{C}}^{\{x,\ e,\ f\}}$. Since $K_{\mathbf{C}}^{\{x,\ e,\ f\}}$ is reductive, a generic $K_{\mathbf{C}}^{\{x,\ e,\ f\}}$ orbit on $\mathfrak{m}$ is closed, so that $\mathbf{S}$ is reductive. Also,

$$\dim K_{\mathbf{C}}^{\{x,\ e,\ f\}} - \dim \mathbf{S} = \dim \text{orbit of maximal dimension of } K_{\mathbf{C}}^{\{x,\ e,\ f\}} \text{ on } \mathfrak{m}.$$

Let $\mathfrak{s}_{\mathbf{C}}$ denote the Lie algebra of $\mathbf{S}$. Since $\mathfrak{s}_{\mathbf{C}}$ is stable under σ and τ, it is the complexification of a Lie subalgebra $\mathfrak{s}_{\mathbf{R}}$ which is contained in $\mathfrak{k}^{\{x,\ e,\ f\}}$. Let $\mathbf{S}_{\mathbf{R}}$ denote the corresponding connected compact subgroup of $K^{\{x,\ e,\ f\}}$. $\mathbf{S}_0$ denotes the identity component of $\mathbf{S}$. $B(\mathbf{S})$ will denote a Borel subgroup of $\mathbf{S}$.

Suppose that X is a variety with $K_{\mathbf{C}}$ action and B_k is a Borel subgroup of $K_{\mathbf{C}}$.

Definition 4.6. The complex codimension of a generic B_k orbit is called the complexity of X, denoted $c_{K_{\mathbf{C}}}(X)$ or $c(X)$ (when the reductive group $K_{\mathbf{C}}$ is understood).

Remark 4.7. $c(X)$ is also the transcendence degree (over $\mathbf{C}$) of the B_k invariant functions in the field of rational functions (with complex coefficients) on X.

Definition 4.8. If X is irreducible, we say that X is spherical for $K_{\mathbf{C}}$ if $c(X) = 0$. That is, some (and hence any) Borel subgroup of $K_{\mathbf{C}}$ has a dense orbit on X.

Remark 4.9. If $\overline{X}$ is the Zariski closure of X, then $c(\overline{X}) = c(X)$.

Corollary 4.10. *If $e \in \mathcal{N}[\mathfrak{p}_{\mathbf{C}}]$, then $K_{\mathbf{C}} \cdot e$ is spherical $\iff$ $\overline{K_{\mathbf{C}} \cdot e}$ is spherical.*

Proposition 4.11. *If $e \in \mathcal{N}[\mathfrak{p}_{\mathbf{C}}]$ and $K_{\mathbf{C}} \cdot e$ is spherical so is each $K_{\mathbf{C}}$ orbit in $\overline{K_{\mathbf{C}} \cdot e}$.*

Proof. This follows from Corollaire 3.5 in [2]. ∎

Proposition 4.12.

$$c_{K_{\mathbf{C}}}(K_{\mathbf{C}} \cdot e) = c_{K_{\mathbf{C}}^x}(K_{\mathbf{C}}^x / {K_{\mathbf{C}}}^{\{x,\ e,\ f\}}) + c_S(Z).$$

Proof. This is Theorem 1.2(a) of [22]. (See also Theorem 2.3(a) of [22].) ∎

Corollary 4.13. *$\mathcal{O}$ is spherical if and only if $K_{\mathbf{C}}^x / {K_{\mathbf{C}}}^{\{x,\ e,\ f\}}$ is spherical and a Borel subgroup of $\mathbf{S}$ has an open orbit on Z.*

Proof. This is Corollary 1.4 of [22]. ∎

Lemma 4.14. *Suppose $\mathfrak{k}_{\mathbf{C}}$-height($e$)$\leq 3$. Then $\mathfrak{k}_{\mathbf{C}}^{\{x,\ e,\ f\}}$ is a symmetric subalgebra of $\mathfrak{k}_{\mathbf{C}}^x$.*

Proof. This is proved like Proposition 3.3 in [22]. ∎

Corollary 4.15. *Let $\{x,\ e,\ f\}$ be a normal triple such that height(e)$= 2$. Then $\mathcal{O}$ is spherical for $K_{\mathbf{C}}$.*

Proof. Since height$(e)=2$ and $Z=(0)$, apply Lemma 4.14 and Corollary 4.13. ∎

Since all spherical nilpotent $G_{\mathbf{C}}$-orbits for $\mathfrak{g}_{\mathbf{C}}$ of type A_n or C_n have height 2 (see [22]), we have the following result.

Corollary 4.16. *Suppose that $\mathfrak{g}_{\mathbf{C}}$ is of type A_n or C_n. If Θ is a spherical nilpotent orbit for $G_{\mathbf{C}}$, and $\mathcal{O}$ is a $K_{\mathbf{C}}$ orbit in $\Theta \cap \mathfrak{p}_{\mathbf{C}}$, then $\mathcal{O}$ is spherical for $K_{\mathbf{C}}$.*

Panyushev has shown that for a nilpotent e in $\mathfrak{g}_{\mathbf{C}}$, height$(e) \geq 4$ implies that $G_{\mathbf{C}} \cdot e$ is not spherical. In addition he has shown that:

Proposition 4.17. *If $\mathfrak{p}_{\mathbf{C}}$-height(e) > 3, or $\mathfrak{k}_{\mathbf{C}}$-height(e) > 4, then $K_{\mathbf{C}} \cdot e$ is not spherical.*

Proof. This is equivalent to Theorem 5.6 in [21]. ∎

Remark 4.18. If $\mathcal{O}$ is spherical for $K_{\mathbf{C}}$, then $\dim_{\mathbf{C}} \mathcal{O} \leq \dim_{\mathbf{C}} B(K_{\mathbf{C}})$, where $B(K_{\mathbf{C}})$ denotes any Borel subgroup of $K_{\mathbf{C}}$.

Remark 4.19. If $e,\ e' \in \mathcal{N}[\mathfrak{p}_{\mathbf{C}}]$ are conjugate under $G^{\theta}_{\mathbf{C}}$, then $K_{\mathbf{C}} \cdot e$ is spherical $\iff K_{\mathbf{C}} \cdot e'$ is spherical.

5. Parametrizing nilpotent $K_{\mathbf{C}}$-orbits in $\mathfrak{p}_{\mathbf{C}}$

Suppose that $\mathfrak{g}$ is a simple classical real Lie algebra. Because of the Kostant-Sekiguchi correspondence, we will generally use the signed partition description of nilpotent conjugacy classes in $\mathcal{N}[\mathfrak{g}]/G$ (see [3]) to describe the conjugacy classes in $\mathcal{N}[\mathfrak{p}_{\mathbf{C}}]/K_{\mathbf{C}}$. The signed partition description is equivalent to the description in terms of "$ab-$ diagrams" given by Ohta ([19]) and others.

Throughout this section and the next $\{x,\ e,\ f\}$ is a KS-triple in $\mathfrak{g}_{\mathbf{C}}$. We assume that $x \in i\mathfrak{t}$, where $\mathfrak{t}$ is a fixed maximal torus of $\mathfrak{k}$. $\mathfrak{a}_{\mathbf{C}}$ is the $sl(2,\ \mathbf{C})$ algebra spanned by triple $\{x,\ e,\ f\}$. For each $\mathfrak{g}$, we will give a recipe for computing x from the signed partition description of $K_{\mathbf{C}} \cdot e$. The idea behind the various recipes is as follows. For each $\mathfrak{g}$ there is a finite dimensional complex vector space $V = V(\mathfrak{g})$ carrying the 'natural' or 'basic' representation of $\mathfrak{g}$. (If $\mathfrak{g}$ is not $sl(s,\ \mathbf{R})$ or $su^*(2n)$, V is the complex vector space carrying the corresponding $\mathfrak{g}$-invariant bilinear form.) V is a completely reducible $\mathfrak{a}_{\mathbf{C}}$-module. An '$ab$'-diagram for the conjugacy class of the nilpotent $e \in \mathfrak{p}_{\mathbf{C}}$ describes a basis for V consisting of eigenvectors of the neutral element x. The eigenvalues of x on this basis determine the values taken by a system of simple roots of $\mathfrak{k}_{\mathbf{C}}$ on the element x. This gives the weighted Dynkin diagram of x which determines the orbit $K_{\mathbf{C}} \cdot e$. In each case, x is dominant with respect to the system of simple roots of $\mathfrak{k}_{\mathbf{C}}$. The recipes below are undoubtedly known to many experts but, to the author's knowledge, have not been published before. Proofs will be given only for $\mathfrak{g} = su(p,\ q)$ and $\mathfrak{g} = sp(n,\ \mathbf{R})$.

5.1. $sl(s,\ \mathbf{R})$.
Let $s = 2n$ or $2n+1$. $\mathfrak{k} = so(s)$. Let $E_{i,\ j}$ denote the $s \times s$ matrix whose $(i,\ j)$ entry is 1 and whose other entries are 0. We define a torus $\mathfrak{t}$ of $\mathfrak{k}$ to be the real span of the matrices $Z_i = E_{2i-1,\ 2i} - E_{2i,\ 2i-1}$, $i = 1, \ldots,\ n$. Then the linear functional e_j, $j = 1, \ldots,\ n$ is defined by $e_j(Z_i) = -\sqrt{-1}\delta_{ij}$ (Kronecker delta).

If $s = 2n+1$, we label the following set of simple roots for $\mathfrak{k}$:

$$e_1 - e_2,\ e_2 - e_3, \ldots,\ e_{n-1} - e_n,\ e_n.$$

For $s = 2n$, $\mathfrak{k} = so(2n)$. We label the following set of simple roots for $\mathfrak{k}$:

$$e_1 - e_2,\ e_2 - e_3, \ldots,\ e_{n-2} - e_{n-1},\ e_{n-1} - e_n,\ e_{n-1} + e_n.$$

Proposition 5.1. *($\mathfrak{g} = sl(2n+1, \mathbf{R})$, or $sl(n.\ \mathbf{R})$) Here is the algorithm for determining the weighted Dynkin diagram of x. Let $\Lambda = m_1 + m_2 + m_3 + \ldots m_r$ be the partition of $2n+1$ or $2n$ determined by e, with $m_1 \geq m_2 \geq \ldots \geq m_r$. Each occurrence of m_j corresponds to an $m_j \times m_j$ Jordan block, that is an m_j-dimensional $\mathfrak{a}_{\mathbf{C}}$-module with basis: $v,\ e \cdot v,\ e^2 \cdot v,\ \ldots,\ e^{m_j - 1} \cdot v$ for some $v \in V$. The eigenvalues of x in this basis are the integers: $(m_j - 1), \ldots, -(m_j - 1)$.*

Case (1) Assume that not all the m_j are even. Form the multiset $A_\Lambda = A_\Lambda(K_{\mathbf{C}} \cdot e)$ by "joining" all such sequences $(m_j - 1), \ldots, -(m_j - 1)$ for a given partition of $2n+1$ or $2n$. Assume that the elements of A_Λ are arranged in descending order. Take the first n non-negative integers from A_Λ. These are (respectively) the values $e_1(x), \ldots, e_n(x)$, which give the weighted Dynkin diagram of x.

Case (2) Assume that all the m_j are even. This is possible only for $\mathfrak{g} = sl(2n,\ \mathbf{R})$. Λ corresponds to two distinct $K_{\mathbf{C}}$ nilpotent orbits that are conjugate under $G^\theta_{\mathbf{C}}$. To record this fact we label one copy of the partition Λ with a roman numeral "I" and a second copy of Λ with roman numeral "II" (Theorem 9.3.3 in [3]) to distinguish the orbits. Applying the procedure in Case (1) we obtain a weighted Dynkin diagram in which $e_n(x) = 1$. We associate this weighted diagram to the orbit Λ_I. We then use the values $e_1(x),\ \ldots e_{n-1}(x),\ e_n(x) = -1$ to form a second weighted Dynkin diagram. This diagram is assigned to orbit Λ_{II}.

Example 5.2. (a) $\Lambda = 4+3+2$ for $sl(9,\ \mathbf{R})$. The sequences corresponding to the parts 4, 3, and 2 respectively $\{3, 1, -1, -3\}$; $\{2, 0, -2\}$;and $\{1, -1\}$. Thus $A_\Lambda = \{3,\ 2,\ 1,\ 0,\ -1,\ -2,\ -3\}$. So $e_1(x) = 3$, $e_2(x) = 2$, $e_3(x) = 1$ and $e_4(x) = 0$. Hence, we obtain the following labels on the simple roots of $\mathfrak{k} = so(7)$: $e_1 - e_2 = 1,\ e_2 - e_3 = 1,\ e_3 - e_4 = 1,\ e_4 = 0$.

(b) $\Lambda = 4+2+2+2$ for $sl(10,\ \mathbf{R})$. The sequences corresponding to the parts 4 and 2 are respectively $\{3, 1, -1, -3\}$ and $\{1, -1\}$. Thus,

$$A_\Lambda = \{3,\ 1,\ 1,\ 1,\ 1,\ -1,\ -1,\ -1,\ -1,\ -3\}.$$

So the orbit Λ_I has $e_1(x) = 3$, $e_2(x) = 1$, $e_3(x) = 1$, $e_4(x) = 1$, and $e_5(x) = 1$. The orbit Λ_{II} has $e_1(x) = 3$, $e_2(x) = 1$, $e_3(x) = 1$, $e_4(x) = 1$, and $e_5(x) = -1$.

5.2. $su^*(2n)$. We use the notation of Helgason [10], chapter 10 , section 2 for $\mathfrak{g}$ and $\mathfrak{k}$. That is, $\mathfrak{g}$ is the space of $2n \times 2n$ matrices of the form $\begin{pmatrix} Z_1 & Z_2 \\ -\bar{Z}_2 & \bar{Z}_1 \end{pmatrix}$

where Z_1 and Z_2 are $n \times n$ complex matrices and $Tr(Z_1) + Tr(\bar{Z}_1) = 0$. ('$\bar{Z}$' denotes complex conjugation.) $\mathfrak{k}$ is the subspace of skew Hermitian matrices in $\mathfrak{g}$. $\mathfrak{k}$ is isomorphic to $sp(n)$. The space of diagonal matrices in $\mathfrak{k}$ is a maximal torus $\mathfrak{t}$ of $\mathfrak{k}$. The linear functional e_i $(1 \leq i \leq n)$ on $\mathfrak{t}$ is defined so that its value on the diagonal matrix $diag(\sqrt{-1}y_1, \ldots, \sqrt{-1}y_n, -\sqrt{-1}y_1, \ldots, -\sqrt{-1}y_n)$ is $\sqrt{-1}y_i$. Then we have the following set π of simple roots for $\mathfrak{k}$:

$$e_1 - e_2,\ e_2 - e_3, \ldots,\ e_{n-1} - e_n,\ 2e_n$$

Nilpotent conjugacy classes are parametrized by partitions of n.

Proposition 5.3. *Suppose $\Lambda = m_1 + m_2 + m_3 + \ldots + m_r$ is a partition of n $(m_1 \geq m_2 \geq \ldots \geq m_r)$. "Double" Λ to obtain a partition Λ' of $2n$. That is*

$$\Lambda' = m_1 + m_1 + m_2 + m_2 + m_3 + +m_3 + \ldots + m_r + m_r$$

Form the multiset $A_{\Lambda'}$ by joining the integer strings corresponding to each of the integers $m_1,\ m_1,\ \ldots,\ m_r,\ m_r$. Arrange the integers in $A_{\Lambda'}$ in descending order and choose the first n non-negative integers in this list. These integers are the values $e_1(x),\ e_2(x),\ \ldots,\ e_n(x)$.

5.3. $su(p,\ q)$**.** Let $n = p + q$ and $V = \mathbf{C}^n$. Let $z = (z_1,\ \ldots,\ z_n)$ and $w = (w_1,\ \ldots,\ w_n)$ be n-tuples in V. Consider the Hermitian form

$$\langle z,\ w\rangle_{p,\ q} = z_1\bar{w}_1 + \ldots + z_p\bar{w}_p - z_{p+1}\bar{w}_{p+1} - \ \ldots - z_{p+q}\bar{w}_{p+q}.$$

('$\bar{z}$' denotes the complex conjugate of z.) $su(p,\ q)$ consists of all the trace zero $n \times n$ complex matrices which leave the form $\langle \cdot,\ \cdot \rangle$ invariant. Let $I_{p,\ q}$ be the block diagonal matrix $diag(I_p,\ -I_q)$ where I_p (resp., I_q) is the $p \times p$ (resp., $q \times q$) identity matrix. Let V^+ (resp. V^-) be the $+1$ (resp. -1) eigenspace of $I_{p,\ q}$. $\mathfrak{k} = su(p) \oplus su(q) \oplus T_1$. Since the restriction of $\langle \cdot,\ \cdot \rangle$ is positive definite on V^+ and negative definite on V^-, $\mathfrak{k}$ can be viewed as trace zero matrices which preserve V^+ and V^-, which we denote $s(u(V^+) \oplus u(V^-))$.

Let $E_{i,\ j}$ denote the $n \times n$ matrix defined in subsection 5.. We define a torus $\mathfrak{t}$ of $\mathfrak{k}$ to be the real span of the matrices $H_i = \sqrt{-1}(E_{i,\ i} - E_{i+1,\ i+1})$, $i = 1, \ldots,\ n-1$. Then the linear functional e_j, for $j = 1, \ldots,\ n$, is defined by $e_j(\sqrt{-1}E_{i,\ i}) = \delta_{ij}$.

The neutral element of a nilpotent conjugacy class will be described by giving its values at the following compact simple roots of $\mathfrak{k}$ and at the noncompact root $-\psi = -(e_1 - e_{p+q})$:

$$\{e_1 - e_2,\ e_2 - e_3,\ \ldots,\ e_{p-1} - e_p\} \cup \{e_{p+1} - e_{p+2},\ \ldots,\ e_{p+q-1} - e_{p+q}\} \quad (3)$$

Nilpotent conjugacy classes are parametrized by signed partitions of signature $(p,\ q)$. See [3].

Proposition 5.4. *Let Λ be a signed partition of signature $(p,\ q)$. See [3]. Let $m_1, \ldots,\ m_d$ be the distinct part sizes of Λ arranged in descending order. Let r_j^+ (resp., r_j^-) be the number of times m_j occurs labelled with a '+' (resp., '-') sign.*

That is, r_j^+ (resp., r_j^-) denotes the number of rows of length m_j in Λ which begin with '+ ' (resp., '− '). We can write

$$\Lambda = (+m_1)^{r_1^+}(-m_1)^{r_1^-} \dots \ (+m_d)^{r_d^+}(-m_d)^{r_d^-}.$$

We form two multisets: A_Λ^p and B_Λ^q, by performing the following procedure on each row of Λ. Suppose λ is a row of Λ of length m_j. Label the first sign in λ with the integer $m_j - 1$, the next sign with the integer $m_j - 3$, etc. The last sign in λ is labelled with the integer $-(m_j-1)$. Each integer labelling a plus sign in λ is placed in A_Λ^p and each integer labelling a minus sign in λ is placed in B_Λ^q. By arranging the elements of A_Λ^p in descending order, we obtain the integers $e_1(x), \dots, e_p(x)$. By arranging the elements of B_Λ^q in descending order, we obtain the integers $e_{p+1}(x), \dots, e_{p+q}(x)$.

Proof. (Sketch) Each row of type $(+m_j)$ in Λ corrresponds to an $\mathfrak{a}_\mathbf{C}$-module W_j^+ (of dimension m_j) with basis: $v,\ e\cdot v,\ e^2\cdot v,\ \dots,\ e^{m_j-1}\cdot v$ for some $v \in V$ with $e^{m_j-1}\cdot v \in V^+$. (And $e^{m_j-2}\cdot v \in V^-$, $e^{m_j-3}\cdot v \in V^+$, etc.) Each row of type $(-m_j)$ corrresponds to a basis of an m_j-dimensional $\mathfrak{a}_\mathbf{C}$-module W_j^- with basis: $v,\ e\cdot v,\ e^2\cdot v,\ \dots,\ e^{m_j-1}\cdot v$ for some $v\in V$ with $e^{m_j-1}\cdot v \in V^-$. (And $e^{m_j-2}\cdot v \in V^+$, $e^{m_j-3}\cdot v\in V^-$, etc.) The bases corresponding to different rows of Λ may be taken to be orthogonal with respect to $\langle\cdot,\ \cdot\rangle$. Since $x \in i\mathfrak{k}$ has real eigenvalues and $\langle\cdot,\ \cdot\rangle$ is conjugate linear in the second variable, eigenvectors in V^+ (resp. V^-) for x with distinct eigenvalues are mutually orthogonal relative to the restriction of $\langle\cdot,\ \cdot\rangle$ to V^+ (respectively V^-). Thus by suitable normalization of the vectors in the bases for each row of Λ, we can create an orthonormal basis of each eigenspace in V^+ (using the form $\langle\cdot,\ \cdot\rangle$) and an orthonormal basis of each eigenspace of V^- (using the form -$\langle\cdot,\ \cdot\rangle$). In this way we obtain orthonormal bases of V^+ and V^-. The integers in A_Λ^p (resp., B_Λ^q) give the eigenvalues of x in an orthonormal basis of V^+ (resp., V^-). We can find an element of K which transforms these orthonormal bases into the standard orthonormal bases of V^+ and V^-. $k\cdot x$ is a diagonal matrix with respect to the new bases. The integers in A_Λ^p occupy the first p entries along the main diagonal and those in B_Λ^q occupy the last q positions. Using the Weyl group of K we can rearrange these entries so that $k \cdot x$ becomes dominant with respect to the simple system in equation (3). This establishes the proposition. ■

Example 5.5. (a) $\mathfrak{g} = su(3,\ 1)$ and $\Lambda = \begin{smallmatrix} + & - & + \\ + & & \end{smallmatrix} = (+3)(+1)$

This is the principal nilpotent. The weighted Dynkin diagram is obtained by evaluating the neutral element at the compact simple roots $e_1 - e_2,\ e_2 - e_3$ and at $-\psi = e_4 - e_1$. The integers for the first row of Λ are $2,\ 0,\ -2$. So 2 and -2 are labeled with $+$, and 0 is labelled with $-$. The 0 in the second row of Λ is labelled '+'. Therefore, $A_\Lambda = \{2,\ -2\}$ and $B_\Lambda = \{0,\ 0\}$. This gives $e_1(x) = 2,\ e_2(x) = 0,\ e_3(x) = -2$, and $e_4(x) = 0$. The weighted Dynkin diagram of x is given by $e_1 - e_2 = 2,\ e_2 - e_3 = 2$, and $e_4 - e_1 = -2$.

(b)(i) $\mathfrak{g} = su(2,\ 2)$ and $\Lambda = +-+- = (+4)$

This is one of the principal nilpotents for $\mathfrak{g}$. $A_\Lambda = \{3,\ -1\}$ and $B_\Lambda = \{1,\ -3\}$. Therefore, $e_1(x) = 3,\ e_2(x) = -1,\ e_3(x) = 1$, and $e_4(x) = -3$. The weighted Dynkin diagram of x is given by $e_1 - e_2 = 4$, $e_3 - e_4 = 4$, and $e_4 - e_1 = -6$.

(b)(ii) $\mathfrak{g} = su(2,\ 2)$ and $\Lambda = -+-+ = (-4)$

This is the other principal nilpotent for $\mathfrak{g}$. $A_\Lambda = \{1,\ -3\}$ and $B_\Lambda = \{3,\ -1\}$. Therefore, $e_1(x) = 1,\ e_2(x) = -3,\ e_3(x) = 3$, and $e_4(x) = -1$. The weighted Dynkin diagram of x is given by $e_1 - e_2 = 4$, $e_3 - e_4 = 4$, and $e_4 - e_1 = -2$.

5.4. $so(p,\ q)$**.** $so(p,\ q)$ is the Lie subalgebra of $sl(p+q,\ \mathbf{R})$ which leaves invariant the quadratic form:

$$((x_1, \ldots, x_{p+q}),\ (x_1, \ldots, x_{p+q}))_{p,\ q} = x_1^2 + \ldots x_p^2 - x_{p+1}^2 - \ldots - x_{p+q}^2.$$

Thus $\mathfrak{g}$ is the space of real matrices of the form $\begin{pmatrix} A & B \\ B^\top & D \end{pmatrix}$ where A is $p \times p$ skew symmetric, D is $q \times q$ skew symmetric, B is $p \times q$ and $B^\top$ denotes the transpose of B. $\mathfrak{k}$ is the subspace of matrices of the form $\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$ where A is $p \times p$ skew symmetric and D is $q \times q$ skew symmetric. So $\mathfrak{k}$ is isomorphic to $so(p) \oplus so(q)$.

Let $I_{p,\ q}$, V^+ and V^- be defined in as subsection 5.. Let $s = [\frac{p}{2}]$ and $t = [\frac{q}{2}]$. Let $E_{i,\ j}$ denote the $(p+q) \times (p+q)$ matrix defined in subsection 5.. Let $\mathfrak{t}_1$ be the real span of the matrices $Y_i = E_{2i-1,\ 2i} - E_{2i,\ 2i-1}$, $i = 1, \ldots,\ s$, and $\mathfrak{t}_2$ be the real span of the matrices $Y_i' = E_{p+2i-1,\ p+2i} - E_{p+2i,\ p+2i-1}$, $i = 1, \ldots,\ t$. Then $\mathfrak{t} = \mathfrak{t}_1 \oplus \mathfrak{t}_2$ is a maximal torus of $\mathfrak{k}$. Define linear functionals e_j, $j = 1, \ldots,\ s+t$ on $\mathfrak{t}$, as follows.

$$e_j(Y_i) = -\sqrt{-1}\delta_{ij}, \quad e_j(\mathfrak{t}_2) = 0, \text{ for } 1 \le i \le s,\ 1 \le j \le s;$$
$$e_j(Y_i') = -\sqrt{-1}\delta_{i,\ j-s},\ e_j(\mathfrak{t}_1) = 0 \text{ for } 1 \le i \le t,\ \ s+1 \le j \le s+t.$$

We specify π, the following system of simple of roots, for $\mathfrak{k}$ depending on the parity of p and q.

$$\pi = \{e_1 - e_2,\ \ldots,\ e_{s-2} - e_{s-1}\} \cup \{e_{s+1} - e_{s+2}, \ldots,\ e_{s+t-2} - e_{s+t-1}\} \cup \pi'$$

where π' equals

$$\{e_{s-1} - e_s,\ e_{s-1} + e_s\} \cup \{e_{s+t-1} - e_{s+t},\ e_{s+t-1} + e_{s+t}\}\ (p,\ q \text{ even});$$
$$\{e_{s-1} - e_s,\ e_{s-1} + e_s\} \cup \{e_{s+t-1} - e_{s+t},\ e_{s+t}\}\ \ (p \text{ even},\ q \text{ odd});$$
$$\{e_{s-1} - e_s,\ e_s\} \cup \{e_{s+t-1} - e_{s+t},\ e_{s+t-1} + e_{s+t}\}\ (p \text{ odd},\ q \text{ even});$$
$$\{e_{s-1} - e_s,\ e_s\} \cup \{e_{s+t-1} - e_{s+t},\ e_{s+t}\}\ (p,\ q \text{ odd}).$$

The nilpotent orbits of $so(p,\ q)$. are parametrized by signed Young diagrams of signature $(p,\ q)$ such that rows of even length occur with even multiplicity and have their leftmost boxes labeled '+'. Some of these diagrams get roman numerals attached to them as follows. If Λ is such a diagram and all the rows have even length, then Λ corresponds to two $K_{\mathbf{C}}$ orbits which are conjugate under $G^\theta_{\mathbf{C}}$. Λ_I and Λ_{II} will denote the two $K_{\mathbf{C}}$ orbits. (This situation is possible only if both p and q are even.) The distinction between Λ_I and Λ_{II} is given below in Proposition 5.6. If at least one row of Λ has odd length and all odd rows have an even number of boxes labeled '+', or all odd rows have an even number of boxes labeled '-', then again Λ_I and Λ_{II} denote the corresponding $K_{\mathbf{C}}$ orbits, which are conjugate under $G^\theta_{\mathbf{C}}$ (This situation is possible only if at leas! t one of the integers $p,\ q$ is even.) The distinction between Λ_I and Λ_{II} is given below in Proposition 5.6. Thus if p and q are both odd, no numerals are attached to any signed Young diagram.

Proposition 5.6. *Let Λ be a signed partition of signature $(p,\ q)$. Represent Λ using the notation of Proposition 5.4.*

(A) *Assume that Λ does not have a numeral. We define two multisets: A^s_Λ and B^t_Λ by following the rules below. By arranging the elements of A^s_Λ in descending order, we obtain the integers $e_1(x),\ \ldots, e_s(x)$. By arranging the elements of B^t_Λ in descending order, we obtain the integers $e_{s+1}(x),\ \ldots, e_{s+t}(x)$.*

1. Suppose λ is an odd row (of length $m+1$) of Λ with integer labelling:

$$m,\ m-2,\ \ldots,\ 2,\ 0,\ -2,\ \ldots -m$$

In this case, the first integer m is even. Set $|\lambda|$ equal to the number of integers in the string, $|\lambda|_+$ (resp. $|\lambda|_-$) is the number of integers labelled with a "+" (resp. "-") sign.

case(a) Assume that m is labelled by a plus sign. Then, $|\lambda|_+ = [\frac{m+1}{2}]+1$ and $|\lambda|_- = [\frac{m-1}{2}]+1$.

Let c^+_m denote the number of rows identical to λ in Λ, i.e., the number of rows of length $|\lambda|$ that begin with a '+' sign. Arrange the $c^+_m|\lambda|_+$ integers from these rows which are labelled with a '+' sign in descending order, and assign the first $[\frac{c^+_m|\lambda|_+}{2}]$ of these integers to A^s_Λ. Likewise, arrange the $c_m|\lambda|_-$ integers from these rows that are labelled with a '-' sign in descending order, and assign the first $[\frac{c^+_m|\lambda|_-}{2}]$ to B^t_Λ.

case(b) Assume that m is labelled by a minus sign. Then, $|\lambda|_+ = [\frac{m-1}{2}]+1$ and $|\lambda|_- = [\frac{m+1}{2}]+1$. Let c^-_m denote the number of rows identical to λ in Λ, i.e., the number of rows of length $|\lambda|$ which begin with a '-' sign. Arrange the $c^-_m|\lambda|_+$ integers from these rows which are labelled with a '+' sign in descending order, and assign the first $[\frac{c^-_m|\lambda|_+}{2}]$ of these integers to A^s_Λ. Likewise, arrange the $c^-_m|\lambda|_-$ integers from these rows that are labelled with a '-' sign in descending order, and assign the first $[\frac{c^-_m|\lambda|_-}{2}]$ of them to B^t_Λ.

2. Now suppose that λ is an even row with integer labelling $m,\ m-2,\ \ldots, -m$. Then m is odd. Suppose that there are $2c$ copies of λ in Λ.

Place c copies of the string: $m,\ m-2,\ \ldots,\ 3,\ 1$ in A^s_Λ and place c copies of the same string in B^t_Λ. For example, if $c=1$, then assign the string

$$m,\ m-2,\ m-4,\ \ldots,\ 3,\ 1.$$

to A^s_Λ, and the string

$$m,\ m-2,\ m-4,\ \ldots,\ 3,\ 1.$$

to B^t_Λ.

3. If after performing the procedures in (1) and (2) on all rows, A^s_Λ has less that s elements (counted with multiplicity) place enough extra zeroes in A^s_Λ so that A^s_Λ has cardinality s. And if B^t_Λ has less that t elements (counted with multiplicity) place enough extra zeroes in B^t_Λ so that B^t_Λ has cardinality t.

We must now take into account the assignment of numerals to some of the signed partitions.

(B) *Assume that Λ is a signed partition with roman numeral "I" or "II". Then there are two possibilities.*

1. If all rows of Λ are even then both p and q are even. If we apply the rules for forming A^s_Λ and B^t_Λ in part (A) of the proposition, we find that $e_s = 1$ and $e_{s+t} = 1$. We stipulate that the numeral "I" assigned to Λ will correspond to the unique neutral element coming from the integers $e_1, \ldots, e_{s+t}$. We stipulate that the numeral "II" assigned to Λ will correspond to the unique neutral element coming from the integers $e_1, \ldots, e_{s-1}, -e_s, e_{s+1}, \ldots, e_{s+t-1}, -e_{s+t}$.

2. case(a) Suppose Λ has some odd rows, and all odd rows contain an even number of plus signs. Then p must be even. Apply the rules for forming A^s_Λ and B^t_Λ in part (A). These rules imply that $e_s \in \{1, 2\}$ and $e_{s+t} = 0$. Label Λ with roman numeral "I". Consider the neutral element x' defined by $e_i(x') = e_i$ for $i \neq s$ and $e_s(x') = -e_s$. It will correspond to Λ_{II}.

2. case (b) Suppose Λ has some odd rows, and all odd rows contain an even number of minus signs. Then q must be even. Apply the rules for forming A^s_Λ and B^t_Λ in part (A). These rules imply that $e_s = 0$ and $e_{s+t} \in \{1, 2\}$. Label Λ with roman numeral "I". Consider the neutral element x' defined by $e'_i(x) = e_i$ for $i \neq s+t$ and $e'_{s+t}(x) = -e_{s+t}$. It will correspond to Λ_{II}.

The proof of Proposition 5.6 uses ideas similar to those in the proof of Proposition 5.4 and will be omitted.

5.5. $so^*(2n)$**.** We use the notation of Helgason [10]. $\mathfrak{g}$ is the space of $2n \times 2n$ complex matrices of the form $\begin{pmatrix} Z_1 & Z_2 \\ -\bar{Z}_2 & \bar{Z}_1 \end{pmatrix}$ where Z_1 and Z_2 are complex $n \times n$-matrices, Z_1 is skew symmetric and Z_2 is Hermitian. ('$\bar{\ }$' denotes complex conjugation.) $\mathfrak{k}$ is the subspace of matrices in $\mathfrak{g}$ where Z_1 and Z_2 are real. (Thus Z_2 is symmetric.) The real linear map from $\mathfrak{k}$ to the complex $n \times n$ matrices given by $\begin{pmatrix} Z_1 & Z_2 \\ -\bar{Z}_2 & \bar{Z}_1 \end{pmatrix} \mapsto Z_1 + \sqrt{-1}Z_2$ is a Lie algebra isomorphism onto the space of $n \times n$ skew Hermitian matrices. Thus $\mathfrak{k}$ is isomorphic to $u(n)$. The subspace of matrices in $\mathfrak{k}$ of the form $\begin{pmatrix} 0 & B \\ -B & 0 \end{pmatrix}$, where B is diagonal, is a maximal torus $\mathfrak{t}$ of $\mathfrak{k}$. If $B = diag(y_1, \ldots, y_n)$, the linear functional e_i $(1 \leq i \leq n)$ on $\mathfrak{t}$ is defined so that its value on the $\begin{pmatrix} 0 & B \\ -B & 0 \end{pmatrix}$ is $\sqrt{-1}y_i$.

$\mathfrak{k}$ can also be realized as follows. Let $(\cdot, \cdot)$ be the usual symmetric form on $\mathbf{C}^{2n}$ and let

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \tag{4}$$

where I_n is the $n \times n$ identity matrix. Set V^+ (resp., V^-) equal to the i (resp., $-i$) eigenspace of J. Then, the bilinear form $\langle v, w \rangle = (v, \bar{w})$ is a Hermitian inner product on V^+, and $\mathfrak{k} = u(V^+)$.

For $\mathfrak{g} = so^*(2n)$, the nilpotent orbits are parametrized by signed Young diagrams of size n **and any signature** in which rows of odd length have their left most boxes labeled '+'.

The neutral element of a nilpotent conjugacy class will be described by giving its values at the following simple roots of $\mathfrak{k} = u(n)$ and at the noncompact root $-\psi$ of $\mathfrak{g}$ (where $\psi = e_1 + e_2$):

$$e_1 - e_2,\ e_2 - e_3,\ \ldots,\ e_{n-1} - e_n, \text{ and } -\psi = -e_1 - e_2.$$

Proposition 5.7. *Let Λ be a signed partition. Represent Λ using the notation of Proposition 5.4. We form a multiset A_Λ. Then, arrange the elements of A_Λ in descending order to obtain the integers: $e_1(x), \ldots, e_n(x)$.*

To form A_Λ, we label the signs in each row of Λ with the appropriate integers and then proceed as follows.

Suppose λ is an even row of Λ. Then it must begin with a non-negative odd integer m. Suppose m labels a plus sign, then place two copies of the integer string

$$m,\ m-4,\ m-8,\ \ldots,\ -(m-2)$$

in A_Λ. If m labels a minus sign, then place two copies of the integer string

$$m-2,\ m-6,\ m-10,\ \ldots,\ -m$$

in A_Λ.

Now suppose that λ is an odd row and is labelled by the integers

$$m,\ m-2,\ \ldots,\ 0,\ -2,\ \ldots\ -m$$

then place this entire string in A_Λ.

Proof. The argument is similar to that for Proposition 5.8 below and will be omitted. ∎

5.6. $sp(n,\ \mathbf{R})$. Let J be the matrix in (4). $sp(n,\ \mathbf{C})$ is the space of $2n \times 2n$ complex matrices X such that $X^\top J + JX = 0$. We identify $sp(n,\ \mathbf{R}))$ with the isomorphic Lie algebra $su(n,\ n) \cap sp(n,\ \mathbf{C})$, where $su(n,\ n)$ is defined as in subsection 5.4. (See Chapter VI, section 10 of [12].). Using this identification, $\mathfrak{k} = sp(n,\ \mathbf{R}) \cap u(2n)$ which is isomorphic to $u(n)$. Let $\mathfrak{t}$ be the space of matrices:

$$\{B = diag(\sqrt{-1}y_1, \ldots, \sqrt{-1}y_n, -\sqrt{-1}y_1, \ldots, -\sqrt{-1}y_n) | y_i \in \mathbf{R}\}. \tag{5}$$

Define the linear functionals e_j on $\mathfrak{t}$ by setting $e_j(B) = \sqrt{-1}y_j$ where B is the diagonal matrix above.

The neutral element of a nilpotent conjugacy class will be described by giving its values at the following simple roots of $\mathfrak{k} = u(n)$ and at $-\psi$ (where $\psi = 2e_1$):

$$e_1 - e_2,\ e_2 - e_3,\ \ldots,\ e_{n-1} - e_n, \text{ and } -\psi = -2e_1.$$

In addition, let V^+ (resp. V^-) denote the $+i$ (resp., $-i$) eigenspace of the matrix J. If $\{\cdot,\ \cdot\}$ is the skew symmetric form on $\mathbf{C}^{2n}$ which defines $sp(n,\ \mathbf{C})$, then $\langle v,\ w\rangle = -\{v,\ \overline{Jw}\}$ is the standard positive definite skew Hermitian form on $\mathbf{C}^{2n}$. We will identify $\mathfrak{k}$ with $u(V^+)$, defined relative to the restriction of $\langle \cdot,\ \cdot\rangle$ to V^+.

Nilpotent orbits are parametrized by signed Young diagrams of size $2n$ and any signature in which odd length rows appear with even multiplicity and begin with a '+'. Even length rows may begin with '+' or '-'.

Proposition 5.8. *Let Λ be a signed partition. Represent Λ using the notation of Proposition 5.4. We label the signs in each row of Λ with the appropriate integers. We form a multiset A_Λ as follows.*

(1) Suppose λ is an even row of Λ. Then it must begin with a non-negative odd integer m. Suppose m labels a plus sign, then place the integer string

$$m,\ m-4,\ m-8,\ \ldots,\ -(m-2)$$

in A_Λ. If m labels a minus sign, then place the integer string

$$m-2,\ m-6,\ m-10,\ \ldots,\ -m$$

in A_Λ.

(2) Suppose that λ is an odd row with integer labelling

$$m,\ m-2,\ \ldots,\ 2,\ 0,\ -2,\ \ldots -m$$

In this case, the first integer m is even, and labels a plus sign.

Suppose that there are exactly $2c$ copies of λ in Λ, then place c copies of the set $\{m,\ m-2,\ \ldots,\ 2,\ 0,\ -2,\ \ldots -m\}$ in A_Λ.

Arrange the elements of A_Λ in descending order to obtain the integers: $e_1(x),\ \ldots, e_n(x)$.

Proof. (Sketch) Suppose that λ is an even length row of type $(+m_j)$, then the theory of 'ab'-diagrams shows that there is an irreducible m_j-dimensional $\mathfrak{a}_{\mathbf{C}}$-submodule W_j^+ of V with basis: $v,\ e\cdot v,\ e^2\cdot v,\ \ldots,\ e^{m_j-1}\cdot v$ for some $v\in V$ with $e^{m_j-1}\cdot v\in V^+$. We have $v\in V^-$, $e\cdot v\in V^+$, etc. In addition,

$$(1)\{e^a\cdot v,\ e^b\cdot v\}=0\ \text{ if } a+b\neq m_j-1.$$
$$(2)\{e^a\cdot v,\ e^{m_j-1-a}\cdot v\}=(-1)^a\alpha_{m_j},$$

where α_{m_j} is a nonzero complex number depending on m_j alone.

If λ is an even length row of type $(-m_j)$ then there is an irreducible m_j-dimensional $\mathfrak{a}_{\mathbf{C}}$-submodule W_j^- of V with basis: $v,\ e\cdot v,\ e^2\cdot v,\ \ldots,\ e^{m_j-1}\cdot v$ for some $v\in V$ and $e^{m_j-1}\cdot v\in V^-$. We have $v\in V^+$, $e\cdot v\in V^-$, etc. In addition,

$$(1)\{e^a\cdot v,\ e^b\cdot v\}=0\ \text{ if } a+b\neq m_j-1.$$
$$(2)\{e^a\cdot v,\ e^{m_j-1-a}\cdot v\}=(-1)^a\beta_{m_j},$$

where β_{m_j} is a nonzero complex number depending on m_j alone.

Since $\{v,\ \bar v\}\neq 0$ and m_j is even, W_j^+ and W_j^- are stable under complex conjugation. See the proof of Proposition 2 in [18].

Clearly $\overline{e^a\cdot v}$ is a multiple of $e^{m_j-1-a}\cdot v$. Therefore, $\langle e^a\cdot v,\ e^b\cdot v\rangle=0$ unless $a=b$. Thus, each $(+m_j)$, for m_j even, contributes mutually orthogonal (relative to $\langle\cdot,\ \cdot\rangle$) eigenvectors

$$e^{m_j-3}\cdot v,\ e^{m_j-7}\cdot v,\ldots,\ v$$

for x with eigenvalues $m_j-3,\ m_j-7,\ldots,\ -(m_j-1)$ to an eigenbasis of V^+. Similarly each $(-m_j)$, for m_j even, contributes mutually orthogonal eigenvectors

$$e^{m_j-1}\cdot v,\ e^{m_j-5}\cdot v,\ldots,\ e\cdot v$$

for x with eigenvalues $m_j - 1,\ m_j - 5, \dots,\ -(m_j - 3)$ to an eigenbasis of V^+.

Suppose that λ is an odd length row of type $(+m_j)$ in Λ. Such rows occur in pairs. The theory of 'ab'-diagrams shows that for each such pair, V contains a direct sum of two irreducible m_j-dimensional $\mathfrak{a}_{\mathbf{C}}$-modules W_j' and W_j'' with respective bases:

$$v,\ e \cdot v,\ e^2 \cdot v,\ \dots,\ e^{m_j - 1} \cdot v$$
$$w,\ e \cdot w,\ e^2 \cdot w,\ \dots,\ e^{m_j - 1} \cdot w$$

where $v \in V^+, e \cdot v \in V^-,\ e^2 \cdot v \in V^+$, etc. and $w \in V^-,\ e \cdot w \in V^+,\ e^2 \cdot w \in V^-$, etc. Moreover,

(1) $\{e^a \cdot v,\ e^b \cdot v\} = 0 = \{e^a \cdot w,\ e^b \cdot w\}$ for all a and b.

(2) $\{e^a \cdot v,\ e^b \cdot w\} = 0$ if $a + b \neq m_j - 1$.

(3) $\{e^a \cdot v,\ e^{m_j - 1 - a} \cdot w\} = (-1)^a \delta_{m_j}$.

δ_{m_j} is a nonzero complex number depending on m_j alone. In this case, W_j'' is the complex conjugate of W_j'. Each pair of odd rows $(+m_j)$ contributes mutually orthogonal eigenvectors (relative to $\langle \cdot,\ \cdot \rangle$) for x:

$$v,\ e \cdot w,\ e^2 \cdot v, \dots,\ e^{m_j - 2} \cdot w,\ e^{m_j - 1} \cdot v$$

with eigenvalues $-(m_j - 1),\ -(m_j - 3), \dots,\ (m_j - 1)$ to an eigenbasis of V^+.

The eigenvectors obtained from distinct even rows and distinct pairs of odd rows of Λ are mutually orthogonal with respect to $\langle \cdot,\ \cdot \rangle$. It is clear that (after normalizing the vectors) steps (1) and (2) of Proposition 5.8 will yield the eigenvalues of x on an orthonormal eigenbasis of V^+. This determines x as an element of $u(V^+)$. The remaining details are left to the reader. ∎

5.7. $sp(p,\ q)$**.** We adopt the notation of Helgason [10] (Chapter X, section 2) for $sp(p,\ q)$ inside $sp(p+q,\ \mathbf{C})$. $\{\cdot,\ \cdot\}$ is the skew symmetric form on $\mathbf{C}^{2p+2q}$ which defines $sp(p{+}q,\ \mathbf{C})$. Let $K_{p,\ q}$ be the block diagonal matrix $diag(-I_p,\ I_q,\ -I_p,\ I_q)$ where I_p (resp., I_q) is the $p \times p$ (resp., $q \times q$) identity matrix. V^+ and V^- denote the $+1$ and -1 eigenspaces of $K_{p,\ q}$. $sp(p,\ q)$ is the space of $2(p+q) \times 2(p+q)$ complex matrices X such that $X^\top K_{p,\ q} + K_{p,\ q}\bar{X} = 0$. $\mathfrak{k} = sp(p,\ q)\ \cap\ u(2p+2q)$ which is isomorphic to $sp(p) \oplus sp(q)$. (See Chapter X, section 2, Lemma 2.1 in [10].) Let $\mathfrak{t}$ be the space of matrices defined as in (5) with $n = p + q$. Define the linear functionals $e_j,\ j = 1, \dots, p+q$ on $\mathfrak{t}$ as in subsection 5.

In this case, we choose the following simple system for $\mathfrak{k}$:

$$\{e_1 - e_2, \dots,\ e_{p-1} - e_p,\ 2e_p\} \cup \{e_{p+1} - e_{p+2}, \dots,\ e_{p+q-1} - e_{p+q},\ 2e_{p+q}\}.$$

Nilpotents in $sp(p,\ q)$ are parametrized by signed Young tableaus of signature $(p,\ q)$ in which even rows begin with '+'.

Proposition 5.9. *Let Λ be a signed partition of signature $(p,\ q)$. Represent Λ using the notation of Proposition 5.4. We form two multisets of nonnegative integers: A^p_Λ and B^q_Λ. By arranging the elements of A^p_Λ in descending order,* *we obtain the integers* $e_1(x),\ \ldots, e_p(x)$. *By arranging the elements of* B^q_Λ *in descending order, we obtain the integers* $e_{p+1}(x),\ \ldots, e_{p+q}(x)$.

To obtain A^p_Λ *and* B^q_Λ, *we first label the signs in each row of* Λ *with the appropriate integers.*

Suppose λ *is an even row of* Λ. *Then it must begin with a non-negative odd integer* m. *In this case* m *labels a plus sign. Place one copy of the integer string*

$$m,\ m-2,\ m-4,\ \ldots,\ 1$$

in A^p_Λ *and one copy of the same string in* B^q_Λ.

Next suppose that λ *is an odd row. We have the integer string:*

$$m,\ m-2,\ \ldots,\ 2,\ 0,\ -2,\ \ldots -m$$

In this case, the first integer m *is even, and the row contains* $m+1$ *integers.*

If m *is labelled with a plus sign there are two subcases:*

(a) $m \equiv 0 \pmod 4$, *so* $m = 4k$. *Then the integer string*

$$m,\ m,\ m-4,\ m-4,\ \ldots,\ 4,\ 4,\ 0$$

which contains $2k+1$ *integers is assigned to* A^p_Λ *and the integer string*

$$m-2,\ m-2,\ m-6,\ m-6,\ \ldots,\ 2,\ 2$$

which contains $2k$ *integers is assigned to* B^q_Λ. *(Thus, if* $k=0$, *no integers are assigned to* B^q_Λ.*)*

(b) $m \equiv 2 \pmod 4$, *so* $m = 4k+2$. *Then the integer string*

$$m,\ m,\ m-4,\ m-4,\ \ldots,\ 2,\ 2$$

which contains $2k+2$ *integers is assigned to* A^p_Λ *and the integer string*

$$m-2,\ m-2,\ m-6,\ m-6,\ \ldots,\ 4,\ 4,\ 0$$

which contains $2k+1$ *integers is assigned to* B^q_Λ.

If m *is labelled with a minus sign there are two subcases:*

(a) $m \equiv 0 \pmod 4$, *so* $m = 4k$. *Then the integer string*

$$m-2,\ m-2,\ m-6,\ m-6,\ \ldots,\ 2,\ 2$$

which contains $2k$ *integers is assigned to* A^p_Λ *and the integer string*

$$m,\ m,\ m-4,\ m-4,\ \ldots,\ 4,\ 4,\ 0$$

which contains $2k+1$ *integers is assigned to* B^q_Λ.

(b) $m \equiv 2 \pmod 4$, *so* $m = 4k+2$. *Then the integer string*

$$m-2,\ m-2,\ m-6,\ m-6,\ \ldots,\ 4,\ 4,\ 0$$

which contains $2k+1$ *integers is assigned to* A^p_Λ *and the integer string*

$$m,\ m,\ m-4,\ m-4,\ \ldots,\ 2,\ 2$$

which contains $2k+2$ *integers is assigned to* B^q_Λ.

Proof. One uses V^+ and V^- in a manner similar to that in the proof of Proposition 5.4. The details are left to the reader. ∎

6. The spherical nilpotent $K_{\mathbf{C}}$-orbits in $\mathfrak{p}_{\mathbf{C}}$ for $\mathfrak{g}$ classical

We will prove the following result in section 8.

Theorem 6.1. *If $\mathfrak{g}$ is a simple real classical Lie algebra then the spherical nilpotent $K_{\mathbf{C}}$-orbits in $\mathfrak{p}_{\mathbf{C}}$ are precisely those corresponding to signed partitions Λ in the following list. The notation for Λ is as in Proposition 5.4. (m, k, k_1, k_2, r, r_1, r_2 are non negative integers.)*

$sl(s,\ \mathbf{R})$: No part size of Λ exceeds 2.

$su^*(2n)$: No part size of Λ exceeds 2.

$su(p,\ q)$: Λ is one the following:

(a) $(+3)(+2)^{k_1}(-2)^{k_2}(+1)^{r_1}(-1)^{r_2};(-3)(+2)^{k_1}(-2)^{k_2}(+1)^{r_1}(-1)^{r_2}$

(b) $(+3)^2(+1)^r$; $(-3)^2(-1)^r$

(c) $(+2)^{k_1}(-2)^{k_2}(+1)^{r_1}(-1)^{r_2}$.

$so(p,\ q)$: Λ (after ignoring numerals) is one of the following:

(a) $(+3)^m(+1)^r$; $(-3)^m(-1)^r$ $(m \leq 2)$

(b) $(+3)^m(+2)^k(+1)^{r_1}(-1)^{r_2}$; $(-3)^m(+2)^k(+1)^{r_1}(-1)^{r_2}$ $(m \leq 1,\ k \text{ even})$.

$so^*(2n)$: Λ is $(+3)(+1)^r$ or $(+2)^{k_1}(-2)^{k_2}(+1)^r$.

$sp(n,\ \mathbf{R})$: No part size of Λ exceeds 2.

$sp(p,\ q)$: Λ is one of the following:

(a) $(+3)(+1)^{r_1}(-1)^{r_2}$; $(+3)(+2)(+1)^{r_1}(-1)^{r_2}$

(b) $(-3)(+1)^{r_1}(-1)^{r_2}$; $(-3)(+2)(+1)^{r_1}(-1)^{r_2}$

(c) $(+2)^k(+1)^{r_1}(-1)^{r_2}$.

Remark 6.2. The argument is case by case. Each algebra is treated in one of the subsections of section 8.

The following proposition greatly reduces the task of classifying the spherical nilpotent $K_{\mathbf{C}}$-orbits in $\mathfrak{p}_{\mathbf{C}}$ for $\mathfrak{g}$ real, classical and simple.

Proposition 6.3. *Assume that $\mathfrak{g}$ is real, classical and simple, and we retain the notation of section 5. If $K_{\mathbf{C}} \cdot e$ is a spherical nilpotent orbit, and Λ is the corresponding signed partition, then $|e_i(x)| \leq 2$ for all i. That is, no part size (i.e., row length) of Λ exceeds 3.*

Proof. The proof is case by case and depends on Proposition 4.17 and precise information about the $\mathfrak{t}_{\mathbf{C}}$-weights of the representation of $\mathfrak{k}_{\mathbf{C}}$ on $\mathfrak{p}_{\mathbf{C}}$ to restrict the values of the $e_i(x)$. We give details only for $\mathfrak{g} = sl(s,\ \mathbf{R})$, $su^*(2n)$, $su(p,\ q)$, and $sp(n,\ \mathbf{R})$.

1. $\mathfrak{g} = sl(s,\ \mathbf{R})$. The weights of $\mathfrak{t}_{\mathbf{C}}$ in $\mathfrak{p}_{\mathbf{C}}$ are of the form $\pm(e_i \pm e_j)$ (and possibly $\pm e_i$) and $\pm 2e_i$. By Proposition 5.1 we have $e_i(x) \geq 0$ for all i. Thus $0 \leq 2e_i(x) \leq 3$ by Proposition 4.17. Hence by integrality each $e_i(x) \leq 1$. By Proposition 5.1 this means that no part size exceeds 2.

2. $\mathfrak{g} = su^*(2n)$. $\mathfrak{p}_{\mathbf{C}}$ is the representation of $\mathfrak{k} = sp(n)$ on the irreducible submodule of $\bigwedge^2 \mathbf{C}^{2n}$ of dimension $2n^2 - n - 1$. The non zero $\mathfrak{t}_{\mathbf{C}}$ weights on $\mathfrak{p}_{\mathbf{C}}$ are of the form $\pm(e_i \pm e_j)$, $1 \leq i < j \leq n$. The non zero $\mathfrak{t}_{\mathbf{C}}$ weights on $\mathfrak{k}_{\mathbf{C}}$ are of the form $\pm(e_i \pm e_j)$, $1 \leq i < j \leq n$, and $\pm 2e_i$, $1 \leq i \leq n$.

By Proposition 5.3 we have $e_i(x) \geq 0$ for all i. By Proposition 4.17, the $\mathfrak{k}_{\mathbf{C}}$-height cannot exceed 4, therefore $0 \leq e_i(x) \leq 2$ for all i. Since the $\mathfrak{p}_{\mathbf{C}}$ height cannot exceed 3, only one $e_i(x)$ may equal 2 and the other $e_i(x)$ must be 0 or 1.

3. $\mathfrak{g} = su(p,\ q)$. The weights of $\mathfrak{t}_{\mathbf{C}}$ in $\mathfrak{p}_{\mathbf{C}}$ are of the form $\pm(e_i - e_j)$, $1 \leq i \leq p$, $p+1 \leq j \leq p+q$. If for some i, $e_i(x) > 2$, then some row of length at least 4 must occur in Λ. If a row of length 4 occurs, then Λ contains either (a)'$+-+-$' or (b) '$-+-+$'. The corresponding integers are $3, 1, -1, -3$. In either case, by Proposition 5.4, there exists $i,\ j$, $1 \leq i \leq p$, $p \leq j \leq p+q$ such that $|e_i(x) - e_j(x)| = 6$. So $\mathfrak{p}_{\mathbf{C}}(6) \neq 0$, contradicting Proposition 4.17. A similar argument shows that no part size can exceed 4.

4. $\mathfrak{g} = sp(n,\ \mathbf{R})$. $\mathfrak{p}_{\mathbf{C}}$ is the sum of the symmetric square of the first fundamental representation of $k = u(n)$ and the symmetric square of the last fundamental representation of $k = u(n)$. So the $\mathfrak{t}_{\mathbf{C}}$-weights of $\mathfrak{p}_{\mathbf{C}}$ have the form $\pm(e_i \pm e_j)$, $1 \leq i, j \leq n$ (where i can equal j). Since $\mathfrak{p}_{\mathbf{C}}$-height cannot exceed 3, $|2e_i(x)| \leq 3$ for all i. Since $e_i(x)$ is integral this implies $|e_i(x)| \leq 1$ for all i. ∎

To classify the spherical nilpotent $K_{\mathbf{C}}$-orbits in $\mathfrak{p}_{\mathbf{C}}$ for $\mathfrak{g}$ real, classical and simple we proceed as follows. First, by Proposition 6.3, we need only consider those signed partitions Λ where no part size exceeds 3. Second, for the corresponding orbit $\mathcal{O} = \mathcal{O}_\Lambda$ we apply the criteria (a) and (b) of Corollary 4.13. To check criterion (a), i.e., whether $K^x_{\mathbf{C}}/K^{\{x,\ e,\ f\}}_{\mathbf{C}}$ is spherical: (1) construct $\mathfrak{k}^x$ and $\mathfrak{k}^{\{x,\ e,\ f\}}$; (2) determine how $\mathfrak{k}^{\{x,\ e,\ f\}}$ sits inside $\mathfrak{k}^x$; and (3) use the results in [14] to check whether $\mathfrak{k}^{\{x,\ e,\ f\}}$ is spherical inside $\mathfrak{k}^x$. If $\mathcal{O}$ satisfies criterion (a), we show that either: (1) $\mathcal{O}$ satisfies criterion (b) of Corollary 4.13; or (2) $\mathcal{O}$ lies in the closure of an orbit we have shown to be spherical; or (3!) the closure of $\mathcal{O}$ contains an orbit which we know is not spherical. (See Proposition 4.11.) These techniques handle all the Λ with no part size exceeding 3.

7. Structure of $\mathfrak{k}^x$ and $\mathfrak{k}^{\{x,\ e,\ f\}}$ for special x

7.1. Notation and conventions. In order to test $K_{\mathbf{C}} \cdot e$ for sphericality we need to know the structure of $\mathfrak{k}^x$, the structure of $\mathfrak{k}^{\{x,\ e,\ f\}}$ and how $\mathfrak{k}^{\{x,\ e,\ f\}}$ is contained in $\mathfrak{k}^x$ when the signed partition corresponding to $K_{\mathbf{C}} \cdot e$ has no part sizes greater than 3. The reductive components of $\mathfrak{k}^x$ can be found from the weighted Dynkin diagram of x that can be found from the recipes in section 5.. The structure of $\mathfrak{g}^{\{x,\ e,\ f\}}$, and hence $\mathfrak{k}^{\{x,\ e,\ f\}}$ can be found by applying results in [24] and [3].

The 'ab'-diagram (signed partition) description of $K_{\mathbf{C}} \cdot e$ allows one to determine how $\mathfrak{k}^{\{x,\ e,\ f\}}$ is embedded in $\mathfrak{k}^x$. This information is summarized below for each classical simple real algebra other than $sl(s,\ \mathbf{R})$ and $sp(n,\ \mathbf{R})$. We will only sketch the proofs in case $\mathfrak{g}$ is $su(p,\ q)$. We use the following conventions in stating the results.

First, if $\mathfrak{q}$ is a reductive Lie algebra, then the containment $\mathfrak{q} \subset 2\mathfrak{q}$ (resp., $\mathfrak{q} \subset 3\mathfrak{q}$) always refers to the diagonal embedding $X \mapsto X \oplus X$ (resp., $X \mapsto X \oplus X \oplus X$). More generally if we have reductive Lie algebras $\mathfrak{q}_0$, $\mathfrak{q}_1$, and $\mathfrak{q}_2$ with $\mathfrak{q}_0 \subset \mathfrak{q}_i$ ($i = 1,\ 2$). Then, we say that $\mathfrak{q}_0$ is diagonally embedded in the direct sum $\mathfrak{q}_1 \oplus \mathfrak{q}_2$ if we have a Lie algebra homorphism $\mathfrak{q}_0 \to \mathfrak{q}_1 \oplus \mathfrak{q}_2$ given by $X \mapsto X \oplus X$.

Suppose $\mathfrak{g}$ is a classical simple real algebra other than $sl(s,\ \mathbf{R})$ or $su^*(2n)$.

Let $V = V(\mathfrak{g})$, be the complex vector space associated to $\mathfrak{g}$ in the corresponding recipe in section 5. In that section subspaces V^+ and V^- were defined for each $\mathfrak{g}$. The notation $\mathfrak{q}^+$ (resp., $\mathfrak{q}^-$) applied to a subalgebra of $\mathfrak{g}$ indicates that $\mathfrak{q}$ consists of linear transformations which preserve the subspace V^+ (resp., V^-) associated to the form on V.(See section 5.) In general, if W is a complex (resp. real) vector space with a positive definite Hermitian (resp. quadratic) form, then $u(W)$ (resp. $o(W)$) denotes the Lie algebra of the unitary (resp. orthogonal) group preserving the form.

Finally we need

Definition 7.1. Let $\mathfrak{g}$ be a classical simple real algebra other than $sl(s, \mathbf{R})$ or $su^*(2n)$. We set $V\{j\}$, $(j = 1,\ 2,\ 3)$ equal to the sum of the irreducible j-dimensional $\mathfrak{a}_{\mathbf{C}}$-submodules of V, $V_+\{j\}$ (resp., $V_-\{j\}$) equal to the sum of the irreducible j-dimensional $\mathfrak{a}_{\mathbf{C}}$-submodules of V whose highest weight vector (i.e., the kernel of the element e) belongs to V^+ (resp. V^-). If $a = 0,\ \pm1,\ \pm2$, or ±3, then $V\{j,\ a\}$ denotes the a-eigenspace of x in $V\{j\}$. $V_+\{j,\ a\}$ and $V_-\{j,\ a\}$ have similar meanings.

7.2. $su^*(2n)$**.** Let $\Lambda = 3^p\ 2^q\ 1^r$. Then

$$\mathfrak{k}^x = u(2p) \oplus u(2q) \oplus sp(p+r);\ \mathfrak{k}^{\{x,\ e,\ f\}} = sp(p) \oplus sp(q) \oplus sp(r) \tag{6}$$

with

$$sp(p) \oplus sp(r) \subset u(2p) \oplus sp(p+r);\ sp(q) \subset u(2q). \tag{7}$$

$sp(p)$ is embedded diagonally in $u(2p)\oplus sp(p+r)$, and $sp(r)$ is embedded naturally in $sp(p+r)$.

Lemma 7.2. *The embedding $sp(p) \oplus sp(r) \subset u(2p) \oplus sp(p+r)$ in equation (7) is spherical if and only if $p \leq 1$.*

Proof. The proof is left to the reader. ∎

7.3. $\mathfrak{g} = su(p,\ q)$**.**

Let $\Lambda = (+3)^{m_1}(-3)^{m_2}(+2)^{n_1}(-2)^{n_2}(+1)^{r_1}(-1)^{r_2}$. Set $\widetilde{\mathfrak{k}} = u(p) \oplus u(q)$. It is more convenient to describe the containment $\widetilde{\mathfrak{k}}^{\{x,\ e,\ f\}} \subset \widetilde{\mathfrak{k}}^x$. This is sufficient since $\mathfrak{k}^x$ (resp., $\mathfrak{k}^{\{x,\ e,\ f\}}$) consists of the trace zero matrices in $\widetilde{\mathfrak{k}}^x$ (resp., $\widetilde{\mathfrak{k}}^{\{x,\ e,\ f\}}$). We have

$$\begin{aligned} \widetilde{\mathfrak{k}}^x &= 2u(m_1)^+ \oplus u(n_1)^+ \oplus u(n_2)^+ \oplus u(r_1+m_2)^+ \\ &\quad \oplus 2u(m_2)^- \oplus u(n_1)^- \oplus u(n_2)^- \oplus u(r_2+m_1)^- \\ \widetilde{\mathfrak{k}}^{\{x,\ e,\ f\}} &= u(m_1) \oplus u(m_2) \oplus u(n_1) \oplus u(n_2) \oplus u(r_1) \oplus u(r_2) \end{aligned} \tag{8}$$

with

$$\begin{aligned} &u(n_1) = D(u(n_1)^+ \oplus u(n_1)^-);\ u(n_2) = D(u(n_2)^+ \oplus u(n_2)^-); \\ &u(m_1) \oplus u(r_2) \subset \oplus(2u(m_1)^+ \oplus u(m_1+r_2)^-); \\ &u(m_2) \oplus u(r_1) \subset (2u(m_2)^- \oplus u(m_2+r_1)^+). \end{aligned} \tag{9}$$

D denotes the diagonal embedding. Note that $u(m_1)$ embeds diagonally in $2u(m_1)^+$ and sits naturally inside $u(m_1+r_2)^-$, while $u(r_2)$ sits naturally inside $u(m_1+r_2)^-$. $u(m_2)$ embeds diagonally in $2u(m_2)^-$ and sits naturally inside $u(m_2+r_1)^-$, while $u(r_1)$ sits naturally inside $u(m_2+r_1)^-$. If $r_1=0$, then $u(m_1)$ embeds diagonally in 3 copies of $u(m_1)$. If $r_2=0$, then $u(m_2)$ embeds diagonally in 3 copies of $u(m_2)$.

We will demonstrate the embeddings $u(n_1) = D(u(n_1)^+ \oplus u(n_1)^-)$ and $u(m_1)\oplus u(r_2) \subset \oplus(2u(m_1)^+ \oplus u(m_1+r_2)^-)$ in (9).

Proof. The form $\langle\cdot,\ \cdot\rangle_{p,\ q}$ is non degenerate and either positive definite or negative definite on each of the complex subspaces $V_\pm\{i,\ j\}$. If $\zeta \in \widetilde{\mathfrak{k}^x}$, ζ preserves the eigenspaces of x and the restriction of ζ to each of these subspaces preserves the restriction of the form $\langle\cdot,\ \cdot\rangle_{p,\ q}$. Thus, ζ corresponds to an element of the direct sum of the Lie algebras in lines (10) and (11) below:

$$u(V_+\{3,2\}), u(V_+\{3,-2\}), u(V_-\{3,0\}+V_+\{1,0\}), u(V_+\{2,1\}),\ u(V_+\{2,-1\}) \tag{10}$$

$$u(V_-\{3,2\}), u(V_-\{3,-2\}), u(V_+\{3,0\}+V_-\{1,0\}), u(V_-\{2,1\}),\ u(V_-\{2,-1\}). \tag{11}$$

Line (10) gives $2u(m_1)^+ \oplus u(m_2+r_1)^+ \oplus u(n_1)^+ \oplus u(n_1)^-$ and line (11) gives $2u(m_2)^- \oplus u(m_1+r_2)^- \oplus u(n_2)^- \oplus u(n_1)^+$. So we have the first half of (8).

Suppose that $\zeta \in \mathfrak{k}^{\{x,\ e,\ f\}}$. Clearly, the maps $e : V_\pm\{3,\ 0\} \to V_\mp\{3,\ 2\}$, $e^2 : V_\pm\{3,\ -2\} \to V_\mp\{3,\ 2\}$, and $e : V_\pm\{2,\ -1\} \to V_\mp\{2,\ 1\}$ are isomorphisms. By redefining the restriction of the form $\langle\cdot,\ \cdot\rangle_{p,\ q}$ on each of the spaces $V_\mp\{3,\ 2\}$, and $V_\mp\{2,\ 1\}$, we can assume that the maps defined by e and e^2 are isometries. ζ preserves the following complex subspaces: the m_1 (resp., m_2) dimensional subspace W_1 (resp., W_1') spanned by the vectors $v+(e^2\cdot v)$ and $v \in V_+\{3,\ -2\}$ (resp., $v \in V_-\{3,\ -2\}$); and the n_1 (resp., n_2) dimensional subspace W_2 (resp. W_2') spanned by $w+e\cdot w$, $w \in V_+\{2,\ -1\}$ (resp., $w \in V_-\{2,\ -1\}$). Thus, $u(W_1)$ maps into the diagonal of $u(V_+\{3,\ 2\}) \oplus u(V_+\{3,\ -2\})$. $u(W_2)$ maps into the diagonal of $u(V_+\{2,\ 1\}) \oplus u(V_+\{2,-1\})$. But in addition there is a map from $u(W_1)$ into $u(V_-\{3,\ 0\} \oplus V_+\{1,\ 0\})$ given as follows: T in $u(W_1)$ is mapped to $e\circ T\circ e^{-1}$ in $u(V_-\{3,\ 0\})$. Similar arguments apply to W_1' and W_2'. ■

Lemma 7.3. *Let $u(p)\oplus u(r) \subset 2u(p)\oplus u(r+p)$ be the embedding in the second line of (9).* (a) *If $p=1$ the embedding is spherical for all r.* (b) *If $p=2$ the embedding is spherical if and only if $r=0$.* (c) *If $p\ge 3$, the embedding is not spherical.*

We leave the proof to the reader.

Corollary 7.4. *If $m_1 \ge 3$ or $m_2 \ge 3$, then $\mathfrak{k}^{\{x,\ e,\ f\}} \subset \mathfrak{k}^x$ in equation (8) is not a spherical embedding.*

7.4. $so(p,\ q)$. Let $\Lambda = (+3)^{m_1}(-3)^{m_2}(+2)^{n}(+1)^{r_1}(-1)^{r_2}$.

$$\begin{gathered}\mathfrak{k}^x = u(m_1)^+ \oplus u(\frac{n}{2})^+ \oplus so(m_2+r_1)^+ \oplus u(m_2)^- \oplus u(\frac{n}{2})^- \oplus so(m_1+r_2)^- \\ \mathfrak{k}^{\{x,\ e,\ f\}} = so(m_1)\oplus so(m_2)\oplus u(\frac{n}{2})\oplus so(r_1)\oplus so(r_2)\end{gathered} \tag{12}$$

where

$$\begin{aligned} u(\frac{n}{2}) &= D(u(\frac{n}{2})^{+} \oplus u(\frac{n}{2})^{-}); \\ so(m_1) \oplus so(r_2) &\subset u(m_1)^{+} \oplus so(m_1+r_2)^{-}; \\ so(m_2) \oplus so(r_1) &\subset u(m_2)^{-} \oplus so(m_2+r_1)^{+}. \end{aligned} \tag{13}$$

$so(m_1)$ (resp., $so(m_2)$) is embedded diagonally in $u(m_1)^{+} \oplus so(m_1+r_2)^{-}$ (resp., $u(m_2)^{-} \oplus so(m_2+r_1)^{+}$). $so(r_2)$ (resp., $so(r_1)$) is embedded naturally in $so(m_1+r_2)^{-}$ (resp., $so(m_1+r_2)^{+}$).

Lemma 7.5. (a) *If $r > 0$ and $m = 2$ or* (b) *$m \geq 3$ the embedding*

$$so(m) \oplus so(r) \subset u(m) \oplus so(m+r))$$

in (13) is not spherical.

We leave the proof to the reader.

Corollary 7.6. *If $m_1 \geq 3$ or $m_2 \geq 3$, then $\mathfrak{k}^{\{x,\, e,\, f\}} \subset \mathfrak{k}^{x}$ in equation (12) is not a spherical embedding.*

7.5. $so^{*}(2n)$**.** Let $\Lambda = (+3)^{p}(+2)^{q_1}(-2)^{q_2}(+1)^{r}$. Then

$$\begin{aligned} \mathfrak{k}^{x} &= 2u(p) \oplus u(p+r) \oplus u(2q_1) \oplus u(2q_2) \\ \mathfrak{k}^{\{x,\, e,\, f\}} &= u(p) \oplus sp(q_1) \oplus sp(q_2) \oplus u(r). \end{aligned} \tag{14}$$

with

$$u(p) \oplus u(r) \subset 2u(p) \oplus u(p+r);\ sp(q_i) \subset u(2q_i),\ i = 1, 2$$

where $u(p) \oplus u(r)$ is embedded in $2u(p) \oplus u(p+r)$ as in equation (9).

Lemma 7.7. *The containment $\mathfrak{k}^{\{x,\, e,\, f\}} \subset \mathfrak{k}^{x}$ in equation (14) is spherical $\iff p \leq 2$.*

Proof. This follows from Lemma 7.3. ∎

7.6. $sp(p,\ q)$**.**

Let $\Lambda = (+3)^{m_1}(-3)^{m_2}(+2)^{n}(+1)^{r_1}(-1)^{r_2}$. Then

$$\begin{aligned} \mathfrak{k}^{x} &= u(2m_1)^{+} \oplus u(n)^{+} \oplus sp(m_2+r_1)^{+} \oplus u(2m_2)^{-} \oplus u(n)^{-} \oplus sp(m_1+r_2)^{-} \\ \mathfrak{k}^{\{x,\, e,\, f\}} &= sp(m_1) \oplus sp(m_2) \oplus u(n) \oplus sp(r_1) \oplus sp(r_2) \end{aligned} \tag{15}$$

where

$$\begin{aligned} u(n) &= D(u(n)^{+} \oplus u(n)^{-}); \\ sp(m_1) \oplus sp(r_2) &\subset u(2m_1)^{+} \oplus sp(m_1+r_2)^{-}; \\ sp(m_2) \oplus sp(r_1) &\subset u(2m_2)^{-} \oplus sp(m_2+r_1)^{+} \end{aligned}$$

$sp(m_1)$ (resp., $sp(m_2)$) is embedded diagonally in $u(2m_1)^{+} \oplus sp(m_1+r_2)^{-}$ (resp., $u(2m_2)^{-} \oplus sp(m_2+r_1)^{+}$). $sp(r_2)$ (resp., $sp(r_1)$) is embedded naturally in $sp(m_1+r_2)^{-}$ (resp., $sp(m_2+r_1)^{+}$).

Lemma 7.8. *If $m_1 \geq 2$ or $m_2 \geq 2$, then $\mathfrak{k}^{\{x,\, e,\, f\}} \subset \mathfrak{k}^{x}$ in equation (15) is not a spherical embedding.*

Proof. This follows from Lemma 7.2. ∎

8. Proof of Theorem 6.1

8.1. $sl(s,\ \mathbf{R})$**.** By the proof of Proposition 6.3 for $sl(s,\ \mathbf{R})$, if $K_{\mathbf{C}} \cdot e$ is spherical then no part size of the corresponding partition Λ exceeds 3. Indeed we have the following.

Proposition 8.1. $(\mathfrak{g} = sl(s,\ \mathbf{R}))$. *$K_{\mathbf{C}} \cdot e$ is spherical if and only if in the corresponding partition Λ no part size exceeds 2.*

Proof. The proof of Proposition 6.3 implies that no part size in Λ can exceed 2. Corollary 4.16 implies that if no part size exceeds 2, then $K_{\mathbf{C}} \cdot e$ is spherical. ∎

8.2. $su^*(2n)$**.** By the proof of the $su^*(2n)$ case of Proposition 6.3 the signed partition for a spherical nilpotent $K_{\mathbf{C}}$-orbit for $su^*(2n)$ has at most one part of size 3. We now show that in fact no part size can exceed 2.

Lemma 8.2. *Suppose $\mathfrak{g} = su^*(2n)$, and $\Lambda = 3^p 2^q 1^r$ is the partition of n corresponding to the orbit $K_{\mathbf{C}} \cdot e$. If $p > 0$ then $K_{\mathbf{C}} \cdot e$ is not spherical.*

Proof. From the discussion preceding this lemma, if Λ is spherical then $p = 0$ or 1. Suppose $p = 1$. By Proposition 5.3, we have that:

$$e_1(x) = 2 = e_2(x);\ e_3(x) = 1 = \ldots = e_{2q+2}(x);\ e_{2q+3}(x) = 0 = \ldots = e_n(x).$$

Since $e_1 + e_2$ is a weight of $\mathfrak{t}_{\mathbf{C}}$ on $\mathfrak{p}_{\mathbf{C}}$, $\mathfrak{p}_{\mathbf{C}}$-height(e) > 3. So by Proposition 4.17, $K_{\mathbf{C}} \cdot e$ is not spherical. ∎

It follows from the previous lemma and Corollary 4.15, that

Proposition 8.3. *If $\mathfrak{g} = su^*(2n)$, then the spherical $K_{\mathbf{C}}$-nilpotent orbits are exactly those corresponding to partitions of n with part sizes not exceeding 2.*

8.3. $su(p,\ q)$**.** Proposition 6.3 implies the following.

Proposition 8.4. *If $K_{\mathbf{C}} \cdot e$ is a spherical nilpotent orbit of $su(p,\ q)$ then the corresponding signed partition has the form*

$$\Lambda = (+3)^{m_1}(-3)^{m_2}(+2)^{n_1}(-2)^{n_2}(+1)^{r_1}(-1)^{r_2}, \tag{16}$$

where m_i, n_i and r_i $(i = 1,\ 2)$ are non negative integers.

Proposition 8.5. *If $\mathfrak{g} = su(p,\ q)$, $K_{\mathbf{C}} \cdot e$ is a spherical nilpotent orbit in $\mathfrak{p}_{\mathbf{C}}$ and Λ is the corresponding signed partition then $K_{\mathbf{C}} \cdot e$ is spherical if and only if Λ has one of the following forms (where $n_1,\ n_2,\ r_1,\ r_2,\ r$ are non-negative integers):*

(a) $(+3)(+2)^{n_1}(-2)^{n_2}(+1)^{r_1}(-1)^{r_2}$; $(-3)(+2)^{n_1}(-2)^{n_2}(+1)^{r_1}(-1)^{r_2}$
(b) $(+3)^2(+1)^r$; $(-3)^2(-1)^r$
(c) $(+2)^{n_1}(-2)^{n_2}(+1)^{r_1}(-1)^{r_2}$.

We prove the proposition in a series of lemmas.

Lemma 8.6. ($\mathfrak{g} = su(p,\ q)$) *Let the Λ be the signed partition in equation (16) corresponding to the nilpotent orbit $K_{\mathbf{C}} \cdot e$. If $m_1 \geq 1$ and $m_2 \geq 1$. Then $K_{\mathbf{C}} \cdot e$ is not spherical.*

Proof. Λ has $\mathfrak{p}_{\mathbf{C}}$-height exceeding 3 so the result follows from Proposition 4.17. ■

Lemma 8.7. ($\mathfrak{g} = su(p,\ q)$) *Let the Λ be the signed partition in equation (16) corresponding to the nilpotent orbit $K_{\mathbf{C}} \cdot e$. If $m_1 \geq 3$ or $m_2 \geq 3$, then $K_{\mathbf{C}} \cdot e$ is not spherical.*

Proof. This follows from Lemma 7.4 and Corollary 4.13. ■

Lemma 8.8. ($\mathfrak{g} = su(p,\ q)$) *Let the signed partition corresponding to the nilpotent orbit $K_{\mathbf{C}} \cdot e$ be $\Lambda = (+2)^{n_1}(-2)^{n_2}(+1)^{r_1}(-1)^{r_2}$. Then $K_{\mathbf{C}} \cdot e$ is spherical.*

Proof. This follows from Corollary 4.15. ■

Lemma 8.9. ($\mathfrak{g} = su(p,\ q)$) *Let the Λ be the signed partition in equation (16) corresponding to the nilpotent orbit $K_{\mathbf{C}} \cdot e$, where m_1 or m_2 is positive. Then Λ is spherical if and only if one of the following is true:*

(a) $m_1 = 1,\ m_2 = 0$.
(a)$'$ $m_2 = 1,\ m_1 = 0$.
(b) $m_1 = 2,\ m_2 = n_1 = n_2 = r_2 = 0$
(b)$'$ $m_2 = 2,\ m_1 = n_1 = n_2 = r_1 = 0$

Proof. By Lemma 8.6 and Lemma 8.7, Λ spherical implies that either $m_1 \in \{1,\ 2\}$ and $m_2 = 0$ or $m_2 \in \{1,\ 2\}$ and $m_1 = 0$.

(a) We first consider the case $m_1 = 1,\ m_2 = 0$. Then $p = n_1 + n_2 + r_1 + 2$ and $q = n_1 + n_2 + r_2 + 1$. After applying Proposition 5.4 we find that $e_j(x) = 0$ for $n_1 + 2 \leq j \leq n_1 + r_1 + 1$ and $p + n_2 + 1 \leq j \leq p + n_2 + r_2 + 1$, and the non-zero $e_j(x)$ are:

$$e_1(x) = 2,\ e_j(x) = 1\ (2 \leq j \leq n_1 + 1),\quad e_j(x) = -1\ (n_1 + r_1 + 2 \leq j \leq p - 1),$$
$$e_p(x) = -2,\ e_j(x) = 1\ (p + 1 \leq j \leq p + n_2),\ e_j(x) = -1\ (p + n_2 + r_2 + 2 \leq j).$$

By equation (8)

$$\begin{aligned} \mathfrak{k}^x &= s(2u(1)^+ \oplus u(n_1)^+ \oplus u(n_2)^+ \oplus u(r_1)^+ \oplus u(n_2)^- \oplus u(n_1)^- \oplus u(r_2+1)^-) \\ \mathfrak{k}^{\{x,\ e,\ f\}} &= s(u(1) \oplus u(n_1) \oplus u(n_2) \oplus u(r_1) \oplus u(r_2)). \end{aligned} \tag{17}$$

The containment relations following equation (8) imply that $K_{\mathbf{C}}^{\{x,\ e,\ f\}}$ is spherical in $K_{\mathbf{C}}^x$. In this case $\mathfrak{s}_{\mathbf{R}} = s(u(1) \oplus t(n_1) \oplus t(n_2) \oplus u(r_1) \oplus u(r_2 - 1))$ where $t(n_1)$ and $t(n_2)$ are maximal tori inside the subalgebras $u(n_1)$ and $u(n_2)$ respectively inside the parentheses in the expression for $\mathfrak{k}^{\{x,\ e,\ f\}}$ in equation (17).

If $\Delta(Z)$ denotes the $\mathfrak{t}_{\mathbf{C}}$-weights of Z, we also have:

$$\Delta(Z) = \{e_1 - e_{p+n_2+r_2+2}, \dots, e_1 - e_{p+q}; e_{p+1} - e_{p+q}, \dots, e_{p+n_2} - e_{p+q}\}.$$

Let V_1 denote the span of the eigenvectors in Z with the first n_1 weights in $\Delta(Z)$ and let V_2 denote the span of the eigenvectors with last n_2 weights in $\Delta(Z)$. One can show that for $i = 1, 2$, V_i is an irreducible representation of $u(n_i)$ of dimension n_i. It is known that the maximal torus of $U(n_i)$ has a dense orbit on V_i. It follows that $B(\mathbf{S})$ has a dense orbit on Z.

(b) Finally, we will consider the case $m_1 = 2$, $m_2 = 0$. ($m_1 = 0$, $m_2 = 2$ is similar.)

We have $\Lambda = (+3)^2(+2)^{n_1}(-2)^{n_2}(+1)^{r_1}(-1)^{r_2}$. From equation (8) we get

$$\mathfrak{k}^x = s(2u(2)^+ \oplus u(n_1)^+ \oplus u(n_2)^+ \oplus u(r_1)^+ \oplus u(n_2)^- \oplus u(n_1)^- \oplus u(r_2+2)^-)$$
$$\mathfrak{k}^{\{x,\ e,\ f\}} = s(u(2) \oplus u(n_1) \oplus u(n_2) \oplus u(r_1) \oplus u(r_2)).$$

By Lemma 7.3, $K_{\mathbf{C}}^x / K_{\mathbf{C}}^{\{x,\ e,\ f\}}$ is spherical if and only if $r_2 = 0$. It follows that

Remark 8.10. If $r_2 > 0$ and n_1, n_2, r_1 are arbitrary, the following nilpotent orbit is not spherical:

$$\Lambda = (+3)^2(+2)^{n_1}(-2)^{n_2}(+1)^{r_1}(-1)^{r_2}.$$

From now on assume that $r_2 = 0$.

By applying Proposition 5.4 to $\Lambda = (+3)^2(+2)^{n_1}(-2)^{n_2}(+1)^{r_1}$, we obtain the values of the $e_i(x)$. Using these values we find $\Delta(Z)$. We obtain the decomposition $\Delta(Z) = \Delta_1 \cup \Delta_2 \cup \Delta_3 \cup \Delta_4$ where

$$\begin{aligned}
\Delta_1 &= \{e_1 - e_{n_1+2n_2+r_1+7}, \dots, e_1 - e_{2n_1+2n_2+r_1+6}\},\\
\Delta_2 &= \{e_2 - e_{n_1+2n_2+r_1+7}, \dots, e_2 - e_{2n_1+2n_2+r_1+6}\},\\
\Delta_3 &= \{e_{n_1+n_2+r_1+5} - e_{n_1+n_2+r_1+3}, \dots, e_{n_1+2n_2+r_1+4} - e_{n_1+n_2+r_1+3}\},\\
\Delta_4 &= \{e_{n_1+n_2+r_1+5} - e_{n_1+n_2+r_1+4}, \dots, e_{n_1+2n_2+r_1+4} - e_{n_1+n_2+r_1+4}\}.
\end{aligned}$$

We also have

$$\dim_{\mathbf{C}} Z = 2n_1 + 2n_2$$
$$\mathfrak{s}_{\mathbf{R}} = s(2u(1) \oplus t(n_1) \oplus t(n_2) \oplus u(r_1))$$

$u(r_1)$ acts trivially on Z. Therefore if $B(\mathbf{S})$ has an open orbit on Z, we must have $n_1 + n_2 + 1 \geq 2n_1 + 2n_2$, i.e., $n_1 + n_2 \leq 1$. But if $n_1 > 0$ or $n_2 > 0$, $\Lambda = (+3)^2(+2)^{n_1}(-2)^{n_2}(+1)^{r_1}$ contains either $(+3)^2(+2)^{n_1-1}(-2)^{n_2}(+1)^{r_1+1}(-1)$ or $(+3)^2(+2)^{n_1}(-2)^{n_2-1}(+1)^{r_1+1}(-1)$ in its closure (by Theorem 2 of [19]). Neither of the latter nilpotent orbits is spherical by Remark 8.10. Therefore Λ is not spherical when $n_1 > 0$ or $n_2 > 0$. If $n_1 + n_2 = 0$, then $Z = 0$ and we conclude that $\Lambda = (+3)^2(+1)^{r_1}$ is spherical. We have shown that cases (a) and (b) of our proposition are the only possibilities when $m_1 > 0$. Likewise by considering $\Lambda = (-3)^2(+2)^{n_1}(-2)^{n_2}(+1)^{r_1}(-1)^{r_2}$ and applying similar arguments we can show that cases $(a)'$ and $(b)'$ are the only ones possible when $m_2 > 0$. ∎

Proposition 8.5 follows from Propositions 8.4, Lemmas 8.6, 8.7, 8.8, and 8.9.

8.4. $so(p,\ q)$**.** Proposition 6.3 implies the following.

Proposition 8.11. ($\mathfrak{g} = so(p,\ q)$) *If $K_{\mathbf{C}} \cdot e$ is spherical, then the corresponding signed partition Λ has the form*

$$\Lambda = (+3)^{m_1}(-3)^{m_2}(+2)^n(+1)^{r_1}(-1)^{r_2},$$

where $m_1,\ m_2,\ n,\ r_1,\ r_2$ are non-negative integers.

For convenience, in the proofs below we consider only nilpotents whose signed partitions are without numerals or have roman numeral "I". Nilpotents whosed signed partitions have roman numeral "II" are handled by remark 4.19.

Lemma 8.12. ($\mathfrak{g} = so(p,\ q)$) *If $K_{\mathbf{C}} \cdot e$ is the orbit corresponding to any of the following signed partitions with or without numerals then $K_{\mathbf{C}} \cdot e$ is spherical:*
(a) $\Lambda = (+3)^m(+2)^n(+1)^{r_1}(-1)^{r_2}$, $1 \geq m \geq 0$, $n \geq 0$, $r_1,\ r_2 \geq 0$;
(a)$'$ $\Lambda = (-3)^m(+2)^n(+1)^{r_1}(-1)^{r_2}$, $1 \geq m \geq 0$, $n \geq 0$, $r_1,\ r_2 \geq 0$;
(b) $\Lambda = (+3)^2(+1)^{r_1}$, $r_1 \geq 0$;
(c) $\Lambda = (-3)^2(-1)^{r_2}$, $r_2 \geq 0$.

Proof. For (a), we use an argument similar to that for part (a) of Lemma 8.9. For (b) and (c), we use an argument similar to that for part (b) of Lemma 8.9. ∎

Lemma 8.13. ($\mathfrak{g} = so(p,\ q)$) *If $K_{\mathbf{C}} \cdot e$ corresponds to any of the following signed partitions with or without numerals then $K_{\mathbf{C}} \cdot e$ is not spherical:*
(a) $\Lambda = (+3)^2(+1)^{r_1}(-1)^{r_2}$, $r_1 \geq 0,\ r_2 \geq 1$;
(b) $\Lambda = (-3)^2(+1)^{r_1}(-1)^{r_2}$, $r_1 \geq 1,\ r_2 \geq 0$;
(c) $\Lambda = (+3)^{p_1}(-3)^{p_2}(+2)^n(+1)^{r_1}(-1)^{r_2}$, $p_1 \geq 1$, $p_2 \geq 1$;

Proof. (a) Here $\mathfrak{g} = so(r_1+4,\ r_2+2)$. By Proposition 5.6: $e_1(x) = 2 = e_2(x)$, and $e_j(x) = 0$ for all other j. If $\{x,\ e,\ f\}$ is the corresponding triple, then:

$$\mathfrak{k}^x = u(2)^+ \oplus so(r_1)^+ \oplus\ so(r_2+2)^-;\quad \mathfrak{k}^{\{x,\ e,\ f\}} = so(2) \oplus so(r_1) \oplus so(r_2).$$

By Lemma 7.5 $K_{\mathbf{C}}^x\ /\ K_{\mathbf{C}}^{\{x,\ e,\ f\}}$ is spherical if and only if $r_2 = 0$. (b) is proven similarly.

(c) This follows from Proposition 4.17. ∎

Lemma 8.14. ($\mathfrak{g} = so(p,\ q)$) *If $K_{\mathbf{C}} \cdot e$ corresponds to any of the following signed partitions with or without numerals then $K_{\mathbf{C}} \cdot e$ is not spherical:*
(a) $\Lambda = (+3)^m(+2)^n(+1)^{r_1}(-1)^{r_2}$, *for $m \geq 2$, and $n \geq 2$ (n even).*
(a)$'$ $\Lambda = (-3)^m(+2)^n(+1)^{r_1}(-1)^{r_2}$, *for $m \geq 2$, and $n \geq 2$ (n even).*
(b) $\Lambda = (+3)^m(+1)^{r_1}(-1)^{r_2}$, *for $m \geq 3$.*
(b)$'$ $\Lambda = (-3)^m(+1)^{r_1}(-1)^{r_2}$, *for $m \geq 3$.*

Proof. Assume that Λ has no numeral attached. In case (a) or (a)$'$, if $n \geq 2$ then the closure of $K_{\mathbf{C}} \cdot e$ contains an orbit $K_{\mathbf{C}} \cdot e'$ where $K_{\mathbf{C}} \cdot e'$ is of type (a) or (b) in Lemma 8.13. (See [9].) Now apply Proposition 4.11. The assertions about (b) and (b)$'$ follow from Corollary 7.6. ∎

Proposition 8.15. *If* $\mathfrak{g} = so(p,\ q)$*, then* $K_{\mathbf{C}} \cdot e$ *is spherical* $\Longleftrightarrow$ *the corresponding signed partition (after ignoring numerals) is of the form:*

(a) $\Lambda = (+3)^m(+1)^{r_1},\ m \leq 2,\ r_1 \geq 0;$

(b) $\Lambda = (-3)^m(-1)^{r_2},\ m \leq 2,\ r_2 \geq 0;$

(c) $\Lambda = (+3)^m(+2)^n(+1)^{r_1}(-1)^{r_2},\ 1 \geq m \geq 0,\ n \geq 0$ *(n even)*, $r_1 \geq 0,\ r_2 \geq 0;$

(d) $\Lambda = (-3)^m(+2)^n(+1)^{r_1}(-1)^{r_2},\ 1 \geq m \geq 0,\ n \geq 0$ *(n even)*, $r_1 \geq 0,\ r_2 \geq 0.$

Proof. The result follows from Proposition 8.11 and Lemmas 8.12, 8.13, and 8.14. ∎

8.5. $so^*(2n)$**.** Proposition 6.3 implies the following.

Proposition 8.16. $(\mathfrak{g} = so^*(2n))$ *If* $K_{\mathbf{C}} \cdot e$ *is spherical, then the corresponding signed partition* Λ *has the form*

$$\Lambda = (+3)^p(+2)^{q_1}(-2)^{q_2}(+1)^r,$$

where $p,\ q_1,\ q_2,\ r$ *are non-negative integers.*

Lemma 8.17. $(\mathfrak{g} = so^*(2n))$ *Let* Λ *be the signed partition for* $K_{\mathbf{C}} \cdot e$. (a) *For all integers* $r \geq 0$*, if* $\Lambda = (+3)(+1)^r$ *, then* $K_{\mathbf{C}} \cdot e$ *is spherical.* (b) *For all integers,* $q_1 \geq 0,\ q_2 \geq 0,\ r \geq 0$*, if* $\Lambda = (+2)^{q_1}(-2)^{q_2}(+1)^r$*, then* $K_{\mathbf{C}} \cdot e$ *is spherical.*

Proof. For (a) first apply Proposition 5.7 to obtain (with $n = r + 3$) to conclude that $e_1(x) = 2$, $e_n(x) = -2$, and $e_j(x) = 0$ for $1 < j < n$. By (14) $\mathfrak{k}^x = 2u(1) \oplus u(r+1)$ and $\mathfrak{k}^{\{x,\ e,\ f\}} = u(1) \oplus u(r)$. By Lemma 7.7 $K_{\mathbf{C}}^x\ /\ K_{\mathbf{C}}^{\{x,\ e,\ f\}}$ is spherical. Since $Z = 0$, Λ is spherical by Corollary 4.13. (b) follows from Corollary 4.15. ∎

Lemma 8.18. $(\mathfrak{g} = so^*(2n))$ *For all integers* $p \geq 2,\ q_1 \geq 0,\ q_2 \geq 0,$ *and* $r \geq 0$*, if* $\Lambda = (+3)^p\ (+2)^{q_1}\ (-2)^{q_2}\ (+1)^r$ *is the signed partition for* $K_{\mathbf{C}} \cdot e$*, then* $K_{\mathbf{C}} \cdot e$ *is not spherical.*

Proof. Applying Proposition 5.7, we find that $e_1(x) = e_2(x) = 2$. Since $e_1 + e_2$ is a weight of $\mathfrak{t}_{\mathbf{C}}$ on $\mathfrak{p}_{\mathbf{C}}$, we can apply Proposition 4.17. ∎

Lemma 8.19. $(\mathfrak{g} = so^*(2n))$ *For all integers* $q_1 \geq 1,\ q_2 \geq 1,\ r \geq 0$*, if the signed partition for* $K_{\mathbf{C}} \cdot e$ *is*

$$\Lambda = (+3)(+2)^{q_1}(-2)^{q_2}(+1)^r,$$

then $K_{\mathbf{C}} \cdot e$ *is not spherical.*

Proof. In this case $n = 2q_1 + 2q_2 + r + 3$. If x is the neutral element corresponding to $\Lambda = (+3)(+2)^{q_1}(-2)^{q_2}(+1)^r$, then:

$$\begin{aligned}\mathfrak{k}^x &= u(2q_1) \oplus u(2q_2) \oplus u(r+1)\\ \mathfrak{k}^{\{x,\ e,\ f\}} &= u(1) \oplus u(r) \oplus sp(q_1) \oplus sp(q_2).\end{aligned}$$

$K_{\mathbf{C}}^x \ / \ K_{\mathbf{C}}^{\{x,\ e,\ f\}}$ is spherical. Applying Proposition 5.7, we find that $e_{2q_1+2}(x) = \ldots = e_{2q_1+r+2}(x) = 0$ and that

$$\Delta(Z) = \{e_1 + e_2, \ldots, e_1 + e_{2q_1+1}, -(e_n + e_{2q_1+r+3}), \ldots, -(e_n + e_{n-1})\}.$$

Using an argument similar to that for part (b) of Lemma 8.9, it can be shown that $B(\mathbf{S})$ does not have a dense orbit in Z. ∎

Proposition 8.20. *If $\mathfrak{g} = so^*(2n)$, then the nilpotent $e \in \mathfrak{p}_{\mathbf{C}}$ is spherical $\iff$ the corresponding signed partition is of the form (1) $\Lambda = (+3)(+1)^r$ or (2) $\Lambda = (+2)^{q_1}(-2)^{q_2}(+1)^r$.*

Proof. Proposition 8.16 and Lemmas 8.17–8.19. ∎

8.6. $sp(n,\ \mathbf{R})$**.**

Proposition 8.21. *If $\mathfrak{g} = sp(n,\ \mathbf{R})$, the spherical nilpotent $K_{\mathbf{C}}$ nilpotent orbits in $\mathfrak{p}_{\mathbf{C}}$ are precisely those corresponding to signed partitions with part sizes not exceeding 2. That is, Λ corresponds to a spherical nilpotent orbit if and only if $\Lambda = (+2)^{q_1}(-2)^{q_2}1^r$ for nonnegative integers q_1, q_2, and r.*

Proof. Corollary 4.16 and the proof of Proposition 6.3. ∎

8.7. $sp(p,\ q)$**.** Proposition 6.3 implies the following.

Proposition 8.22. ($\mathfrak{g} = sp(p,\ q)$) *If $K_{\mathbf{C}} \cdot e$ is spherical, then the corresponding signed partition Λ has the form*

$$\Lambda = (+3)^{m_1}(-3)^{m_2}(+2)^n(+1)^{r_1}(-1)^{r_2}, \tag{18}$$

where $m_1, , m_2,\ n,\ r_1,\ r_2$ are non-negative integers.

If Λ is as in (18), then by applying Proposition 5.9 we find that $e_j(x) = 0$ for $2m_1 + n + 1 \le j \le p$ and $p + 2m_2 + n + 1 \le j \le p + q$ and the non-zero $e_j(x)$ are:

$$\begin{aligned}&e_1(x) = 2 = \ldots = e_{2m_1}(x),\ e_{2m_1+1}(x) = 1 = \ldots = e_{2m_1+n}(x),\\ &e_{p+1}(x) = 2 = \ldots = e_{p+2m_2}(x),\ e_{p+2m_2+1}(x) = 1 = \ldots = e_{p+2m_2+n}(x),\end{aligned} \tag{19}$$

where $p = 2m_1 + m_2 + n + r_1$ and $q = m_1 + 2m_2 + n + r_2$.

Corollary 8.23. ($\mathfrak{g} = sp(p,\ q)$) *If the nilpotent orbit $K_{\mathbf{C}} \cdot e$ is spherical and corresponds to the signed partition Λ in equation (18) , then either,* (a) $m_1 = 0 = m_2 = 0$, (b) $m_1 = 1,\ m_2 = 0$, *or* (c) $m_1 = 0,\ m_2 = 1$.

Proof. By Lemma 7.8 and Corolary 4.13, we must have $m_1 \leq 1$ and $m_2 \leq 1$. If $m_1 = m_2 = 1$, then apply Proposition 4.17. ■

Lemma 8.24. ($\mathfrak{g} = sp(p,\ q)$) *Let* Λ *be the signed partition corresponding to* $K_{\mathbf{C}} \cdot e$.

(a) *If* $\Lambda = (+2)^n(+1)^{r_1}(-1)^{r_2}$, *where* $n \geq 0$, $r_1 \geq 0$, $r_2 \geq 0$, *then* $K_{\mathbf{C}} \cdot e$ *is spherical.*

(b) *If* $\Lambda = (+3)(+2)^n(+1)^{r_1}(-1)^{r_2}$ *or* $\Lambda = (-3)(+2)^n(+1)^{r_1}(-1)^{r_2}$, *where* $0 \leq n \leq 1$ *and* $r_1,\ r_2 \geq 0$, *then* $K_{\mathbf{C}} \cdot e$ *is spherical.*

(c) *If* $\Lambda = (+3)(+2)^n(+1)^{r_1}(-1)^{r_2}$ *or* $\Lambda = (-3)(+2)^n(+1)^{r_1}(-1)^{r_2}$, *where* $n > 1$ *and* $r_1,\ r_2 \geq 0$, *then* $K_{\mathbf{C}} \cdot e$ *is not spherical.*

Proof. (a) follows from Corollary 4.15.

For (b) and (c), first note that if $\Lambda = (+3)(+2)^n(+1)^{r_1}(-1)^{r_2}$, then $p = n + r_1 + 2$, and $q = n + r_2 + 1$.(15) implies that:

$$\mathfrak{k}^x = u(2)^+ \oplus u(n)^+ \oplus sp(r_1)^+ \oplus u(n)^- \oplus sp(r_2+1)^-$$
$$\mathfrak{k}^{\{x,\ e,\ f\}} = sp(1) \oplus u(n) \oplus sp(r_1) \oplus sp(r_2).$$

By Lemma 7.2, $K^x_{\mathbf{C}}\ /\ K^{\{x,\ e,\ f\}}_{\mathbf{C}}$ is spherical.

Now suppose that $n = 0$. Equation (19) implies that $0 \leq e_j(x) + e_k(x) \leq 2$ for all $1 \leq j \leq p$ and $p+1 \leq k \leq p+q$. Therefore, $Z = 0$. Since $Z = 0$, $K_{\mathbf{C}} \cdot e$ is spherical by Corollary 4.13. The proof of sphericality for $\Lambda = (-3)(+1)^{r_1}(-1)^{r_2}$ is similar.

If $n > 0$, an argument similar to that in part (b) of Lemma 8.9 completes the proof. ■

Proposition 8.25. *If* $\mathfrak{g} = sp(p,\ q)$, *then* $K_{\mathbf{C}} \cdot e$ *is spherical* $\iff$ *the corresponding signed partition is one of the following (for* $n \geq 0$, $r_1 \geq 0$, $r_2 \geq 0$*):*

(a) $\Lambda = (+3)(+1)^{r_1}(-1)^{r_2}$ *or* $(+3)(+2)(+1)^{r_1}(-1)^{r_2}$;

(b) $\Lambda = (-3)(+1)^{r_1}(-1)^{r_2}$ *or* $(-3)(+2)(+1)^{r_1}(-1)^{r_2}$;

(c) $\Lambda = (+2)^n(+1)^{r_1}(-1)^{r_2}$.

Proof. Combine Proposition 8.22, Corollary 8.23, and Lemma 8.24. ■

9. The spherical nilpotent $K_{\mathbf{C}}$-orbits in $\mathfrak{p}_{\mathbf{C}}$ for $\mathfrak{g}$ exceptional

The following conventions hold throughout this section. We will use the notation of Helgason [10] to describe the simple real exceptional algebra $\mathfrak{g}$. We use the tables of Đokovic in [4] and [5] to describe the $K_{\mathbf{C}}$-nilpotent conjugacy classes of the exceptional simple algebras and for information about the neutral ement x, $\mathfrak{k}^x$, and $\mathfrak{k}^{\{x,\ e,\ f\}}$ associated to each conjugacy class.

Theorem 9.1. *If* $\mathfrak{g}$ *is a simple real exceptional Lie algebra then the spherical nilpotent* $K_{\mathbf{C}}$*-orbits in* $\mathfrak{p}_{\mathbf{C}}$ *are the following ones as listed in [4] and [5].*

EI: classes 1, 2 and 3.

EII: classes 1, 2, 3, 4 and 5.
EIII: classes 1, 2, 3, 4, 5, 6, 7, 8, and 9.
EIV: class 1.
EV: classes 1, 2, 3, 4, 5, 8 and 9.
EVI: classes 1, 2, 3, 4, and 5.
EVII: classes: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12.
EVIII: classes 1, 2, 3 and 6.
EIX: classes 1, 2, 3, 4, and 5.
FI: classes 1, 2, 3, 4, and 5.
FII: classes 1 and 2.
GI: classes 1 and 2.

The proof is case by case. Detailed arguments will only be given for $\mathfrak{g}$ not equal to EIV, FI, FII or GI.

9.1. EI **(**$\mathfrak{k} = sp(4)$**).** By Remark 4.18 and Proposition 4.17, the $K_{\mathbf{C}}$ classes 1, 2 and 3 are the only ones that could be spherical. Classes 1 and 2 must be spherical by Corollary 4.15.

For class 3, $\mathfrak{k}^x = su(3) \oplus 2u(1)$, $\mathfrak{k}^{\{x,\, e,\, f\}} = su(2) \oplus u(1)$. Therefore, $K_{\mathbf{C}}^x / K_{\mathbf{C}}^{\{x,\, e,\, f\}}$ is spherical. $\mathfrak{s}_{\mathbf{R}} = 2u(1)$. $Z = \mathfrak{p}_{\mathbf{C}}(3)$ and $\dim Z = 1$. By constructing $\mathfrak{s}_{\mathbf{R}}$ and Z one shows directly that $\mathfrak{s}_{\mathbf{R}}$ acts nontrivially on Z. Since $\mathbf{S}_0 = B(\mathbf{S})$, this shows that $B(\mathbf{S})$ has a dense orbit on Z. So class 3 is spherical.

9.2. EII **(**$\mathfrak{k} = su(6) \oplus su(2)$**).** By Remark 4.18 and Proposition 4.17, the $K_{\mathbf{C}}$ classes 1-7, are the only ones that could be spherical. Classes 1, 2 and 3 are spherical by Corollary 4.15.

For class 6, $\mathfrak{k}^x = su(6) \oplus u(1)$ and $\mathfrak{k}^{\{x,\, e,\, f\}} = 2su(3)$. $K_{\mathbf{C}}^x / K_{\mathbf{C}}^{\{x,\, e,\, f\}}$ is not spherical. For class 7, $\mathfrak{k}^x = su(4) \oplus su(2) \oplus 2u(1)$ and $\mathfrak{k}^{\{x,\, e,\, f\}} = 2su(2) \oplus 2u(1)$. By explicit construction of $\mathfrak{k}^x$ and $\mathfrak{k}^{\{x,\, e,\, f\}}$ one finds that $\mathfrak{k}^{\{x,\, e,\, f\}}$ is contained in the $su(4) \oplus 2u(1)$ component of $\mathfrak{k}^x$ hence $K_{\mathbf{C}}^x / K_{\mathbf{C}}^{\{x,\, e,\, f\}}$ is not spherical. It follows that neither class 6 nor class 7 is spherical.

Finally consider classes 4 and 5. For class 4, $\mathfrak{k}^x = 2su(3) \oplus 2u(1)$, and $\mathfrak{k}^{\{x,\, e,\, f\}} = su(3) \oplus u(1)$; for class 5, $\mathfrak{k}^x = 2su(2) \oplus 4u(1)$, and $\mathfrak{k}^{\{x,\, e,\, f\}} = su(2) \oplus 2u(1)$. By Lemma 4.14, $\mathfrak{k}^{\{x,\, e,\, f\}}$ is a symmetric subalgebra of $\mathfrak{k}^x$, so condition (a) of Corollary 4.13 is satisfied by class 4 and class 5. For class 4 and class 5, $\mathfrak{s}_{\mathbf{R}} = 3u(1)$. One verifies directly that in each case $\mathfrak{s}_{\mathbf{R}}$ acts non trivially on $Z = \mathfrak{p}_{\mathbf{C}}(3)$ which is one dimensional. Therefore, $B(\mathbf{S})$, the complex torus corresponding to $\mathfrak{s}_{\mathbf{C}}$, has a dense orbit on Z. So condition (b) of Corollary 4.13 is satisfied and both classes are spherical.

9.3. $EIII$ **(**$\mathfrak{k} = so(10) \oplus \mathbf{R}$**).** By Remark 4.18 and Proposition 4.17, the $K_{\mathbf{C}}$ classes 1-9, are the only ones that could be spherical. In fact, classes 1, 2, 3, 4 and 5 are spherical since the height of each of the corresponding $G_{\mathbf{C}}$ classes is 2.

For class 9, $\mathfrak{k}^x = so(8) \oplus 2u(1)$, and $\mathfrak{k}^{\{x,\, e,\, f\}} = (G_2)_{-14}$. So $K_{\mathbf{C}}^{\{x,\, e,\, f\}}$ is spherical in $K_{\mathbf{C}}^x$. Since $Z = 0$, class 9 is spherical.

By Figure 5 in [6], the closure of class 9 contains the closure of classes 6, 7 and 8 (and classes 1-5). Therefore, the spherical $K_{\mathbf{C}}$ nilpotent classes of $EIII$ are classes 1–9.

9.4. EV **(**$\mathfrak{k} = su(8)$**).** By Remark 4.18 and Proposition 4.17, the $K_{\mathbf{C}}$ classes 1-6, 8 and 9 are the only ones that could be spherical. Classes 1, 2, 3 and 4 are spherical because each of the corresponding $G_{\mathbf{C}}$ classes has height 2.

For class 6, we have $\mathfrak{k}^x = su(6) \oplus 2u(1)$ and $\mathfrak{k}^{\{x,\,e,\,f\}} = 2su(3)$. So $K^x_{\mathbf{C}}/K^{\{x,\,e,\,f\}}_{\mathbf{C}}$ is not spherical.

For class 8 and for class 9, we have $\mathfrak{k}^x = 2su(3) \oplus 3u(1)$ and $\mathfrak{k}^{\{x,\,e,\,f\}} = su(3) \oplus u(1)$. Class 8 and class 9 are conjugate under $G^{\theta}_{\mathbf{C}}$.

Consider class 8. By constructing $\mathfrak{k}^x$ and $\mathfrak{k}^{\{x,\,e,\,f\}}$ one checks that the $su(3)$ summand of $\mathfrak{k}^{\{x,\,e,\,f\}}$ is embedded diagonally in the $2su(3)$ summand of $\mathfrak{k}^x$ and that $K^x_{\mathbf{C}}/K^{\{x,\,e,\,f\}}_{\mathbf{C}}$ is spherical. We also have $\mathfrak{s}_{\mathbf{R}} = 3u(1)$, a maximal torus in $\mathfrak{k}^{\{x,\,e,\,f\}}$. Now $\mathfrak{s}_{\mathbf{C}}$ is a complex three dimensional torus. One can find a basis for Z consisting of noncompact root vectors for a maximal torus of $\mathfrak{k}$ that contains $\mathfrak{s}_{\mathbf{R}}$. Using this information, one verifies that $[\mathfrak{s}_{\mathbf{C}},\ Z] = Z$. So $B(\mathbf{S}) = \mathbf{S}_0$ has a dense orbit on Z, making class 8 spherical. By Remark 4.19, class 9 is spherical.

Figure 6 in [8] shows that class 5 lies in the closure of each of the following classes: 8 and 9. Their sphericality implies that of class 5.

9.5. *EVI* **($\mathfrak{k} = so(12) \oplus su(2)$).** By Remark 4.18 and Proposition 4.17, the $K_{\mathbf{C}}$ classes 1–6, 8 and 9 are the only ones that could be spherical. In fact, classes 1, 2 and 3 are spherical since the height of each of the corresponding $G_{\mathbf{C}}$ classes is 2.

Since $Z = 0$ for classes 6 and 8, it suffices to check whether $K^{\{x,\,e,\,f\}}_{\mathbf{C}}$ is spherical in $K^x_{\mathbf{C}}$. By [14], for $n \geq 3$, $su(n)$ is only spherical in $so(2n)$ when n is odd. For class 6, $\mathfrak{k}^x = so(12) \oplus u(1)$, and $\mathfrak{k}^{\{x,\,e,\,f\}} = su(6)$. For class 8, $\mathfrak{k}^x = so(8) \oplus su(2) \oplus u(1) \oplus u(1)$, and $\mathfrak{k}^{\{x,\,e,\,f\}} = s(u(4) \oplus u(2)) = su(4) \oplus su(2) \oplus u(1)$. Thus for classes 6 and 8 sphericality fails. Since class 8 is not spherical and the closure of class 9 contains class 8 (by results in [7]), class 9 can't be spherical.

For class 4, $\mathfrak{k}^x = su(6) \oplus 2u(1)$, and $\mathfrak{k}^{\{x,\,e,\,f\}} = sp(3) \oplus u(1)$. For class 5, $\mathfrak{k}^x = su(4) \oplus su(2) \oplus 3u(1)$, and $\mathfrak{k}^{\{x,\,e,\,f\}} = sp(2) \oplus su(2) \oplus u(1)$. So for class 4 and class 5, $K^{\{x,\,e,\,f\}}_{\mathbf{C}}$ is spherical in $K^x_{\mathbf{C}}$. In each case, $\mathfrak{s}_{\mathbf{R}} = 3su(2) \oplus u(1)$ and $\mathfrak{s}_{\mathbf{R}}$ contains a maximal torus of $\mathfrak{k}^{\{x,\,e,\,f\}}$. One checks directly that this torus acts non trivially on the one dimensional space Z. This implies that $B(\mathbf{S})$ has an open orbit on Z. Thus classes 4 and 5 are spherical.

9.6. *EVII* **($\mathfrak{k} = e_6 \oplus \mathbf{R}$).** By Remark 4.18 and Proposition 4.17, the $K_{\mathbf{C}}$ classes 1–12 are the only ones that could be spherical. In fact, classes 1–9 are spherical since the height of each of the corresponding $G_{\mathbf{C}}$ classes $=$ 2.

Since $Z = 0$ for class 10, it suffices to check whether $K^{\{x,\,e,\,f\}}_{\mathbf{C}}$ is spherical in $K^x_{\mathbf{C}}$. We have $\mathfrak{k}^x = su(6) \oplus 2u(1)$, and $\mathfrak{k}^{\{x,\,e,\,f\}} = su(5) \oplus u(1)$. By constructing these subalgebras, one checks that sphericality holds for the corresponding pair of reductive groups. Thus class 10 is spherical.

For classes 11 and 12, $\mathfrak{k}^x = su(4) \oplus su(2) \oplus 3u(1)$, and $\mathfrak{k}^{\{x,\,e,\,f\}} = su(4) \oplus u(1) = u(4)$, so $K^{\{x,\,e,\,f\}}_{\mathbf{C}}$ is spherical in $K^x_{\mathbf{C}}$. Consider either class 11 or class 12. In each case $Z = \mathfrak{p}_{\mathbf{C}}(3)$. $\mathfrak{s}_{\mathbf{R}} = 4u(1)$, a maximal torus in $\mathfrak{k}^{\{x,\,e,\,f\}}_{\mathbf{C}}$. A. Noel constructs this maximal torus in [17]. One verifies directly that the intersection of this maximal torus with $su(4)$ does not act trivially on Z. This shows that $\mathfrak{p}_{\mathbf{C}}(3)$ (which has dimension 4) is not a trivial $su(4)$-module. Since, the dimension of the smallest non-trivial $su(4)$ module is 4 this shows that $\mathfrak{p}_{\mathbf{C}}(3)$ is a 4 dimensional irreducible representation of $u(4)$. It follows that $B(\mathbf{S}) = \mathbf{S}_0$ has a dense open orbit on $\mathfrak{p}_{\mathbf{C}}(3)$. This establishes the sphericality of classes 11 and 12.

9.7. *EVIII* **(**$\mathfrak{k} = so(16)$**).** By Remark 4.18 and Proposition 4.17, the $K_{\mathbf{C}}$ classes 1–4 and 6 are the only ones that could be spherical. In fact classes 1 and 2 are spherical since each has height $= 2$.

For class 6, $\mathfrak{k}^{\{x,\,e,\,f\}} = u(4)$ is diagonally embedded in $\mathfrak{k}^x = u(4) \oplus u(4)$, so that $K_{\mathbf{C}}^{\{x,\,e,\,f\}}$ is spherical inside $K_{\mathbf{C}}^x$. $\mathfrak{s}_{\mathbf{R}} = 4u(1)$, and an argument similar to that for classes 11 and 12 in *EVII* shows that $B(\mathbf{S})$ has a dense orbit on $\mathfrak{p}_{\mathbf{C}}(3)$. So class 6 is spherical. Since class 3 lies in the closure of class 6, class 3 is also spherical.

For class 4, $\mathfrak{k}^x = so(12) \oplus su(2) \oplus u(1)$, and $\mathfrak{k}^{\{x,\,e,\,f\}} = su(6) \oplus su(2)$. Therefore, so $K_{\mathbf{C}}^{\{x,\,e,\,f\}}$ is not spherical in $K_{\mathbf{C}}^x$. Hence class 4 is not spherical.

9.8. *EIX* **(**$\mathfrak{k} = e_7 \oplus su(2)$**).** By Remark 4.18 and Proposition 4.17, the $K_{\mathbf{C}}$ classes 1–6, 8 and 9 are the only ones that could be spherical. In fact classes 1, 2, and 3 are spherical because they have height equal to 2.

Since $Z = 0$ for classes 6 and 8 for sphericality, it suffices to check whether $K_{\mathbf{C}}^{\{x,\,e,\,f\}}$ is spherical in $K_{\mathbf{C}}^x$. For class 6, $\mathfrak{k}^x = (E_7)_{-133} \oplus u(1)$, and $\mathfrak{k}^{\{x,\,e,\,f\}} = (E_6)_{-78}$, so sphericality fails. For class 8, $\mathfrak{k}^x = so(12) \oplus su(2) \oplus u(1)$, and $\mathfrak{k}^{\{x,\,e,\,f\}} = so(10) \oplus u(1)$. One verifies that $\mathfrak{k}^{\{x,\,e,\,f\}}$ is contained in the $so(12) \oplus su(2)$ component of $\mathfrak{k}^x$. But $K_{\mathbf{C}}^{\{x,\,e,\,f\}}$ is not spherical in $K_{\mathbf{C}}^x$. Therefore, class 8 is not spherical. The closure of class 9 contains class 8. Hence by Proposition 4.11, class 9 is not spherical.

For class 4, $\mathfrak{k}^x = E_{6(-78)} \oplus 2u(1)$, and $\mathfrak{k}^{\{x,\,e,\,f\}} = F_{4(-52)} \oplus u(1)$. So sphericality holds for the reductive groups. $\mathfrak{s}_{\mathbf{R}} = so(8) \oplus u(1)$. One checks that the maximal torus in $\mathbf{S}$ acts non-trivially on the one dimensional space Z which establishes the sphericality of class 4. A similar argument establishes the sphericality of class 5.

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