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Hierarchy of Closures of Matrix Pencils

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Abstract. The focus of this paper is the standard linear representation of the group $\mathrm{SL}_n(\mathbb{C})\times\mathrm{SL}_m(\mathbb{C})\times\mathrm{SL}_2(\mathbb{C})$, that is, the tensor product of the corresponding tautological representations. Classification of its orbits is a classic problem, which goes back to the works of Kronecker and Weierstrass. Here, we summarize some known results about standard linear representations of $\mathrm{SL}_n(\mathbb{C}) \times \mathrm{SL}_m(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$, $\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_m(\mathbb{C})$, $\mathrm{SL}_n(\mathbb{C}) \times \mathrm{SL}_m(\mathbb{C})$, and $\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_m(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C})$, classify the orbits and describe their degenerations. For the case $n \neq m$, we prove that the algebras of invariants of the actions of $\mathrm{SL}_n(\mathbb{C}) \times \mathrm{SL}_m(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$ and $\mathrm{SL}_n(\mathbb{C}) \times \mathrm{SL}_m(\mathbb{C})$ are generated by one polynomial of degree $nm/|n-m|$, if $d=|n-m|$ divides n (or m), and are trivial otherwise. It is also shown that the null cone of $\mathrm{SL}_n(\mathbb{C}) \times \mathrm{SL}_m(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$ is irreducible and contains an open orbit if $n \neq m$. We relate the degenerations of orbits of matrix pencils to the degenerations of matrix pencil bundles and prove that the closure of a matrix pencil bundle consists of closures of the corresponding orbits and closures of their amalgams. From this fact we derive the degenerations of orbits of the four groups listed above. All degenerations are cofined to the list of minimal degenerations, which are summarized as transformations of Ferrer diagrams. We tabulate the orbits of matrix pencils up to seventh order and portray the hierarchy of closures of 2×2, 3×3, 4×4, 5×5, 5×6 and 6×6 matrix pencil bundles.

1. Introduction

Although classification of orbits of the standard linear representation of $\mathrm{SL}_{n_1}(\mathbb{C}) \times \cdots \times \mathrm{SL}_{n_s}(\mathbb{C})$ is trivial for $s \leq 2$, it is no longer feasible in any reasonable sense even for $s=3$. The simplest nontrivial case of this generic problem is when one of the tensor factors is $\mathrm{SL}_2(\mathbb{C})$. Here, we investigate this case, the standard linear representation of $\mathrm{SL}_{n,m,2} = \mathrm{SL}_n(\mathbb{C}) \times \mathrm{SL}_m(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$ in $\mathbb{C}^{n,m,2} = \mathbb{C}^n \otimes \mathbb{C}^m \otimes \mathbb{C}^2$, and describe its orbits and their degenerations.

The space $\mathbb{C}^{n,m,2}$ is endowed with the natural actions of three other groups: $\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_m(\mathbb{C})$, $\mathrm{SL}_n(\mathbb{C}) \times \mathrm{SL}_m(\mathbb{C})$, and $\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_m(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C})$. They are denoted by $\mathrm{GL}_{n,m}$, $\mathrm{SL}_{n,m}$, and $\mathrm{GL}_{n,m,2}$, respectively. If bases in $\mathbb{C}^n$, $\mathbb{C}^m$, and $\mathbb{C}^2$ are chosen then the components T^{ijk} of a tensor $T \in \mathbb{C}^{n,m,2}$ form two $n \times m$ matrices, whose entries are T^{ij1} and T^{ij2}, respectively. An element of $\mathbb{C}^{n,m,2}$ can

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be regarded as a pair of complex $n \times m$ matrices A and B. Then it is called the *matrix pencil* and is denoted by $A + \lambda B$, where λ is a varying coefficient. If $n = m$ then the matrix pencil is called *square*; otherwise it is called *rectangular*. The actions of $\mathrm{GL}_{n,m,2}$ and $\mathrm{SL}_{n,m,2}$ on matrix pencils are given by formula

$$(P, Q, R) \circ (A + \lambda B) = (r_{11} PAQ^{-1} + r_{12} PBQ^{-1}) + \lambda (r_{21} PAQ^{-1} + r_{22} PBQ^{-1}), \quad (1)$$

where r_{ij} are the entries of R^{-1}. Two matrix pencils are said to be *G-equivalent* if one of them is mapped to the other by a transformation that belongs to G, where G is one of the groups listed above. Two matrix pencils are said to be *equivalent* (no prefix used), if they are $\mathrm{GL}_{n,m}$-equivalent.

Description of equivalence classes of matrix pencils under the action of $\mathrm{GL}_{n,m}$ was obtained by Weierstrass and Kronecker [15, 25]. Here we state some of their results. The direct sum of a $n_1 \times m_1$ matrix pencil $\mathcal{P}_1 = X_1 + \lambda Y_1$ and a $n_2 \times m_2$ matrix pencil $\mathcal{P}_2 = X_2 + \lambda Y_2$ is the $(n_1 + n_2) \times (m_1 + m_2)$ matrix pencil $\mathcal{P}_1 \oplus \mathcal{P}_2 = (X_1 \oplus X_2) + \lambda (Y_1 \oplus Y_2)$, where $Z_1 \oplus Z_2$ denotes the diagonal block matrix composed of Z_1 and Z_2. A matrix pencil is said to be *indecomposable* if it cannot be represented as a direct sum of two non-trivial matrix pencils. We also consider $n \times 0$ and $0 \times m$ matrix pencils; by this we mean that if such pencil is present in a direct sum then the corresponding matrices are given rows or columns of zeroes. Every indecomposable matrix pencil is equivalent to one of the following matrix pencils:

$$\mathcal{L}_k = \underbrace{\begin{pmatrix} 1 & \lambda & & & \\ & 1 & \lambda & & \\ & & \ddots & \ddots & \\ & & & 1 & \lambda \end{pmatrix}}_{k+1}, \quad \mathcal{R}_k = \left.\begin{pmatrix} 1 & & & \\ \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix}\right\} k+1, \quad k \geq 0,$$

$$\mathcal{D}_k(\mu) = E_k + \lambda J_k(\mu), \quad k > 0,$$
$$\mathcal{D}_k(\infty) = \lambda J_k(1), \quad k > 0,$$

where E_k is the k-th order identity matrix, and $J_k(\mu)$ is the k-th order Jordan matrix with eigenvalue μ. Every matrix pencil is equivalent to the direct sum

$$\mathcal{L}_{k_1} \oplus \cdots \oplus \mathcal{L}_{k_p} \oplus \mathcal{R}_{l_1} \oplus \cdots \oplus \mathcal{R}_{l_q} \oplus \mathcal{D}_{n_1}(\mu_1) \oplus \cdots \oplus \mathcal{D}_{n_s}(\mu_s), \quad (2)$$

where $k_1, \ldots, k_p$ and $l_1, \ldots, l_q$ are the *minimal indices* of *rows* and *columns*, respectively, and $\mu_1, \ldots, \mu_s$ are the *eigenvalues* of the matrix pencil. The decomposition (2) is called the *Kronecker canonical form* of a matrix pencil. The set of indecomposable matrix pencils in (2) is defined unambiguously up to a transposition. A matrix pencil $A + \lambda B$ is said to be *regular*, if it is square and $\det(A + \lambda B)$ is not identically zero. Otherwise, it is called *singular*. A matrix pencil is said to be *completely singular*, if its Kronecker canonical form has no regular blocks. Every matrix pencil $\mathcal{P}$ is decomposed into sum of regular and completely singular matrix pencils $\mathcal{P}^{reg}$ and $\mathcal{P}^{sing}$. They are called the *regular part* and the *singular part* of the matrix pencil, respectively. A matrix pencil is said to be *perfect*, if its Kronecker canonical form is $\mathcal{L}_k \oplus \cdots \oplus \mathcal{L}_k$ or $\mathcal{R}_k \oplus \cdots \oplus \mathcal{R}_k$. Otherwise, it is said to be *imperfect*.

Ja'ja extended the results of Kronecker and Weierstrass to the action of $\mathrm{GL}_{n,m,2}$ [13]. In particular, he proved that completely singular matrix pencils are $\mathrm{GL}_{n,m,2}$-equivalent if and only if they are $\mathrm{GL}_{n,m}$-equivalent. Thus, the classification of matrix pencils under the action of $\mathrm{GL}_{n,m,2}$ consists of two separate problems: classification of completely singular matrix pencils under the action of $\mathrm{GL}_{n,m}$, and classification of regular matrix pencils with respect to the action of $\mathrm{GL}_{n,m,2}$.

It took hundred years since the works of Kronecker and Weierstrass to get the description of closures of $\mathrm{GL}_{n,m}$-orbits. During this time, investigations in the theory of matrix pencils were motivated mainly by applications to differential equations and linear-quadratic optimization problems [23, 8]. Several algoritms for calculation of eigenvalues and Kronecker canonical form of matrix pencils were designed [10, 24, 2]. However, as in the case of calculation of the Jordan normal form of a matrix, these algorithms have a common problem: they cannot distinguish between close points that belong to different orbits. The description of closures of $\mathrm{GL}_{n,m}$-orbits was obtained in 1986 by Pokrzywa [21] and later by Bongartz [3]. Pokrzywa has found and systematically described the main types of degenerations of orbits and has shown that the other degenerations are the combinations of those main ones. Bongartz has got his result by using the theory of representations of quivers. However, as the number of orbits is infinite, it is more convenient to describe the closures of *bundles* of matrix pencils. This has been done by Edelman *et al* [6]. In §7. we give a careful proof for the criterion that has been stated in their work.

Classification of orbits is closely related to description of invariants. A pair of matrices A and B that are defined up to the action of $\mathrm{GL}_{n,m}$ is a class of equivalent representations of the quiver of type $\tilde{\mathbf{A}}_1$. In fact, description of invariants of $\mathrm{SL}_{n,m}$ for arbitrary n and m can be derived from the general results on the invariants of representations of tame quivers. For rectangular matrix pencils $\mathrm{GL}_{n,m}$ the description of $\mathrm{SL}_{n,m}$-invariants can be obtained from [11]. In case of square matrix pencils the coefficients of the binary form $\det(\lambda A + \mu B)$ coincide with the semi-invariants constructed in [22], which generate the algebra of $\mathrm{SL}_{n,m}$-invariants according to [12, theorem 2.3]. Hence, the algebra of invariants of the standard linear representation of $\mathrm{SL}_{n,n,2}$ is isomorphic to the algebra of invariants of binary forms of degree n.

Thus, the works related to the standard linear representations of $\mathrm{GL}_{n,m}$, $\mathrm{SL}_{n,m}$, $\mathrm{GL}_{n,m,2}$, and $\mathrm{SL}_{n,m,2}$, can be summarized in the following table (by asterics we denote the questions addressed in this work):

Group	Classification of orbits	Orbits' closures	Invariants
$\mathrm{GL}_{n,m}$	[15, 25]	[21], [3], [6]	trivial
$\mathrm{GL}_{n,m,2}$	[13]	$\star$	trivial
$\mathrm{SL}_{n,m}$	$\star$	$\star$	[22], [12], [16]
$\mathrm{SL}_{n,m,2}$	$\star$	$\star$	$\star$

For several small n and m the standard linear representation of $\mathrm{SL}_n(\mathbb{C})\times\mathrm{SL}_m(\mathbb{C})\times\mathrm{SL}_2(\mathbb{C})$ has good properties, which make it more tractable than in generic case. For instance, if $n = m = 4$ then the representation is visible, that is, the number of nilpotent orbits is finite [14]. This case was extensively studied in [19, 20]. Other "exceptional cases" were investigated in works of Ehrenborg and Nurmiev on $2 \times 2 \times 2$ and $3 \times 3 \times 3$ matrices [7, 17]. In regard of visible representations it is also worth of mentioning the work of Parfenov [18] and its generalization to the case of real matrices that was obtained recently by Dokovic and Tingley [5].

Now we ouline the structure of the paper. We classify the orbits of $\mathrm{GL}_{n,m}$, $\mathrm{SL}_{n,m}$, $\mathrm{GL}_{n,m,2}$, and $\mathrm{SL}_{n,m,2}$ separately for regular (section 2.), imperfect singular (section 3.), and perfect (section 4.) matrix pencils. In section 5. we summarize the results of the classification. In section 4. we also prove that the algebra of invariants of rectangular $n \times m$ matrix pencils is generated by one polynomial of degree nm/d, if $d = |n-m|$ divides n and m and is trivial otherwise. In section 6. we introduce a notion of matrix pencil bundle, which is a union of orbits of $\mathrm{GL}_{n,m}$ over all possible eigenvalues given that the sets of minimal indices of rows, minimal indices of columns, and multiplicites of eigenvalues are fixed. When we take limits of elements of a matrix pencil bundle some of the eigenvalues may coalesce; this process is called amalgamation. In section 7. we prove that the closure of a matrix pencil bundle consists of closures of the corresponding orbits and closures of their amalgams (theorem 7.5). In sections 8. and 9. we study closures of orbits of $\mathrm{GL}_{n,m,2}$ and $\mathrm{SL}_{n,m,2}$. In order to describe the hierarchy of closures (also called the Hasse diagram), we come up with a list of minimal degenerations, which are summarized in section 10. as a set of rules for transformations of Ferrer diagrams. In section 11. we focus on the geometry of the null cones of $\mathrm{SL}_{n,m}$ and $\mathrm{SL}_{n,m,2}$. In particular, if $n \neq m$ then the null cone of $\mathrm{SL}_{n,m,2}$ is irreducible and contains an open orbit. If $n = m$ then the null cone of $\mathrm{SL}_{n,m,2}$ is also irreducible but does not contain an open orbit for $n > 4$. In section 12. we tabulate the orbits of matrix pencils (up to seventh order) and produce figures for the hierarchy of closures of 2×2, 3×3, 4×4, 5×5, 5×6 and 6×6 matrix pencil bundles.

In what follows, the base field is the field $\mathbb{C}$ of complex numbers. If V is a vector space over $\mathbb{C}$ then V^*, $\mathrm{L}(V)$, $\mathrm{L}_0(V)$, $\Lambda^k V$, $\mathrm{S}^k V$, and $\otimes^k V$ denote the dual vector space, the space of all linear endomorphisms of V, the space of linear endomorphisms of V with zero trace, the k-th exterior, symmetric, and tensor powers of V, respectively. The space $\Lambda^k V$ is identified with the subspace of $\otimes^k V$ in a natural way. If $e_1, \ldots, e_n$ is a base in V then $e^1, \ldots, e^n$ denotes the dual base of V^*. The components of tensors $T \in \otimes^k V$ and $T^* \in \otimes^k V^*$ are denoted by $T^{i_1 \ldots i_k}$ and $T^*_{i_1 \ldots i_k}$, respectively. The mapping $\pi\colon \Lambda^k V \to (\Lambda^{n-k} V)^*$ given by

$$(\pi v^{i_1 \ldots i_k})(q^{i_{k+1} \ldots i_n}) = \det{}_{i_1 \ldots i_k i_{k+1} \ldots i_n} v^{i_1 \ldots i_k} q^{i_{k+1} \ldots i_n}, \tag{3}$$

where $\det(e_1, \ldots, e_n) = 1$, is an isomorphism between $\Lambda^k V$ and $(\Lambda^{n-k} V)^*$.

Let G be a group acting on a set X. Denote by Gx (or $\mathcal{O}(x)$, if choice of G is clear from the context) the orbit of $x \in X$ under the action of G. If X is a vector space over $\mathbb{C}$, and G is an algebraic group then $\mathbb{C}[X]$ and $\mathbb{C}[X]^G$ denote the algebra of polynomial functions on X and its subalgebra of G-invariant polynomial functions, respectively. The spectrum of $\mathbb{C}[X]^G$ is denoted by $X/\!/G$. An element $x \in X$ is called *nilpotent* if the closure of its orbit contains zero element. The set of all nilpotent elements of X is called the *null cone* and is denoted by $\mathfrak{N}_X$. The dimension and the codimension of $Y \subset X$ are denoted by $\dim Y$ and $\mathrm{cod}(X, Y)$, respectively. The closure of Y in X (in the topology of $\mathbb{C}$) is denoted by $\overline{Y}$. The number of elements of a finite set S is denoted by $|S|$.

2. Regular matrix pencils

Regular matrix pencils are square. Every regular matrix pencil is $\mathrm{SL}_{n,n}$-equivalent to the matrix pencil $c\cdot\mathcal{P}$, where $\mathcal{P}$ is the Kronecker canoical form. Obviously, $c_1\cdot\mathcal{P}$ and $c_2\cdot\mathcal{P}$ are $\mathrm{SL}_{n,n}$-equivalent if and only if $c_1^n=c_2^n$, where n is the order of $\mathcal{P}$. Every regular matrix pencil is $\mathrm{SL}_{n,m,2}$-equivalent to $E+\lambda D$, where E is the identity matrix and D is a Jordan matrix. The matrix pencils $E+\lambda D$ and $E+\lambda D'$ are $\mathrm{GL}_{n,m,2}$-equivalent if and only if there exist $P,Q\in\mathrm{GL}_n(\mathbb{C})$ and $R\in\mathrm{GL}_2(\mathbb{C})$ such that

$$\begin{cases} P(r_{11}E+r_{12}D)Q^{-1}=E \\ P(r_{21}E+r_{22}D)Q^{-1}=D'. \end{cases} \tag{4}$$

Equivalently, there exist $P\in\mathrm{GL}_n(\mathbb{C})$ and $R\in\mathrm{GL}_2(\mathbb{C})$ such that $P(\varphi_R(D))P^{-1}=D'$. Here φ_R is the linear fractional transformation defined by $\varphi_R(X)=(r_{21}E+r_{22}X)(r_{11}E+r_{12}X)^{-1}$. If we want $E+\lambda D$ and $E+\lambda D'$ to be $\mathrm{SL}_{n,m,2}$-equivalent then we need to have $\det(r_{11}+r_{12}D)=\det R=1$. It is clear that proportional matrices define equal fractional transformations. Therefore we can assume that $R\in\mathrm{GL}_2(\mathbb{C})$ has the following property:

$$\det(r_{11}+r_{12}D)=(\det R)^{n/2}. \tag{5}$$

We now state some of our previous results [20]. The matrix pencils $E+\lambda D$ and $E+\lambda D'$ are $\mathrm{GL}_{n,m,2}$-equivalent (respectively, $\mathrm{SL}_{n,m,2}$-equivalent) if and only if the eigenvalues of D are mapped to the eigenvalues of D' by a linear fractional transformation (respectively, by a linear fractional transformation that satisfies (5)) that preserves the multiplicities of the corresponding eigenvalues. The polynomial mapping $\theta\colon\mathbb{C}^{n,n,2}\to Y=\mathrm{S}^n(\mathbb{C}^{2^*})$, that takes each matrix pencil $A+\lambda B$ to the binary form $f(\alpha,\beta)=\det(\alpha A+\beta B)$ is dominant and $\mathrm{SL}_{n,n,2}$-equivariant assuming that the first and the second factor of $\mathrm{SL}_{n,n,2}$ act trivially on Y. The algebra $\mathbb{C}[\mathbb{C}^{n,n,2}]^{\mathrm{SL}_{n,n}}$ is generated by coefficients of the binary form $\det(\alpha A+\beta B)$. It follows from this theorem that the algebra of $\mathrm{SL}_{n,n,2}$-invariants of $n\times n$ matrix pencils is isomorphic to the algebra of $\mathrm{SL}_2(\mathbb{C})$-invariants of binary forms of degree n, and a square matrix pencil $\mathcal{P}$ is nilpotent under the action of $\mathrm{SL}_{n,n,2}$ if and only if the binary form $\theta(\mathcal{P})$ is nilpotent.

3. Imperfect singular matrix pencils

It follows from [13, theorem 3] that all perfect matrix pencils are $\mathrm{GL}_{n,m,2}$-equivalent. Imperfect singular matrix pencils are $\mathrm{GL}_{n,m,2}$-equivalent if and only if their regular components are $\mathrm{GL}_{n,m,2}$-equivalent and their singular components are $\mathrm{GL}_{n,m}$-equivalent.

Now we focus on $\mathrm{SL}_{n,m}$- and $\mathrm{SL}_{n,m,2}$-equivalence of imperfect singular matrix pencils. The following lemma is an analog of [20, lemma 2] for rectangular matrix pencils.

Lemma 3.1. *Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be matrix pencils, and let $n_1\times m_1$ and $n_2\times m_2$ be the sizes of $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively. If the pairs (n_1,m_1) and (n_2,m_2) are not proportional then $\mathcal{P}_1\oplus\mathcal{P}_2$ and $(\alpha\cdot\mathcal{P}_1)\oplus(\beta\cdot\mathcal{P}_2)$ are $\mathrm{SL}_{n,m}$-equivalent for any pair of non-zero complex numbers α and β.*

Proof. Put $P = \mathrm{diag}(\alpha_1 E_{n_1}, \beta_1 E_{n_2})$ and $Q = \mathrm{diag}(\alpha_2 E_{m_1}, \beta_2 E_{m_2})$, where E_k is the k-th order identity matrix. Then $\mathcal{P}_1 \oplus \mathcal{P}_2$ and $(\alpha \cdot \mathcal{P}_1) \oplus (\beta \cdot \mathcal{P}_2)$ are equivalent under the action of $(P, Q) \in \mathrm{SL}_{n,m}$ if and only if $\alpha_1^{n_1}\beta_1^{n_2} = 1$, $\alpha_2^{m_1}\beta_2^{m_2} = 1$, $\alpha_1\alpha_2^{-1} = \alpha$ and $\beta_1\beta_2^{-1} = \beta$. Obviously, these equations have a common solution if the pairs (n_1, m_1) and (n_2, m_2) are not proportional. ∎

Corollary 3.2. *The following matrix pencils*

(i) $\mathcal{P}_1 = \mathcal{L}_p \oplus \mathcal{R}_q$ *and* $\mathcal{P}_2 = (\alpha \cdot \mathcal{L}_p) \oplus (\beta \cdot \mathcal{R}_q)$,

(ii) $\mathcal{P}_1 = \mathcal{L}_p \oplus \mathcal{L}_q$ *and* $\mathcal{P}_2 = (\alpha \cdot \mathcal{L}_p) \oplus (\beta \cdot \mathcal{L}_q)$ *for* $p \neq q$,

(iii) $\mathcal{P}_1 = \mathcal{R}_p \oplus \mathcal{R}_q$ *and* $\mathcal{P}_2 = (\alpha \cdot \mathcal{R}_p) \oplus (\beta \cdot \mathcal{R}_q)$ *for* $p \neq q$,

(iv) $\mathcal{P}_1 = \mathcal{L}_p \oplus \mathcal{Q}$ *and* $\mathcal{P}_2 = (\alpha \cdot \mathcal{L}_p) \oplus (\beta \cdot \mathcal{Q})$,

(v) $\mathcal{P}_1 = \mathcal{R}_p \oplus \mathcal{Q}$ *and* $\mathcal{P}_2 = (\alpha \cdot \mathcal{R}_p) \oplus (\beta \cdot \mathcal{Q})$

are $\mathrm{SL}_{n,m}$*-equivalent for any square* $l \times l$ *matrix pencil* $\mathcal{Q}$ *and any pair of non-zero complex numbers* α *and* β.

Theorem 3.3. *Imperfect singular matrix pencils are* $\mathrm{SL}_{n,m}$*-equivalent if and only if they are* $\mathrm{GL}_{n,m}$*-equivalent.*

Proof. Suppose that $\mathcal{P}_1$ and $\mathcal{P}_2$ are equivalent. Then $\mathcal{P}_1$ and $k \cdot \mathcal{P}_2$ are $\mathrm{SL}_{n,m}$-equivalent for some $k \in \mathbb{C}$. We may replace $\mathcal{P}_1$ and $\mathcal{P}_2$ with their Kronecker canonical forms. If $\mathcal{P}_2$ is a square singular matrix pencil then $\mathcal{P}_2 = \mathcal{L}_p \oplus \mathcal{R}_q \oplus \mathcal{Q}$. It is clear that $k \cdot \mathcal{P}_2$ and $(k' \cdot \mathcal{L}_p) \oplus (k' \cdot \mathcal{R}_q) \oplus \mathcal{Q}$ are equivalent under the action of $\mathrm{SL}_{n,m}$. Then $\mathcal{P}_1$ and $\mathcal{P}_2$ are $\mathrm{SL}_{n,m}$-equivalent by corollary 3.2. If $\mathcal{P}_2$ is an imperfect rectangular matrix pencil then its Kronecker canonical form contains two left blocks of different sizes or two right blocks of different sizes or a non-trivial regular block. It follows from corollary 3.2 that $\mathcal{P}_1$ and $\mathcal{P}_2$ are $\mathrm{SL}_{n,m}$-equivalent. The proof of the converse is trivial. ∎

Theorem 3.4. *Imperfect singular matrix pencils are* $\mathrm{SL}_{n,m,2}$*-equivalent if and only if they are* $\mathrm{GL}_{n,m,2}$*-equivalent.*

Proof. Suppose that $\mathcal{P}_1$ and $\mathcal{P}_2$ are $\mathrm{GL}_{n,m,2}$-equivalent. By [13, theorem 3], their singular components are $\mathrm{GL}_{n,m}$-equivalent, and their regular components are $\mathrm{SL}_{n,m,2}$-equivalent up to a multiplicative factor. Therefore, $\mathcal{P}_1$ is $\mathrm{SL}_{n,m,2}$-equivalent to $\mathcal{P}_1^{sing} \oplus (k \cdot \mathcal{P}_2^{reg})$ that is, in turn, $\mathrm{SL}_{n,m}$-equivalent to $(k' \cdot \mathcal{P}_1^{sing}) \oplus \mathcal{P}_2^{reg}$. Then $\mathcal{P}_1$ and $\mathcal{P}_2$ are $\mathrm{SL}_{n,m,2}$-equivalent by previous theorem. ∎

4. Perfect matrix pencils

Without loss of generality we consider only perfect $n \times m$ matrix pencils for $m > n$. Then $d = m - n$ divides n and every perfect matrix pencil is $\mathrm{SL}_{n,m}$-equivalent to $\mathcal{L}_k \oplus \cdots \oplus \mathcal{L}_k$ up to a multiplicative factor. Here $\mathcal{L}_k \oplus \cdots \oplus \mathcal{L}_k$ denotes the sum of d indecomposable $k \times l$ matrix pencils, where $k = n/d$ and

$l = m/d = k+1$. The matrix pencil $\mathcal{L}_k \oplus \cdots \oplus \mathcal{L}_k$ is $\mathrm{SL}_{n,m}$-equivalent to the matrix pencil $\mathcal{P} = [E_n|0_{nd}] + \lambda[0_{nd}|E_n]$, where E_n is the n-th order identity matrix, 0_{nd} is the zero $n \times d$ matrix, and $[A|B]$ denotes the matrix obtained by attachment of the matrix B to the right of the matrix A. The following lemma shows that the statement of theorem 3.3 is correct only for imperfect matrix pencils, that is, it is impossible to multiply a perfect matrix pencil by an arbitrary complex number by a transformation from $\mathrm{SL}_{n,m}$.

Theorem 4.1. *Let $\mathcal{P}$ be a matrix pencil. Then $\mathcal{P}$ and $c \cdot \mathcal{P}$ are $\mathrm{SL}_{n,m}$-equivalent iff $c^{km} = 1$.*

Proof. Assume that $\mathcal{P} = [E_n|0_{nd}] + \lambda[0_{nd}|E_n]$. Let matrices $P \in \mathrm{SL}_n(\mathbb{C})$ and $Q \in \mathrm{SL}_m(\mathbb{C})$ be such that $P\mathcal{P}Q^{-1} = c \cdot \mathcal{P}$. Consider the following partition of the matrix Q

$$\begin{pmatrix} Q' & Q_1 \\ Q_2 & Q_0 \end{pmatrix},$$

where Q' is a square $n \times n$ matrix, Q_1, Q_2, and Q_0 are the corresponding complementary matrices. Then $P\,[E|0]\,Q^{-1} = [E|0]$ implies $P = Q'$ and $Q_1 = 0$. Applying similar arguments to $P\,[0|E]\,Q^{-1} = [0|E]$ we get that Q is composed of square $d \times d$-blocks

$$\begin{pmatrix} Q_{11} & Q_{12} & \dots & Q_{1k} & 0 \\ 0 & \vdots & & \vdots & 0 \\ 0 & Q_{l2} & \dots & Q_{lk} & Q_{ll} \end{pmatrix},$$

and

$$P = \begin{pmatrix} Q_{11} & Q_{12} & \dots & Q_{1k} \\ 0 & \vdots & & \vdots \\ 0 & Q_{k2} & \dots & Q_{kk} \end{pmatrix} = \begin{pmatrix} Q_{22} & \dots & Q_{2k} & 0 \\ \vdots & & \vdots & 0 \\ Q_{l2} & \dots & Q_{lk} & Q_{ll} \end{pmatrix}.$$

Therefore, $Q_{32} = \cdots = Q_{l2} = 0$, $Q_{1k} = \cdots = Q_{k-1\,k} = 0$ and $Q_{11} = \cdots = Q_{ll}$. Iterating this process k times we get that $P = \mathrm{diag}(A, \dots, A)$ and $Q = \mathrm{diag}(cA, \dots, cA)$, where A is a $d \times d$ matrix. Therefore, $(\det A)^k = 1$ and $c^m (\det A)^{k+1} = 1$. This completes the proof. ∎

It follows from theorem 4.1 that $\mathrm{SL}_{n,m}$ has infinite number of orbits in $\mathbb{C}^{n,m,2}$. This fact indicates that the algebra of invariants is non-trivial. It turns out that $\mathbb{C}[\mathbb{C}^{n,m,2}]^{\mathrm{SL}_{n,m}}$ has only one generator, which is also $\mathrm{SL}_{n,m,2}$-invariant. For simplicity we now assume that $d = 1$. Then, $n = k$ and $m = k+1$. Denote by Δ_i the n-th order minor of the matrix $A + \lambda B$ that doesn't contain the i-th column; they are polynomial functions of λ

$$\Delta_i(\lambda) = \sum_{s=0}^{n} \Delta_{is} \lambda^{n-s},$$

and their coefficients Δ_{is} are n-th degree polynomials of the entries of A and B. The numbers Δ_{is} form a square $(n+1) \times (n+1)$ matrix Δ, whose determinant we denote by $\omega(\mathcal{P})$.

Lemma 4.2. *The polynomial ω is invariant under the action of $\mathrm{SL}_{n,m,2}$.*

Proof. It is clear that $\omega(\mathcal{P})$ is invariant under the left action of $\mathrm{SL}_n(\mathbb{C})$. Consider the matrix $A+\lambda B$ as a collection of columns $a_1(\lambda),\dots,a_m(\lambda)$. If we add the j-th column multiplied by some factor c to the i-th column then Δ_j is converted to $\Delta_j + c(-1)^{j-i}\Delta_i$ and the other minors don't change. This proves that $\det(\Delta)$ is invariant under the right action of $\mathrm{SL}_m(\mathbb{C})$. Now consider the transformation $A \mapsto A+cB$. It corresponds to the transformation $\lambda \mapsto \lambda+c$ of the varying coefficient, which adds a linear combination of the successive columns to each column of Δ. Transposition of A and B, being combined with simultaneous multiplication of B by -1, converts $\Delta_i(\lambda)$ to $\lambda^n\Delta_i(-\lambda^{-1})$. These transformations don't affect $\det(\Delta)$. Thus, $\omega(\mathcal{P})$ is $\mathrm{SL}_{n,m,2}$-invariant. ∎

In order to construct the invariant for an arbitrary d we will use the values of $\Delta_i(\lambda)$ instead of their coefficients. Let $e_1,\dots,e_m$ be a base in $\mathbb{C}^m$, and let $e^1,\dots,e^m$ be the corresponding dual base. Denote the tautological linear representation of $\mathrm{SL}_p(\mathbb{C})$ in $\mathbb{C}^p$ by ρ_p and consider the morphism η of the linear representations $\rho_n\otimes\rho_m$ and $\Lambda^n\rho_m^*$, that takes each matrix A to the n-vector

$$\eta(A) = \sum_{i_1<\dots<i_n} \Delta_{i_1\dots i_n}(A)\, e^{i_1}\wedge\dots\wedge e^{i_n}.$$

Here $\Delta_{i_1\dots i_n}(A)$ is the n-th order minor of the matrix A that contains the columns $i_1,\dots,i_n$. One can show that the right multiplication of A by a unimodular matrix Q corresponds to the left multiplication of $\eta(A)$ by $Q^\top$. Obviously, the left multiplication of A by a unimodular matrix doesn't affect the n-th order minors. Therefore, η is $\mathrm{SL}_{n,m}$-invariant (we assume that $\mathrm{SL}_n(\mathbb{C})$ acts on $(\Lambda^n\mathbb{C}^m)^*$ trivially). Recall that $(\Lambda^n\mathbb{C}^m)^*$ is identified with $\Lambda^d\mathbb{C}^m$ by the $\mathrm{SL}_{n,m}$-equivariant morphism (3). Then η can be regarded as the following chain of morphisms: $\mathrm{Hom}((\mathbb{C}^n)^*,\mathbb{C}^m) \to \mathrm{Hom}(\Lambda^n(\mathbb{C}^n)^*,\Lambda^n\mathbb{C}^m) \to \mathrm{Hom}(\Lambda^n(\mathbb{C}^m)^*,\Lambda^n\mathbb{C}^n) = \Lambda^d\mathbb{C}^m$. Now chose some distinct complex numbers $\lambda_1,\dots,\lambda_l$ and define ω by formula

$$\omega(\mathcal{P}) = (-1)^{\frac{d(d-1)}{2}\cdot\frac{l(l-1)}{2}}\,\eta\big(\mathcal{P}(\lambda_1)\big)\wedge\dots\wedge\eta\big(\mathcal{P}(\lambda_l)\big). \tag{6}$$

The morphism ω takes each matrix pencil $\mathcal{P}$ to the element of $\Lambda^m\mathbb{C}^m \simeq \mathbb{C}$ and is invariant under the action of $\mathrm{SL}_{n,m}$.

Theorem 4.3. *The morphism ω is invariant under the action of* $\mathrm{SL}_{n,m,2}$.

Proof. It is easy to prove that if $\mathcal{P} = \mathcal{L}_k\oplus\dots\oplus\mathcal{L}_k$ then $\omega(\mathcal{P})$ is equal to Wandermond determinant of $\lambda_1,\dots,\lambda_l$ and otherwise it is equal to zero. The action of $\mathrm{SL}_2(\mathbb{C})$ on $\mathbb{C}^{n,m,2}$ is generated by transformations $A+\lambda B \mapsto (A+cB)+\lambda B$ and $A+\lambda B \mapsto B-\lambda A$. They correspond to the following transformations of λ: $\mathcal{P}(\lambda)\mapsto\mathcal{P}(\lambda+c)$ and $\mathcal{P}(\lambda)\mapsto\lambda\mathcal{P}(-\lambda^{-1})$. If we add the same constant to all $\lambda_1,\dots,\lambda_l$ then the Wandermond determinant will not change. The transformation $\mathcal{P}(\lambda)\mapsto\lambda\mathcal{P}(-\lambda^{-1})$ doesn't affect ω, too, as it inverses the order of the columns in the Wandermond determinant and multiplies the odd columns by -1. Therefore, $\omega(\mathcal{P})$ is $\mathrm{SL}_{n,m,2}$-invariant. ∎

Corollary 4.4. *Perfect matrix pencils are* $\mathrm{SL}_{n,m,2}$*-equivalent if and only if they are* $\mathrm{SL}_{n,m}$*-equivalent.*

It follows from this theorem that the set of perfect matrix pencils is an open and dense subset in the set of all $n\times m$ matrix pencils, and the orbits of a perfect matrix pencil under the actions of $\mathrm{SL}_{n,m}$ and $\mathrm{SL}_{n,m,2}$ are closed.

Theorem 4.5. *If $d = m-n$ divides n, then $\mathbb{C}[\mathbb{C}^{n,m,2}]^{\mathrm{SL}_{n,m}} \simeq \mathbb{C}[\mathbb{C}^{n,m,2}]^{\mathrm{SL}_{n,m,2}} \simeq \mathbb{C}[\omega]$. Otherwise, $\mathbb{C}[\mathbb{C}^{n,m,2}]^{\mathrm{SL}_{n,m}} \simeq \mathbb{C}[\mathbb{C}^{n,m,2}]^{\mathrm{SL}_{n,m,2}} \simeq \mathbb{C}$.*

Proof. If d doesn't divide n then all elements of $\mathbb{C}^{n,m,2}$ are imperfect singular matrix pencils. Then by theorems 3.3 and 3.4 all orbits are nilpotent and therefore $\mathbb{C}[\mathbb{C}^{n,m,2}]^{\mathrm{SL}_{n,m}} \simeq \mathbb{C}[\mathbb{C}^{n,m,2}]^{\mathrm{SL}_{n,m,2}} \simeq \mathbb{C}$. Now let d divide n. Consider the one-dimensional subspace L spanned over the perfect matrix pencil $\mathcal{L}_k \oplus \dots \oplus \mathcal{L}_k$. The set $\mathrm{SL}_{n,m} \cdot L$ is a Zarisski open set, which is cut by $\omega(\mathcal{P}) \neq 0$. Then L is a Chevallier section, that is, the morphism of restriction of invariants on L is an isomorphism, and therefore the transcendence degree of $\mathbb{C}[\mathbb{C}^{n,m,2}]^{\mathrm{SL}_{n,m}}$ is equal to one. By theorem 4.1, we have $|W| = km = \deg(\omega)$. Thus, ω is a generator of $\mathbb{C}[\mathbb{C}^{n,m,2}]^{\mathrm{SL}_{n,m}}$ and, since the generic $\mathrm{SL}_{n,m}$-orbits and $\mathrm{SL}_{n,m,2}$-orbits coincide, also a generator of $\mathbb{C}[\mathbb{C}^{n,m,2}]^{\mathrm{SL}_{n,m,2}}$. ∎

5. Classification of matrix pencils

A *partition* of an integer n is a non-decresing infinite sequence of integers $\mathbf{n} = (n_1, n_2, \dots)$ that sums up to n. It is obvious that the number of non-zero terms in such sequence is finite. Let $\mathbf{n}$ and $\mathbf{m}$ be the partitions of the integers n and m, respectively. The *sum* $\mathbf{n}+\mathbf{m}$ is the termwise sum of the sequences of n and m. The *union* $\mathbf{n}\cup\mathbf{m}$ is the union of terms of n and m sorted in descending order. The *conjugate* partition $\mathbf{n}^*$ is defined by $n^*_k = |\{i \mid n_i \geq k\}|$. For any partitions $\mathbf{n}$ and $\mathbf{m}$ we have $(\mathbf{n}+\mathbf{m})^* = \mathbf{n}^* \cup \mathbf{m}^*$ and $(\mathbf{n}^*)^* = \mathbf{n}$. We say that the partition $\mathbf{n}$ *dominates* the partition $\mathbf{m}$ and write $\mathbf{n} \geq \mathbf{m}$, if $n_1+\dots+n_k \geq m_1+\dots+m_k$ for all k. If $\mathbf{n} = (n_1, n_2, \dots)$ and a is an integer then $\mathbf{n}+a$ denotes $(n_1+a, n_2+a, \dots)$. If $\mathbf{n}$ and $\tilde{\mathbf{n}}$ are the partitions of the same integer n and $\mathbf{n} \geq \tilde{\mathbf{n}}$ then $\tilde{\mathbf{n}}^* \geq \mathbf{n}^*$. The partition $\mathbf{m}$ (respectively, $\mathbf{n}$) is said to be the *lowering* (respectively, the *heightening*) of the partition $\mathbf{n}$ (respectively, $\mathbf{m}$), if $\mathbf{n} > \mathbf{m}$, that is, $\mathbf{n} \geq \mathbf{m}$ and $\mathbf{n} \neq \mathbf{m}$. The lowering (respectively, the heightening) is said to be *minimal*, if there is no partition $\mathbf{k}$ such that $\mathbf{n} > \mathbf{k} > \mathbf{m}$. The partitions are usually illustrated by Ferrer diagrams. The i-th column of the Ferrer diagram contains n_i cells. The Ferrer diagram of $\mathbf{n}^*$ is obtained from the Ferrer diagram of $\mathbf{n}$ by transposition. The Ferrer diagram of a minimal lowering of the partition $\mathbf{n}$ is obtained from the Ferrer diagram of $\mathbf{n}$ by crumbling or deletion of its rightmost cell (if it exists).

Let $\mathcal{P}$ be a matrix pencil. Denote the number of blocks $\mathcal{L}_k$, $\mathcal{R}_k$ and $\mathcal{D}_k(\mu)$ in the Kronecker canonical form of $\mathcal{P}$ by $l_k(\mathcal{P})$, $r_k(\mathcal{P})$ and $d_k(\mu, \mathcal{P})$, respectively. Define $l(\mathcal{P}) = \sum l_k(\mathcal{P})$ and $r(\mathcal{P}) = \sum r_k(\mathcal{P})$. The number $\dim U - l(\mathcal{P}) = \dim V - r(\mathcal{P})$ is called the *normal rank* of $\mathcal{P}$ and is denoted by $\mathrm{nrk}(\mathcal{P})$. Define the partitions $\mathfrak{D}(\mu, \mathcal{P}) = (\mathfrak{d}_1(\mu, \mathcal{P}), \mathfrak{d}_2(\mu, \mathcal{P}), \dots)$, $\mathfrak{L}(\mathcal{P}) = (\mathfrak{l}_0(\mathcal{P}), \mathfrak{l}_1(\mathcal{P}), \dots)$, and $\mathfrak{R}(\mathcal{P}) = (\mathfrak{r}_0(\mathcal{P}), \mathfrak{r}_1(\mathcal{P}), \dots)$ by

$$\mathfrak{l}_i(\mathcal{P}) = \sum_{k \geq i} l_k(\mathcal{P}) \tag{7}$$

$$\mathfrak{r}_i(\mathcal{P}) = \sum_{k \geq i} r_k(\mathcal{P}) \tag{8}$$

$$\mathfrak{d}_i(\mu, \mathcal{P}) = \sum_{k \geq i} d_k(\mu, \mathcal{P}). \tag{9}$$

The sequences $l_i(\mathcal{P})$, $r_i(\mathcal{P})$ and $d_i(\mu,\mathcal{P})$ are determined uniquely by partitions $\mathfrak{D}(\mu,\mathcal{P})$, $\mathfrak{L}(\mathcal{P})$ and $\mathfrak{R}(\mathcal{P})$. Note that the partition $\mathfrak{D}(\mu,\mathcal{P})$ is conjugate to the set of orders of regular blocks corresponding to the eigenvalue μ. For instance, if $\mathfrak{D}(\mu,\mathcal{P})=(2,1,1)$ and $\mathfrak{L}(\mathcal{P})=(1,1)$ then $\mathcal{P}=\mathcal{D}_1(\mu)+\mathcal{D}_3(\mu)+\mathcal{L}_1$.

Now the classes of equivalent matrix pencils are described as follows:

1. A class of $\mathrm{GL}_{n,m}$-equivalent matrix pencils is given by the set $\{\mu_1,\dots,\mu_s\}$ of (distinct) eigenvalues and the partitions $\mathfrak{L}(\mathcal{P})$, $\mathfrak{R}(\mathcal{P})$, and $\mathfrak{D}(\mu_j,\mathcal{P})$.

2. A calss of $\mathrm{GL}_{n,m,2}$-equivalent matrix pencils is given by the set of eigenvalues that are defined up to a linear fractional transformation and the partitions $\mathfrak{L}(\mathcal{P})$, $\mathfrak{R}(\mathcal{P})$, and $\mathfrak{D}(\mu_j,\mathcal{P})$.

3. A class of $\mathrm{SL}_{n,m}$-equivalent matrix pencils is given by

   (a) the set of eigenvalues and the partitions $\mathfrak{L}(\mathcal{P})$, $\mathfrak{R}(\mathcal{P})$, and $\mathfrak{D}(\mu_j,\mathcal{P})$, if the matrix pencil is singular and imperfect;

   (b) the proportionality factor (between matrix pencil and its Kronecker canonical form) that is defined up to multiplication by km-th root of unity, if the matrix pencil is perfect;

   (c) the set of eigenvalues, the partitions $\mathfrak{D}(\mu_j,\mathcal{P})$, and the proportionality factor (as before) that is defined up to multiplication by n-th root of unity, if the matrix pencil is regular;

4. A class of $\mathrm{SL}_{n,m,2}$-equivalent matrix pencils is given by

   (a) the set of eigenvalues that are defined up to a linear fractional transformation and the partitions $\mathfrak{L}(\mathcal{P})$, $\mathfrak{R}(\mathcal{P})$, and $\mathfrak{D}(\mu_j,\mathcal{P})$, if the matrix pencil is singular and imperfect;

   (b) the proportionality factor that is defined up to multiplication by km-th root of unity, if the matrix pencil is perfect;

   (c) the set of eigenvalues that are defined up to a linear fractional transformation, which satisfies (5) and the partitions $\mathfrak{D}(\mu_j,\mathcal{P})$, if the matrix pencil is regular.

Remark 5.1. In all tables and figures we use the following simplified notation of matrix pencils. If $\mathfrak{D}(\mu_i,\mathcal{P})=(n_{i1},n_{i2},\dots)$ then the regular component of $\mathcal{P}$ is denoted by $\mathcal{J}_{n_1}(n_{11},n_{12},\dots)\oplus\dots\oplus\mathcal{J}_{n_s}(n_{s1},n_{s2},\dots)$. The integers n_{ij} that are equal to 0 or 1 are omitted in this expression. We assume that the eigenvalues that correspond to different terms in this sum are distinct complex numbers defined up to a certain transformation (depending on which group is acting). The proportionality factor between matrix pencil and its Kronecker canonical form is also omitted. For instance, $\mathcal{D}_2(\mu_1)\oplus\mathcal{D}_1(\mu_1)\oplus\mathcal{D}_1(\mu_2)$ corresponds to $\mathcal{J}_3(2)\oplus\mathcal{J}_1$. The matrix pencil $\mathcal{P}\oplus\dots\oplus\mathcal{P}$ is shortly denoted by $n\mathcal{P}$, where n is the number of terms in the sum. The matrix pencils $\mathcal{L}_0$ and $\mathcal{R}_0$ are not shown in the canonical form.

6. Matrix pencil bundles

A set $\mathbf{X} = \{X_\alpha\}$ of subsets of a topological space X is called the *stratification*, if $\cup X_\alpha = X$, the closure of each X_α is a union of elements of $\mathbf{X}$, and $X_\alpha \cap X_\beta = \varnothing$ for $\alpha \neq \beta$. The sets X_α are called *strata*. We say that X_α *covers* X_β, if $\overline{X_\alpha} \supset \overline{X_\beta}$ and there is no X_γ other than X_α and X_β such that $\overline{X_\alpha} \supset \overline{X_\gamma} \supset \overline{X_\beta}$.

Now let X be an algebraic manifold with an action of some algebraic group. We are interested in stratifications that have invariant strata. Certainly, such strata are the unions of orbits, and the orbital decomposition, that is, the set of all orbits is the finest possible invariant stratification. However, it typically has infinite number of strata. In the next few paragraphs we introduce the notion of matrix pencil bundle, which gives a natural stratification that has finite number of strata.

The term "bundle" originates in works of Arnold and deals with linear operators [1]. A *bundle of linear operators* is a set of linear operators that have fixed Jordan structure and varying eigenvalues. In other words, the linear operators $\oplus J_{k_i}(\alpha_i)$ and $\oplus J_{k_i}(\beta_i)$ belong to the same bundle if and only if $\alpha_i = \alpha_j$ implies $\beta_i = \beta_j$ and *vice versa* for all i and j. Obviously, the number of bundles of linear operators is finite.

It is clear how to give similar definition for matrix pencils. Let $\mathcal{P}$ be a matrix pencil that is defined up to a transformation from $\mathrm{GL}_{n,m}$, and let φ be a one-to-one mapping of $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ into itself. The matrix pencil $\varphi(\mathcal{P})$ is defined by

$$\mathfrak{L}(\varphi(\mathcal{P})) = \mathfrak{L}(\mathcal{P}), \tag{10}$$

$$\mathfrak{R}(\varphi(\mathcal{P})) = \mathfrak{R}(\mathcal{P}), \tag{11}$$

$$\mathfrak{D}(\mu, \varphi(\mathcal{P})) = \mathfrak{D}(\varphi^{-1}(\mu).\mathcal{P}) \tag{12}$$

up to a a transformation from $\mathrm{GL}_{n,m}$. The Kronecker canonical form of $\varphi(\mathcal{P})$ is obtained from the Kronecker canonical form of $\mathcal{P}$ by replacement of the eigenvalues of $\mathcal{P}$ with their images under φ. Denote by Φ the set of one-to-one mappings of $\overline{\mathbb{C}}$ into itself. The set

$$\mathrm{B}(\mathcal{P}) = \bigcup_{\varphi \in \Phi} \varphi(\mathcal{P}) \tag{13}$$

is called a *matrix pencils bundle*. In what follows, the matrix pencils bundles are shortly referred to as *bundles*. Similarly to (13), we introduce the following notation for the unions of closures of orbits:

$$\overline{\mathrm{B}}(\mathcal{P}) = \bigcup_{\varphi \in \Phi} \overline{\varphi(\mathcal{P})}. \tag{14}$$

Obviously, $\mathrm{B}(\mathcal{P}) \subset \overline{\mathrm{B}}(\mathcal{P}) \subset \overline{\mathrm{B}(\mathcal{P})}$.

Now let ψ be a mapping of $\overline{\mathbb{C}}$ into itself (not necesserily one-to-one). Define $\psi(\mathcal{P})$ by (10), (11) and

$$\mathfrak{D}(\mu, \psi(\mathcal{P})) = \bigcup_{\nu \in \psi^{-1}(\mu)} \mathfrak{D}(\nu, \mathcal{P}). \tag{15}$$

The matrix pencil $\psi(\mathcal{P})$ is defined up to a transformation from $\mathrm{GL}_{n,m}$ and is called the *amalgam* of $\mathcal{P}$. In what follows we often use two special types of amalgams.

Let $\gamma(z)$ be the mapping that takes all $\overline{\mathbb{C}}$ to the point a, and let $\gamma_{z_0}(z)$ be the mapping that takes $\overline{\mathbb{C}}\backslash\{z_0\}$ to the point a and z_0 to some other point b for some $z_0 \in \overline{\mathbb{C}}$. If $\mathcal{P}$ is a matrix pencil and μ is an eigenvalue of $\mathcal{P}$ then the amalgams $\gamma(\mathcal{P})$ and $\gamma_\mu(\mathcal{P})$ are called the *main* and the *submain* amalgams, respectively.

Assume that $\mathcal{P}$ is a Kronecker canonical form, and let $\mathfrak{S}(\mathcal{P}) \subset \overline{\mathbb{C}}$ be the set of eigenvalues of $\mathcal{P}$. Consider the set $Z = \{(x_1, \dots, x_s) \in \mathbb{C}^s \mid x_i \neq x_j \text{ if } i \neq j\}$, where $s = |\mathfrak{S}(\mathcal{P})|$, and the mapping $f : Z \times \mathrm{GL}_{n,m} \to \mathbb{C}^{n,m,2}$, that takes each pair $((x_1, \dots, x_s), g)$ to a matrix pencil obtained from $\mathcal{P}$ by replacement of the eigenvalues of $\mathcal{P}$ with $x_1, \dots, x_s$ and consequent application of the action of $g \in \mathrm{GL}_{n,m}$. This mapping is polynomial and has constant rank. Since the dimension of stabilizer of a matrix pencil depends on the partitions $\mathfrak{D}(\mu, \mathcal{P})$ but not on the eigenvalues themselves, the image of f, that is, $\mathrm{B}(\mathcal{P})$ is an irreducible regular algebraic manifold, whose codimension is given by

$$\mathrm{cod}(\mathbb{C}^{n,m,2}, \mathrm{B}(\mathcal{P})) = \mathrm{cod}(\mathbb{C}^{n,m,2}, \mathcal{O}(\mathcal{P})) - |\mathfrak{S}(\mathcal{P})|. \tag{16}$$

The codimension of orbit, which appears on the right side of (16), can be calculated as follows [4]. Let $\mathcal{P}$ and $\mathcal{Q}$ be $n_1 \times m_1$ and $n_2 \times m_2$ matrix pencils. Define

$$\langle \mathcal{P}, \mathcal{Q} \rangle = \dim\{(A, B) \in (\mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2}) \times (\mathbb{C}^{m_1} \otimes \mathbb{C}^{m_2}) \mid \mathcal{P}A = B\mathcal{Q}\}.$$

Then the dimension of stabilizer of $n \times m$ matrix pencil $\mathcal{P}$ in $\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_m(\mathbb{C})$ is equal to $\langle \mathcal{P}, \mathcal{P} \rangle$, and the codimension of its orbit in $\mathbb{C}^{n,m,2}$ is equal to $\langle \mathcal{P}, \mathcal{P} \rangle - (n-m)^2$. One can prove that $\langle \mathcal{P}, \mathcal{Q} \rangle = \sum \langle \mathcal{P}_i, \mathcal{Q}_j \rangle$, if $\mathcal{P} = \oplus \mathcal{P}_i$ and $\mathcal{Q} = \oplus \mathcal{Q}_j$. For $\mu_1 \neq \mu_2$ we have $\langle \mathcal{D}_k(\mu_1), \mathcal{D}_j(\mu_2) \rangle = 0$; for the other pairs of indecomposable matrix pencils the values of $\langle \mathcal{P}, \mathcal{Q} \rangle$ are given in the following table.

	$\mathcal{L}_k$	$\mathcal{R}_k$	$\mathcal{D}_k(\mu)$
$\mathcal{L}_j$	$(j-k+1)_+$	$j+k$	j
$\mathcal{R}_j$	0	$(k-j+1)_+$	0
$\mathcal{D}_j(\mu)$	0	k	$\min(j,k)$

(17)

This gives an efficient method for computation of orbit's codimension from the Kronecker canonical form. By dropping $\mathcal{L}_0$ and $\mathcal{R}_0$ in the Kronecker canonical form we implicitly use the natural embedding of $\mathbb{C}^{n,m,2}$ into $\mathbb{C}^{N,M,2}$ for $N \geq n$ and $M \geq m$. Codimensions of orbits, however, depend on the dimension of the enveloping space. This motivates the following definition. A matrix pencil $\mathcal{P}$ is said to be a *veritable* $n \times m$ *matrix pencil*, if $l_0(\mathcal{P}) = r_0(\mathcal{P}) = 0$. Veritability of a matrix pencil depends on the dimension of the representation. The $n \times m$ matrix pencils that are not veritable $n \times m$ matrix pencils are veritable matrix pencils of smaller orders.

It is convenient to reduce the calculation of the orbit's codimension to the case of veritable matrix pencils. Let $\mathcal{P}$ be a veritable $n \times m$ matrix pencil , $N \geq n$, $M \geq m$, $D = N - M$, $d = n - m$. Simple algebra proves that the codimension of $\mathrm{GL}_{N,M}$-orbit of $\mathcal{P}$ is given by

$$\mathrm{cod}(\mathbb{C}^{N,M,2}, \mathrm{GL}_{N,M}\mathcal{P}) - \mathrm{cod}(\mathbb{C}^{n,m,2}, \mathrm{GL}_{n,m}\mathcal{P}) = d^2 - D^2 + N(N-n) + M(M-m). \tag{18}$$

7. Degenerations of matrix pencil bundles

For any partition $\mathbf{n}$, put $J(\mu,\mathbf{n}) = \oplus J_{n_i}(\mu)$, $\mu \in \mathbb{C}$ and $\mathcal{J}(\mu,\mathbf{n}) = \oplus \mathcal{D}_{n_i}(\mu)$, $\mu \in \overline{\mathbb{C}}$; $J(\mu,\mathbf{n})$ is a matrix, $\mathcal{J}(\mu,\mathbf{n})$ is a matrix pencil.

Lemma 7.1. *Let A be an upper triangular block matrix with blocks A_1, $A_2,\dots,A_m$ on the diagonal. If A_i is conjugate to $J(\mu,\mathbf{n}_i)$ for all i then A belongs to the closure of $\mathrm{SL}_n(\mathbb{C})$-orbit of $J(\mu,\sum \mathbf{n}_i)$. If A_i is conjugate to $J(\mu_i,\mathbf{n}_i)$ for all i and $\mu_i \neq \mu_j$ for $i \neq j$ then A is conjugate to $\oplus J(\mu_i,\mathbf{n}_\mathbf{i})$.*

Proof. The first statement is found in [9]. The proof of the second statement is trivial. ∎

Corollary 7.2. *Let $\mathcal{P}$ be an upper triangular block matrix pencil with blocks $\mathcal{P}_i$ on the diagonal. If $\mathcal{P}_i$ is equivalent to $\mathcal{J}(\mu,\mathbf{n}_i)$ for all i then $\mathcal{P}$ belongs to the closure of $\mathrm{GL}_{n,m}$-orbit of $\mathcal{J}(\mu,\sum \mathbf{n}_i)$. If $\mathcal{P}_i$ is equivalent to $\mathcal{J}(\mu_i,\mathbf{n}_i)$ for all i and $\mu_i \neq \mu_j$ for $i \neq j$ then $\mathcal{P}$ and $\oplus\mathcal{J}(\mu_i,\mathbf{n}_\mathbf{i})$ are equivalent.*

Lemma 7.3. *Let $\gamma_1(\varepsilon),\dots,\gamma_s(\varepsilon)$ be a set of complex numbers that are distinct for all $\varepsilon \in (0,\varepsilon_0]$ and $\gamma_s(\varepsilon) \to 0$ when $\varepsilon \to 0$ for all i. There exist matrices C_ε such that $C_\varepsilon\big(J_{k_1}(\gamma_1(\varepsilon)) \oplus \dots \oplus J_{k_s}(\gamma_s(\varepsilon))\big)C_\varepsilon^{-1} \to J_{k_1+\dots+k_s}(0)$ when $\varepsilon \to 0$.*

Proof. Find $a_1(\varepsilon),\dots,a_n(\varepsilon)$ from the equation $\lambda^n - a_1(\varepsilon)\lambda^{n-1} - a_2(\varepsilon)\lambda^{n-2} - \dots - a_n(\varepsilon) = (\lambda-\gamma_1(\varepsilon))^{k_1}\dots(\lambda-\gamma_s(\varepsilon))^{k_s}$ and define

$$B_\varepsilon = \begin{pmatrix} a_1(\varepsilon) & 1 & & \\ a_2(\varepsilon) & 0 & \dots & \\ \dots & & \dots & 1 \\ a_n(\varepsilon) & & & 0 \end{pmatrix}.$$

Clearly, $\det(\lambda E - B_\varepsilon) = (\lambda-\gamma_1(\varepsilon))^{k_1}\dots(\lambda-\gamma_s(\varepsilon))^{k_s}$ and the rank of $\lambda E - B_\varepsilon$ is greater then or equal to $n-1$ for all $\lambda \in \mathbb{C}$. Therefore, the Jordan normal form of B_ε consists of blocks $J_{k_1}(\gamma_1(\varepsilon)),\dots,J_{k_s}(\gamma_s(\varepsilon))$, i.e. $B_\varepsilon = C_\varepsilon^{-1}J_{k_1}(\gamma_1(\varepsilon)) \oplus \dots \oplus J_{k_s}(\gamma_s(\varepsilon))C_\varepsilon$ for some $C_\varepsilon \in \mathrm{SL}_n(\mathbb{C})$. To complete the proof, it remains to note that all $a_i(\varepsilon)$ tend to 0 when $\varepsilon \to 0$. ∎

Lemma 7.4. *If $\mathcal{P}$ is an upper triangular block matrix pencil with blocks $\mathcal{P}^l$, $\mathcal{P}^{reg}$, $\mathcal{P}^r$ in the diagonal (in this order!), $\mathcal{P}^{reg}$ is a regular matrix pencil, $\mathcal{P}^l$ contains only the blocks $\mathcal{L}_i$, and $\mathcal{P}^r$ contains only the blocks $\mathcal{R}_j$ then $\mathcal{P}$ is equivalent to $\mathcal{P}^l \oplus \mathcal{P}^{reg} \oplus \mathcal{P}^r$.*

Proof. For simplicity, assume that $\mathcal{P}$ is

$$\begin{pmatrix} \mathcal{P}^l & \mathcal{A} \\ 0 & \mathcal{P}^{reg} \end{pmatrix}.$$

To prove the equivalence of $\mathcal{P}$ and $\mathcal{P}^l \oplus \mathcal{P}^{reg}$, it is enough to find matrices X and Y such that $\mathcal{P}^l X + Y\mathcal{P}^{reg} + \mathcal{A} = 0$. We prove this statement for $\mathcal{P}^l = \mathcal{L}_i$ and $\mathcal{P}^{reg} = \mathcal{D}_j(\mu)$. Consider the linear operator that takes each pair of matrices (X,Y) to the matrix pencil $\mathcal{L}_i X + Y\mathcal{D}_j(\mu)$. This operator acts on $(2ij+j)$-dimensional complex vector space. According to (17), the dimension of its kernel is j. Therefore the dimension of its image is $2ij$, and, therefore, the desired matrices X and Y exist for any $i \times j$ matrix pencil $\mathcal{A}$. ∎

Theorem 7.5. *The closure of* $\mathrm{B}(\mathcal{P})$ *is the union of closures of its orbits and closures of their amalgams. That is,*

$$\overline{\mathrm{B}(\mathcal{P})} = \bigcup_{\psi} \overline{\mathrm{B}}(\psi(\mathcal{P})),$$

where ψ *runs over the set of all mappings of* $\overline{\mathbb{C}}$ *into itself.*

Proof. By lemma 7.3, $\mathrm{B}(\psi(\mathcal{P})) \subset \overline{\mathrm{B}(\mathcal{P})}$. Therefore, $\overline{\mathrm{B}}(\psi(\mathcal{P})) \subset \overline{\mathrm{B}(\mathcal{P})}$ for any amalgam ψ. Now we need to prove that if $\mathcal{P}_n \in \mathrm{B}(\mathcal{P})$ is a sequence of matrix pencils that converges to $\mathcal{P}_*$ then $\mathcal{P}_*$ belongs to $\overline{\mathrm{B}}(\psi(\mathcal{P}))$ for some ψ. Assume that $\mathcal{P}_1 = \mathcal{P}$ and denote the eigenvalues of $\mathcal{P}_n$ by $\mu_{n1}, \dots, \mu_{ns}$. Since $\overline{\mathbb{C}}$ is compact, we may assume that μ_{ni} converges to some μ_i for all $i = 1 \dots s$. Consider the mapping ψ (defined on the set of eigenvalues of $\mathcal{P}_1$) that takes each μ_{1i} to μ_i. It follows from Ivasawa decomposition in the groups $\mathrm{GL}_n(\mathbb{C})$ and $\mathrm{GL}_m(\mathbb{C})$ that there exist unitary matrices P_n and Q_n such that

$$P_n \mathcal{P}_n Q_n^{-1} = \begin{pmatrix} \mathcal{P}_n^l & * & * \\ 0 & \mathcal{P}_n^{reg} & * \\ 0 & 0 & \mathcal{P}_n^r \end{pmatrix}, \tag{21}$$

where $\mathcal{P}_n^{reg}$ is an upper triangular matrix with numbers $a_{in} + \lambda b_{in}$ in the diagonal, $b_{in}/a_{in} = \mu_{in}$, where μ_{in} that are equal to each other run in succesion, $\mathcal{P}^l$ contains only the blocks $\mathcal{L}_i$, $\mathcal{P}^r$ contains only the blocks $\mathcal{R}_j$, and $\mathcal{P}_n^l \oplus \mathcal{P}_n^{reg} \oplus \mathcal{P}_n^r$ is equivalent to $\mathcal{P}_n$. Decomposition (21) is called the *generalized Shur form* of a matrix pencil [24].

Since P_n and Q_n are unitary, we may assume that their limits exist. The numbers a_{in} and b_{in} are uniformely bounded by i, as $\mathcal{P}_n$ is converging. Therefore, we may assume that $a_{in} \to a_i$ and $b_{in} \to b_i$. Put $\alpha_{in} = a_{in} - a_i$, $\beta_{in} = b_{in} - b_i$ and consider the matrix pencil

$$\mathcal{P}'_n = \begin{pmatrix} \mathcal{P}_n^l & * & * \\ 0 & \mathcal{P}_n^{reg} - \mathcal{Q}_n & * \\ 0 & 0 & \mathcal{P}_n^r \end{pmatrix},$$

where $\mathcal{Q}_n$ is a diagonal matrix pencil with $\alpha_{in} + \lambda\beta_{in}$ on the diagonal. By lemma 7.4 $\mathcal{P}'_n$ is equivalent to $\mathcal{P}_n^l \oplus (\mathcal{P}_n^{reg} - \mathcal{Q}_n) \oplus \mathcal{P}_n^r$. By corollary 7.2, we have $\mathcal{P}'_n \in \overline{\mathcal{O}(\psi(\mathcal{P}))}$ for all n and, since the limit of $\mathcal{Q}_n$ is 0, $(\lim P_n)\mathcal{P}_*(\lim Q_n)^{-1} \in \overline{\mathcal{O}(\psi(\mathcal{P}))}$. Therefore, $\mathcal{P}_* \in \overline{\mathrm{B}}(\psi(\mathcal{P}))$. The proof is now completed. ∎

Remark 7.6. The sequences a_{in} and b_{in} are uniformely bounded by i since $\mathcal{P}_n$ converges. Therefore, a_i and b_i are both finite. What if a_i and b_i are both equal to zero? Then $\mathcal{P}_n^{reg} - \mathcal{Q}_n$ would not be a regular matrix pencil and the premise of lemma 7.4 would not hold. In this case we multiply $\mathcal{P}_n$ by a diagonal matrix $T_n = \mathrm{diag}(t_{in})$ to prevent a_{in} and b_{in} from tending to zero simultaneously. Then, $P_n \mathcal{P}_n Q_n^{-1} - T_n^{-1}\mathcal{Q}_n \in \overline{\mathrm{B}}(\psi(\mathcal{P}))$, but $T_n^{-1}\mathcal{Q}_n \to 0$ since t_{in} must tend to infinity.

8. Degenerations of orbits of $\mathrm{GL}_{n,m}$ and $\mathrm{GL}_{n,m,2}$

Description of closures of matrix pencil bundles is now reduced to description of closures of $\mathrm{GL}_{n,m}$-orbits. The closures of $\mathrm{GL}_{n,m}$-orbits have been described in [21]. In particular, the matrix pencil $\mathcal{Q}$ belongs to the closure of $\mathrm{GL}_{n,m}$-orbit of the matrix pencil $\mathcal{P}$ if and only if

$$\begin{aligned} \mathfrak{R}(\mathcal{P}) + \mathrm{nrk}(\mathcal{P}) &\geq \mathfrak{R}(\mathcal{Q}) + \mathrm{nrk}(\mathcal{Q}), \\ \mathfrak{L}(\mathcal{P}) + \mathrm{nrk}(\mathcal{P}) &\geq \mathfrak{L}(\mathcal{Q}) + \mathrm{nrk}(\mathcal{Q}), \\ \mathfrak{D}(\mu, \mathcal{P}) + r_0(\mathcal{P}) &\leq \mathfrak{D}(\mu, \mathcal{Q}) + r_0(\mathcal{Q}) \end{aligned}$$

hold for all $\mu \in \overline{\mathbb{C}}$ [21]. The Kronecker canonical form of $\mathcal{Q}$ can be obtained from the Kronecker canonical fom of $\mathcal{P}$ by the following transformations [21]:

Ia. $\mathcal{R}_j \oplus \mathcal{R}_k \to \mathcal{R}_{j-1} \oplus \mathcal{R}_{k+1}\,,\ 1 \leq j \leq k\,,$

Ib. $\mathcal{L}_j \oplus \mathcal{L}_k \to \mathcal{L}_{j-1} \oplus \mathcal{L}_{k+1}\,,\ 1 \leq j \leq k\,,$

IIa. $\mathcal{R}_{j+1} \oplus \mathcal{D}_k(\mu) \to \mathcal{R}_j \oplus \mathcal{D}_{k+1}(\mu)\,,\ j,k \geq 0, \quad \mu \in \overline{\mathbb{C}},$

IIb. $\mathcal{L}_{j+1} \oplus \mathcal{D}_k(\mu) \to \mathcal{L}_j \oplus \mathcal{D}_{k+1}(\mu)\,,\ j,k \geq 0, \quad \mu \in \overline{\mathbb{C}},$

III. $\mathcal{D}_{j-1}(\mu) \oplus \mathcal{D}_{k+1}(\mu) \to \mathcal{D}_j(\mu) \oplus \mathcal{D}_k(\mu)\,,\ 1 \leq j \leq k\,,\ \mu \in \overline{\mathbb{C}},$

IV. $\bigoplus_{i=1}^{s} \mathcal{D}_{k_i}(\mu_i) \to \mathcal{L}_p \oplus \mathcal{R}_q\,,\ \mu_i \neq \mu_j$ for $i \neq j\,,\ \mu_i \in \overline{\mathbb{C}},\ p+q+1 = \sum\limits_{i=1}^{s} k_i\,.$

The transformations I–IV are not minimal, but the list of minimal transformations is easily derived from them [6].

In section 2. we have shown that $\mathrm{GL}_{n,m,2}$-orbit of a matrix pencil $\mathcal{P}$ is a union of $\mathrm{GL}_{n,m}$-orbits $\varphi(\mathcal{P})$ over all linear fractional transformations of $\overline{\mathbb{C}}$. The following lemma describes the limits of sequences of linear fractional transformations.

Lemma 8.1. *For any sequence of linear fractional transformations of $\overline{\mathbb{C}}$ there exist a subsequence that converges in pointwise topology. The limit of a sequence of linear fractional transformations is either a linear fractional transformation or a mapping that is constant on $\overline{\mathbb{C}}$ except for, may be, one point.*

Proof. Let $\varphi_{R_k}(z) = (r^k_{21} + r^k_{22}z)/(r^k_{11} + r^k_{12}z)$ be a sequence of linear fractional transformations. Since r^k_{ij} are defined up to proportionality, we may assume that r^k_{ij} converges to some r_{ij} for all i and j and r_{ij} are not all equal to zero. Passing to $\varphi_{R_k}(1/z)$ or $(\varphi_{R_k}(z))^{-1}$, we may assume that $r_{12} \neq 0$. If matrix (r_{ij}) is degenerate, then the limits of $\varphi_{R_k}(z)$ are equal to each other for all $z \in \overline{\mathbb{C}}$ except for, may be, $z = -r_{11}/r_{12}$. If matrix (r_{ij}) is non-degenerate, then the limit is a linear fractional transformation. ∎

Now let us go back to the definition of matrix pencil bundles. If in (13) we allowed φ to be only a linear fractional transformation then the bundles would be exactly $\mathrm{GL}_{n,m,2}$-orbits. This observation allows us to use the proof of theorem 7.5 for the following theorem.

Theorem 8.2.

$$\overline{\mathrm{B}(\mathcal{P})} = \overline{\mathrm{B}}(\mathcal{P}) \cup \left(\bigcup_{\mu \in \mathfrak{S}(\mathcal{P})} \overline{\mathrm{B}}(\gamma_\mu(\mathcal{P})) \right) \cup \overline{\mathrm{B}}(\gamma(\mathcal{P})),$$

where $\mathrm{B}(\mathcal{P}) = \mathrm{GL}_{n,m,2}\mathcal{P}$ *and* $\overline{\mathrm{B}}(\mathcal{P}) = \mathrm{GL}_{n,m,2}(\overline{\mathrm{GL}_{n,m}\mathcal{P}})$.

Proof. Let $\mu_1, \ldots, \mu_s$ be the set of eigenvalues of $\mathcal{P}$. It follows from lemma 7.3 that $\mathrm{B}(\gamma(\mathcal{P})) \subset \overline{\mathrm{B}(\mathcal{P})}$ since $\mu_1, \ldots, \mu_s$ are mapped to $\varepsilon\mu_1, \ldots, \varepsilon\mu_s$ by the linear fractional transformation $z \mapsto \varepsilon z$. Similarly, $\mathrm{B}(\gamma_\mu(\mathcal{P})) \subset \overline{\mathrm{B}(\mathcal{P})}$. The proof of the converse is similar to one of theorem 7.5; the difference is that if two eigenvalues coalesce then all eigenvalues do except for, may be, one (lemma 8.1). ∎

Thus, the regular parts of $\mathrm{GL}_{n,m,2}$-orbits either coalesce all together, or coalese all but one, or do not coalese at all.

9. Degenerations of orbits of $\mathrm{SL}_{n,m}$ and $\mathrm{SL}_{n,m,2}$.

We have shown that $\mathrm{SL}_{n,m}$-orbits of perfect matrix pencils are closed, while $\mathrm{SL}_{n,m}$-orbits of imperfect singular matrix pencils coincide with their $\mathrm{GL}_{n,m}$-orbits (theorem 3.3). We now have to describe degenerations of $\mathrm{SL}_{n,m}$-orbits of regular matrix pencils. It is enough to determine which of the transformations I–IV (section 8.) can be executed under the action of $\mathrm{SL}_{n,m}$. First of all, transformations I–II are not applicable to regular matrix pencils. The transformation IV is also not applicable, since the coefficients of the binary form $\theta(\mathcal{P})$ are invariant and therefore the closure of $\mathrm{SL}_{n,m}$-orbit of a regular matrix pencil consists of regular matrix pencils only. Transformation III, indeed, is applicable to regular matrix pencils; it corresponds to regrouping of Jordan blocks under the adjoint representation of $\mathrm{SL}_n(\mathbb{C})$.

Now we focus on $\mathrm{SL}_{n,m,2}$-orbits. As before, we may confine ourselves to $\mathrm{SL}_{n,m,2}$-orbits of regular matrix pencils because $\mathrm{SL}_{n,m}$-orbits and $\mathrm{GL}_{n,m}$-orbits of imperfect singular matrix pencils coincide (theorem 3.4). Recall that $\mathrm{SL}_{n,m,2}$-orbit of a regular matrix pencil is defined uniquely by the partitions $\mathfrak{D}(\mu_j, \mathcal{P})$ and the set of eigenvalues, which are defined up to a linear fractional transformation that satisfies (5). The condition (5) is equivalent to

$$\prod_{i=1}^{n}(r_{11} + r_{12}\mu_i) = (\det R)^{n/2} \tag{23}$$

for a regular matrix pencil $\mathcal{P} = E + \lambda D$, where D is an upper triangular matrix with $\mu_1, \ldots, \mu_n$ on the diagonal. Let $\varphi_R(z)$ be a linear fractional transformation that satisfies (23). Since

$$\varphi_R(z) - \varphi_R(w) = \frac{\det R\,(z-w)}{(r_{11} + r_{12}z)(r_{11} + r_{12}w)} \tag{24}$$

holds for any z and w, we have

$$W(\varphi_R(\mu_1), \ldots, \varphi_R(\mu_n)) = \frac{(\det R)^{\frac{n(n-1)}{2}} W(\mu_1, \ldots, \mu_n)}{\left(\prod(r_{11} + r_{12}\mu_i)\right)^{n-1}}, \tag{25}$$

where $W(z_1, \ldots, z_n) = \prod_{i>j}(z_i - z_j)$ is the Wandermond determinant. It follows from (23) that

$$W(\varphi_R(\mu_1), \ldots, \varphi_R(\mu_n)) = W(\mu_1, \ldots, \mu_n). \tag{26}$$

Theorem 9.1. $\mathrm{SL}_{n,m,2}$*-orbit of a regular* $n \times n$ *matrix pencil that has exactly* n *distinct eigenvalues is closed.*

Proof. Let $g_k = (P_k, Q_k, R_k)$ be a sequence of elements of $\mathrm{SL}_{n,m,2}$ such that $\lim g_k\mathcal{P} = \mathcal{P}^*$. Denote the eigenvalues of $\mathcal{P}$ by $\mu_1^*, \ldots, \mu_n^*$, and let μ_i^* be the limits of $\varphi_{R_k}(\mu_i)$ when $k \to \infty$. We may assume that μ_i and μ_i^* are finite for all $i = 1 \ldots n$. It follows that $W(\mu_1^*, \ldots, \mu_n^*) = W(\mu_1, \ldots, \mu_n)$ and, therefore, $\mu_1^*, \ldots, \mu_n^*$ are distinct and the limit of φ_{R_k} is a linear fractional transformation. Since $\mathrm{SL}_{n,m}$-orbit of $\mathcal{P}$ is closed, it follows that $\mathcal{P}$ and $\mathcal{P}^*$ are $\mathrm{SL}_{n,m,2}$-equivalent. ∎

This theorem has a simplier proof. The orbit of a regular matrix pencil whose eigenvalues are distinct is a pre-image of the orbit of a binary form that doesn't have multiple factors. Such orbits are known to be closed. However, this proof is applicable only when the eigenvalues are distinct, but there are other closed orbits. In the proof of theorem 9.1 we essentially used the fact that $W(\mu_1, \ldots, \mu_n)$ is non-zero. Now we introduce another approach that works for matrix pencils with multiple eigenvalues.

Lemma 9.2. *Let* $\mathbf{n} = (n_1, \ldots, n_s)$ *be a partition of an even integer* n. *The integer* n_1 *is smaller than or equal to the half of* n *if and only if there exist a non-oriented graph whose vertices are* $n_1, \ldots, n_s$ *and the number of edges coming from* n_i *is equal to* n_i *for all* i.

Proof. Let us translate this statement to the language of Ferrer diagrams. The Ferrer diagram of $\mathbf{n}$ can be built step by step from an empty diagram by picking any two columns and adding one new cell to each of them. Equivalently, the Ferrer diagram of $\mathbf{n}$ can be sequentially disassembled by picking any two columns on each step and deleting one cell in each of them. Clearly, if each column gets one cell at one time then none of the columns gets more than a half of the total number of cells. This proves the necessary part. The proof of the converse is by induction over n. Consider three cases:

1. $n_1 > n_2$. Delete the upper cells in the first column and (some) other column without loss of monotonicity of $\mathbf{n}$. The partition obtained has the same property: the number of cells in its first column is smaller than or equal to the total number of cells.

2. $n_1 = n_2 > n_3$. Delete the upper cells in the first two columns. Then the number of cells in the first column is again smaller than or equal to the total number of cells.

3. $n_1 = n_2 = n_3 = n_s > n_{s+1}$. In this case $n_1 \leq n/3$. Therefore, $n_1 < n/2 - 1$ for $n > 6$. Now delete the upper cells in the $(s-1)$-th and the s-th columns. The number of cells in the first column of the partition obtained is smaller than or equal to the total number of cells. The proof is trivial for $n \leq 6$.

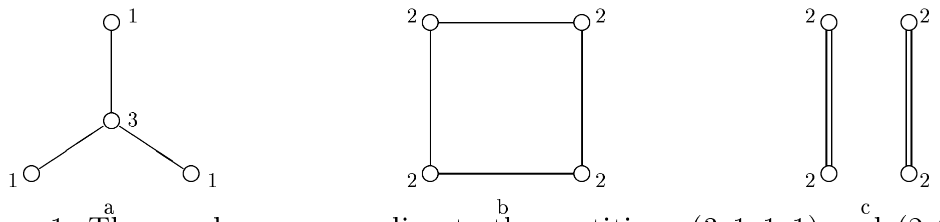

Figure 1: The graphs corresponding to the partitions $(3,1,1,1)$ and $(2,2,2,2)$.

■

The non-oriented graph whose existence is guaranteed by previous lemma is not always well-defined. Figure 1 illustrates this for the partitions $(3,1,1,1)$ (a) and $(2,2,2,2)$ (b and c). A non-oriented graph is said to be a *star*, if all its edges share a common node. Obviously, the graph built in lemma 9.2 is a star if and only if $n_1 = n/2$ and in this case it is defined unambiguously.

In what follows, only the main and the submain amalgams of $\mathrm{SL}_{n,m,2}$-orbits are considered. The number of distinct eigenvalues of these amalgams is smaller than or equal to two. Therefore, $\mathrm{SL}_{n,m,2}$-orbits of $\gamma_\mu(\mathcal{P})$ and $\gamma(\mathcal{P})$ are well-defined, since every pair of complex numbers can be taken to another pair by a linear fractional transformation that satisfies (5).

We have shown that the regular matrix pencil $\mathcal{P}$ is nilpotent if and only if the binary form $\theta(\mathcal{P})$ is nilpotent. It is known that the binary form is nilpotent if and only if it has a linear factor whose multiplicity is greater than the half of the degree of the form. Hence, the regular matrix pencil $\mathcal{P}$ is nilpotent if and only if it has an eigenvalue whose multiplicity is greater than the half of the order of the matrix pencil. The following theorem describes closures of non-nilpotent $\mathrm{SL}_{n,m,2}$-orbits.

Theorem 9.3. *Put* $\mathrm{B}(\mathcal{P}) = \mathrm{SL}_{n,m,2}\mathcal{P}$ *and* $\overline{\mathrm{B}}(\mathcal{P}) = \mathrm{SL}_{n,m,2}(\overline{\mathrm{SL}_{n,m}\mathcal{P}})$. *If the multiplicities of all eigenvalues of a regular matrix pencil* $\mathcal{P}$ *are strictly smaller than the half of its order then* $\overline{\mathrm{B}(\mathcal{P})} = \overline{\mathrm{B}}(\mathcal{P})$. *If the multiplicitiy of the eigenvalue* μ *is equal to the half of the order of* $\mathcal{P}$ *then* $\overline{\mathrm{B}(\mathcal{P})} = \overline{\mathrm{B}}(\mathcal{P}) \cup \overline{\mathrm{B}}(\gamma_\mu(\mathcal{P}))$.

Proof. Recall the proof of theorem 9.1. Consider the sequence $\mathcal{P}_k$ of elements of $\mathrm{SL}_{n,m,2}$-orbit of $\mathcal{P}$ that converges to $\mathcal{P}^*$ and denote by μ_i^* the limit of $\varphi_{R_k}(\mu_i)$. Since $\mathcal{P}$ is not nilpotent, the sequence of binary forms $\theta(\mathcal{P}_k)$ converges to a non-zero limit. Therefore, μ_i^* are the eigenvalues of $\mathcal{P}^*$. Assume that $\mu_1, \ldots, \mu_s$ are distinct eigenvalues of $\mathcal{P}$. Their multiplicities form a partition, which we denote by $\mathbf{n} = (n_1, n_2, \ldots)$.

(i) Case $n_1 < n/2$. If n is an odd number then multiply $\mathbf{n}$ by 2. Denote by E the set of edges of the graph Γ that was constructed in lemma 9.2. We now follow the proof of theorem 9.1 with one exception: instead of W we consider

$$\widehat{W}(z_1, \ldots, z_s) = \prod_{[i,j]\in E} (z_i - z_j),$$

where $[i,j]$ denotes the edge connecting the i-th and the j-th nodes, $i < j$. It follows from (24) that

$$\widehat{W}(\varphi_R(\mu_1), \ldots, \varphi_R(\mu_s)) = \frac{(\det R)^n \widehat{W}(\mu_1, \ldots, \mu_s)}{\left(\prod (r_{11} + r_{12}\mu_i)^{n_i}\right)^2}.$$

For any fractioal transformation $\varphi_R(z)$ that satisfies (23) we have $\widehat{W}(\varphi_R(\mu_1),\ldots,\varphi_R(\mu_s)) = \widehat{W}(\mu_1,\ldots,\mu_s)$. Therefore, $\widehat{W}(\mu_1^*,\ldots,\mu_s^*) = \widehat{W}(\mu_1,\ldots,\mu_s)$. Suppose that the limit of φ_{R_k} is not a linear fractional transformation. Then all μ_i^* are equal to each other but may be one. Note that Γ is not a star when $n_1 < n/2$. If we delete one node along with all its edges then there will remain at least one edge. Since this edge corresponds to a factor in $\widehat{W}(\mu_1^*,\ldots,\mu_s^*)$, we have $\widehat{W}(\mu_1^*,\ldots,\mu_s^*) = 0$, and therefore $\widehat{W}(\mu_1,\ldots,\mu_s) = 0$. This contradicts to the assumption that $\mu_1,\ldots,\mu_s$ are distinct. Thus, the limit of φ_{R_k} is a linear fractional transformation, and $\mathcal{P}^*$ is $\mathrm{SL}_{n,m,2}$-equivalent to a matrix pencil from $\overline{\mathrm{SL}_{n,m}\mathcal{P}}$.

(ii) Case $n_1 = n/2$. Using the same arguments one can prove that the eigenvalue μ whose multiplicity is $n/2$ cannot coalesce with the others. It follows that $\overline{\mathrm{B}(\mathcal{P})} \subset \overline{\mathrm{B}}(\mathcal{P}) \cup \overline{\mathrm{B}}(\gamma_\mu(\mathcal{P}))$. It now remains to prove that, indeed, $\overline{\mathrm{B}}(\gamma_\mu(\mathcal{P})) \subset \overline{\mathrm{B}(\mathcal{P})}$. It is convenient to change the notation now. Let μ be equal to 0, and denote the other eigenvalues by $\mu_1,\ldots,\mu_s$. Consider the linear fractional transformation

$$\varphi_k(z) = \frac{b_k c_k z}{1 + c_k z}. \tag{27}$$

Then (23) is equivalent to

$$b_k^{n/2} = \frac{(1 + c_k\mu_1)\ldots(1 + c_k\mu_s)}{c_k^{n/2}}. \tag{28}$$

The degree of the denominator is equal to the degree of the numerator ($s = n/2$). If $c_k \to \infty$ when $k \to \infty$, then $b_k \to (\mu_1 \ldots \mu_s)^{2/n}$ and $\varphi_k(z) \to (\mu_1 \ldots \mu_s)^{2/n} \neq 0$ for any $z \in \mathbb{C}\backslash\{0\}$, and $\varphi(0) = 0$. The proof is now completed. ∎

Theorem 9.4. *Let* $\mathrm{B}(\mathcal{P})$ *and* $\overline{\mathrm{B}}(\mathcal{P})$ *be as in theorem 9.3, and let* $\mathbf{R}$ *and* $\mathbf{S}$ *denote the sets of regular and singular* $n \times n$ *matrix pencils, respectively, and consider a nilpotent regular matrix pencil* $\mathcal{P}$*. Then* $\overline{\mathrm{B}(\mathcal{P})} \cap \mathbf{R} = \overline{\mathrm{B}}(\mathcal{P}) \cup \overline{\mathrm{B}}(\gamma(\mathcal{P}))$ *and* $\overline{\mathrm{B}(\mathcal{P})} \cap \mathbf{S} = \overline{\mathrm{GL}_{n,m}(\gamma(\mathcal{P}))} \cap \mathbf{S}$.

Proof. First we prove that if some of the eigenvalues coalesce then they all do. Suppose that the eigenvalue that has the greatest multiplicity is equal to zero, and denote the other eigenvalues by $\mu_1,\ldots,\mu_s$. We may assume that $\varphi_{R_k}(0) = 0$ for all k. Then φ_{R_k} has form (27). The limit of φ_{R_k} is not a linear fractional transformation if and only if the limit of $\det(R_k)$ is equal to zero or is infinite. If it is equal to zero then all μ_i^* are equal to zero. If it is infinite then it follows from (28) that $c_k \to \infty$ and $b_k \to 0$, since the degree of the nominator is smaller than the degree of the denominator ($s < n/2$). Thus, $\varphi_k(z) \to 0$ for all $z \in \mathbb{C}$.

If the limit of φ_{R_k} is a regular matrix pencil that doesn't belong to $\overline{\mathrm{B}}(\mathcal{P})$ then its eigenvalues are all equal to each other. That is, $\overline{\mathrm{B}(\mathcal{P})} \cap \mathbf{R} \subset \overline{\mathrm{B}}(\gamma(\mathcal{P}))$. It is clear that $\overline{\mathrm{B}}(\gamma(\mathcal{P})) \subset \overline{\mathrm{B}(\mathcal{P})} \cap \mathbf{R}$. If the limit of φ_{R_k} is a singular matrix pencil then there exist unitary transformations P_n and Q_n that take $\mathcal{P}_n$ to the generalized Shur form. As in theorem 7.5 we can substract a matrix pencil $\mathcal{Q}_n$ from $P_n\mathcal{P}_nQ_n^{-1}$ to make the difference belong to $\overline{\mathrm{GL}_{n,m}(\gamma(\mathcal{P}))}$. Now it remains to prove that $\overline{\mathrm{GL}_{n,m}(\gamma(\mathcal{P}))} \cap \mathbf{S} \subset \overline{\mathrm{B}(\mathcal{P})}$. It is enough to prove that $\overline{\mathrm{GL}_{n,m}(\gamma(\mathcal{P}))} \cap \mathbf{S} \subset$

$\overline{\mathrm{B}(\gamma(\mathcal{P}))}$. The Kronecker canonical form of a singular matrix pencil that is covered by $\mathrm{GL}_{n,m}$-orbit of a regular matrix pencil can be obtained using transformation IV (section 8.). The matrix pencils

$$\begin{pmatrix} 1 & \lambda & & & & \\ & \ddots & \ddots & & & \\ & & 1 & \lambda & & \\ & & & \ddots & \ddots & \\ & & & & 1 & \lambda \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & \lambda & & & & \\ & \ddots & \ddots & & & \\ & & \varepsilon & \lambda & & \\ & & & \ddots & \ddots & \\ & & & & 1 & \lambda \end{pmatrix}$$

are $\mathrm{SL}_{n,m,2}$-equivalent for all $\varepsilon \neq 0$ (the transformation consists in multiplication of the first matrix by $\operatorname{diag}(\varepsilon, \varepsilon, \ldots, \varepsilon^{1-n}, \ldots, \varepsilon)$ and consequent application of the action of $\operatorname{diag}(\varepsilon^{-1}, \varepsilon)$). Thus, the degeneration $\mathcal{D}_n(0) \to \mathcal{L}_p \oplus \mathcal{R}_q$ can be performed by the action of $\mathrm{SL}_{n,m,2}$. ■

10. Minimal degenerations

So far we have obtained the criteria for a matrix pencil to belong to the closure of an orbit (or bundle) of another matrix pencil. In order to describe the hierarchy of closures, we need to come up with a description of the covering relationship for all cases discussed in sections 7.–9.. For $\mathrm{GL}_{n,m}$-orbits and matrix pencil bundles it has been done in [6].

The orbit of $\mathcal{P}$ covers the orbit of $\mathcal{Q}$ if and only if the partitions $\mathfrak{D}(\mu, \mathcal{Q})$, $\mathfrak{R}(\mathcal{Q})$, and $\mathfrak{L}(\mathcal{Q})$ are obtained from $\mathfrak{D}(\mu, \mathcal{P})$, $\mathfrak{R}(\mathcal{P})$, and $\mathfrak{L}(\mathcal{P})$, respectively, by one of the following transformations:

A1. Minimal lowering of $\mathfrak{L}(\mathcal{P})$ or $\mathfrak{R}(\mathcal{P})$ that doesn't affect the leftmost column.

A2. Deletion of the rightmost standalone cell of $\mathfrak{R}(\mathcal{P})$ or $\mathfrak{L}(\mathcal{P})$ and simultaneous addition of a new cell to $\mathfrak{D}(\mu, \mathcal{P})$ for some $\mu \in \overline{\mathbb{C}}$.

A3. Minimal hightening of $\mathfrak{D}(\mathcal{P})$.

A4. Deletion of all cells in the lowest raw of each $\mathfrak{D}(\mu, \mathcal{P})$ and simultaneous addition of one new cell to each $\mathfrak{l}_p(\mathcal{P})$, $p = 0, \ldots, t$ and $\mathfrak{r}_q(\mathcal{P})$, $q = 0, \ldots, k-t-1$, where k is the number of deleted cells, such that every non-zero column is given at least one cell.

The rule A1 corresponds to the transformations Ia and Ib, the rule A2 corresponds to the transformations IIa and IIb, the rule A3 corresponds to the transformation III and the rule A4 corresponds to the transformation IV. A good set of figures illustrating the respective transformations of the Ferrer diagrams is found in [6].

The bundle of $\mathcal{P}$ covers the bundle of $\mathcal{Q}$ if and only if the partitions $\mathfrak{D}(\mu, \mathcal{Q})$, $\mathfrak{R}(\mathcal{Q})$, and $\mathfrak{L}(\mathcal{Q})$ are obtained from $\mathfrak{D}(\mu, \mathcal{P})$, $\mathfrak{R}(\mathcal{P})$, and $\mathfrak{L}(\mathcal{P})$, respectively, by the rules A1, B2, A3, B4 and B5:

B2. Same as rule A2, but start with a new set of cells for a new eigenvalue (otherwise it could have been done using A2 and then B5).

B4. Same as rule A4, but appply only if there is just one eigenvalue or if all eigenvalues have at least 2 Jordan blocks (otherwise it could have been done using B5 and then A4).

B5. Union of any two $\mathfrak{D}(\mu, \mathcal{P})$.

Here we list the similar set of rules for minimal degenerations of $\mathrm{SL}_{n,m,2}$ and $\mathrm{GL}_{n,m,2}$-orbits. No proof is necessary as they are obtained from ones above in a very straightforward way.

The $\mathrm{GL}_{n,m,2}$-orbit of $\mathcal{P}$ covers the $\mathrm{GL}_{n,m,2}$-orbit of $\mathcal{Q}$ if and only if $\mathfrak{D}(\mu, \mathcal{Q})$, $\mathfrak{R}(\mathcal{Q})$, and $\mathfrak{L}(\mathcal{Q})$ are obtained from $\mathfrak{D}(\mu, \mathcal{P})$, $\mathfrak{R}(\mathcal{P})$, and $\mathfrak{L}(\mathcal{P})$, respectively, by the rules A1, C2, A3, C4, and C5:

C2. Same as rule A2, but start a new set of cells for a new eigenvalue if the matrix pencil has less than three eigenvalues.

C4. Same as rule A4, but apply only if there is just one eigenvalue or if all eigenvalues have at least 2 Jordan blocks.

C5. Union of all $\mathfrak{D}(\mu, \mathcal{P})$ but one.

The $\mathrm{SL}_{n,m,2}$-orbit of $\mathcal{P}$ covers the $\mathrm{SL}_{n,m,2}$-orbit of $\mathcal{Q}$ if and only if $\mathfrak{D}(\mu, \mathcal{Q})$, $\mathfrak{R}(\mathcal{Q})$, and $\mathfrak{L}(\mathcal{Q})$ are obtained from $\mathfrak{D}(\mu, \mathcal{P})$, $\mathfrak{R}(\mathcal{P})$, and $\mathfrak{L}(\mathcal{P})$, respectively, by the following rules:

1. rules A1, C2, A3, C4 and C5 , if the matrix pencils is imperfect and singular;

2. rules A3, D4 and D5

   D4. Rule A4 and consequent union of all $\mathfrak{D}(\mu, \mathcal{P})$.

   D5. Union of all $\mathfrak{D}(\mu, \mathcal{P})$,

   if the matrix pencils is nilpotent and regular;

3. rules A3 and D6

   D6. Union of all $\mathfrak{D}(\mu, \mathcal{P})$ except for ones, whose eigenvalue's multiplicity is $n/2$.

   if the matrix pencil is regular and has an eigenvalue with mutiplicity $n/2$;

4. rule A3 for all other regular matrix pencils.

11. Null cone

Recall that the null cone is the union of all nilpotent orbits. The main focus of this paragraph is the null cone of $\mathrm{SL}_{n,m,2}$.

Theorem 11.1. *The null cone of $\mathrm{SL}_{n,m,2}$ is irreducible and contains an open orbit for $n \neq m$.*

Proof. Suppose $m > n$ and put $d = m - n$. Condiser the matrix pencil $\mathcal{P} = d\mathcal{L}_k \oplus (n \mod d)\mathcal{L}_{k+1}$, where $k = [n/d]$. If d divides n then $\mathcal{P}$ is a perfect matrix pencil and the null cone is the border of its $\mathrm{GL}_{n,m,2}$-orbit. The Ferrer diagram of $\mathfrak{L}(\mathcal{P})$ is a rectangle with k columns and d raws. If $d > 1$ then only the rule A1 is applicable to this diagram. If $d = 1$ then the only applicable rule is C2. The corresponding degenerations are $(d-2)\mathcal{L}_k \oplus \mathcal{L}_{k-1} \oplus \mathcal{L}_{k+1}$ and $(d-1)\mathcal{L}_{k-1} \oplus \mathcal{J}_1$. Therefore, the orbit of $\mathcal{P}$ covers only one $\mathrm{SL}_{n,m,2}$-orbit, since the other $\mathrm{GL}_{n,m,2}$-orbits and $\mathrm{SL}_{n,m,2}$-orbits coincide (theorem 3.4). Then the null cone is irreducible and has an open orbit. If d doesn't divide n then $\mathrm{GL}_{n,m,2}$-orbits and $\mathrm{SL}_{n,m,2}$-orbits coincide and the null cone is $\mathbb{C}^{n,m,2}$. The reader will easily prove that the closure of the orbit of $\mathcal{P}$ contains all other orbits. ■

Theorem 11.2. *The null cone of* $\mathrm{SL}_{n,n,2}$ *is irreducible.*

Proof. A regular matrix pencil is nilpotent if and only if multiplicity of one of its eigenvalues is greater than the half of its order. Then the null cone of $\mathrm{SL}_{n,n,2}$ is the union of the set of the matrix pencil bundles that have an eigenvalue with multiplicity $n/2$ and the set of singular matrix pencils. There is only one bundle (namely, $\mathcal{J}_{[n/2]+1} \oplus (n - [n/2] - 1)\mathcal{J}_1$) whose closure is the entire null cone. It now remains to note that the bundles are irreducible. ■

12. Hierarchy of closures

It turns out that the hierarchy of closures of matrix pencil bundles has an interesting property: the matrix pencil bundles can be subdivided into three classes — *left*, *central* and *right*, which are defined by $\mathfrak{L}(\mathcal{P}) > \mathfrak{R}(\mathcal{P})$, $\mathfrak{L}(\mathcal{P}) = \mathfrak{R}(\mathcal{P})$, and $\mathfrak{L}(\mathcal{P}) < \mathfrak{R}(\mathcal{P})$, respectively, such that if $\mathcal{P}_1$ is left, $\mathcal{P}_2$ is right, and $\overline{\mathrm{B}(\mathcal{P}_1)} \supset \overline{\mathrm{B}(\mathcal{P}_2)}$ then there is a central matrix pencil $\mathcal{Q}$ such that $\overline{\mathrm{B}(\mathcal{P}_1)} \supset \overline{\mathrm{B}(\mathcal{Q})} \supset \overline{\mathrm{B}(\mathcal{P}_2)}$. In other words, matrix pencil bundles from the left and the right classes do not cover each other. Also, $\mathrm{B}(\mathcal{Q})$ covers $\mathrm{B}(\mathcal{P})$ if and only if $\mathrm{B}(\mathcal{Q}^\top)$ covers $\mathrm{B}(\mathcal{P}^\top)$. Thus, we can save some space in figures illustrating the closures' hierarchy by showing only left and central matrix pencils. Sadly, this symmetry breaks for 6×7 matrix pencils (see #223 in table 1). Moreover, in higher dimensions there exist matrix pencils such that $\mathfrak{L}(\mathcal{P})$ and $\mathfrak{R}(\mathcal{P})$ are not comparable at all (for example, $\mathcal{L}_0 \oplus \mathcal{L}_3 \oplus 2\mathcal{R}_1$).

The classification of bundles and $\mathrm{GL}_{n,m,2}$- and $\mathrm{SL}_{n,m,2}$-orbits of matrix pencils is summarized in table 1 with the following conventions. The consecutive indexing for matrix pencils of all sizes is used. Only the left and the central veritable matrix pencils (except for #223) are shown. The right veritable matrix pencils are denoted by $k^\top$, where k is the index of the corresponding transposed left matrix pencil. For example, $7^\top$ stands for $\mathcal{R}_2$. The columns "n" and "m" have obvious meaning. The column "c" contains codimensions of matrix pencil bundles in $\mathbb{C}^{n,m,2}$ (that is in the space where they are veritable). The column "bundle" lists indices of matrix pencil bundles that are covered by the given matrix pencil bundle; k^* means that the given bundle covers both k and $k^\top$. For instance, $\mathcal{R}_1 \oplus \mathcal{L}_1 \oplus \mathcal{J}_1$ (# 25) covers $\mathcal{L}_1 \oplus 2\mathcal{J}_1$ and $\mathcal{R}_1 \oplus 2\mathcal{J}_1$. The column "$\mathrm{GL}_{n,m,2}$-orbit" lists the indices of $\mathrm{GL}_{n,m,2}$-orbits that are covered by the given $\mathrm{GL}_{n,m,2}$-orbit. If a cell of this colums is empty then its content was the same as in "bundle". The last column lists the indices of $\mathrm{SL}_{n,m,2}$-orbits that are covered by $\mathrm{SL}_{n,m,2}$-orbit of $\mathcal{P}$, where $\mathcal{P}$ is a (veritable) $n \times m$ matrix pencil. The indices of $\mathrm{SL}_{n,m,2}$-orbits that are covered by the orbit of of $\mathcal{P}$ under the action of $\mathrm{SL}_{n,m,2}$ for larger n and m are found in the column "$\mathrm{GL}_{n,m,2}$-orbit". For instance, the $\mathrm{SL}_{n,m,2}$-orbit of the 6×7 matrix pencil #228 covers the $\mathrm{SL}_{n,m,2}$-orbit of the 6×6 matrix pencil #168. The symbol of empty set stands for the closed orbits.

The hierarchy of closures of matrix pencil bundles is shown in figures 2–5. The dots correspond to the matrix pencil bundles; if the canonical form is not shown in the figure then it is found in table 1. In all figures except for figure 2 (a) only the left and the central matrix pencils are shown. Two nodes are connected with an edge if and only if the matrix pencil bundle that corresponds to the node drawn above covers the matrix pencil bundle that corresponds to the node drawn below. The matrix pencil bundles, whose corresponding nodes are drawn on the same level, have equal codimensions, which are indicated by framed numbers on the side of the diagram.

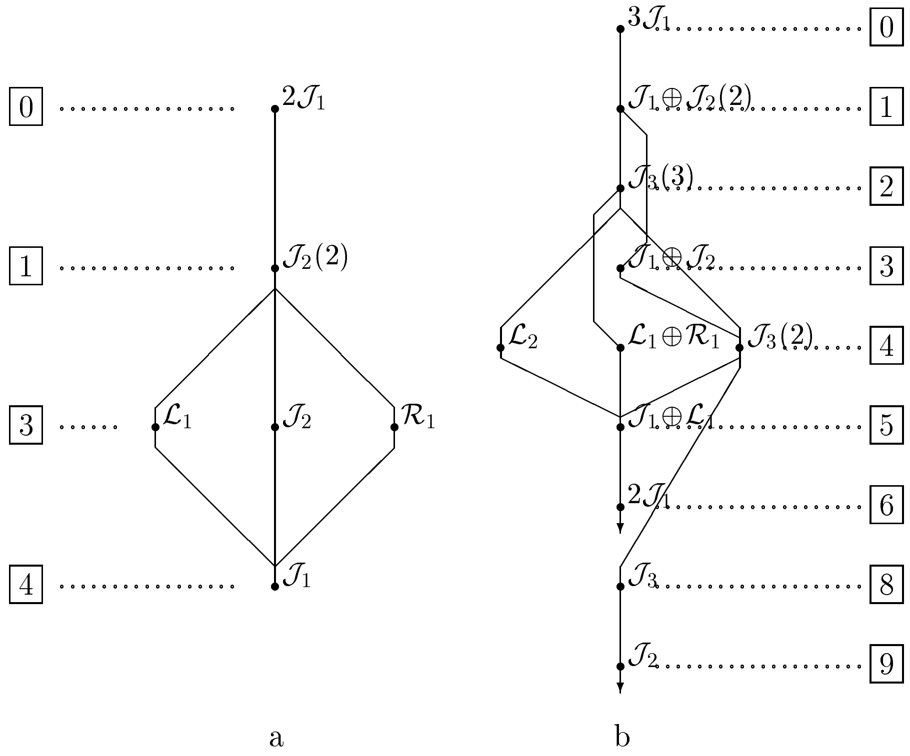

Figure 2: Hierarchy of closures of 2×2 (a) and 3×3 (b) bundles.

#	Canonical form	n	m	c	Bundle	$\mathrm{GL}_{n,m,\mathcal{Z}}$-orbit	$\mathrm{SL}_{n,m,\mathcal{Z}}$-orbit
1	$\mathcal{J}_1$	1	1	0			$\varnothing$
2	$\mathcal{L}_1$	1	2	0	1		$\varnothing$
3	$\mathcal{J}_2$	2	2	3	1		
4	$\mathcal{J}_2(2)$	2	2	1	$3, 2^*$		
5	$2\mathcal{J}_1$	2	2	0	4		$\varnothing$
6	$\mathcal{L}_1{+}\mathcal{J}_1$	2	3	1	5		
7	$\mathcal{L}_2$	2	3	0	6		$\varnothing$
8	$\mathcal{J}_3$	3	3	8	3		
9	$\mathcal{R}_1{+}\mathcal{L}_1$	3	3	4	6^*		
10	$\mathcal{J}_3(2)$	3	3	4	$8, 6^*$		
11	$\mathcal{J}_2{+}\mathcal{J}_1$	3	3	3	10		
12	$\mathcal{J}_3(3)$	3	3	2	$10, 9, 7^*$		
13	$\mathcal{J}_2(2){+}\mathcal{J}_1$	3	3	1	$12, 11$		
14	$3\mathcal{J}_1$	3	3	0	13		$\varnothing$
15	$2\mathcal{L}_1$	2	4	0	7		$\varnothing$
16	$\mathcal{L}_1{+}\mathcal{J}_2$	3	4	5	11		
17	$\mathcal{L}_1{+}\mathcal{J}_2(2)$	3	4	3	$16, 15, 13$		
18	$\mathcal{L}_1{+}2\mathcal{J}_1$	3	4	2	$17, 14$		
19	$\mathcal{L}_2{+}\mathcal{J}_1$	3	4	1	18		
20	$\mathcal{L}_3$	3	4	0	19		$\varnothing$
21	$\mathcal{J}_4$	4	4	15	8		
22	$\mathcal{J}_4(2)$	4	4	9	$21, 16^*$		
23	$\mathcal{J}_3{+}\mathcal{J}_1$	4	4	8	22		
24	$\mathcal{J}_4(2,2)$	4	4	7	$22, 17^*$		
25	$\mathcal{R}_1{+}\mathcal{L}_1{+}\mathcal{J}_1$	4	4	6	18^*		
26	$2\mathcal{J}_2$	4	4	6	$24, 18^*$		$\varnothing$
27	$\mathcal{R}_1{+}\mathcal{L}_2$	4	4	5	$25, 19$		
28	$\mathcal{J}_4(3)$	4	4	5	$25, 24, 19^*$		
29	$\mathcal{J}_3(2){+}\mathcal{J}_1$	4	4	4	$28, 23$		
30	$\mathcal{J}_2(2){+}\mathcal{J}_2$	4	4	4	$28, 26$		26
31	$\mathcal{J}_2{+}2\mathcal{J}_1$	4	4	3	$30, 29$		30
32	$\mathcal{J}_4(4)$	4	4	3	$28, 27^*, 20^*$		
33	$\mathcal{J}_3(3){+}\mathcal{J}_1$	4	4	2	$32, 29$		
34	$2\mathcal{J}_2(2)$	4	4	2	$32, 30$		30
35	$\mathcal{J}_2(2){+}2\mathcal{J}_1$	4	4	1	$34, 33, 31$		$34, 31$
36	$4\mathcal{J}_1$	4	4	0	35	33	$\varnothing$
37	$\mathcal{J}_1{+}2\mathcal{L}_1$	3	5	2	19		
38	$\mathcal{L}_2{+}\mathcal{L}_1$	3	5	0	$37, 20$		
39	$\mathcal{L}_1{+}\mathcal{J}_3$	4	5	11	23		
40	$\mathcal{R}_1{+}2\mathcal{L}_1$	4	5	8	$37, 27$		
41	$\mathcal{L}_1{+}\mathcal{J}_3(2)$	4	5	7	$39, 37, 29$		
42	$\mathcal{L}_1{+}\mathcal{J}_2{+}\mathcal{J}_1$	4	5	6	$41, 31$		
43	$\mathcal{L}_1{+}\mathcal{J}_3(3)$	4	5	5	$41, 40, 38, 33$		
44	$\mathcal{L}_2{+}\mathcal{J}_2$	4	5	5	42		
45	$\mathcal{L}_1{+}\mathcal{J}_2(2){+}\mathcal{J}_1$	4	5	4	$43, 42, 35$		
46	$\mathcal{L}_1{+}3\mathcal{J}_1$	4	5	3	$45, 36$		
47	$\mathcal{L}_2{+}\mathcal{J}_2(2)$	4	5	3	$45, 44$		
48	$\mathcal{L}_2{+}2\mathcal{J}_1$	4	5	2	$47, 46$		
49	$\mathcal{L}_3{+}\mathcal{J}_1$	4	5	1	48		
50	$\mathcal{L}_4$	4	5	0	49		$\varnothing$
51	$\mathcal{J}_5$	5	5	24	21		

Table 1: (continued)

#	Canonical form	n	m	c	Bundle	$GL_{n,m,2}$-orb.	$SL_{n,m,2}$-orb.
52	$\mathcal{J}_5(2)$	5	5	16	$51, 39^*$		
53	$\mathcal{J}_4{+}\mathcal{J}_1$	5	5	15	52		
54	$\mathcal{J}_5(2,2)$	5	5	12	$52, 41^*$		
55	$\mathcal{R}_1{+}\mathcal{L}_1{+}\mathcal{J}_2$	5	5	11	42^*		
56	$\mathcal{J}_3{+}\mathcal{J}_2$	5	5	11	$54, 42^*$		54
57	$\mathcal{J}_5(3)$	5	5	10	$55, 54, 44^*$		
58	$\mathcal{R}_1{+}\mathcal{L}_1{+}\mathcal{J}_2(2)$	5	5	9	$55, 45^*$		
59	$\mathcal{J}_4(2){+}\mathcal{J}_1$	5	5	9	$57, 53$		
60	$\mathcal{J}_3{+}\mathcal{J}_2(2)$	5	5	9	$57, 56$		
61	$\mathcal{R}_1{+}\mathcal{L}_1{+}2\mathcal{J}_1$	5	5	8	$58, 46^*$		
62	$\mathcal{J}_3{+}2\mathcal{J}_1$	5	5	8	$60, 59$		57
63	$\mathcal{J}_5(3,2)$	5	5	8	$58, 57, 47^*$		
64	$\mathcal{R}_1{+}\mathcal{L}_2{+}\mathcal{J}_1$	5	5	7	$61, 48$		
65	$\mathcal{J}_4(2,2){+}\mathcal{J}_1$	5	5	7	$63, 59$		
66	$\mathcal{J}_3(2){+}\mathcal{J}_2$	5	5	7	$63, 61, 56, 48^*$		$63, 56$
67	$\mathcal{R}_1{+}\mathcal{L}_3$	5	5	6	$64, 49$		
68	$\mathcal{R}_2{+}\mathcal{L}_2$	5	5	6	64^*		
69	$\mathcal{J}_1{+}2\mathcal{J}_2$	5	5	6	$66, 65$		$\varnothing$
70	$\mathcal{J}_5(4)$	5	5	6	$64^*, 63, 49^*$		
71	$\mathcal{J}_4(3){+}\mathcal{J}_1$	5	5	5	$70, 65$		
72	$\mathcal{J}_3(2){+}\mathcal{J}_2(2)$	5	5	5	$70, 66, 60$		
73	$\mathcal{J}_3(3){+}\mathcal{J}_2$	5	5	5	$70, 66$		
74	$\mathcal{J}_3(2){+}2\mathcal{J}_1$	5	5	4	$72, 71, 62$		$70, 62$
75	$\mathcal{J}_2(2){+}\mathcal{J}_2{+}\mathcal{J}_1$	5	5	4	$73, 72, 71, 69$		69
76	$\mathcal{J}_5(5)$	5	5	4	$70, 68, 67^*, 50^*$		
77	$\mathcal{J}_2{+}3\mathcal{J}_1$	5	5	3	$75, 74$	$73, 71$	$\varnothing$
78	$\mathcal{J}_4(4){+}\mathcal{J}_1$	5	5	3	$76, 71$		
79	$\mathcal{J}_3(3){+}\mathcal{J}_2(2)$	5	5	3	$76, 73, 72$		
80	$\mathcal{J}_3(3){+}2\mathcal{J}_1$	5	5	2	$79, 78, 74$		$76, 74$
81	$\mathcal{J}_1{+}2\mathcal{J}_2(2)$	5	5	2	$79, 78, 75$		75
82	$\mathcal{J}_2(2){+}3\mathcal{J}_1$	5	5	1	$81, 80, 77$	$79, 78, 77$	77
83	$5\mathcal{J}_1$	5	5	0	82	78	$\varnothing$
84	$3\mathcal{L}_1$	3	6	0	38		$\varnothing$
85	$\mathcal{J}_2{+}2\mathcal{L}_1$	4	6	7	44		
86	$\mathcal{J}_2(2){+}2\mathcal{L}_1$	4	6	5	$85, 84, 47$		
87	$2\mathcal{L}_1{+}2\mathcal{J}_1$	4	6	4	$86, 48$		
88	$\mathcal{L}_2{+}\mathcal{L}_1{+}\mathcal{J}_1$	4	6	2	$87, 49$		
89	$\mathcal{L}_3{+}\mathcal{L}_1$	4	6	1	$88, 50$		
90	$2\mathcal{L}_2$	4	6	0	89		$\varnothing$
91	$\mathcal{L}_1{+}\mathcal{J}_4$	5	6	19	53		
92	$\mathcal{L}_1{+}\mathcal{J}_4(2)$	5	6	13	$91, 85, 59$		
93	$\mathcal{L}_1{+}\mathcal{J}_3{+}\mathcal{J}_1$	5	6	12	$92, 62$		
94	$\mathcal{R}_1{+}\mathcal{J}_1{+}2\mathcal{L}_1$	5	6	11	$87, 64$		
95	$\mathcal{L}_1{+}\mathcal{J}_4(2,2)$	5	6	11	$92, 86, 65$		
96	$\mathcal{L}_2{+}\mathcal{J}_3$	5	6	11	93		
97	$\mathcal{R}_2{+}2\mathcal{L}_1$	5	6	10	$94, 68$		
98	$\mathcal{L}_1{+}2\mathcal{J}_2$	5	6	10	$95, 87, 69$		
99	$\mathcal{R}_1{+}\mathcal{L}_2{+}\mathcal{L}_1$	5	6	9	$94, 88, 67$		
100	$\mathcal{L}_1{+}\mathcal{J}_4(3)$	5	6	9	$95, 94, 88, 71$		
101	$\mathcal{L}_1{+}\mathcal{J}_3(2){+}\mathcal{J}_1$	5	6	8	$100, 93, 74$		
102	$\mathcal{L}_1{+}\mathcal{J}_2(2){+}\mathcal{J}_2$	5	6	8	$100, 98, 75$		

Table 1: (continued)

#	Canonical form	n	m	c	Bundle	$GL_{n,m,2}$-orb.	$SL_{n,m,2}$-orb.
103	$\mathcal{L}_1+\mathcal{J}_2+2\mathcal{J}_1$	5	6	7	$102, 101, 77$		
104	$\mathcal{L}_1+\mathcal{J}_4(4)$	5	6	7	$100, 99, 97, 89, 78$		
105	$\mathcal{L}_2+\mathcal{J}_3(2)$	5	6	7	$101, 96$		
106	$\mathcal{L}_1+\mathcal{J}_3(3)+\mathcal{J}_1$	5	6	6	$104, 101, 80$		
107	$\mathcal{L}_1+2\mathcal{J}_2(2)$	5	6	6	$104, 102, 81$		
108	$\mathcal{L}_2+\mathcal{J}_2+\mathcal{J}_1$	5	6	6	$105, 103$		
109	$\mathcal{L}_1+\mathcal{J}_2(2)+2\mathcal{J}_1$	5	6	5	$107, 106, 103, 82$		
110	$\mathcal{L}_2+\mathcal{J}_3(3)$	5	6	5	$106, 105, 90$		
111	$\mathcal{L}_3+\mathcal{J}_2$	5	6	5	108		
112	$\mathcal{L}_1+4\mathcal{J}_1$	5	6	4	$109, 83$	$106, 83, 82$	
113	$\mathcal{L}_2+\mathcal{J}_2(2)+\mathcal{J}_1$	5	6	4	$110, 109, 108$		
114	$\mathcal{L}_2+3\mathcal{J}_1$	5	6	3	$113, 112$		
115	$\mathcal{L}_3+\mathcal{J}_2(2)$	5	6	3	$113, 111$		
116	$\mathcal{L}_3+2\mathcal{J}_1$	5	6	2	$115, 114$		
117	$\mathcal{L}_4+\mathcal{J}_1$	5	6	1	116		
118	$\mathcal{L}_5$	5	6	0	117		$\varnothing$
119	$\mathcal{J}_6$	6	6	35	51		
120	$\mathcal{J}_6(2)$	6	6	25	$119, 91^*$		
121	$\mathcal{J}_5+\mathcal{J}_1$	6	6	24	120		
122	$\mathcal{J}_6(2,2)$	6	6	19	$120, 92^*$		
123	$\mathcal{R}_1+\mathcal{L}_1+\mathcal{J}_3$	6	6	18	93^*		
124	$\mathcal{J}_4+\mathcal{J}_2$	6	6	18	$122, 93^*$		122
125	$\mathcal{J}_6(3)$	6	6	17	$123, 122, 96^*$		
126	$\mathcal{J}_6(2,2,2)$	6	6	17	$122, 95^*$		
127	$2\mathcal{R}_1+2\mathcal{L}_1$	6	6	16	97^*		
128	$\mathcal{J}_5(2)+\mathcal{J}_1$	6	6	16	$125, 121$		
129	$\mathcal{J}_4+\mathcal{J}_2(2)$	6	6	16	$125, 124$		
130	$2\mathcal{J}_3$	6	6	16	$126, 98^*$		$\varnothing$
131	$\mathcal{J}_4+2\mathcal{J}_1$	6	6	15	$129, 128$		125
132	$\mathcal{R}_1+\mathcal{L}_1+\mathcal{J}_3(2)$	6	6	14	$123, 101^*$		
133	$\mathcal{R}_1+\mathcal{L}_1+\mathcal{J}_2+\mathcal{J}_1$	6	6	13	$132, 103^*$		
134	$\mathcal{J}_6(3,2)$	6	6	13	$132, 126, 125, 105^*$		
135	$\mathcal{R}_1+\mathcal{L}_1+\mathcal{J}_3(3)$	6	6	12	$132, 127, 106^*$		
136	$\mathcal{R}_1+\mathcal{L}_2+\mathcal{J}_2$	6	6	12	$133, 108$		
137	$\mathcal{J}_5(2,2)+\mathcal{J}_1$	6	6	12	$134, 128$		
138	$\mathcal{J}_4(2)+\mathcal{J}_2$	6	6	12	$134, 133, 124, 108^*$		$134, 124$
139	$\mathcal{J}_3(2)+\mathcal{J}_3$	6	6	12	$134, 133, 130, 108^*$		130
140	$\mathcal{R}_1+\mathcal{L}_1+\mathcal{J}_2(2)+\mathcal{J}_1$	6	6	11	$135, 133, 109^*$		
141	$\mathcal{J}_3+\mathcal{J}_2+\mathcal{J}_1$	6	6	11	$139, 138, 137$		139
142	$\mathcal{J}_6(4)$	6	6	11	$136^*, 134, 111^*$		
143	$\mathcal{J}_6(3,3)$	6	6	11	$135, 134, 110^*$		
144	$\mathcal{R}_1+\mathcal{L}_1+3\mathcal{J}_1$	6	6	10	$140, 112^*$		
145	$\mathcal{R}_1+\mathcal{L}_2+\mathcal{J}_2(2)$	6	6	10	$140, 136, 113$		
146	$\mathcal{J}_5(3)+\mathcal{J}_1$	6	6	10	$142, 137$		
147	$\mathcal{J}_4(2)+\mathcal{J}_2(2)$	6	6	10	$142, 138, 129$		
148	$\mathcal{J}_4(2,2)+\mathcal{J}_2$	6	6	10	$143, 140, 138, 113^*$		$143, 138$
149	$\mathcal{J}_3(3)+\mathcal{J}_3$	6	6	10	$142, 139$		139
150	$\mathcal{R}_1+\mathcal{L}_2+2\mathcal{J}_1$	6	6	9	$145, 144, 114$		
151	$\mathcal{J}_4(2)+2\mathcal{J}_1$	6	6	9	$147, 146, 131$		$142, 131$
152	$\mathcal{J}_3+\mathcal{J}_2(2)+\mathcal{J}_1$	6	6	9	$149, 147, 146, 141$		$149, 141$
153	$3\mathcal{J}_2$	6	6	9	$148, 144, 114^*$		$\varnothing$

Table 1: (continued)

#	Can. form	n	m	c	Bundle	$GL_{n,m,\mathcal{Z}}$-orb.	$SL_{n,m,\mathcal{Z}}$-orb.
154	$\mathcal{J}_6(4,2)$	6	6	9	145*, 143, 142, 115*		
155	$\mathcal{R}_1+\mathcal{L}_3+\mathcal{J}_1$	6	6	8	150, 116		
156	$\mathcal{R}_2+\mathcal{L}_2+\mathcal{J}_1$	6	6	8	150*		
157	$\mathcal{J}_3+3\mathcal{J}_1$	6	6	8	152, 151	149, 146	149
158	$\mathcal{J}_5(3,2)+\mathcal{J}_1$	6	6	8	154, 146		
159	$\mathcal{J}_4(2,2)+\mathcal{J}_2(2)$	6	6	8	154, 148, 147		
160	$\mathcal{J}_4(3)+\mathcal{J}_2$	6	6	8	154, 150*, 148, 116*		154, 148
161	$2\mathcal{J}_3(2)$	6	6	8	154, 150*, 139, 116*		139
162	$\mathcal{R}_1+\mathcal{L}_4$	6	6	7	155, 117		
163	$\mathcal{R}_2+\mathcal{L}_3$	6	6	7	156, 155		
164	$\mathcal{J}_4(2,2)+2\mathcal{J}_1$	6	6	7	159, 158, 151		154, 151
165	$\mathcal{J}_3(2)+\mathcal{J}_2+\mathcal{J}_1$	6	6	7	161, 160, 158, 141		161, 141
166	$\mathcal{J}_2(2)+2\mathcal{J}_2$	6	6	7	160, 159, 153		153
167	$\mathcal{J}_6(5)$	6	6	7	156, 155*, 154, 117*		
168	$2\mathcal{J}_2+2\mathcal{J}_1$	6	6	6	166, 165, 164	160, 158	∅
169	$\mathcal{J}_5(4)+\mathcal{J}_1$	6	6	6	167, 158		
170	$\mathcal{J}_4(3)+\mathcal{J}_2(2)$	6	6	6	167, 160, 159		
171	$\mathcal{J}_3(3)+\mathcal{J}_3(2)$	6	6	6	167, 161, 149		161, 149
172	$\mathcal{J}_4(4)+\mathcal{J}_2$	6	6	6	167, 160		
173	$\mathcal{J}_4(3)+2\mathcal{J}_1$	6	6	5	170, 169, 164		167, 164
174	$\mathcal{J}_3(2)+\mathcal{J}_2(2)+\mathcal{J}_1$	6	6	5	169–171, 165, 152		171, 165, 152
175	$\mathcal{J}_3(3)+\mathcal{J}_2+\mathcal{J}_1$	6	6	5	171, 172, 169, 165		171, 165
176	$\mathcal{J}_2+2\mathcal{J}_2(2)$	6	6	5	172, 170, 166		166
177	$\mathcal{J}_6(6)$	6	6	5	167, 163*, 162*, 118*		
178	$\mathcal{J}_3(2)+3\mathcal{J}_1$	6	6	4	174, 173, 157	171, 169, 157	171, 157
179	$\mathcal{J}_2(2)+\mathcal{J}_2+2\mathcal{J}_1$	6	6	4	173–176, 168	168–172	168
180	$\mathcal{J}_5(5)+\mathcal{J}_1$	6	6	4	177, 169		
181	$\mathcal{J}_4(4)+\mathcal{J}_2(2)$	6	6	4	177, 172, 170		
182	$2\mathcal{J}_3(3)$	6	6	4	177, 171		171
183	$\mathcal{J}_2+4\mathcal{J}_1$	6	6	3	179, 178	172, 169	∅
184	$\mathcal{J}_4(4)+2\mathcal{J}_1$	6	6	3	181, 180, 173		177, 173
185	$\mathcal{J}_3(3)+\mathcal{J}_2(2)+\mathcal{J}_1$	6	6	3	180–182, 175, 174		182, 175, 174
186	$3\mathcal{J}_2(2)$	6	6	3	181, 176		176
187	$\mathcal{J}_3(3)+3\mathcal{J}_1$	6	6	2	185, 184, 178	182, 180, 178	182, 178
188	$2\mathcal{J}_2(2)+2\mathcal{J}_1$	6	6	2	186, 185, 184, 179	181, 180, 179	179
189	$\mathcal{J}_2(2)+4\mathcal{J}_1$	6	6	1	188, 187, 183	183, 181, 180	183
190	$6\mathcal{J}_1$	6	6	0	189	180	∅
191	$\mathcal{J}_1+3\mathcal{L}_1$	4	7	3	88		
192	$\mathcal{L}_2+2\mathcal{L}_1$	4	7	0	191, 90		
193	$\mathcal{J}_3+2\mathcal{L}_1$	5	7	14	96		
194	$\mathcal{R}_1+3\mathcal{L}_1$	5	7	12	191, 99		
195	$\mathcal{J}_3(2)+2\mathcal{L}_1$	5	7	10	193, 191, 105		
196	$\mathcal{J}_2+\mathcal{J}_1+2\mathcal{L}_1$	5	7	9	195, 108		
197	$\mathcal{J}_3(3)+2\mathcal{L}_1$	5	7	8	195, 194, 192, 110		
198	$\mathcal{J}_2(2)+\mathcal{J}_1+2\mathcal{L}_1$	5	7	7	197, 196, 113		
199	$\mathcal{L}_2+\mathcal{L}_1+\mathcal{J}_2$	5	7	7	196, 111		
200	$3\mathcal{J}_1+2\mathcal{L}_1$	5	7	6	198, 114		
201	$\mathcal{L}_2+\mathcal{L}_1+\mathcal{J}_2(2)$	5	7	5	199, 198, 115		
202	$\mathcal{L}_2+\mathcal{L}_1+2\mathcal{J}_1$	5	7	4	201, 200, 116		
203	$\mathcal{L}_3+\mathcal{L}_1+\mathcal{J}_1$	5	7	3	202, 117		
204	$\mathcal{L}_4+\mathcal{L}_1$	5	7	2	203, 118		

Table 1: (continued)

#	Can. form	n	m	c	Bundle	$GL_{n,m,2}$-,$SL_{n,m,2}$-orbit
205	$\mathcal{J}_1$+2$\mathcal{L}_2$	5	7	2	203	
206	$\mathcal{L}_3$+$\mathcal{L}_2$	5	7	0	205, 204	
207	$\mathcal{L}_1$+$\mathcal{J}_5$	6	7	29	121	
208	$\mathcal{L}_1$+$\mathcal{J}_5(2)$	6	7	21	207, 193, 128	
209	$\mathcal{L}_1$+$\mathcal{J}_4$+$\mathcal{J}_1$	6	7	20	208, 131	
210	$\mathcal{L}_2$+$\mathcal{J}_4$	6	7	19	209	
211	$\mathcal{R}_1$+$\mathcal{J}_2$+2$\mathcal{L}_1$	6	7	17	196, 136	
212	$\mathcal{L}_1$+$\mathcal{J}_5(2,2)$	6	7	17	208, 195, 137	
213	$\mathcal{L}_1$+$\mathcal{J}_3$+$\mathcal{J}_2$	6	7	16	212, 196, 141	
214	$\mathcal{R}_1$+$\mathcal{J}_2(2)$+2$\mathcal{L}_1$	6	7	15	211, 198, 145	
215	$\mathcal{L}_1$+$\mathcal{J}_5(3)$	6	7	15	212, 211, 199, 146	
216	$\mathcal{R}_1$+2$\mathcal{L}_1$+2$\mathcal{J}_1$	6	7	14	214, 200, 150	
217	$\mathcal{L}_1$+$\mathcal{J}_4(2)$+$\mathcal{J}_1$	6	7	14	215, 209, 151	
218	$\mathcal{L}_1$+$\mathcal{J}_3$+$\mathcal{J}_2(2)$	6	7	14	215, 213, 152	
219	$\mathcal{R}_2$+$\mathcal{J}_1$+2$\mathcal{L}_1$	6	7	13	216, 156	
220	$\mathcal{L}_1$+$\mathcal{J}_3$+2$\mathcal{J}_1$	6	7	13	218, 217, 157	
221	$\mathcal{L}_1$+$\mathcal{J}_5(3,2)$	6	7	13	215, 214, 201, 158	
222	$\mathcal{L}_2$+$\mathcal{J}_4(2)$	6	7	13	217, 210	
223	$\mathcal{R}_3$+2$\mathcal{L}_1$	6	7	12	219, $\mathbf{163}^\top$	
224	$\mathcal{R}_1$+$\mathcal{L}_2$+$\mathcal{L}_1$+$\mathcal{J}_1$	6	7	12	216, 202, 155	
225	$\mathcal{L}_1$+$\mathcal{J}_4(2,2)$+$\mathcal{J}_1$	6	7	12	221, 217, 164	
226	$\mathcal{L}_1$+$\mathcal{J}_3(2)$+$\mathcal{J}_2$	6	7	12	221, 216, 213, 202, 165	
227	$\mathcal{L}_2$+$\mathcal{J}_3$+$\mathcal{J}_1$	6	7	12	222, 220	
228	$\mathcal{L}_1$+$\mathcal{J}_1$+2$\mathcal{J}_2$	6	7	11	226, 225, 168	226, 225, 168, 166
229	$\mathcal{R}_2$+$\mathcal{L}_2$+$\mathcal{L}_1$	6	7	11	224, 219, 163	
230	$\mathcal{R}_1$+$\mathcal{L}_3$+$\mathcal{L}_1$	6	7	11	224, 203, 162	
231	$\mathcal{L}_1$+$\mathcal{J}_5(4)$	6	7	11	224, 221, 219, 203, 169	
232	$\mathcal{L}_2$+$\mathcal{J}_4(2,2)$	6	7	11	225, 222	
233	$\mathcal{L}_3$+$\mathcal{J}_3$	6	7	11	227	
234	$\mathcal{L}_1$+$\mathcal{J}_4(3)$+$\mathcal{J}_1$	6	7	10	231, 225, 173	
235	$\mathcal{L}_1$+$\mathcal{J}_3(2)$+$\mathcal{J}_2(2)$	6	7	10	231, 226, 218, 174	
236	$\mathcal{L}_1$+$\mathcal{J}_3(3)$+$\mathcal{J}_2$	6	7	10	231, 226, 175	
237	$\mathcal{R}_1$+2$\mathcal{L}_2$	6	7	10	230, 205	
238	$\mathcal{L}_2$+2$\mathcal{J}_2$	6	7	10	232, 228	
239	$\mathcal{L}_1$+$\mathcal{J}_3(2)$+2$\mathcal{J}_1$	6	7	9	235, 234, 220, 178	
240	$\mathcal{L}_1$+$\mathcal{J}_2(2)$+$\mathcal{J}_2$+$\mathcal{J}_1$	6	7	9	234–236, 228, 179	234–236, 228, 179, 176
241	$\mathcal{L}_1$+$\mathcal{J}_5(5)$	6	7	9	229–231, **223**, 204, 180	
242	$\mathcal{L}_2$+$\mathcal{J}_4(3)$	6	7	9	234, 232, 205	
243	$\mathcal{L}_1$+$\mathcal{J}_2$+3$\mathcal{J}_1$	6	7	8	240, 239, 183	236, 234, 183, 179, 178
244	$\mathcal{L}_1$+$\mathcal{J}_4(4)$+$\mathcal{J}_1$	6	7	8	241, 234, 184	
245	$\mathcal{L}_1$+$\mathcal{J}_3(3)$+$\mathcal{J}_2(2)$	6	7	8	241, 236, 235, 185	
246	$\mathcal{L}_2$+$\mathcal{J}_3(2)$+$\mathcal{J}_1$	6	7	8	242, 239, 227	
247	$\mathcal{L}_2$+$\mathcal{J}_2(2)$+$\mathcal{J}_2$	6	7	8	242, 240, 238	
248	$\mathcal{L}_1$+$\mathcal{J}_3(3)$+2$\mathcal{J}_1$	6	7	7	245, 244, 239, 187	
249	$\mathcal{L}_1$+$\mathcal{J}_1$+2$\mathcal{J}_2(2)$	6	7	7	245, 244, 240, 188	245, 244, 240, 188, 186
250	$\mathcal{L}_2$+$\mathcal{J}_2$+2$\mathcal{J}_1$	6	7	7	247, 246, 243	
251	$\mathcal{L}_2$+$\mathcal{J}_4(4)$	6	7	7	244, 242, 237, 206	
252	$\mathcal{L}_3$+$\mathcal{J}_3(2)$	6	7	7	246, 233	
253	$\mathcal{L}_1$+$\mathcal{J}_2(2)$+3$\mathcal{J}_1$	6	7	6	249, 248, 243, 189	243–245, 189, 188, 187
254	$\mathcal{L}_2$+$\mathcal{J}_3(3)$+$\mathcal{J}_1$	6	7	6	251, 248, 246	
255	$\mathcal{L}_2$+2$\mathcal{J}_2(2)$	6	7	6	251, 249, 247	

Table 1: (continued)

#	Canonical form	n	m	c	Bundle	$\mathrm{GL}_{n,m,\mathcal{Z}}$-orb.	$\mathrm{SL}_{n,m,\mathcal{Z}}$-orb.
256	$\mathcal{L}_3+\mathcal{J}_2+\mathcal{J}_1$	6	7	6	$252, 250$		
257	$\mathcal{L}_1+5\mathcal{J}_1$	6	7	5	$253, 190$	$244, 190, 189$	
258	$\mathcal{L}_2+\mathcal{J}_2(2)+2\mathcal{J}_1$	6	7	5	$255, 254, 253, 250$		
259	$\mathcal{L}_3+\mathcal{J}_3(3)$	6	7	5	$254, 252$		
260	$\mathcal{L}_4+\mathcal{J}_2$	6	7	5	256		
261	$\mathcal{L}_2+4\mathcal{J}_1$	6	7	4	$258, 257$	$257, 254, 253$	
262	$\mathcal{L}_3+\mathcal{J}_2(2)+\mathcal{J}_1$	6	7	4	$259, 258, 256$		
263	$\mathcal{L}_3+3\mathcal{J}_1$	6	7	3	$262, 261$		
264	$\mathcal{L}_4+\mathcal{J}_2(2)$	6	7	3	$262, 260$		
265	$\mathcal{L}_4+2\mathcal{J}_1$	6	7	2	$264, 263$		
266	$\mathcal{L}_5+\mathcal{J}_1$	6	7	1	265		
267	$\mathcal{L}_6$	6	7	0	266		$\varnothing$
268	$\mathcal{J}_7$	7	7	48	119		
269	$\mathcal{J}_7(2)$	7	7	36	$268, 207^*$		
270	$\mathcal{J}_6+\mathcal{J}_1$	7	7	35	269		
271	$\mathcal{J}_7(2,2)$	7	7	28	$269, 208^*$		
272	$\mathcal{R}_1+\mathcal{L}_1+\mathcal{J}_4$	7	7	27	209^*		
273	$\mathcal{J}_5+\mathcal{J}_2$	7	7	27	$271, 209^*$		271
274	$\mathcal{J}_7(3)$	7	7	26	$272, 271, 210^*$		
275	$\mathcal{J}_6(2)+\mathcal{J}_1$	7	7	25	$274, 270$		
276	$\mathcal{J}_5+\mathcal{J}_2(2)$	7	7	25	$274, 273$		
277	$\mathcal{J}_5+2\mathcal{J}_1$	7	7	24	$276, 275$		274
278	$\mathcal{J}_7(2,2,2)$	7	7	24	$271, 212^*$		
279	$\mathcal{J}_4+\mathcal{J}_3$	7	7	23	$278, 213^*$		278
280	$\mathcal{R}_1+\mathcal{L}_1+\mathcal{J}_4(2)$	7	7	21	$272, 217^*$		
281	$\mathcal{J}_1+2\mathcal{R}_1+2\mathcal{L}_1$	7	7	20	219^*		
282	$\mathcal{R}_1+\mathcal{L}_1+\mathcal{J}_3+\mathcal{J}_1$	7	7	20	$280, 220^*$		
283	$\mathcal{J}_7(3,2)$	7	7	20	$280, 278, 274, 222^*$		
284	$\mathcal{R}_1+\mathcal{L}_1+\mathcal{J}_4(2,2)$	7	7	19	$280, 225^*$		
285	$\mathcal{R}_1+\mathcal{L}_2+\mathcal{J}_3$	7	7	19	$282, 227$		
286	$\mathcal{J}_6(2,2)+\mathcal{J}_1$	7	7	19	$283, 275$		
287	$\mathcal{J}_5(2)+\mathcal{J}_2$	7	7	19	$283, 282, 273, 227^*$		$283, 273$
288	$\mathcal{J}_4+\mathcal{J}_3(2)$	7	7	19	$283, 282, 279, 227^*$		$283, 279$
289	$\mathcal{L}_2+\mathcal{L}_1+2\mathcal{R}_1$	7	7	18	$281, 229, \mathbf{223}^{\top}$		
290	$\mathcal{R}_1+\mathcal{L}_1+2\mathcal{J}_2$	7	7	18	$284, 228^*$		
291	$\mathcal{J}_4+\mathcal{J}_2+\mathcal{J}_1$	7	7	18	$288, 287, 286$		283
292	$\mathcal{J}_7(4)$	7	7	18	$285^*, 283, 233^*$		
293	$\mathcal{J}_7(3,2,2)$	7	7	18	$284, 283, 232^*$		
294	$\mathcal{R}_1+\mathcal{L}_1+\mathcal{J}_4(3)$	7	7	17	$284, 281, 234^*$		
295	$\mathcal{J}_6(3)+\mathcal{J}_1$	7	7	17	$292, 286$		
296	$\mathcal{J}_6(2,2,2)+\mathcal{J}_1$	7	7	17	$293, 286$		
297	$\mathcal{J}_5(2)+\mathcal{J}_2(2)$	7	7	17	$292, 287, 276$		
298	$\mathcal{J}_4+\mathcal{J}_3(3)$	7	7	17	$292, 288$		
299	$\mathcal{J}_4(2)+\mathcal{J}_3$	7	7	17	$293, 290, 279, 238^*$		$293, 279$
300	$\mathcal{R}_1+\mathcal{L}_1+\mathcal{J}_3(2)+\mathcal{J}_1$	7	7	16	$294, 282, 239^*$		
301	$\mathcal{R}_1+\mathcal{L}_1+\mathcal{J}_2(2)+\mathcal{J}_2$	7	7	16	$294, 290, 240^*$		
302	$\mathcal{J}_5(2)+2\mathcal{J}_1$	7	7	16	$297, 295, 277$		$292, 277$
303	$\mathcal{J}_4+\mathcal{J}_2(2)+\mathcal{J}_1$	7	7	16	$298, 297, 295, 291$		$292, 291$
304	$\mathcal{J}_1+2\mathcal{J}_3$	7	7	16	$299, 296$		$\varnothing$
305	$\mathcal{J}_7(3,3)$	7	7	16	$294, 293, 242^*$		
306	$\mathcal{R}_1+\mathcal{L}_1+\mathcal{J}_2+2\mathcal{J}_1$	7	7	15	$301, 300, 243^*$		

Table 1: (continued)

#	Canonical form	c	Bundle	$\mathrm{GL}_{n,m,2}$-orbit	$\mathrm{SL}_{n,m,2}$-orb.
307	$\mathcal{R}_1+\mathcal{L}_1+\mathcal{J}_4(4)$	15	$294, 289^*, 244^*$		
308	$\mathcal{R}_1+\mathcal{L}_2+\mathcal{J}_3(2)$	15	$300, 285, 246$		
309	$\mathcal{J}_4+3\mathcal{J}_1$	15	$303, 302$	$298, 295$	292
310	$\mathcal{J}_5(2,2)+\mathcal{J}_2$	15	$305, 300, 287, 246^*$		$305, 287$
311	$\mathcal{J}_4(2,2)+\mathcal{J}_3$	15	$305, 301, 299, 247^*$		$305, 299$
312	$\mathcal{R}_1+\mathcal{L}_1+\mathcal{J}_3(3)+\mathcal{J}_1$	14	$307, 300, 248^*$		
313	$\mathcal{R}_1+\mathcal{L}_1+2\mathcal{J}_2(2)$	14	$307, 301, 249^*$		
314	$\mathcal{R}_1+\mathcal{L}_2+\mathcal{J}_2+\mathcal{J}_1$	14	$308, 306, 250$		
315	$\mathcal{J}_3+2\mathcal{J}_2$	14	$311, 310, 306, 250^*$		$\varnothing$
316	$\mathcal{J}_7(4,2)$	14	$308^*, 305, 292, 252^*$		
317	$\mathcal{R}_1+\mathcal{L}_1+\mathcal{J}_2(2)+2\mathcal{J}_1$	13	$313, 312, 306, 253^*$		
318	$\mathcal{R}_1+\mathcal{L}_2+\mathcal{J}_3(3)$	13	$312, 308, 254$		
319	$\mathcal{R}_1+\mathcal{L}_3+\mathcal{J}_2$	13	$314, 256$		
320	$\mathcal{R}_2+\mathcal{L}_2+\mathcal{J}_2$	13	314^*		
321	$\mathcal{J}_6(3,2)+\mathcal{J}_1$	13	$316, 296, 295$		
322	$\mathcal{J}_5(2,2)+\mathcal{J}_2(2)$	13	$316, 310, 297$		
323	$\mathcal{J}_5(3)+\mathcal{J}_2$	13	$316, 314^*, 310, 256^*$		$316, 310$
324	$\mathcal{J}_4(2)+\mathcal{J}_3(2)$	13	$316, 314^*, 299, 288, 256^*$		$316, 299, 288$
325	$\mathcal{J}_4(3)+\mathcal{J}_3$	13	$316, 314^*, 311, 256^*$		$316, 311$
326	$\mathcal{R}_1+\mathcal{L}_1+4\mathcal{J}_1$	12	$317, 257^*$	$312, 257^*, 253^*$	
327	$\mathcal{R}_1+\mathcal{L}_2+\mathcal{J}_2(2)+\mathcal{J}_1$	12	$318, 317, 314, 258$		
328	$\mathcal{J}_5(2,2)+2\mathcal{J}_1$	12	$322, 321, 302$		$316, 302$
329	$\mathcal{J}_4(2)+\mathcal{J}_2+\mathcal{J}_1$	12	$324, 323, 321, 291$		$316, 291$
330	$\mathcal{J}_3(2)+\mathcal{J}_3+\mathcal{J}_1$	12	$325, 324, 321, 304$		304
331	$\mathcal{J}_3+\mathcal{J}_2(2)+\mathcal{J}_2$	12	$325, 323, 322, 315$		315
332	$\mathcal{J}_7(5)$	12	$320, 319^*, 316, 260^*$		
333	$\mathcal{J}_7(4,3)$	12	$318^*, 316, 259^*$		
334	$\mathcal{R}_1+\mathcal{L}_2+3\mathcal{J}_1$	11	$327, 326, 261$		
335	$\mathcal{R}_1+\mathcal{L}_3+\mathcal{J}_2(2)$	11	$327, 319, 262$		
336	$\mathcal{R}_2+\mathcal{L}_2+\mathcal{J}_2(2)$	11	$327^*, 320$		
337	$\mathcal{J}_3+\mathcal{J}_2+2\mathcal{J}_1$	11	$331, 330, 329, 328$	$325, 323, 321$	$\varnothing$
338	$\mathcal{J}_6(4)+\mathcal{J}_1$	11	$332, 321$		
339	$\mathcal{J}_6(3,3)+\mathcal{J}_1$	11	$333, 321$		
340	$\mathcal{J}_5(3)+\mathcal{J}_2(2)$	11	$332, 323, 322$		
341	$\mathcal{J}_5(3,2)+\mathcal{J}_2$	11	$333, 327^*, 323, 262^*$		$333, 323$
342	$\mathcal{J}_4(2)+\mathcal{J}_3(3)$	11	$332, 324, 298$		
343	$\mathcal{J}_4(2,2)+\mathcal{J}_3(2)$	11	$333, 327^*, 324, 311, 262^*$		$333, 324, 311$
344	$\mathcal{J}_4(4)+\mathcal{J}_3$	11	$332, 325$		
345	$\mathcal{R}_1+\mathcal{L}_3+2\mathcal{J}_1$	10	$335, 334, 263$		
346	$\mathcal{R}_2+\mathcal{L}_2+2\mathcal{J}_1$	10	$336, 334^*$		
347	$\mathcal{J}_5(3)+2\mathcal{J}_1$	10	$340, 338, 328$		$332, 328$
348	$\mathcal{J}_4(2)+\mathcal{J}_2(2)+\mathcal{J}_1$	10	$342, 340, 338, 329, 303$		$332, 329, 303$
349	$\mathcal{J}_4(2,2)+\mathcal{J}_2+\mathcal{J}_1$	10	$343, 341, 339, 329$		$333, 329$
350	$\mathcal{J}_3(3)+\mathcal{J}_3+\mathcal{J}_1$	10	$344, 342, 338, 330$		330
351	$\mathcal{J}_3+2\mathcal{J}_2(2)$	10	$344, 340, 331$		331
352	$\mathcal{J}_3(2)+2\mathcal{J}_2$	10	$343, 341, 334^*, 315, 263^*$		315
353	$\mathcal{J}_7(5,2)$	10	$336, 335^*, 333, 332, 264^*$		
354	$\mathcal{R}_1+\mathcal{L}_4+\mathcal{J}_1$	9	$345, 265$		
355	$\mathcal{R}_2+\mathcal{L}_3+\mathcal{J}_1$	9	$346, 345$		
356	$\mathcal{J}_4(2)+3\mathcal{J}_1$	9	$348, 347, 309$	$342, 338, 309$	$332, 309$
357	$\mathcal{J}_3+\mathcal{J}_2(2)+2\mathcal{J}_1$	9	$351, 350, 348, 347, 337$	$344, 340, 338, 337$	337
358	$\mathcal{J}_1+3\mathcal{J}_2$	9	$352, 349$	$341, 339, 334^*, 263^*$	$\varnothing$
359	$\mathcal{J}_6(4,2)+\mathcal{J}_1$	9	$353, 339, 338$		
360	$\mathcal{J}_5(3,2)+\mathcal{J}_2(2)$	9	$353, 341, 340$		

Table 1: (continued)

#	Can. form	c	Bundle	$GL_{n,m,2}$-orbit	$SL_{n,m,2}$-orb.
361	$\mathcal{J}_5(4)+\mathcal{J}_2$	9	$353, 346, 345^*, 341, 265^*$		$353, 341$
362	$\mathcal{J}_4(2,2)+\mathcal{J}_3(3)$	9	$353, 343, 342$		
363	$\mathcal{J}_4(3)+\mathcal{J}_3(2)$	9	$353, 346, 345^*, 343, 325, 265^*$		$353, 343, 325$
364	$\mathcal{R}_1+\mathcal{L}_5$	8	$354, 266$		
365	$\mathcal{R}_2+\mathcal{L}_4$	8	$355, 354$		
366	$\mathcal{R}_3+\mathcal{L}_3$	8	355^*		
367	$\mathcal{J}_3+4\mathcal{J}_1$	8	$357, 356$	$344, 338$	$\varnothing$
368	$\mathcal{J}_5(3,2)+2\mathcal{J}_1$	8	$360, 359, 347$		$353, 347$
369	$\mathcal{J}_4(2,2)+\mathcal{J}_2(2)+\mathcal{J}_1$	8	$362, 360, 359, 349, 348$		$353, 349, 348$
370	$\mathcal{J}_4(3)+\mathcal{J}_2+\mathcal{J}_1$	8	$363, 361, 359, 349$		$353, 349$
371	$\mathcal{J}_1+2\mathcal{J}_3(2)$	8	$363, 359, 330$		330
372	$\mathcal{J}_3(2)+\mathcal{J}_2(2)+\mathcal{J}_2$	8	$363, 361, 360, 352, 331$		$352, 331$
373	$\mathcal{J}_3(3)+2\mathcal{J}_2$	8	$362, 361, 352$		352
374	$\mathcal{J}_7(6)$	8	$355^*, 354^*, 353, 266^*$		
375	$\mathcal{J}_4(2,2)+3\mathcal{J}_1$	7	$369, 368, 356$	$362, 359, 356$	$356, 353$
376	$\mathcal{J}_2(2)+\mathcal{J}_1+2\mathcal{J}_2$	7	$373, 372, 370, 369, 358$	$361, 360, 359, 358$	358
377	$\mathcal{J}_3(2)+\mathcal{J}_2+2\mathcal{J}_1$	7	$372, 371, 370, 368, 337$	$363, 361, 359, 337$	337
378	$\mathcal{J}_6(5)+\mathcal{J}_1$	7	$374, 359$		
379	$\mathcal{J}_5(4)+\mathcal{J}_2(2)$	7	$374, 361, 360$		
380	$\mathcal{J}_5(5)+\mathcal{J}_2$	7	$374, 361$		
381	$\mathcal{J}_4(3)+\mathcal{J}_3(3)$	7	$374, 363, 362$		
382	$\mathcal{J}_4(4)+\mathcal{J}_3(2)$	7	$374, 363, 344$		
383	$3\mathcal{J}_1+2\mathcal{J}_2$	6	$376, 377, 375$	$361, 359$	$\varnothing$
384	$\mathcal{J}_4(4)+\mathcal{J}_2+\mathcal{J}_1$	6	$382, 380, 378, 370$		$374, 370$
385	$\mathcal{J}_4(3)+\mathcal{J}_2(2)+\mathcal{J}_1$	6	$381, 379, 378, 370, 369$		$374, 370, 369$
386	$\mathcal{J}_5(4)+2\mathcal{J}_1$	6	$379, 378, 368$		$374, 368$
387	$\mathcal{J}_3(3)+\mathcal{J}_3(2)+\mathcal{J}_1$	6	$382, 381, 378, 371, 350$		$371, 350$
388	$\mathcal{J}_3(2)+2\mathcal{J}_2(2)$	6	$382, 379, 372, 351$		$372, 351$
389	$\mathcal{J}_3(3)+\mathcal{J}_2(2)+\mathcal{J}_2$	6	$381, 380, 379, 373, 372$		$373, 372$
390	$\mathcal{J}_7(7)$	6	$374, 366, 365^*, 364^*, 267^*$		
391	$\mathcal{J}_4(3)+3\mathcal{J}_1$	5	$385, 386, 375$	$381, 378, 375$	$375, 374$
392	$\mathcal{J}_3(2)+\mathcal{J}_2(2)+2\mathcal{J}_1$	5	$385–388, 377, 357$	$382, 377–379, 357$	$377, 357$
393	$\mathcal{J}_3(3)+\mathcal{J}_2+2\mathcal{J}_1$	5	$389, 387, 384, 386, 377$	$381, 380, 378, 377$	377
394	$\mathcal{J}_2+\mathcal{J}_1+2\mathcal{J}_2(2)$	5	$389, 388, 384, 385, 376$	$380, 379, 378, 376$	376
395	$\mathcal{J}_6(6)+\mathcal{J}_1$	5	$390, 378$		
396	$\mathcal{J}_5(5)+\mathcal{J}_2(2)$	5	$390, 380, 379$		
397	$\mathcal{J}_4(4)+\mathcal{J}_3(3)$	5	$390, 382, 381$		
398	$\mathcal{J}_3(2)+4\mathcal{J}_1$	4	$392, 391, 367$	$382, 378, 367$	367
399	$\mathcal{J}_2(2)+\mathcal{J}_2+3\mathcal{J}_1$	4	$394, 393, 392, 391, 383$	$383, 380, 379, 378$	383
400	$\mathcal{J}_5(5)+2\mathcal{J}_1$	4	$396, 395, 386$		$390, 386$
401	$\mathcal{J}_4(4)+\mathcal{J}_2(2)+\mathcal{J}_1$	4	$397, 396, 395, 384, 385$		$390, 384, 385$
402	$\mathcal{J}_1+2\mathcal{J}_3(3)$	4	$397, 395, 387$		387
403	$\mathcal{J}_3(3)+2\mathcal{J}_2(2)$	4	$397, 396, 389, 388$		$389, 388$
404	$\mathcal{J}_2+5\mathcal{J}_1$	3	$399, 398$	$380, 378$	$\varnothing$
405	$\mathcal{J}_4(4)+3\mathcal{J}_1$	3	$401, 400, 391$	$397, 395, 391$	$391, 390$
406	$\mathcal{J}_3(3)+\mathcal{J}_2(2)+2\mathcal{J}_1$	3	$400–403, 393, 392$	$395–397, 393, 392$	$393, 392$
407	$\mathcal{J}_1+3\mathcal{J}_2(2)$	3	$403, 401, 394$	$396, 395, 394$	394
408	$\mathcal{J}_3(3)+4\mathcal{J}_1$	2	$406, 405, 398$	$398, 397, 395$	398
409	$3\mathcal{J}_1+2\mathcal{J}_2(2)$	2	$407, 406, 405, 399$	$399, 396, 395$	399
410	$\mathcal{J}_2(2)+5\mathcal{J}_1$	1	$409, 408, 404$	$404, 396, 395$	404
411	$7\mathcal{J}_1$	0	410	395	$\varnothing$

Table 1: Bundles and orbits of matrix pencils.

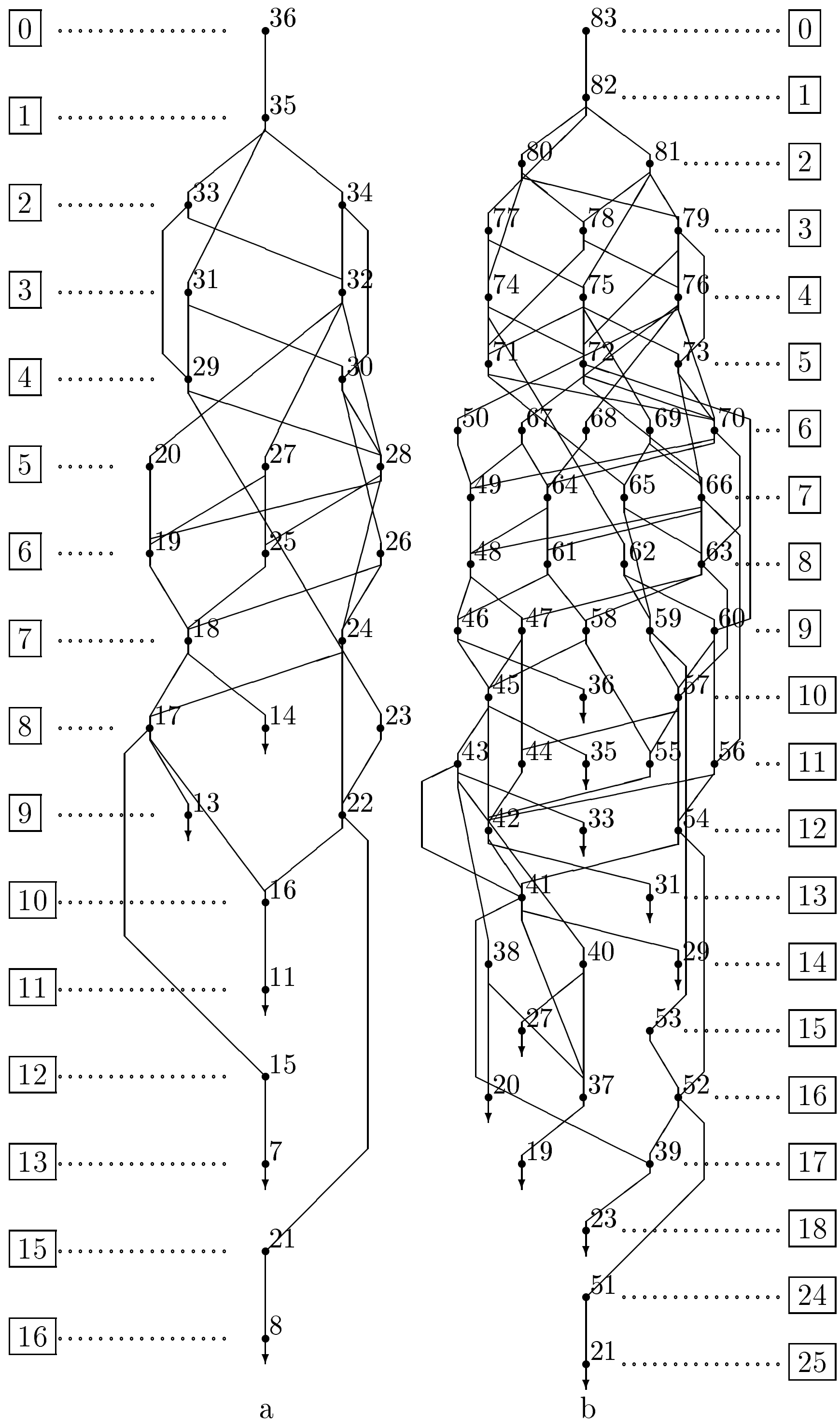

Figure 3: Hierarchy of closures of 4×4 (a) and 5×5 (b) bundles.

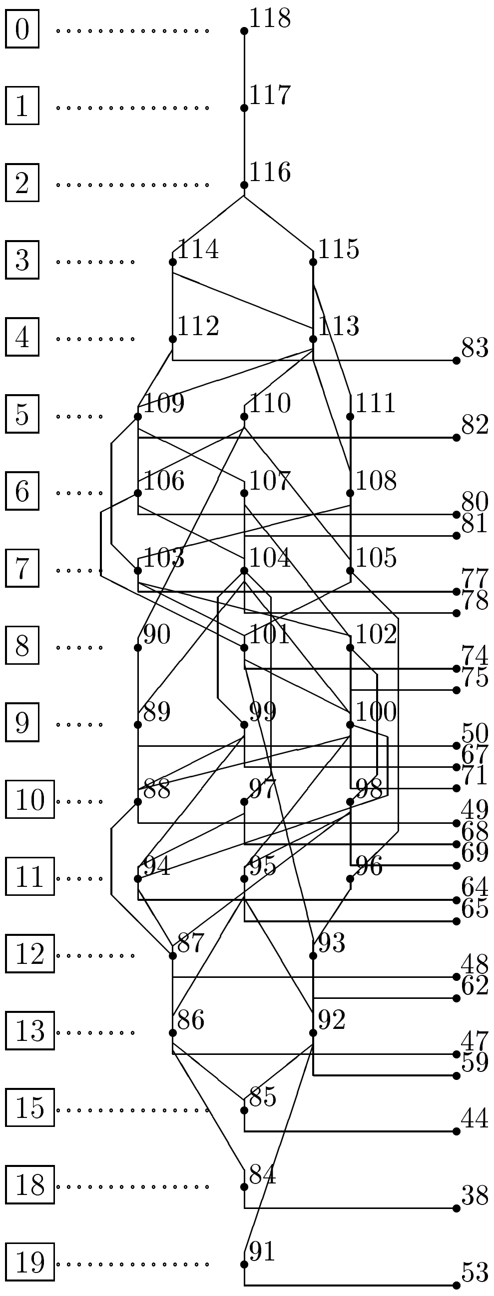

Figure 4: Hierarchy of closures of 5×6 bundles.

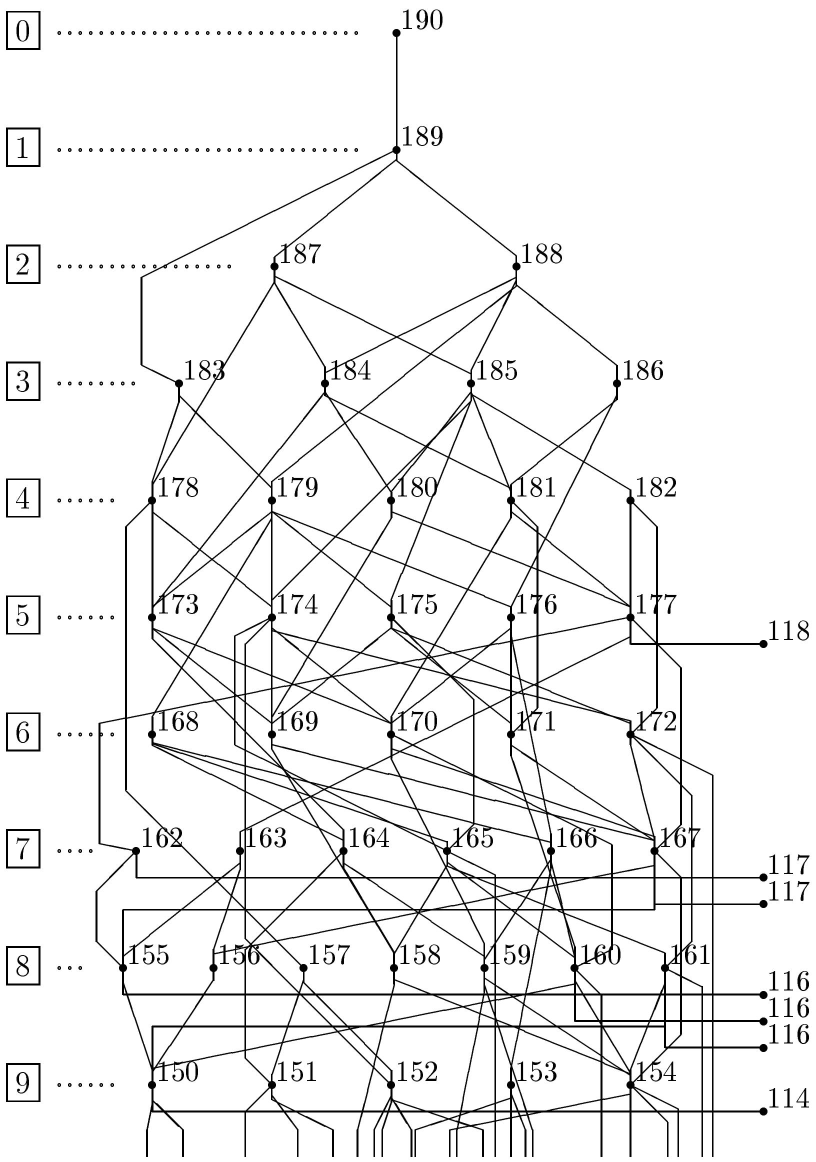

Figure 5: Hierarchy of closures of 6×6 bundles.

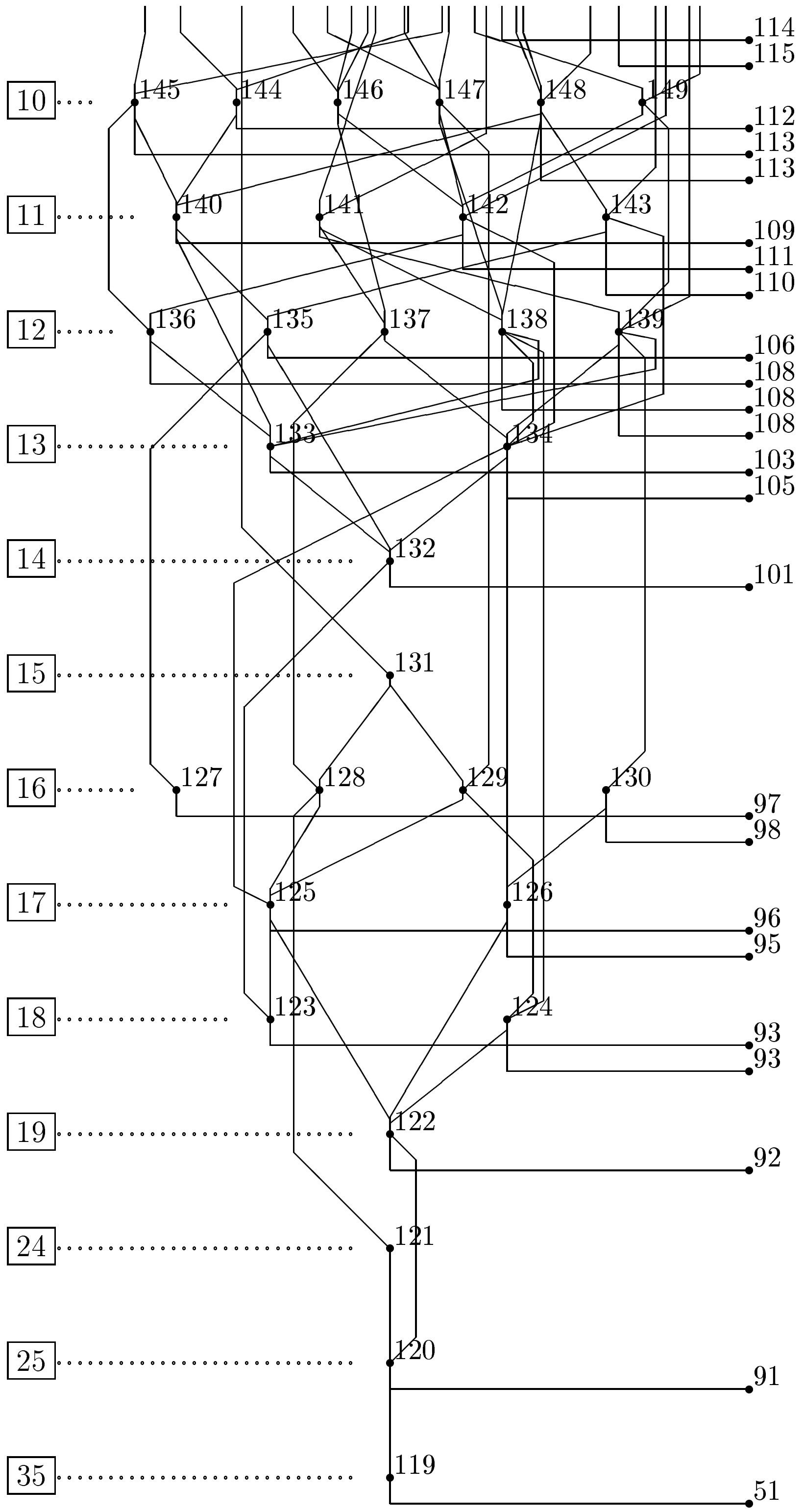

Figure 5: (continued)

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