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# A Real Analog of Kostant's Version of the Bott–Borel–Weil Theorem

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**Abstract.** We show how to describe the cohomology of the nilradical of a parabolic subalgebra a semisimple Lie algebra with coefficients in an irreducible representation of $\mathfrak{g}$. The situation in the complex case is well–known, Kostant's result (see below) gives an explicit description of a representation of a proper reductive subalgebra on the space of the complex cohomology. The aim of this work is to determine the structure of the real cohomology from the structure of the complex one. We will use the notation of Dynkin and Satake diagrams for the description of semisimple and parabolic real and complex Lie algebras and their representations.
*Keywords:* semisimple Lie algebra, Lie algebra cohomology, parabolic subalgebra, real form, real cohomology.

**0. Introduction.** The description of the real cohomology is based on the structure of the complex case. Each standard parabolic subalgebra $\mathfrak{q} \subseteq \mathfrak{f}$ of the complex semisimple Lie algebra $\mathfrak{f}$ determines a decomposition $\mathfrak{f} = \mathfrak{f}_- \oplus \mathfrak{f}_0 \oplus \mathfrak{q}_+$ where $\mathfrak{q} = \mathfrak{f}_0 \oplus \mathfrak{q}_+$. Given a representation $\pi : \mathfrak{q}_+ \longrightarrow \mathfrak{gl}(V)$, we define the differential $d : \mathrm{Hom}\ (\bigwedge^n \mathfrak{q}_+; V) \longrightarrow \mathrm{Hom}\ (\bigwedge^{n+1} \mathfrak{q}_+; V)$ in the usual way. The corresponding cohomology will be denoted by $H^n(\mathfrak{q}_+, V)$. We will be interested only in cases where $\pi$ is a restriction of some irreducible representation of $\mathfrak{f}$.

Following Kostant (see [3]), we define a natural representation $\mathfrak{f}_0 \longrightarrow \mathfrak{gl}(H^n(\mathfrak{q}_+, V))$ of the reductive subalgebra $\mathfrak{f}_0$ on the cohomology space. The main result of [3] is the description of highest weights of irreducible components of this representation. The construction of these weights yields the ordering on the cohomology and a corresponding Hasse diagram. The algorithmic description of these weights (which uses the notation of Dynkin diagrams) is well known, cf. [9], which also includes the description of cohomologies $H^n(\mathfrak{f}_-, V)$ as a dual to $H^n(\mathfrak{q}_+, V^*)$.

Let us consider the real semisimple Lie algebra $\mathfrak{g}$ and its complexification

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$\mathfrak{f} = \mathfrak{g}(\mathbb{C})$. The algebra $\mathfrak{g}$ is called a real form of $\mathfrak{f}$. The real forms, up to isomorphism, are in 1–1 correspondence with involutive automorphisms of $\mathfrak{f}$ up to conjugacy [4, 5]. Such an automorphism induces a symmetry of the Dynkin diagram of $\mathfrak{f}$ which is important for the description of irreducible representations of $\mathfrak{g}$ in terms of their complexification.

There is a classification of parabolic subalgebras of $\mathfrak{f}$ based on crossing out some vertices of the Dynkin diagram of $\mathfrak{f}$. Similarly, parabolic subalgebras of $\mathfrak{g}$ can be given by crossing out some vertices of the Satake diagram of $\mathfrak{g}$. [11] describes which vertices can be crossed out in this case.

The structure of the representation of $\mathfrak{f}_0$ on the complex cohomology $H^n(\mathfrak{q}_+, V)$ (see [3]) follows, that the cohomology of the complexification is isomorphic to the complexification of the real cohomology. The representation on the cohomology in the complex case is well understood, see [3]. The remaining problem is to describe the representation on the real cohomology in terms of its complexification. The description of real representations with the help of its complexifications is well–known for semisimple Lie algebras. We generalize it to the reductive Lie algebras and we show its connection with Satake diagrams. Then we describe the relation between the real and complex cohomologies. We will see that Hasse diagrams of real and complex cohomologies are often the same.

It is useful to compute the results from [3] by computers. The web implementation, which computes both real and complex cohomologies, is available on the address `www.math.muni.cz/~silhan/lac`. It is based on the software package *LiE* (see [6]) which offers the data structures and corresponding procedures for the computation with semisimple Lie algebras.

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## 1. Known results: complex cohomology and real algebras

**1.1. Weyl group and weights.** Let us consider a complex semisimple Lie algebra $\mathfrak{f}$ with a Cartan subalgebra $\mathfrak{h}$, sets of simple roots, positive roots and roots $\Pi \subseteq \Delta_+ \subseteq \Delta$ and Weyl group $W$. The group $W$ is generated by *simple reflections* i.e. the reflections corresponding to the simple roots. The number of positive roots $\alpha \in \Delta_+$ which are transformed to $w(\alpha) \in \Delta_- = -\Delta_+$ is called the *length* of $w$ for which we write $|w|$. Equivalently (see [2]), the length of $w$ is the minimal number of simple reflections in any expression for $w$ in terms of simple reflections.

The weights of $\mathfrak{f}$ can be described by labelling the nodes of the Dynkin diagram by the integer coefficients referring to the linear combination of fundamental weights. The weight is dominant for $\mathfrak{f}$ if and only if all the coefficients are nonnegative (such a labeled Dynkin diagram describes an irreducible representation of $\mathfrak{f}$).

The *affine action* of the Weyl group is defined by

$$w.\Lambda = w(\Lambda + \mathrm{R}) - \mathrm{R}$$

for the weight $\Lambda$ where $\mathrm{R} = \frac{1}{2}\sum_{\alpha\in\Delta_+}\alpha$ is the lowest strictly dominant weight of $\mathfrak{f}$. It means (in the terms of the Dynkin diagram) to add one over each node, then act with $w$ and finally subtract one over each node.

**1.2. Parabolic subalgebras.** The standard parabolic subalgebra $\mathfrak{q} \subseteq \mathfrak{f}$ is defined by some set of simple roots $\Sigma \subseteq \Pi$ and it is generated by the Cartan subalgebra, root spaces corresponding to the positive roots and root spaces corresponding to the negative roots which can be expressed as a negative linear combination of roots from $\Pi \setminus \Sigma$. The corresponding Dynkin diagram for $\mathfrak{q}$ is obtained from the Dynkin diagram for $\mathfrak{f}$ by crossing out nodes corresponding to the simple roots from $\Sigma$. Using Satake diagrams, a similar notation can be established for the real case. Each parabolic subalgebra is conjugate to some standard parabolic subalgebra so we will deal only with standard parabolics. The set $\Sigma$ induces the decomposition $\mathfrak{f} = \mathfrak{f}_- \oplus \mathfrak{f}_0 \oplus \mathfrak{q}_+$ where $\mathfrak{q} = \mathfrak{f}_0 \oplus \mathfrak{q}_+$. The reductive part $\mathfrak{f}_0$ includes the semisimple part of $\mathfrak{q}$ and the rest of the Cartan subalgebra; $\mathfrak{q}_+$ is the nilradical of $\mathfrak{q}$.

It follows from the standard parabolic theory that irreducible representations of $\mathfrak{q}$ are irreducible representations of $\mathfrak{f}_0$ with the trivial action of $\mathfrak{q}_+$. Weights of representations of $\mathfrak{f}_0$ can be described by a labeled Dynkin diagram, where coefficients over non–crossed nodes are integers. Such a weight is dominant for $\mathfrak{q}$ (or, equivalently, the highest weight of $\mathfrak{f}_0$) if and only if the coefficients over non–crossed nodes are nonnegative.

For each set $\Sigma \subseteq \Pi$, and the corresponding parabolic subalgebra $\mathfrak{q} \subseteq \mathfrak{f}$, we define $W^{\mathfrak{q}} \subseteq W$ as a subset of all elements, which map the weights dominant for $\mathfrak{f}$ into the weights dominant for $\mathfrak{q}$. Equivalently, $W^{\mathfrak{q}}$ is the set of all elements $w$ for which the set $\Phi_w = w(\Delta_-) \cap \Delta_+$ contains only roots corresponding to $\mathfrak{q}_+$ i.e. the positive roots of $\mathfrak{f}$ which are not roots of the semisimple part of $\mathfrak{f}_0$ (see [3]).

**1.3. Cohomology of Lie algebras.** For a representation $\pi : \mathfrak{a} \longrightarrow \mathfrak{gl}(V)$ of a Lie algebra $\mathfrak{a}$ we define the differential $d : \mathrm{Hom}\,(\bigwedge^n \mathfrak{a}; V) \longrightarrow \mathrm{Hom}\,(\bigwedge^{n+1} \mathfrak{a}; V)$ by the formula

$$\begin{aligned}(dp)(X_0, \dots, X_n) = &\sum_{i<j}(-1)^{i+j}p([X_i, X_j], X_0, \dots \hat{X}_i \dots \hat{X}_j \dots, X_n)\\ &+\sum_i(-1)^i\pi(X_i)p(X_0, \dots \hat{X}_i \dots, X_n).\end{aligned}$$

The differential $d$ induces the cohomology $H^n(\mathfrak{a}; V)$, called the cohomology of $\mathfrak{a}$ with the coefficients in $V$ because $d^2 = 0$. We set $\mathrm{Hom}\,(\bigwedge^n \mathfrak{a}; V) = 0$ for $n < 0$ and $n > \dim \mathfrak{a}$.

On the complex level, we are interested only in the case, where $\mathfrak{a} = \mathfrak{q}_+$ and $\pi = \lambda'|\mathfrak{q}_+$ for some representation $\lambda' : \mathfrak{f} \longrightarrow \mathfrak{gl}(V)$ on a complex vector space $V$. This is completely solved in [3] and we will further use a lot of results therein without a specification. We have a natural representation $\beta' : \mathfrak{q} \longrightarrow \mathfrak{gl}(H(\mathfrak{q}_+; V))$ on the cohomology (for details see 1.5). This representation is completely reducible and thus it suffices to investigate the restriction $\beta' : \mathfrak{f}_0 \longrightarrow \mathfrak{gl}(H(\mathfrak{q}_+; V))$.

**Theorem 1.4.** [3] **Kostant's result.** *For a finite dimensional representation $\lambda' : \mathfrak{f} \longrightarrow \mathfrak{gl}(V)$ with the highest weight $\Lambda$ and the restriction $\pi = \lambda'|\mathfrak{q}_+$, the irreducible components of $\beta'$ are in bijective correspondence with the set $W^{\mathfrak{q}}$ and the multiplicity of each component is one. The highest weight of the irreducible component of the representation $\beta'$ corresponding to $w \in W^{\mathfrak{q}}$ is $w.\Lambda = w(\Lambda + \mathrm{R}) - \mathrm{R}$ and it occurs in degree $|w|$.*

**1.5. Complexification of the real cohomology.** Now we will consider a real semisimple algebra $\mathfrak{g}$ with the complexification denoted by $\mathfrak{f} = \mathfrak{g}(\mathbb{C})$ and a parabolic subalgebra $\mathfrak{p} \subseteq \mathfrak{g}$. That is the complexification $\mathfrak{q} = \mathfrak{p}(\mathbb{C})$ is a parabolic subalgebra of $\mathfrak{f}$. We have the decomposition $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{p}_+$ and $\mathfrak{f} = \mathfrak{f}_- \oplus \mathfrak{f}_0 \oplus \mathfrak{q}_+$ where $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{p}_+$ and $\mathfrak{q} = \mathfrak{f}_0 \oplus \mathfrak{q}_+$ such that the complex decomposition determines the real decomposition. Let us consider an irreducible representation $\lambda : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ on a real vector space $V$ with the complexification $\lambda(\mathbb{C}) : \mathfrak{f} \longrightarrow \mathfrak{gl}(V(\mathbb{C}))$. Our aim is to describe the cohomology $H(\mathfrak{p}_+; V)$ with respect to the restriction $\lambda|\mathfrak{p}_+$.

It follows from the structure of parabolic subalgebras that we have the natural action of the elements from $\mathfrak{q}$ on $\mathfrak{q}_+^*$ (the dual of the adjoint action) and on $V(\mathbb{C})$ (the restriction of $\lambda(\mathbb{C})$). This induces the representation of $\mathfrak{q}$ on $\mathrm{Hom}\,(\bigwedge^n \mathfrak{q}_+; V(\mathbb{C}))$, $n \in \mathbb{N}$ and there is a factorization $\beta' : \mathfrak{q} \longrightarrow \mathfrak{gl}(H(\mathfrak{q}_+, V(\mathbb{C})))$ on the cohomology, see [3]. It is completely reducible so we can consider $\beta' : \mathfrak{f}_0 \longrightarrow \mathfrak{gl}(H(\mathfrak{q}_+, V(\mathbb{C})))$. Moreover, there exists (see [3]) a subspace $H' \subseteq \bigwedge \mathfrak{q}_+^* \otimes V(\mathbb{C})$ which is isomorphic to $H(\mathfrak{q}_+, V(\mathbb{C}))$ as an $\mathfrak{f}_0$–module. Here we consider a tensor product of the dual of the adjoint action and $\lambda(\mathbb{C})|\mathfrak{f}_0$ on $H'$. This isomorphism $i : H' \longrightarrow H(\mathfrak{q}_+, V(\mathbb{C}))$ is given by the projection to the factor space (i.e. the cohomology space).

Similarly, we can define the completely reducible representation of $\mathfrak{p}$ on $H(\mathfrak{p}_+; V)$ i.e. the representation $\beta : \mathfrak{g}_0 \longrightarrow \mathfrak{gl}(H(\mathfrak{p}_+, V))$. Due to the isomorphism $i$, we can consider $\beta : \mathfrak{g}_0 \longrightarrow \mathfrak{gl}(H)$ where $H = H' \cap \bigwedge \mathfrak{p}_+^* \otimes V$. Since $\beta$ is the restriction of $\beta'$ and $H' = H(\mathbb{C})$, we get $\beta' = \beta(\mathbb{C})$. Therefore, we have shown that the complexification of the real cohomology is the cohomology of the complexified algebra and its representation.

**1.6. Satake diagrams and parabolic subalgebras.** We describe here the real form $\mathfrak{g}$ of the complex semisimple Lie algebra $\mathfrak{f} = \mathfrak{g}(\mathbb{C})$ in more details. Recall that the real simple algebra can be categorized in two ways: real form of a complex simple algebra and realification of a complex simple algebra. Let us remind that a realification of a complex vector space is the same set understood as a real vector space. A real Lie algebra is called *compact* if it admits an invariant scalar product. Each complex semisimple Lie algebra has a compact real form which is unique up to isomorphism [4, 5].

There is a 1–1 correspondence between the classes of real forms of a complex semisimple algebra $\mathfrak{f}$ up to isomorphism, the classes of involutive antiautomorphisms of $\mathfrak{f}$ up to conjugacy and the classes of involutive automorphisms of $\mathfrak{f}$ up to conjugacy. Involutive antiautomorphisms are called the *real structures* and they are just the complex conjugations given by real forms. Let us denote the involutive antiautomorphism and automorphism of $\mathfrak{f}$ corresponding to $\mathfrak{g}$ by $\sigma$ and $\theta$, respectively. This can be chosen in such a way that there exists a *compact* *structure* (i.e. a real structure corresponding to a compact real form) $\tau$ such that $\theta = \sigma\tau$ and $\theta$, $\sigma$, $\tau$ commute. See [4] for details. The involutive automorphism $\theta$ is then called the *Cartan involution*. Clearly $\theta(\mathfrak{g}) = \mathfrak{g}$ and it induces the *Cartan decomposition* $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{r}$ where $\mathfrak{l}$ is the $+1$–eigenspace and $\mathfrak{r}$ the $-1$–eigenspace of $\theta|\mathfrak{g}$. The Killing form of $\mathfrak{f}$ is negative definite on $\mathfrak{l}$ and positive definite on $\mathfrak{r}$. This implies that $\mathfrak{l}$ is a compact Lie algebra.

There is a diagramatic description of real semisimple Lie algebras, so called *Satake diagrams*. We remind their construction, for details see [8, 5]. There exists a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ such that $\theta(\mathfrak{h}) = \mathfrak{h}$ and $\mathfrak{h}_\mathfrak{r} = \mathfrak{h} \cap \mathfrak{r}$ is a maximal abelian subspace of $\mathfrak{r}$. It yields the decomposition $\mathfrak{h} = \mathfrak{h}_\mathfrak{l} \oplus \mathfrak{h}_\mathfrak{r}$ to the compact part $\mathfrak{h}_\mathfrak{l}$ and the real part $\mathfrak{h}_\mathfrak{r}$. The Cartan subalgebra $\mathfrak{h}(\mathbb{C}) \subseteq \mathfrak{f}$ yields the system of roots $\Delta$. Since $\sigma$ is an antiautomorphism of $\mathfrak{f}$, one can show that the mapping $\sigma^* : \Delta \longrightarrow \Delta$ given by the formula $\sigma^*\alpha(H) = \overline{\alpha(\sigma H)}$ for $\alpha \in \Delta$ and $H \in \mathfrak{h}(\mathbb{C})$ is an involutive automorphism of $\Delta$. The roots, which satisfy $\sigma^*\alpha = -\alpha$ are called *compact* and the ones which do not, are called *non–compact* roots. Let us denote the set of compact roots by $\Delta_c$; clearly $\Delta_c = \{\alpha \in \Delta \mid \alpha|\mathfrak{h}_\mathfrak{r} = 0\}$. A system of positive roots $\Delta_+$ can be found in such a way that $\sigma^*(\alpha) \in \Delta_+$ for each non–compact root $\alpha \in \Delta_+$. To obtain such a system $\Delta_+$, we consider the lexicographical ordering with respect to a base $H_1, \dots, H_n$ of $\mathfrak{h}$ such that the first elements $H_1, \dots, H_p$ constitute the base of $\mathfrak{h}_\mathfrak{r}$. The set of simple roots $\Pi$ then has the following property: if $\alpha \in \Pi$ is a non–compact root then there exists a unique non–compact root $\alpha' \in \Pi$ such that $(\sigma^*\alpha - \alpha')|\mathfrak{h}_\mathfrak{r} = 0$. (This is equivalent to the property $(\sigma^*\alpha' - \alpha)|\mathfrak{h}_\mathfrak{r} = 0$). The Satake diagram of $\mathfrak{g}$ is the Dynkin diagram of $\mathfrak{f}$ given by $\Pi$ where the compact roots are denoted by a black dot $\bullet$, non–compact roots by a white dot $\circ$ and if, for a non–compact root $\alpha \in \Pi$, the unique $\alpha' \in \Pi$ such that $(\sigma^*\alpha - \alpha')|\mathfrak{h}_\mathfrak{r} = 0$ is different from $\alpha$, then the two corresponding dots are joined by an arrow.

Parabolic subalgebras of $\mathfrak{g}$ can be again described by crossing out some vertices of the Satake diagram but there are restrictions, see [11]: we cannot cross out the compact roots and if we cross out some non–compact root $\alpha$, we must cross out any non–compact root $\alpha'$ connected to $\alpha$ by an arrow.

**1.7. Representations of real semisimple Lie algebras.** Facts about representations of real (semisimple) Lie algebras can be found in [4, 5], we will use the notation from [4]. First we consider an arbitrary real Lie algebra $\mathfrak{g}$. A *complex structure* on a real vector space $V$ is an automorphism $J : V \longrightarrow V$ such that $J^2 = -\mathrm{id}$. A *real (quaternionic) structure* on a complex vector space $V$ is an antiautomorphism $J : V \longrightarrow V$ such that $J^2 = \mathrm{id}$ ($J^2 = -\mathrm{id}$). Having a complex vector space $V$, we will denote the set $V$, understood as a real vector space, by $V_\mathbb{R}$ (the underlying real vector space of $V$). Let us consider a representation $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ on the complex vector space $V$ viewed as a space $V_\mathbb{R}$ with the complex structure $J$. We define a complex space $\bar{V}$ as the space $V_\mathbb{R}$ with the complex structure $-J$ and the *complex–conjugate* representation $\bar{\rho} : \mathfrak{g} \longrightarrow \mathfrak{gl}(\bar{V})$ on the complex space $\bar{V}$ such that $\rho = \bar{\rho}$ on the space $V_\mathbb{R} = \bar{V}_\mathbb{R}$. Let us fix a base on $V$ and denote $x \mapsto C(x)$, $x \in \mathfrak{g}$ the matrix form of $\rho$. Then the same base can be regarded as a base of $\bar{V}$ and the corresponding matrix form of $\bar{\rho}$ is given by the complex conjugate matrix $x \mapsto \overline{C(x)}$, $x \in \mathfrak{g}$. For a representation $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ on a real vector space $V$, we will denote its extension to the (complex) space $V(\mathbb{C})$

by $\rho^{\mathbb{C}} : \mathfrak{g} \longrightarrow \mathfrak{gl}(V(\mathbb{C}))$ and its complexification by $\rho(\mathbb{C}) : \mathfrak{g}(\mathbb{C}) \longrightarrow \mathfrak{gl}(V(\mathbb{C}))$. For a representation $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ on a complex vector space $V$, we will denote its extension to the (complex) algebra $\mathfrak{g}(\mathbb{C})$ by $\rho_{\mathbb{C}} : \mathfrak{g}(\mathbb{C}) \longrightarrow \mathfrak{gl}(V)$. The *complex–conjugate* representation of $\rho : \mathfrak{f} = \mathfrak{g}(\mathbb{C}) \longrightarrow \mathfrak{gl}(V)$ on a complex space $V$ with respect to the real form $\mathfrak{g}$ is the representation $\bar{\rho} = (\overline{\rho|\mathfrak{g}})_{\mathbb{C}}$ on a complex space $\bar{V}$. A representation $\rho$ of $\mathfrak{g}$ or $\mathfrak{f}$ on a complex vector space $V$ is called *self–conjugated* if $\rho \sim \bar{\rho}$ where $\sim$ denotes the isomorphism of representations in both real and complex case. Moreover, we will denote the realification of $\rho$ by $\rho^{\mathbb{R}} : \mathfrak{g} \longrightarrow \mathfrak{gl}(V_{\mathbb{R}})$.

Let us consider a representation $\lambda : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ on a real vector space $V$. The representation $\lambda$ is

- *quaternionic* (or of the *quaternionic type*) if there exists a complex structure on $V$ and a quaternionic structure on $V$ (understood as a complex space), both commuting with the action of $\mathfrak{g}$
- *complex* (or of the *complex type*) if there exists a complex structure on $V$ commuting with the action of $\mathfrak{g}$ and $\lambda$ is not quaternionic
- *real* (or of the *real type*) if there is no complex structure on $V$ commuting with the action of $\mathfrak{g}$.

Let us suppose that $\lambda$ is irreducible. The complexification depends on the type in the following way. If $\lambda$ is real then $\lambda(\mathbb{C})$ is irreducible too and $\lambda(\mathbb{C}) \sim \overline{\lambda(\mathbb{C})}$. If $\lambda$ is complex (quaternionic) then the space $V$ can be understood as a complex vector space and $\lambda(\mathbb{C}) \sim \lambda_{\mathbb{C}} \oplus \bar{\lambda}_{\mathbb{C}}$ and $\lambda_{\mathbb{C}} \not\sim \bar{\lambda}_{\mathbb{C}}$ $(\lambda_{\mathbb{C}} \sim \bar{\lambda}_{\mathbb{C}})$.

The self–conjugacy condition appears in both real and quaternionic representationans so we need some other tool to distinguish these two types. The irreducible self–conjugate representation $\gamma : \mathfrak{f} \longrightarrow \mathfrak{gl}(V)$ on the complex vector space $V$ with the highest weight $\Gamma$ admits an antiautomorphism $J : V \longrightarrow V$ commuting with $\gamma|\mathfrak{g}$ such that $J^2 \in \{+\mathrm{id}, -\mathrm{id}\}$. We define the *index* $\varepsilon(\mathfrak{g}, \gamma)$ as the sign. A correctness of this definition is shown in [5, 4]. (An index $+1$ indicates that $\gamma$ can be obtained as the complexification of some real irreducible representation of $\mathfrak{g}$ and $-1$ indicates that $\gamma$ is a part of the complexification of some quaternionic irreducible representation of $\mathfrak{g}$.) In the case of semisimple (reductive) algebras, we will sometimes write $\varepsilon(\mathfrak{g}; \Gamma)$ where $\Gamma$ is the highest weight of $\gamma$.

Henceforth we will suppose that $\mathfrak{g}$ is a real form of a complex semisimple Lie algebra $\mathfrak{f}$. Each involutive automorphism of $\mathfrak{f}$ induces a symmetry of the Dynkin diagram of $\mathfrak{f}$ and thus we can consider the symmetry $s$ induced by the Cartan involution $\theta$ corresponding to $\mathfrak{g}$. We will describe irreducible representations of $\mathfrak{f}$ with the help of their highest weights as given by the vector of coefficients over the Dynkin diagram. Furthermore, we will denote the symmetry of the Dynkin diagram which realizes dual weights by $\nu$. Since coefficients of the highest weights correspond to nodes of the Dynkin diagram, we can consider symmetries of the diagram on highest weights too. Section 2. shows how to identify $s$ and $\nu$ on the Satake diagram. These two symmetries commute and allow us to describe the relation between the irreducible representations $\lambda$ and $\bar{\lambda}$ of $\mathfrak{g}$ on complex spaces $V$ and $\bar{V}$. Denoting the highest weights of $\lambda_{\mathbb{C}}$ $(\bar{\lambda}_{\mathbb{C}})$ by $\Lambda$ $(\bar{\Lambda})$, it turns out $\bar{\Lambda} = s\nu(\Lambda)$. These facts and formulas for indices of semisimple Lie algebras can be found in [4, 5], for details see Section 5.

**1.8. Hasse graph on the cohomology.** We can define a structure of a Hasse graph on the cohomology. In the complex case given by a parabolic subalgebra $\mathfrak{q}$ of the semisimple algebra $\mathfrak{f}$, the set of vertices the Hasse graph is $W^{\mathfrak{q}}$ (or equivalently, the set of irreducible components in the cohomology, see 1.4). Further, there is an arrow $w_1 \longrightarrow w_2$, $w_1, w_2 \in W^{\mathfrak{q}}$ if and only if $w_2 = s_\alpha w_1$ where $s_\alpha$ is a reflection corresponding to a root $\alpha \in \Delta$ and $|w_2| = |w_1| + 1$.

In the case of the real cohomology, given by a parabolic subalgebra $\mathfrak{p}$ of a semisimple algebra $\mathfrak{g}$, the Hasse graph also depends on the representation $\lambda : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$. In particular it depends on its type (see Section 6. for details). We define the set of vertices as the set of irreducible components in the cohomology. Furthermore, there is an arrow $\beta_1 \longrightarrow \beta_2$ where $\beta_1, \beta_2$ are irreducible components of the representation $\beta : \mathfrak{g} \longrightarrow \mathfrak{gl}(H(\mathfrak{p}_+; V))$ if and only if:
(a) $\lambda$ is complex or quaternionic and there is an arrow between $(\beta_1)_{\mathbb{C}}$ and $(\beta_2)_{\mathbb{C}}$ in the complex cohomology $H(\mathfrak{p}_+(\mathbb{C}); V)$ in the given direction, or
(b) $\lambda$ is real and there is some arrow between the component(s) $\beta_1(\mathbb{C})$ and $\beta_2(\mathbb{C})$ in the complex cohomology $H(\mathfrak{p}_+(\mathbb{C}); V(\mathbb{C}))$ in the given direction.
(The correctness of this definition follows from 1.5 for (b) and from 6.2 for (a).)

## 2. Symmetries of diagrams

Now we show how to see the discussed symmetries of the Satake diagrams. Let us consider a real form $\mathfrak{g}$ of a complex semisimple Lie algebra $\mathfrak{f}$ and the corresponding Cartan involution $\theta$ which induces the symmetry $s$ of the Dynkin diagram. Further, let us denote the symmetry which realizes the dual weights of $\mathfrak{f}$ by $\nu$. We can consider the system of simple roots $\Pi$ in such a way that $\Pi$ induces both the Satake diagram of $\mathfrak{g}$ and the Dynkin diagram of $\mathfrak{f}$ (i.e. $\Pi$ satisfies the properties from 1.6). Then we can see all these symmetries on both diagrams. Moreover, let us denote the symmetry induced by the arrows of the Satake diagram by $a$. We will show that except for the exceptional cases to be described, it turns out that $a = s\nu$. Let us note that each involutive symmetry of the Dynkin diagram induces an involution of $\mathfrak{f}$.

**2.1.** We will use the notation from 1.6. Further we denote the set of non–compact roots $\Delta \setminus \Delta_c$ by $\Delta_{nc}$ and the corresponding sets of positive roots by $\Delta_{nc}^+$ and $\Delta_c^+$. This determines a decomposition on the set of simple roots to $\Pi_c = \Pi \cap \Delta_c$ and $\Pi_{nc} = \Pi \cap \Delta_{nc}$. Let us consider the compact subalgebra $\mathfrak{l}_c \subseteq \mathfrak{l}$ corresponding to the root system $\Delta_c$. The Weyl group $W_c$ of $\mathfrak{l}_c(\mathbb{C})$ can be understood as a subgroup of the Weyl group $W$ of $\mathfrak{f}$. There is a unique element $w_0 \in W$ ($w_0^c \in W_c$) such that $w_0(\Pi) = -\Pi$ ($w_0^c(\Pi_c) = -\Pi_c$). The dual of the involution $\hat{\nu}$ induced by $\nu$ satisfies $\hat{\nu}^* = -w_0$, see [4]. Similarly, $\hat{\nu}_c^* = -w_0^c$ where $\nu_c$ is the corresponding symmetry of $\Pi_c$.

Now we "improve" the involution $\theta$ to see the induced symmetry $s$ on the system of simple roots $\Pi$. The automorphism $\theta$ induces a dual mapping on $\Delta$ defined as $(\theta^*\alpha)(H) = \alpha(\theta H)$ for $\alpha \in \Delta$ and $H \in \mathfrak{h}(\mathbb{C})$. Clearly $\theta^*|\Delta_c = \mathrm{id}$ and it follows form properties of a special basis used for the definition of $\Delta_+$ (see 1.6) that $\theta^*(\Delta_{nc}^+) \subseteq \Delta_{nc}^-$. Let us remind that a group of *inner automorphisms* of $\mathfrak{f}$ is a subgroup of Aut $\mathfrak{f}$ generated by all $\exp(\mathrm{ad}X)$, $X \in \mathfrak{f}$. The symmetry induced on $\Pi$ by inner automorphisms of $\mathfrak{f}$ is the identity [4, 5]. Since the elements of the Weyl group are induced by inner automorphism of $\mathfrak{f}$, the composition $w_0^c\theta^*$ induces the same symmetry as $\theta^*$. The form of the Weyl reflections corresponding to the root $\alpha$ is $S_\alpha(\beta) = \beta - 2\frac{(\beta,\alpha)}{(\alpha,\alpha)}\alpha$ for $\beta \in \Delta$ and thus $S_\alpha(\beta) \in \Delta_{nc}^+$ for $\alpha \in \Delta_c$ and $\beta \in \Delta_{nc}^+$. This implies that $w_0^c(\Delta_{nc}^+) = \Delta_{nc}^+$. Since $w_0^c(\Delta_c^+) = \Delta_c^-$, we have shown that $w_0^c\theta^*(\Delta_+) = \theta^* w_0^c(\Delta_+) = \Delta_-$. It follows that $w_0 w_0^c \theta^*$ fixes the set $\Pi$ and induces the symmetry $w_0 w_0^c\theta^*|\Pi = s$.

Now we describe the relation between $w_0 w_0^c\theta^*$ and $\sigma^*$. It is easy to see that $\sigma^* = -\theta^*$. We have $w_0^c\sigma^* = -w_0^c\theta^* = (\hat{\nu}^* w_0)w_0^c\theta^*$ and thus $w_0^c\sigma^*|\Pi = \nu s$. Considering a simple root $\alpha \in \Pi_{nc}$, it is easy to see that $w_0^c(\alpha) = \alpha + \sum_{\beta\in\Pi_c} c_\beta\beta$ where $c_\beta \geq 0$ for $\beta \in \Pi_c$. Further, it follows from the construction of Satake diagrams that $\sigma^*(\alpha) = \alpha' + \sum_{\beta\in\Pi_c} d_\beta\beta$, where $\alpha' \in \Pi_{nc}$ and $d_\beta \geq 0$ for $\beta \in \Pi_c$; $\alpha$ and $\alpha'$ are connected by an arrow in the Satake diagram if $\alpha \neq \alpha'$. Since $w_0^c\sigma^*(\alpha) = \nu s(\alpha)$, this root is simple and from the expressions for $\sigma^*$ and $w_0^c$ it follows that $w_0^c\sigma^*(\alpha) = \alpha' = \nu s(\alpha)$. This shows that the symmetry $\nu s$ coincides with the arrows of the Satake diagram on non–compact simple roots. Next, we consider a simple root $\alpha \in \Pi_c$. Since $\sigma^*(\alpha) = -\alpha$, we have $w_0^c\sigma^*(\alpha) = -w_0^c(\alpha) = \nu_c(\alpha) = \nu s(\alpha)$.

**Theorem 2.2.** *Suppose $\mathfrak{g}$ is a real semisimple Lie algebra and let us consider the corresponding symmetries $s, \nu, a$ on its Satake diagram.*
*(i) Upon restriction to the non-compact roots we have $s\nu = a$ and upon restriction to the compact roots we have $s\nu = \nu_c$.*
*(ii) If $\mathfrak{g}$ is simple, then $a = s\nu$ for all (simple) real Lie algebras with the following exceptions: $\mathfrak{su}_n$, $\mathfrak{so}_{k,2n-k}$ where $k, n$ have different parity and the compact form of $E6$. (Note that the Satake diagrams of $\mathfrak{su}_n$ and the compact form of $E6$ have only the compact roots and in the case of $\mathfrak{so}_{n,2n-k}$ where $k, n$ have different parity, the number of the compact roots is even.)*

**Proof.** It remains to prove (ii). Let us suppose that $\mathfrak{g}$ is a real simple Lie algebra. If $s\nu$ is non–trivial on non–compact roots then $a = s\nu$ because there is only one possibility to extending this symmetry from the non–compact roots to the whole Satake diagram. If $\mathfrak{f}$ is simple then it is easy to check it case by case. If $\mathfrak{f}$ is not simple then it follows from the definition of the Satake diagrams that $\Pi_{nc} = \Pi$. If $s\nu$ is trivial on the non–compact roots (i.e. $a = \mathrm{id}$), there are some exceptions which satisfy $a \neq s\nu$: $\mathfrak{su}_n$, $\mathfrak{so}_{k,2n-k}$ where $k, n$ have different parity and the compact form of $E6$. It is easy to verify this fact case-by-case. ■

## 3. Formulation of the problem

**3.1. Notation.** Henceforth we will use the following notation. We will consider sets of roots $\Pi \subseteq \Delta_+ \subseteq \Delta$ as described in 1.6 and a parabolic subalgebra given by the set $\Sigma \subseteq \Pi$ (see 1.2 and 1.6). As in 1.5, we will consider a real semisimple Lie algebra $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{p}_+$ with the complexification $\mathfrak{f} = \mathfrak{f}_- \oplus \mathfrak{f}_0 \oplus \mathfrak{q}_+$ where $\mathfrak{f}_- = \mathfrak{g}_-(\mathbb{C})$, $\mathfrak{f}_0 = \mathfrak{g}_0(\mathbb{C})$, $\mathfrak{q}_+ = \mathfrak{p}_+(\mathbb{C})$ and parabolic subalgebras are $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{p}_+$ and $\mathfrak{p}(\mathbb{C}) = \mathfrak{q} = \mathfrak{f}_0 \oplus \mathfrak{q}_+$. The decompositions $\mathfrak{g}_0 = \mathfrak{g}_0^{ss} \oplus \mathfrak{z}$ and $\mathfrak{f}_0 = \mathfrak{f}_0^{ss} \oplus \mathfrak{z}(\mathbb{C})$ give the semisimple part and the center of $\mathfrak{g}_0$ and $\mathfrak{f}_0$, respectively. The construction of the Satake diagram from 1.6 gives the Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ with the complexification $\mathfrak{h}(\mathbb{C}) \subseteq \mathfrak{f}$. The Weyl group of $f$ will be denoted by $W$.

The set of simple roots $\Pi$ induces both the Satake diagram of $\mathfrak{g}$ and the Dynkin diagram of $\mathfrak{f}$. Therefore we can consider all symmetries on both diagrams. Let us denote by $\sigma$ ($\theta$) the real structure (Cartan involution) corresponding to the real form $\mathfrak{g}$ of $\mathfrak{f}$ and $s$ and $\nu$ symmetries of a diagram where $s$ is the symmetry induced by $\theta$ and $\nu$ is the symmetry which realizes dual weights of $\mathfrak{f}$, see Section 2. Furthermore, let us consider the symmetry $a$ induced by arrows of the Satake diagram. Let us denote by $s', \nu', a'$ the corresponding symmetries of the diagrams of $\mathfrak{g}_0^{ss}$ and $\mathfrak{f}_0^{ss}$ with respect to the real form $\mathfrak{g}_0^{ss}$ of $\mathfrak{f}_0^{ss}$. Then $s'\nu'$ is the restriction of $s\nu$. This follows from the fact that $s\nu = a$ in many cases and the symmetries $a$ and $a'$ satisfy this condition. The exceptions must be discussed case by case. Furthermore, we will not distinguish between $s\nu$ and $s'\nu'$ and both of them will be denoted by $s\nu$. Let us note that the symmetry $s'$ is not a restriction of $s$ and similarly, $\nu'$ is not a restriction of $\nu$.

**3.2. Complexification and realification of the cohomology for complex and quaternionic $\mathfrak{g}$–representations.** Let us start with an arbitrary real Lie algebra $\mathfrak{a}$ with a representation $\pi : \mathfrak{a} \longrightarrow \mathfrak{gl}(V)$ on a complex vector space $V$ and consider the representations $\pi^{\mathbb{R}} : \mathfrak{a} \longrightarrow \mathfrak{gl}(V_{\mathbb{R}})$ and $\pi_{\mathbb{C}} : \mathfrak{a}(\mathbb{C}) \longrightarrow \mathfrak{gl}(V)$. Our aim is to compare all three induced cohomologies. The mapping $\_{}_{\mathbb{C}} : \operatorname{Hom}(\bigwedge \mathfrak{a}; V) \longrightarrow \operatorname{Hom}(\bigwedge \mathfrak{a}(\mathbb{C}); V)$ which extends a given multilinear mapping from $\mathfrak{a}$ to $\mathfrak{a}(\mathbb{C})$, is an isomorphism of complex vector spaces. Similarly, we have the identification $\operatorname{Hom}(\bigwedge \mathfrak{a}; V)_{\mathbb{R}} = \operatorname{Hom}(\bigwedge \mathfrak{a}; V_{\mathbb{R}})$ because the complex structure on $\operatorname{Hom}(\bigwedge \mathfrak{a}; V)_{\mathbb{R}}$ is determined by the complex structure on $V$. It follows from the definition of the differential $d$, that $d(p_{\mathbb{C}}) = (dp)_{\mathbb{C}}$ and so the cohomologies $H(\mathfrak{a}; V)$ and $H(\mathfrak{a}(\mathbb{C}); V)$ are isomorphic as complex vector spaces. Similarly, the definition of $d$ follows that $H(\mathfrak{a}; V)$ and $H(\mathfrak{a}; V_{\mathbb{R}})$ are isomorphic as real spaces.

Now we consider the case $\mathfrak{a} = \mathfrak{p}_+$ with the representations $\pi = \lambda|\mathfrak{p}_+$, $\pi_{\mathbb{C}} = \lambda_{\mathbb{C}}|\mathfrak{q}_+$ and $\pi^{\mathbb{R}} = \lambda^{\mathbb{R}}|\mathfrak{p}_+$ where $\lambda : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ is a representation on a complex vector space $V$. We are interested in the representation $\beta : \mathfrak{g}_0 \longrightarrow \mathfrak{gl}(H(\mathfrak{p}_+; V))$ on the cohomology corresponding to $\pi$ where we understand $H(\mathfrak{p}_+; V)$ as a complex space. Since the actions of $X \in \mathfrak{g}_0$ on spaces $\operatorname{Hom}(\bigwedge \mathfrak{p}_+; V)$ and $\operatorname{Hom}(\bigwedge \mathfrak{q}_+; V)$ commute with the isomorphism $\_{}_{\mathbb{C}}$, the induced representations of $\mathfrak{g}_0$ on $H(\mathfrak{p}_+; V)$ and $H(\mathfrak{q}_+; V)$ are isomorphic. This follows that we can understand the representation of $\mathfrak{f}_0$ on $H(\mathfrak{q}_+; V)$ as the representation $\beta_{\mathbb{C}} : \mathfrak{f}_0 \longrightarrow \mathfrak{gl}(H(\mathfrak{q}_+; V))$. Similarly, we can consider the representation of $\mathfrak{g}_0$ on $H(\mathfrak{p}_+; V_{\mathbb{R}})$ as the representation $\beta^{\mathbb{R}} : \mathfrak{g}_0 \longrightarrow \mathfrak{gl}(H(\mathfrak{p}_+; V_{\mathbb{R}}))$.

Kostant's result 1.4 gives the explicit description of $\beta_{\mathbb{C}}$ and we need to describe $\beta^{\mathbb{R}} = (\beta_{\mathbb{C}}|\mathfrak{g}_0)^{\mathbb{R}}$. The following simple lemma says when the restriction $(\_|\mathfrak{g})^{\mathbb{R}}$ preserves the irreducibility. It is the opposite result to the complexification of an irreducible $\mathfrak{g}$-representation in 1.7.

**Lemma 3.3.** *Let us consider an arbitrary real Lie algebra $\mathfrak{g}$ with the complexification $\mathfrak{f} = \mathfrak{g}(\mathbb{C})$ and an irreducible representation $\gamma : \mathfrak{f} \longrightarrow \mathfrak{gl}(V)$. Then the following holds:*
*(i) $(\gamma|\mathfrak{g})^{\mathbb{R}}$ is irreducible if and only if $\gamma \not\sim \bar{\gamma}$ or $\gamma \sim \bar{\gamma}$ and $\varepsilon(\mathfrak{g}, \gamma) = -1$*
*(ii) $(\gamma|\mathfrak{g})^{\mathbb{R}}$ is reducible if and only if $\gamma \sim \bar{\gamma}$ and $\varepsilon(\mathfrak{g}, \gamma) = 1$.*

**Proof.** It is sufficient to prove only the assertion $(ii)$. The implication $\Longleftarrow$ is clear. If $(\gamma|\mathfrak{g})^{\mathbb{R}}$ is reducible then a $\gamma|\mathfrak{g}$-invariant subspace $W \subsetneq V_{\mathbb{R}}$ exists. The subspace $iW$ is also invariant. Therefore $W \cap iW$ and $W + iW$ are $\gamma$-invariant subspaces. It shows that $W \cap iW = 0$ and so $V = W(\mathbb{C})$. ■

**3.4.** Using the previous observation, we are going to formulate explicitly what we need to know for a description of the real cohomology. In the case of complex and quaternionic $\mathfrak{g}$–representations we use Lemma 3.3 and the paragraph before that lemma. Therefore we need to catalogue complex–conjugate representations and indices for the reductive algebras $\mathfrak{g}_0$. Moreover, we will determine the relation between $\varepsilon(\mathfrak{g}; \Lambda)$ for a self–conjugate $\mathfrak{f}$–dominant weight $\Lambda$ and $\varepsilon(\mathfrak{g}_0; w.\Lambda)$ for a self–conjugate $\mathfrak{f}_0$–dominant weight $w.\Lambda$ where $w$ is an element of the Weyl group of $\mathfrak{f}$. In the case of a real $\mathfrak{g}$–representation, we describe the real cohomology with the help of its complexification along the lines described in 1.5. That is we show which irreducible components (couples of irreducible components) in the complex cohomology correspond to an irreducible component in the real cohomology. We will again need to know the complex–conjugation for the Lie algebra $\mathfrak{g}_0$.

## 4. Conjugate representations of reductive algebras

Considering an irreducible representation $\gamma : \mathfrak{f}_0 \longrightarrow \mathfrak{gl}(V)$ on a complex vector space $V$, our aim is to describe the complex–conjugate representation $\bar{\gamma} : \mathfrak{f}_0 \longrightarrow \mathfrak{gl}(\bar{V})$ with respect to the real form $\mathfrak{g}_0$. We denote the highest weight of $\gamma$ and $\bar{\gamma}$ by $\Gamma$ and $\bar{\Gamma}$ and we will understand them as vectors of coefficients over diagrams. Recall that in the case of $\mathfrak{f}_0$–dominant weights, the coefficients over non–crossed nodes are nonnegative integers and the coefficients over crosses are arbitrary real numbers.

The $i$th fundamental weight of $\mathfrak{f}$ or $\mathfrak{f}_0$ is a weight with the coefficient 1 over the $i$th node and the coefficient 0 over the remaining nodes. The corresponding representation is called the $i$th fundamental representation of $\mathfrak{f}$ or $\mathfrak{f}_0$. That is the $i$th fundamental representation of $\mathfrak{f}_0$ is an irreducible component of the $i$th fundamental representation of $\mathfrak{f}$ restricted to $\mathfrak{f}_0$ generated by the vector of the highest weight. In particular, the fundamental representations of $\mathfrak{f}_0$ corresponding to crosses (i.e. with the coefficient 1 over a cross) are one dimensional. Clearly the center of $\mathfrak{g}_0$ acts by some (possibly complex) scalar in irreducible representations.

**4.1.** First we claim that it is sufficient to treat complex conjugation for the fundamental representations of $\mathfrak{f}_0$. This follows since $\overline{\Gamma_1 + \Gamma_2} = \bar{\Gamma}_1 + \bar{\Gamma}_2$ for $\mathfrak{f}_0$–dominant weights $\Gamma_1$ and $\Gamma_2$. This relation holds for coefficients over non-crossed nodes because after the restriction to the semisimple part, the complex conjugation is given by the symmetry $s\nu$. Coefficients over crosses are given by the action of the center. This action on the vector of the highest weight is given directly by the highest weight understood as a form on the Cartan subalgebra of $\mathfrak{f}$ (which includes the center). The relation above now follows from the fact that the scalar action of the center $\mathfrak{g}_0$ in a complex–conjugate representation is given by the complex–conjugation in $\mathbb{C}$ (see 1.7). The same argument implies that if $\rho_i$ is a fundamental representation corresponding to a cross then $\overline{r\rho_i} = r\bar{\rho}_i$, $r \in \mathbb{R}$.

The description of the complex–conjugation for the fundamental weights corresponding to the non–crossed vertices is given by the symmetry $s\nu$ so we need to consider only fundamental representations corresponding to crosses. In this case, we use the structure of the zero cohomology. Let us consider a fundamental representation $\rho_i : \mathfrak{f}_0 \longrightarrow \mathfrak{gl}(V_i)$ with a highest weight $\Gamma_i$ corresponding to the $i$th cross. Understanding $\Gamma_i$ as an $\mathfrak{f}$–dominant weight, we get corresponding fundamental representation $\rho_i' : \mathfrak{f} \longrightarrow \mathfrak{gl}(V_i')$ with this highest weight and we can suppose $V_i \subseteq V_i'$. Vertices in the Satake diagram of $\mathfrak{f}$ corresponding to the crosses in the Satake diagram of $\mathfrak{f}_0$ are non–compact and so we have two possibilities according to the arrows:

I. First let us suppose that the $i$th cross has no arrow. Then the representation $\rho_i'$ is self–conjugate because its highest weight is symmetric according to the symmetry $s\nu$ (the coefficient 1 is over a vertex without an arrow and all remaining coefficients are 0, cf. Theorem 2.2). The formulas in 5.1 show that in that case, the index is +1 because all "quaternionic" vertices have the coefficient 0. Thus $\rho_i'$ is the complexification of some representation of $\mathfrak{g}$ and according to 1.5, the same happens with the representation of $\mathfrak{f}_0$ on the zero cohomology $H^0(\mathfrak{q}_+; V_i')$. This is a representation with the highest weight the same as in the representation $\rho_i$, see 1.4. It shows that $\rho_i$ is self–conjugate.

II. Now let us suppose that the $i$th vertex has an arrow. Then the representation $\rho_i'$ is not self–conjugate because in this case, the symmetry $s\nu$ (for $\mathfrak{f}$) coincides with arrows according to Theorem 2.2 and there is an arrow connecting vertices with coefficients 1 and 0. We use the (reducible) representation $\rho_i' \oplus \overline{\rho_i'} : \mathfrak{f} \longrightarrow \mathfrak{gl}(V_i' \oplus \overline{V_i'})$ with the highest weights $\Gamma_i$ and $\bar{\Gamma}_i = s\nu(\Gamma_i) \neq \Gamma_i$ which is the complexification of an irreducible representation of $\mathfrak{g}$, see 1.7. In the zero cohomology $H^0(\mathfrak{q}_+; V_i' \oplus \overline{V_i'})$, which is again the complexification of some representation of $\mathfrak{g}_0$, we have the components with the highest weights $\Gamma_i$ and $s\nu(\Gamma_i)$ understood as weights of $\mathfrak{f}_0$. This implies that the representation $\rho_i$ is either self–conjugate or $\bar{\rho}_i$ has the highest weight $s\nu(\Gamma_i)$. But we will show below that there is an element $W \in \mathfrak{g}_0$ which acts by some non–zero non–real scalar in $\rho_i$ (or, equivalently, on the vector of the highest weight in $\rho_i'$ because $\rho_i$ is one dimensional). It follows from the definiton of complex–conjugation that in $\bar{\rho}_i$, this element must act by a complex–conjugate i.e. by a different scalar. Hence $\bar{\rho}_i \not\simeq \rho_i$ and thus $\bar{\rho}_i$ has the highest weigth $s\nu(\Gamma_i)$.

Now we will show the existence of $W \in \mathfrak{g}_0$ with the required properties. Let us denote the root corresponding to the discussed cross by $\alpha$ and the root connected by an arrow by $\alpha'$; the corresponding root elements will be denoted by $H_\alpha$ and $H_{\alpha'}$. The real structure $\sigma$ is just complex conjugation corresponding to the real form $\mathfrak{g} \subseteq \mathfrak{f}$. We describe some properties of $H_\alpha - H_{\alpha'} \in \mathfrak{f}$. Denoting $\langle , \rangle$ the Killing form on $\mathfrak{f}$, we can compute, for any simple root $\tilde{\alpha} \in \Pi$, that $\tilde{\alpha}(H_\alpha - H_{\alpha'}) = \langle \tilde{\alpha}, \alpha \rangle - \langle \tilde{\alpha}, \alpha' \rangle$ and after applying $\sigma$ to $H_\alpha - H_{\alpha'}$ we get

$$\tilde{\alpha}(\sigma(H_\alpha - H_{\alpha'})) = (\sigma^* \tilde{\alpha})(H_\alpha - H_{\alpha'}) = \langle \sigma^* \tilde{\alpha}, \alpha \rangle - \langle \sigma^* \tilde{\alpha}, \alpha' \rangle = \langle \tilde{\alpha}, \sigma^* \alpha \rangle - \langle \tilde{\alpha}, \sigma^* \alpha' \rangle$$

because $\sigma^*$ is an involutive automorphism of the root system. Since $\alpha$ is a non–compact root, it follows from the construction of the Satake diagrams that $\sigma^* \alpha = \alpha' + \sum_{\beta \in \Pi_c} d_\beta \beta$ where $\Pi_c$ denotes the system of compact simple roots and the coefficients $d_\beta$ are nonnegative integers. This follows that $\sigma^* \alpha' = \alpha + \sum_{\beta \in \Pi_c} d_\beta \beta$ because $\sigma^* \beta = -\beta$ for each compact root $\beta$ and $\sigma^*$ is an involution. Thus $\tilde{\alpha}(\sigma(H_\alpha - H_{\alpha'})) = -\tilde{\alpha}(H_\alpha - H_{\alpha'})$. Since this holds for each simple root $\tilde{\alpha} \in \Pi$, it shows that $\sigma(H_\alpha - H_{\alpha'}) = -(H_\alpha - H_{\alpha'})$. Now we put $W := i(H_\alpha - H_{\alpha'}) \in \mathfrak{g}_0$; this element acts by the scalar $i$ in $\rho_i$ because it is a fundamental representation corresponding to $\alpha$.

Summarizing, we have proved the following theorem:

**Theorem 4.2.** *Let us consider a semisimple Lie algebra $\mathfrak{g}$ with complexification $\mathfrak{f} = \mathfrak{g}(\mathbb{C})$ and their reductive subalgebras $\mathfrak{g}_0 \subseteq \mathfrak{f}_0 = \mathfrak{g}_0(\mathbb{C})$ given by the Satake diagram with crosses. If an irreducible representation of $\mathfrak{f}_0$ has the highest weight $\Gamma$, understood as a vector of coefficients over the diagram, then the complex–conjugate representation (with respect to $\mathfrak{g}_0$) has the highest weight $s\nu(\Gamma)$ where the latter symmetries are given by $\mathfrak{g} \subseteq \mathfrak{f}$, see Section 2..* ∎

**Remark 4.3.** In this section, we have considered only reductive Levi factors of parabolic subalgebras. Their complex–conjugate representations were described by using of the structure of the zero cohomology. Another possibility is to evaluate the action of the center. This is useful for arbitrary reductive algebra.

## 5. Indices

**5.1. Indices of semisimple algebras.** Using the notation from 3.1, we show how to compute the indices $\varepsilon(\mathfrak{g}, \Lambda)$ for an irreducible self–conjugate $\mathfrak{f}$–representation with the highest weight $\Lambda$. First let us note that indices of semisimple cases are products of indices of the simple parts (and corresponding restrictions of $\Lambda$). There is a general procedure for computing the indices which gives formulas for all (semi)simple Lie algebras. This can be found in [4]. Another way how to compute the indices (which we will need in the proof of the next theorem) is to begin with the fundamental representations. The set of vertices of the Dynkin diagram of $\mathfrak{f}$ can be identified to the set of the fundamental weights $F$ and let us denote, by $Q \subseteq F$ the set of quaternionic fundamental weights. Then the self–conjugate representation with the highest weight $\Lambda = (\Lambda_i)$, $i \in F$ is real (quaternionic) if the sum $\sum_{i \in Q} \Lambda_i$ is an even (odd) number. This fact and the indices of fundamental weights can be found in [10].

We are going to demonstrate both the resulting formulae and the types of fundamental representations in the summary below. This case–by–case discussion follows that all quaternionic fundamental representation correspond to the compact roots of the Satake diagram. In all cases, the parameter $l$ denotes the number of nodes of a diagram. The fundamental representation $\rho_i$ corresponds to the coefficient $\Lambda_i$ i.e. it denotes the representation with the highest weight $(0, \dots, 0, 1, 0, \dots, 0)$ where the 1 is in the position $i \in F$ and 0 on the remaining positions. Simple real forms not mentioned below have index +1 for each self–conjugate $\mathfrak{f}$–representation i.e. they do not admit quaternionic representations. The notation of real forms and the corresponding Satake diagrams used, is as in Tables in [5]. The index is given by a formula on the end of the first line of each item.

- $\mathfrak{su}_{p,l+1-p}$, $l$ odd, $l+1 = 2m$, $s\nu \neq \mathrm{id}$ ........................ $(-1)^{(m-p)\Lambda_m}$
  The unique self–conjugate representation is $\rho_m$. This representation is real

(quaternionic) if $m-p$ is even (odd). Note that $m-p$ is the "bigger half" of the (odd) number of compact nodes in the Satake diagram.

- $\mathfrak{sl}_m(\mathbb{H})$, $l=2m-1$, $s\nu = \mathrm{id}$ ............................ $(-1)^{\Lambda_1+\Lambda_3+\cdots+\Lambda_l}$
All fundamental representations are self–conjugate. The representation $\rho_i$ is real (quaternionic) if $i$ is even (odd). In the other words, fundamental representations corresponding to compact nodes are quaternionic and the remaining are real.

- $\mathfrak{so}_{p,2l+1-p}$, $s\nu = \mathrm{id}$, $p=2k$, ............................... $(-1)^{(k+\frac{l(l-1)}{2})\Lambda_l}$
First let us note the the assumption $p=2k$ is no restriction due to the isomorphism $\mathfrak{so}_{q,r} \simeq \mathfrak{so}_{r,q}$. All fundamental representations are self–conjugate. Representations $\rho_1,\dots,\rho_{l-1}$ are real. The representation $\rho_l$ is real (quaternionic) if $l-p \equiv 0$ or $3$ ($l-p \equiv 1$ or $2$), all cases modulo $4$. Note that $l-p$ is the number of compact nodes in the Satake diagram.

- $\mathfrak{sp}_{p,l-p}$, $s\nu = \mathrm{id}$ ................................. $(-1)^{\Lambda_1+\Lambda_3+\cdots+\Lambda_{2[\frac{1}{2}(l+1)]-1}}$
Here we denote the integer part of $a \in \mathbb{R}$ by $[a]$. All fundamental representations are self–conjugate. The representation $\rho_i$ is real (quaternionic) if $i$ is even (odd).

- $\mathfrak{so}_{p,2l-p}$. The fundamental representations $\rho_1,\dots,\rho_{l-2}$ are self–conjugate and real. Futher we distinguish two possibilities according to the parity of $l-p$. Note that $l-p$ is the number of compact nodes in the Satake diagram with the exception of the case $\mathfrak{so}_{l-1,l+1}$. In this case, the diagram has an arrow and no compact node and $l-p=1$.
  - $l-p$ even, $s\nu = \mathrm{id}$ ................................. $(-1)^{\frac{l-p}{2}(\Lambda_{l-1}+\Lambda_l)}$
  The fundamental representations $\rho_{l-1}$, $\rho_l$ are self–conjugate. They are both real (quaternionic) if $4 \mid l-p$ ($4 \nmid l-p$).
  - $l-p$ odd, $s\nu \neq \mathrm{id}$ ............................................. $+1$
  The fundamental representations $\rho_{l-1}$ and $\rho_l$ are mutually conjugate.

- $\mathfrak{u}_l^*(\mathbb{H})$. The fundamental representations $\rho_1,\dots,\rho_{l-2}$ are self–conjugate; the representation $\rho_i$, $i \le l-2$ is real (quaternionic) if $i$ is even (odd). We distinguish two possibilities according to the parity of $l$. Note that if $l$ is even then one of "legs" of the Satake diagram is compact and the second is non–compact and if $l$ is odd then both "legs" are non–compact and connected by an arrow.
  - $l$ even, $s\nu = \mathrm{id}$ .................................. $(-1)^{\Lambda_1+\Lambda_3+\ldots+\Lambda_{l-1}}$
  The fundamental representations $\rho_{l-1}$ and $\rho_l$ are self–conjugate. The representation, $\rho_{l-1}$, corresponding to the compact node is quaternionic and the other is real.
  - $l$ odd, $s\nu \neq \mathrm{id}$ ................................... $(-1)^{\Lambda_1+\Lambda_3+\ldots+\Lambda_{l-2}}$
  The fundamental representations $\rho_{l-1}$ and $\rho_l$ are mutually conjugate.

In other words, the fundamental representations corresponding to the compact nodes are quaternionic and the fundamental representations corresponding to the non–compact nodes without arrows are real.

- The compact form and the real form $EVI$ of $E_7$, $s\nu = \mathrm{id}$ $\ldots (-1)^{\Lambda_1+\Lambda_3+\Lambda_7}$ All fundamental representations are self–conjugate. The quaternionic representations $\rho_1, \rho_3, \rho_7$ correspond to the compact roots of the Satake digram of $EVI$.

**5.2. Indices of reductive algebras.** Let us consider an irreducible representation $\gamma : \mathfrak{k}_0 \longrightarrow \mathfrak{gl}(V)$ on the complex vector space $V$ which is self–conjugate i.e. the highest weight of $\gamma$ satisfies $\Gamma = \bar{\Gamma}$. We understand the highest weights as vectors of coefficients over a diagram. Then there exists a $\gamma$–invariant antiautomorphism $J$ on $V$ such that $J^2 \in \{+\mathrm{id}, -\mathrm{id}\}$. This follows that each element of the center of $\mathfrak{g}_0$ must act by some real scalar since non–real scalars do not commute with any antiautomorphism. But real scalars commute with any antiautomorphism and thus the action of the center has no effect on the index. Denoting by $\mathfrak{g}_0^{ss}$ the semisimple part of $\mathfrak{g}_0$, we have shown that $\varepsilon(\mathfrak{g}_0, \gamma) = \varepsilon(\mathfrak{g}_0^{ss}, \gamma|\mathfrak{g}_0^{ss})$. The index on the right hand side can be easily computed using the formulas above.

If the representation $\gamma$ is a self–conjugate component in the cohomology, we can determine the index more precisely. Let us suppose that the highest weight $\Gamma = \bar{\Gamma}$ of $\gamma$ is of the form $\Gamma = w.\Lambda$ for a self–conjugate $\mathfrak{k}$–dominant weight $\Lambda$ and an element $w \in W$. The following theorem states that the indices $\varepsilon(\mathfrak{g}_0, \Gamma)$ and $\varepsilon(\mathfrak{g}, \Lambda)$ are same. Denoting by $F_0 \subseteq F$ the set of fundamental weights corresponding to the non–crossed roots, it follows from the above list and the corresponding Satake diagrams that all quaterionic fundamental representations correspond to compact roots and thus $Q \subseteq F_0$. We observe that therefore $\varepsilon(\mathfrak{g}, \Lambda) = \varepsilon(\mathfrak{g}_0, \Lambda)$ where $\Lambda$ on the right hand side is understood as an $\mathfrak{k}_0$–dominant weight.

**Theorem 5.3.** *Let us consider a real semisimple Lie algebra $\mathfrak{g}$ and its reductive subalgebra $\mathfrak{g}_0$ with complexification $\mathfrak{k}_0 \subseteq \mathfrak{k}$. If $\Lambda$ is a self–conjugate $\mathfrak{k}$–dominant weight and $\Gamma$ is a self–conjugate $\mathfrak{k}_0$–dominant weight $\Gamma = w(\Lambda)$ for $w \in W$ then $\varepsilon(\mathfrak{g}, \Lambda) = \varepsilon(\mathfrak{g}_0, \Gamma)$. The same assertion holds if we replace the Weyl action by the affine Weyl action i.e. $\Gamma = w.\Lambda$.*

**Proof.** It follows from the list above that $\varepsilon(\mathfrak{g}_0; \Lambda_1 + \Lambda_2) = \varepsilon(\mathfrak{g}_0; \Lambda_1)\varepsilon(\mathfrak{g}_0; \Lambda_2)$ for $\mathfrak{g}_0$–dominant weights $\Lambda_1$ and $\Lambda_2$. This proves the last claim in the theorem. We prove the theorem case by case for simple real Lie algebras. It is sufficient to consider only the real forms discussed in 5.1 because the remaining ones admit only the index $+1$ for the self–conjugate highest weights. For most of them the proof is easy if we observe the Weyl actions in terms of the Dynkin diagram notation. The simple reflection $w_i \in W$ corresponding to $\alpha_i \in \Pi$ acting on a weight $\Lambda'$ of $\mathfrak{k}$ has the following form:

*Let $a$ be the coefficient over the $i$th node in the expression of $\Lambda'$. In order to get the coefficients over the nodes corresponding to $w_i(\Lambda')$, add $a$ to the adjacent coefficients, with the apropriate multiplicity if there is a multiple edge directed towards the adjacent node, and replace $a$ by $-a$.* (This algorithm for computing with the Dynkin diagrams was established in [1]).

Let us begin with the real form $\mathfrak{sl}_m(\mathbb{H})$ where $l = 2m - 1$. The index of $\Lambda$ is given by the parity of the sum $\Lambda_1 + \Lambda_3 + \ldots + \Lambda_l$. It is easy to see that the simple reflections do not change the parity of this sum. The index of $\mathfrak{so}_{p,2l+1-p}$ is either always $+1$ (for both $\Lambda$ and $\Gamma$) or depends on the parity of the last coefficient. But the first $l-2$ reflections do not change the last coefficient and the last two reflections do not change its parity. Similar considerations prove the theorem for the algebras $\mathfrak{sp}_{p,l-p}$, $\mathfrak{so}_{p,2l-p}$, the compact form and the real form $EVI$ of $E_7$ and $\mathfrak{u}_l^*(\mathbb{H})$ for $l$ even.

The remaining cases $\mathfrak{su}_{p,l+1-p}$, $l=2m-1$ and $\mathfrak{u}_l^*(\mathbb{H})$ for $l$ odd must be discussed more carefully. We will consider the usual matrix presentation (see e.g. [7]) of the (complex) algebras $\mathfrak{sl}_{2m}(\mathbb{C})$ and $\mathfrak{so}_{2l}(\mathbb{C})$. Remind that $\Lambda$ is a vector of coefficients of the expression of the highest weight (denoted by $\Lambda$ as well) in the basis of fundamental weights. We will write $\Lambda_s$ to express the weight $\Lambda$ as a vector of coefficients with respect to the basis of simple roots. On the other hand we write $\Lambda_e$ to express the same weight in terms of the "matrix" base. (In the case of $\mathfrak{sl}_n(\mathbb{C})$, we will use the matrix base $e^1,\dots,e^{l+1}$ where $e^i$ extracts the $i$th element of the diagonal i.e. the matrix base of the whole algebra $\mathfrak{gl}_n(\mathbb{C})$. In the case of $\mathfrak{so}_{2l}(\mathbb{C})$, the matrix base is $e^1,\dots,e^l$ where $e^i$ extracts the $2i$th element of the diagonal.) Similarly, we will consider the vectors $\Gamma$, $\Gamma_s$ and $\Gamma_e$. We shall see below that the proof of the theorem follows from properties of expressions in "matrix" basis. The Cartan matrices will be denoted by $C(A_l)$ and $C(D_l)$.

**I.** Let us start with the algebra $A_l$, $l=2m-1$ with the real form $\mathfrak{su}_{p,l+1-p}$ (i.e. the non–trivial symmetry $s\nu$). We will consider $\Lambda_s=\Lambda\cdot(C(A_l))^{-1}$ with respect to the system of simple roots $\Pi=\{e^1-e^2,\dots,e^l-e^{l+1}\}$. The structure of the matrix $(C(A_l))^{-1}$ implies that the vector $\Lambda$ is symmetric if and only if the vector $\Lambda_s$ is symmetric. If we denote $\Lambda_s=(a_1,\dots,a_l)$ we get $\Lambda_e=(a_1,a_2-a_1,\dots,a_l-a_{l-1},-a_l)$. This implies that the symmetric vector $\Lambda_s$ corresponds to the antisymmetric vector $\Lambda_e$. We will further consider the form $\Lambda_e=(b_1,\dots,b_m,-b_m,\dots,-b_1)$. Starting with a (symmetric) highest weight $\Lambda=(\Lambda_1,\dots,\Lambda_m,\dots,\Lambda_l)$, a short computations reveals that $b_k=\Lambda_k+\dots+\Lambda_{m-1}+\frac{1}{2}\Lambda_m$ for $1\le k\le m$.

From this point of view, the Weyl group $W$ is realized as $\mathrm{Sym}_{2m}$. Considering the form of elements of $\Lambda_e$ we see that $\Lambda_m$ is even (odd) if and only if the double of an arbitrary element of $\Lambda_e$ is even (odd). But the last property is not changed by permutation. We have shown that $\Lambda_m$ and $\Gamma_m$ have the same parity. (We do not need to do the backward transformation if the weight $\Gamma$ is symmetric, which is the case according to our assumptions.) Since the indices of $\Lambda$ and $\Gamma$ depend on this parity in the same way, we have proved the theorem for the real form $\mathfrak{su}_{p,l+1-p}$ for $l$ odd.

**II.** The case of the algebra $D_l$, $l$ odd with the real form $\mathfrak{u}_l^*(\mathbb{H})$ (i.e. the non–trivial symmetry $s\nu$) is similar. We consider $\Lambda_s=\Lambda\cdot(C(D_l))^{-1}$ with respect to the system of simple roots $\Pi=\{e^1-e^2,\dots,e^{l-1}-e^l,e^{l-1}+e^l\}$. The structure of the matrix $(C(D_l))^{-1}$ implies that the vector $\Lambda$ is symmetric if and only if the vector $\Lambda_s$ is symmetric (and this symmetry means the equality of the last two elements). If we denote $\Lambda_s=(a_1,\dots,a_l)$ we get $\Lambda_e=(a_1,a_2-a_1,\dots,a_{l-2}-a_{l-1},a_{l-1}+a_l-a_{l-2},a_l-a_{l-1})$. This implies that a symmetric vector $\Lambda_s$ corresponds to a vector $\Lambda_e$ with zero in the last position. We will further consider the form $\Lambda_e=(b_1,\dots,b_{l-1},0)$. Starting with a (symmetric) highest weight $\Lambda=(\Lambda_1,\dots,\Lambda_{l-1},\Lambda_{l-1})$, a short computation reveals that $b_k=\Lambda_k+\dots+\Lambda_{l-1}$ for $1\le k\le l-1$.

From this point of view, the Weyl group $W$ is generated by permutations

$(i,j)$ and permutations $(i,j)$ following by a sign change of the elements at positions $i$ and $j$. Considering the form of elements of $\Lambda_e$, we see that the parity of the sum $\Lambda_1+\Lambda_3+\cdots+\Lambda_{l-2}$ is the same as the parity of the sum $\sum_{i=1}^{l-1}(\Lambda_e)_i = \Lambda_1+2\Lambda_2+\cdots+(l-2)\Lambda_{l-2}+(l-1)\Lambda_{l-1}$. But the parity of the last sum is not changed by the permutations and sign changes, if the resulting vector $\Gamma$ has zero as the last component. We have shown that the sums $\Lambda_1+\Lambda_3+\cdots+\Lambda_{l-2}$ and $\Gamma_1+\Gamma_3+\cdots+\Gamma_{l-2}$ have the same parity. Since the indices of $\Lambda$ and $\Gamma$ depend on this parity in the same way, this proves the theorem for the real form $\mathfrak{u}_l^*(\mathbb{H})$ for $l$ odd. ∎

## 6. Relation between real and complex cohomologies

Now we know how to identify the couples of conjugate representations of $\mathfrak{f}_0$ (with respect to $\mathfrak{g}_0$) and how to compute the index of a self–conjugate representation of $\mathfrak{f}_0$ (with respect to $\mathfrak{g}_0$) and thus we can finish the considerations from the Section 3.. Using the same notation, we describe the real cohomology for irreducible representations $\lambda:\mathfrak{g}\longrightarrow\mathfrak{gl}(V)$ of all types. First we show that we can consider symmetries of diagrams as isomorphisms of the Weyl group.

**6.1. Symmetries as isomorphisms of the Weyl group.** For an arbitrary symmetry $a'$ of the Dynkin diagram of $\mathfrak{f}$, we define the isomorphism $a':W\longrightarrow W$ in the following way: the image of the simple reflection $w_i$, corresponding to the simple root $\sigma_i\in\Pi$, will be the simple reflection $w_{a'(i)}$, corresponding to the simple root $a'(\sigma_i)$. In the other words, an element $w=w_{i_1}\dots w_{i_k}$ given by the sequence $i_1,\dots,i_k$ with respect to the (ordered) set $\Pi$, is mapped to the element $a'(w)$ given by the same sequence with respect to the set $a'(\Pi)$. It also shows the correctness of the definition of $a'$ (an expression of $w\in W$ as a composition of simple reflections is not unique).

**6.2. Real cohomology for complex and quaternionic representations.** Let us suppose that $\lambda$ is complex or quaternionic. Thus $V$ is a complex vector space and we denote the highest weight of $\lambda_{\mathbb{C}}:\mathfrak{f}\longrightarrow\mathfrak{gl}(V)$ by $\Lambda$. We can consider the representation on the complex cohomology $\beta_{\mathbb{C}}:\mathfrak{f}_0\longrightarrow\mathfrak{gl}(H(\mathfrak{p}_+(\mathbb{C}));V)$ and recall the representation $\beta^{\mathbb{R}}:\mathfrak{g}_0\longrightarrow\mathfrak{gl}(H(\mathfrak{p}_+;V_{\mathbb{R}}))$ on the real cohomology is given by the relation $\beta^{\mathbb{R}}=(\beta_{\mathbb{C}}|\mathfrak{g}_0)^{\mathbb{R}}$, see Section 3.. We will show that the irreducible components of $\beta_{\mathbb{C}}$ and $\beta^{\mathbb{R}}$ are in 1–1 correspondence. Recall that the highest weights of the components of $\beta_{\mathbb{C}}$ are of the form $w.\Lambda$, $w\in W^{\mathfrak{q}}$, see Theorem 1.4.

I. First let us suppose that $\lambda$ is complex i.e. $\Lambda\neq s\nu(\Lambda)$. We claim the following useful property. For each $w\in W$, an arbitrary symmetry $a'$ of the Dynkin diagram and a highest weight $\Lambda'$, the following relation holds:

$$a'(w.\Lambda')=w.\Lambda' \iff \Lambda'=a'(\Lambda') \text{ and } w=a'(w). \qquad (*)$$

The equality $w.\Lambda'=a'(w.\Lambda')=a'(w).a'(\Lambda')$ implies that the irreducible component with the highest weight $a'(w.\Lambda')$ is a cohomology component in the cohomology induced by $\mathfrak{f}$–representations with highest weights $\Lambda'$ and $a'(\Lambda')$. This implies $\Lambda'=a'(\Lambda')$ because $\Lambda'$ and $a'(\Lambda')$ are $\mathfrak{f}$–dominant weights on the same orbit of the affine Weyl action. Thus $w.\Lambda'=a'(w).\Lambda'$ and since $\Lambda'+R$ is inside the Weyl chamber and the Weyl group acts simply transitively on Weyl chambers, it follows $w = a'(w)$.

Now putting $a' := s\nu$ and $\Lambda' := \Lambda$, it follows from Theorem 4.2 that each component in $\beta^{\mathbb{R}}$ is of the complex type.

II. If $\lambda$ is quaternionic then $\Lambda = s\nu(\Lambda)$ and $\varepsilon(\mathfrak{g}, \Lambda) = -1$. If there is a component in $\beta_{\mathbb{C}}$ with the highest weight $w.\Lambda$, $w \in W^{\mathfrak{q}}$ such that $s\nu(w.\Lambda) = w.\Lambda$ then $\varepsilon(\mathfrak{g}_0, w.\Lambda) = \varepsilon(\mathfrak{g}, \Lambda) = -1$, see Theorem 5.3. This shows that each component in $\beta_{\mathbb{C}}$ is complex or quaternionic.

In summary, the space of the real cohomology $H(\mathfrak{p}_+; V)$, for a complex or quaternionic representation $\lambda$, is just the space of the complex cohomology $H(\mathfrak{p}_+(\mathbb{C}); V)$ understood as a real vector space. Moreover, Lemma 3.3 implies that the irreducible components of $\beta_{\mathbb{C}}$ and $\beta^{\mathbb{R}}$ are the same and from the definition of arrows in 1.8 it follows that the Hasse graphs are the same too.

**6.3. Real cohomology for real $\mathfrak{g}$–representations.** If the representation $\lambda$ is real then we start with its (irreducible) complexification $\lambda(\mathbb{C}) : \mathfrak{f} \longrightarrow \mathfrak{gl}(V(\mathbb{C}))$ with the highest weight $\Lambda$. It follows from 1.5 that the complexification of the required representation $\beta : \mathfrak{g}_0 \longrightarrow \mathfrak{gl}(H(\mathfrak{p}_+; V))$ on the real cohomology is just the representation $\beta(\mathbb{C}) : \mathfrak{f}_0 \longrightarrow \mathfrak{gl}(H(\mathfrak{p}_+(\mathbb{C}); V(\mathbb{C})))$ on the complex cohomology for the $\mathfrak{f}$–representation $\lambda(\mathbb{C})$. Now, our aim is to say which irreducible components of $\beta(\mathbb{C})$ correspond to irreducible components in $\beta$ and which couples of irreducible components of $\beta(\mathbb{C})$ correspond to irreducible components in $\beta$, cf. 1.7.

The description of the highest weights of $\beta(\mathbb{C})$ is given by Theorem 1.4. Moreover, each highest weight has the multiplicity one. Using the Theorem 4.2, this implies that there is no quaternionic component in $\beta$. Thus, the type of components in $\beta$ is fully determined by the symmetry $s\nu$. In particular, there can be complex components in $\beta$ only if $s\nu$ is non–trivial.

(a) $s\nu = \mathrm{id}$. All components in $H(\mathfrak{p}_+, V)$ are clearly of the real type and the Hasse graphs of $H(\mathfrak{p}_+(\mathbb{C}), V(\mathbb{C}))$ and $H(\mathfrak{p}_+, V)$ are isomorphic. Considering simple Lie algebras, this case includes all real forms of algebras $B_l$, $C_l$, $E_7$, $E_8$, $F_4$, $G_2$ and real algebras $\mathfrak{sl}_l(\mathbb{R})$, $\mathfrak{sl}_{2m}(\mathbb{H})$, $\mathfrak{so}_{p,2l-p}$ for $l-p$ even, $u^*_{2l}(\mathbb{H})$, $E$I and $E$IV.

(b) $s\nu \neq \mathrm{id}$. There can be real and complex components in $H(\mathfrak{p}_+, V)$ (and one can easily show that both these types of representations in the cohomology exists for each choice of a parabolic subalgebra). A component in $\beta(\mathbb{C})$ with a highest weight $w.\Lambda$, $w \in W^{\mathfrak{q}}$ corresponds to a component in $\beta$ if and only if $s\nu(w.\Lambda) = w.\Lambda$ (or equivalently $s\nu(w) = w$) and a couple of components in $\beta(\mathbb{C})$ with highest weights $w_1.\Lambda \neq w_2.\Lambda$, $w_1, w_2 \in W^{\mathfrak{q}}$ corresponds to a component in $\beta$ if and only if $s\nu(w_1.\Lambda) = w_2.\Lambda$ (or equivalently $s\nu(w_1) = w_2$). The Hasse graph on irreducible components $H(\mathfrak{p}_+, V)$ is obtained from the Hasse graph on irreducible components of $H(\mathfrak{p}_+(\mathbb{C}), V(\mathbb{C}))$ by connecting these couples of components which correspond to one component in the real case. Considering simple Lie algebras, this case includes $\mathfrak{su}_{p,l-p}$, $\mathfrak{so}_{p,2l-p}$ for $l-p$ odd, $u^*_{2l+1}(\mathbb{H})$, $E$II, $E$III and all simple complex Lie algebras understood as real ones.

**Example 6.4. Cohomology for $\mathfrak{su}_{3,1}$ and the adjoint representation.** The Satake diagram of $\mathfrak{su}_{3,1}$ is  and it admits only the parabolic subalgebra given by the diagram  . The adjoint representation of $\mathfrak{su}_{3,1}$ is real because

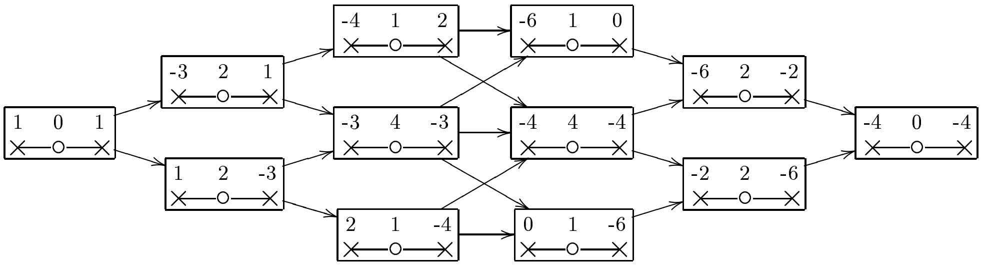


Figure 1: The cohomology of the adjoint representation of $\mathfrak{sl}_4(\mathbb{C}) = (\mathfrak{su}_{3,1})(\mathbb{C})$.

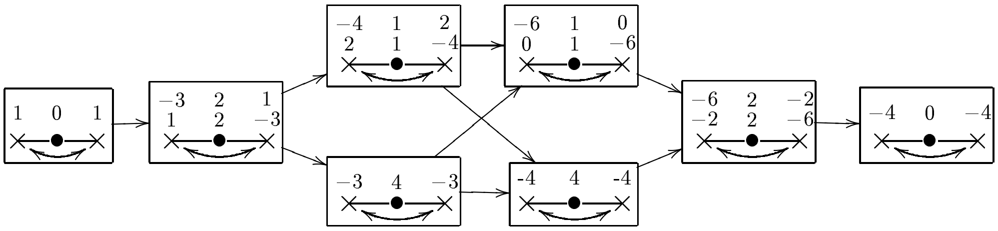


Figure 2: The real cohomology of $\mathfrak{su}_{3,1}$.

its complexification is the adjoint representation of $\mathfrak{sl}_4(\mathbb{C})$ and this is an irreducible representation.

The adjoint representation in the complex case is described by the Dynkin diagram with coefficients $\overset{1}{\circ}\!-\!\overset{0}{\circ}\!-\!\overset{1}{\circ}$ . The irreducible components of complex cohomology and the structure of the Hasse graph are shown in the Figure 1. (The way to use Kostant's result for the algorithmic computation of the cohomology is described in [9].)

In general, the components of the real cohomology are described by the Satake diagram with the highest weight of the complexification (for the components of the real type) or by the Satake diagram with a couple of the highest weights of the complexification (for the components of the complex and quaternionic type). The Satake diagram with one highest weight denotes an $\mathfrak{g}_0$–invariant real subspace of $\mathfrak{f}_0$–representation with the given highest weight. The Satake diagram with two highest weights denotes a representation of $\mathfrak{g}_0$ obtained as the realification of the $\mathfrak{f}_0$–representation with any of the given highest weights.

According to 6.3, there can be only real and complex components in the real cohomology of $\mathfrak{su}_{3,1}$. Since the complex Hasse graph is symmetric (with respect to the highest weights of its components) according to the middle line, the real Hasse graph is just the "upper part" of the complex one, see Figure 2.

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