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Stable Affine Models for Algebraic Group Actions

Zinovy Reichstein[1] **and Nikolaus Vonessen**[2]

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Abstract. Let G be a reductive linear algebraic group defined over an algebraically closed base field k of characteristic zero. A G-variety is an algebraic variety with a regular action of G, defined over k. An affine G-variety is called stable if its points in general position have closed G-orbits. We give a simple necessary and sufficient condition for a G-variety to have a stable affine birational model.

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1. Introduction

Let G be a linear algebraic group, defined over an algebraically closed base field k of characteristic zero. We shall refer to a reduced but not necessarily irreducible algebraic variety X (defined over k), with a regular action of G (also defined over k) as a G-variety. By a morphism $X \longrightarrow Y$ of G-varieties, we shall mean a G-equivariant morphism. The notions of isomorphism, rational map, birational isomorphism, etc. of G-varieties are defined in a similar manner. As usual, given a G-action on X, we shall denote the orbit of $x \in X$ by Gx and the stabilizer subgroup of x by $G_x \subseteq G$. Finally, we shall say that a property holds for $x \in X$ in general position if it holds for every point x of some dense open subset of X.

In this note we will be interested in studying G-varieties up to birational isomorphism. In this context it is natural to ask whether or not a given G-variety X has an affine model. Indeed, there are numerous results and constructions in invariant theory that are available for affine G-varieties but not in general, especially if G is reductive; cf. [5].

Recall that an affine G-variety X is called *stable*, if the orbit Gx is closed for $x \in X$ in general position. If G is reductive, these varieties have many nice properties; for a summary, see, e.g., [10, Section 8]. The question we will address

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in this note is: Which G-varieties have a stable affine birational model? Our main result is the following:

Theorem 1.1. *Let G be a reductive linear algebraic group and X be a G-variety. Then the following are equivalent:*

(a) *X is birationally equivalent to a stable affine G-variety.*

(b) *The stabilizer G_x is reductive for x in general position in X.*

In the case where $X = G/H$ is a homogeneous space, Theorem 1.1 reduces to a theorem of Matsushima [3] which says that G/H is affine if and only if H is reductive. Moreover, the implication (a) $\Longrightarrow$ (b) of Theorem 1.1 is an immediate consequence of Matsushima's theorem. Indeed, after replacing X by a stable affine model, we see that for $x \in X$ in general position the orbit $Gx \simeq G/G_x$ is affine, so that G_x is reductive.

Our proof of the implication (b) $\Longrightarrow$ (a) will be based on the following more general result:

Theorem 1.2. *Let G be a linear algebraic group and X a G-variety. Denote by G_x the stabilizer of $x \in X$ in G. Assume that one of the following two conditions is satisfied:*

(i) *$G_x = \{1\}$ for $x \in X$ in general position (i.e., the G-action on X is generically free), or*

(ii) *the normalizer $\mathrm{N}_G(G_x)$ is reductive for $x \in X$ in general position.*

Then X is birationally isomorphic to a stable affine G-variety.

Note that if G and G_x are both reductive then so is the normalizer $\mathrm{N}_G(G_x)$; see [2, Lemma 1.1]. Thus Theorem 1.2(ii) proves the implication (b) $\Longrightarrow$ (a) of Theorem 1.1.

The rest of this note will be devoted to proving Theorem 1.2. Our proof of part (ii) will be based on part (i) and a theorem of Richardson [7, Theorem 9.3.1] about the existence of stabilizers in general position.

We remark that the theorems of Matsushima and Richardson mentioned above were originally proved only for $k = \mathbb{C}$ (by analytic methods). An algebraic proof of Matsushima's theorem over an algebraically closed field k of characteristic zero can be found in [1, Section 2]. Richardson's theorem is also valid over such k by the Lefschetz principle; it is stated in this form in [5, Theorem 7.1]. Nevertheless, it would be interesting to find a direct algebraic proof.

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2. Proof of Theorem 1.2(i)

We begin with a simple lemma.

Lemma 2.1. *Every linear algebraic group G has a stable generically free linear representation.*

Proof. We first embed G as a closed subgroup in SL_n for some $n \geq 1$. The linear action of SL_n on $V = \mathrm{M}_n(k)$ by left multiplication is easily seen to be stable and generically free. So for $v \in V$ in general position, the stabilizer of v in SL_n is trivial, and $\mathrm{SL}_n v$ is closed in V. For such a v, the map $h \mapsto hv$ gives an isomorphism of SL_n onto $\mathrm{SL}_n v$. Hence, Gv is closed in $\mathrm{SL}_n v$ and thus in V. Consequently, the induced linear action of G on V is stable and generically free. ■

We are now ready to proceed with the proof of Theorem 1.2(i). Recall that a G-variety is called *primitive* if G transitively permutes the irreducible components of X. It is easy to see that every X is birationally isomorphic to a disjoint union of primitive G-varieties; cf. [6, Lemma 2.2]. Hence, we may assume that X is primitive.

By a theorem of Rosenlicht there exists a rational quotient map

$$\pi_{\mathrm{rat}} \colon X \dashrightarrow Z \,,$$

separating the G-orbits in general position in X; see [8] (for the case where X is irreducible) and [9] (for general X). Here Z is only defined up to birational isomorphism, so we may assume without loss of generality that Z is affine. After replacing X by a dense open G-invariant subset, we may assume that $G_x = \{1\}$ for every $x \in X$, and that π_{rat} is regular and separates the G-orbits in X. Since X is primitive, Z is irreducible.

By Lemma 2.1 there exists a stable generically free linear representation V of G. Let V_0 be a G-stable dense open subset of V such that every point $v \in V_0$ has a closed orbit (in V) and trivial stabilizer. By [6, Proposition 7.1] there is a G-equivariant rational map $f \colon X \dashrightarrow V$ whose image contains a point $v \in V_0$. Let Y be the closure of the image of $f \times \pi_{\mathrm{rat}} \colon X \dashrightarrow V \times Z$. Note that Y is G-primitive and affine. Moreover, $U = Y \cap (V_0 \times Z)$ is a G-invariant non-empty (and hence, dense) open subset of Y, and every point of U has a trivial stabilizer in G and a closed G-orbit in $V \times Z$. Thus Y is a stable affine generically free G-variety.

It remains to show that $f \times \pi_{\mathrm{rat}}$ is a birational isomorphism between X and Y. Since we are working in characteristic zero, since X is primitive, and since $f \times \pi_{\mathrm{rat}} \colon X \dashrightarrow Y$ is dominant, it suffices to check that $f \times \pi_{\mathrm{rat}}$ is injective on a dense open subset of X. Indeed, let $W = (f \times \pi_{\mathrm{rat}})^{-1}(U)$. Then W is a G-stable nonempty (and thus dense) open subset of X. Now assume that $y = (f \times \pi_{\mathrm{rat}})(x_1) = (f \times \pi_{\mathrm{rat}})(x_2)$ for some $x_1, x_2 \in W$. Since π_{rat} separates the orbits in X, $x_2 = g(x_1)$ for some $g \in G$. But then $g \in G_y = \{1\}$. We conclude that $x_1 = x_2$. ■

3. Homogeneous Fiber Spaces

Let N be a closed subgroup of a linear algebraic group G, and let W be an N-variety. We first recall the definition of the G-variety $G *_N W$. Consider the action of $G \times N$ on the variety $G \times W$ given by

$$(g,n) \cdot (h,w) = (ghn^{-1}, nw)\,. \tag{1}$$

The variety $G *_N W$ is, by definition, the rational quotient of $G \times W$ for the N-action given by the above formula (where we identify N with the subgroup $\{1\} \times N$ of $G \times N$). Moreover, one can choose a particular model for $G *_N W$ such that the G-action on $G \times N$ descends to a regular G-action on $G *_N W$ making the rational quotient map $G \times W \dashrightarrow G *_N W$ G-equivariant. The variety $G *_N W$ is often referred to as a *homogeneous fiber space*. (If W is a point with trivial N-action, then $G *_N W$ is the homogeneous space G/N.) Note that above definition of $G *_N W$, taken from [6, 2.12], is somewhat different from the one given in [5, 4.8], where the categorical, rather than the rational quotient is considered. As a result, our $G *_N W$ is only defined up to birational isomorphism (of G-varieties); however, there is no restriction on W (compare with [5, Theorem 4.19]).

For lack of a reference, we include the following easy lemma.

Lemma 3.1. *Let N be a closed subgroup of a linear algebraic group G. If W_1 and W_2 are birationally isomorphic N-varieties, then $G *_N W_1$ and $G *_N W_2$ are birationally isomorphic G-varieties.*

Proof. Say $\varphi\colon W_1 \dashrightarrow W_2$ is a birational isomorphism of N-varieties. Then $\mathrm{id}_G \times \varphi\colon G \times W_1 \dashrightarrow G \times W_2$ is a birational isomorphism of $G \times N$-varieties, so induces a birational isomorphism of the rational quotients by N. One easily checks that this birational isomorphism $G *_N W_1 \dashrightarrow G *_N W_2$ is G-equivariant. ■

We now recall that a subgroup $S \subseteq G$ is called *a stabilizer in general position* for a G-variety X if G_x is conjugate to S for $x \in X$ in general position. Note that if a G-variety X has a stabilizer in general position, it is unique up to conjugacy.

Lemma 3.2. *Let G be a linear algebraic group, and let X be a primitive G-variety. Assume that X has a stabilizer $S \subseteq G$ in general position. Set $N = \mathrm{N}_G(S)$, and denote by X^S the set of S-fixed points in X. Let Y be the union of the irreducible components of X^S of maximal dimension. Then X is birationally isomorphic to $G *_N Y$ as a G-variety.*

Our proof is closely related to arguments in [4, Section 1.7]; however, for the sake of completeness, and because we are assuming that X is primitive but not necessarily irreducible, our proof will be self-contained.

Proof. After replacing X by a G-invariant dense open subset, we may assume that G_x is conjugate to S for every $x \in X$. By comparing stabilizers, we see that

$$GX^S = X \tag{2}$$

and

$$\text{if } gx_1 = x_2 \text{ for some } x_1, x_2 \in X^S \text{ and } g \in G, \text{ then } g \in N. \tag{3}$$

We first show that GY is dense in X. Consider the map $\Psi\colon G \times X^S \longrightarrow X$ given by $(g,x) \mapsto gx$. By (2), Ψ is surjective. By (3), the fibers of Ψ are precisely the N-orbits in $G \times X^S$, where N acts by $n \cdot (g, x) = (gn^{-1}, nx)$. Since this action is free (i.e., the stabilizer of every point is trivial), every fiber has the same dimension $\dim(N)$. Since X is G-primitive, the fiber dimension theorem implies that GY is dense in X.

Since Ψ sends N-orbits in $G \times Y$ to points in X, the universal property of rational quotients of N-varieties (see, e.g., [6, Remark 2.4]) says that $\Psi|_{G\times Y}$ descends to a rational map $\psi\colon G *_N Y \dashrightarrow X$ of G-varieties. Since GY is dense in X, ψ is dominant. We claim that ψ is a birational isomorphism.

Since the irreducible components of Y have the same dimension, and since the dimension of the fibers of Ψ is constant, we conclude that the irreducible components of $G *_N Y$ are also of the same dimension (namely, of dimension, $\dim(G) + \dim(Y) - \dim(N)$). Thus in order to show that ψ is a birational isomorphism, we only need to check that ψ is generically one-to-one. More precisely, we will show that if $\Psi(g_1, y_1) = \Psi(g_2, y_2)$, then (g_1, y_1) and (g_2, y_2) lie in the same N-orbit in $G \times Y$.

Indeed, $\Psi(g_1, y_1) = \Psi(g_2, y_2)$ can, by definition, be rewritten as $y_1 = g_1^{-1}g_2y_2$. By (3), $g_1^{-1}g_2 \in N$. Setting $n = g_1^{-1}g_2$, we see that $(g_1, y_1) = (g_2n^{-1}, ny_2)$, so that (g_1, y_1) and (g_2, y_2) are, indeed, in the same N-orbit. This completes the proof of Lemma 3.2. ∎

4. Proof of Theorem 1.2(ii)

We begin with several preliminary reductions. First note that if $\mathrm{N}_G(G_x)$ is reductive then G_x itself must be reductive. Indeed, the unipotent radical $R_u(G_x)$ is trivial, because it is a normal unipotent subgroup of $\mathrm{N}_G(G_x)$.

Secondly, we may assume, as we did in the previous section, that X is primitive, i.e., G transitively permutes the irreducible components of X.

Thirdly, by a theorem of Richardson (see [7, Theorem 9.3.1] or [5, Theorem 7.1]), X has a stabilizer $S \subseteq G$ in general position. (If X_0 is an irreducible component of X, and if $G_0 = \{g \in G \,|\, g(X_0) = X_0\}$, Richardson's theorem immediately implies that the G_0 action on X_0 has a stabilizer in general position; one easily checks that this stabilizer is also a stabilizer in general position for the G-action on the primitive G-variety X.) Now, after replacing X by a G-invariant dense open subset, we may assume that G_x is conjugate to S for every $x \in X$. By assumption, $N = \mathrm{N}_G(S)$ is reductive. Denote by X^S the set of S-fixed points in X. Let Y be the union of the irreducible components of X^S of maximal dimension. By Lemma 3.2, X is birationally isomorphic to $G *_N Y$ as G-variety.

Since S acts trivially on Y, we can think of Y as an N/S-variety. By our assumption, G_x is conjugate to S for every $x \in X$. In particular, $G_x = S$ for every $x \in X^S$. Hence, the N/S-action on Y is generically free and, by Theorem 1.2(i), there is a stable affine N/S-variety Z, birationally equivalent to Y. By Lemma 3.1, $X \simeq G *_N Y$ is birationally isomorphic to $G *_N Z$ as G-variety. So it suffices to show that $G *_N Z$ has a stable affine model, namely the categorical quotient $X' = (G \times Z) /\!\!/ N$. Here we identify, as before, N with the subgroup $\{1\} \times N$ of $G \times N$, and the N-action on $G \times Z$ is given by (1).

Since $G \times Z$ is affine and N is reductive, X' is affine. Since the N-action on $G \times Z$ is free, it is stable. Since N is reductive, the categorical quotient map

$\pi_{\mathrm{cat}}\colon G \times Z \longrightarrow X'$ separates closed orbits, so all orbits, see, e.g., [5, p. 189, Corollary]. This implies that X' is birationally isomorphic to $G *_N Z$, the rational quotient of $G \times Z$ by N, see, e.g., [6, Remark 2.5].

It remains to show that the G-action on X' is stable. We first note that the $G \times N$-action on $G \times Z$, given by (1) is stable. Indeed, the $G \times N$-orbit of $(g, z) \in G \times Z$ is $G \times (Nz)$. Since the N-action on Z is stable, this orbit is closed for z in general position in Z.

Finally, the G-orbits in X' are images, under π_{cat}, of $G \times N$-orbits in $G \times Z$. Since π_{cat} maps N-invariant closed sets in $G \times Z$ to closed sets in X' (cf. [5, p. 188, Corollary]), it follows that the G-action on X' is stable. This completes the proof of Theorem 1.2(ii). ■

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Zinovy Reichstein
Department of Mathematics
University of British Columbia
Vancouver, BC V6T 1Z2
Canada
reichst@math.ubc.ca
www.math.ubc.ca/~reichst

Nikolaus Vonessen
Department of Mathematical Sciences
University of Montana
Missoula, MT 59812-0864
USA
Nikolaus.Vonessen@umontana.edu
www.math.umt.edu/~vonessen

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