Journal of Lie Theory
Volume **14** (2004) 555–561

A Splitting Criterion for Two-Dimensional Semi-Tori

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Communicated by F. Knop

Abstract. We investigate conditions under which a two-dimensional complex semi-torus splits into a direct product of $\mathbb{C}^*$ and a one-dimensional compact complex torus.

1. Introduction

A semi-torus is a complex Lie group arising as a quotient of the additive group of a complex vector space V by a discrete subgroup Γ with the property that Γ generates V as complex vector space. Semi-tori without non-constant holomorphic functions are also known as *Cousin groups*. They have been studied by many mathematicians since the work of Cousin about a century ago [1] and continue to be the focus of ongoing research.

In this article we investigate a special aspect of two-dimensional semi-tori. Let Γ be a discrete subgroup of $\mathbb{Z}$-rank three in $(\mathbb{C}^2,+)$. Then $T = \mathbb{C}^2/\Gamma$ is a *semi-torus*. Let $\Gamma_{\mathbb{R}}$ denote the $\mathbb{R}$-span of Γ and $H = \Gamma_{\mathbb{R}} \cap i\Gamma_{\mathbb{R}}$. Then for every $\gamma \in \Gamma \setminus H$ the quotient $S_\gamma = \mathbb{C}\gamma/(\mathbb{C}\gamma \cap \Gamma)$ is a closed complex Lie subgroup of T which is isomorphic to $\mathbb{C}^*$. The quotient $E_\gamma = T/S_\gamma$ is an elliptic curve. Usually, the isomorphism class and even the isogeny class of this elliptic curve E_γ will depend on γ. For example, if T is the quotient of $\mathbb{C}^2$ by the lattice

$$\Gamma = \left\langle \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}\tau\\\sigma\end{pmatrix} \right\rangle_{\mathbb{Z}},$$

with $\tau, \sigma \in H^+ = \{z \in \mathbb{C} : \Im(z) > 0\}$, then there is a quotient elliptic curve isomorphic to $\mathbb{C}/\langle 1, \sigma\rangle$ as well as one isomorphic to $\mathbb{C}/\langle 1, \tau\rangle$. On the other hand, if $T \simeq \mathbb{C}^* \times E$ for some elliptic curve E, then the restriction of the projection map $T \to E_\gamma$ to $\{1\} \times E$ yields an isogeny between E and E_γ. Thus in this case all these quotient elliptic curves E_γ must be isogenous. In this paper we investigate to what extent this property characterizes those T which split into a direct product of $\mathbb{C}^*$ and an elliptic curve. It turns out that besides such a splitting also a certain arithmetic property may cause all the quotient elliptic curves to be isogenous.

[*] The author wants to thank the Korea Institute for Advanced Study in Seoul. The research for this article was done during the stay of the author at this institute.

ISSN 0949–5932 / \$2.50

Theorem 1.1. *Let Γ be a discrete subgroup of $\mathbb{Z}$-rank three in $(\mathbb{C}^2,+)$ and $T = \mathbb{C}^2/\Gamma$. Assume that all elliptic curves that are images of surjective homomorphisms from T have the same isogeny class.*

Then one of the following conditions hold:

1. *T is isomorphic to a direct product of $\mathbb{C}^*$ and an elliptic curve, or*
2. *There is a number field k of degree 3 over $\mathbb{Q}$ such that $\Gamma \subset k^2$ after some linear change of coordinates on $\mathbb{C}^2$.*

Conversely, if one of these two conditions is fulfilled, then there exists an elliptic curve E_0 such that E is isogenous to E_0 for every surjective complex Lie group homomorphism from T to an elliptic curve E.

It should be remarked that $T = \mathbb{C}^2/\Gamma$ must be a direct product of $\mathbb{C}^*$ and an elliptic curve if $\Gamma \subset k^2$ for some *quadratic* number field k (see lemma 2.1).

On the other hand, if k is a *cubic* number field, then there do exist examples of such $\Gamma \subset k^2$ such that $\mathbb{C}^2/\Gamma$ is not isomorphic to a direct product of $\mathbb{C}^*$ and an elliptic curve (see example 2.5 below).

One might inquire what happens if one asks for isomorphisms instead of mere isogenies between all quotient elliptic curves. As it turns out this simply is too much to ask for: There are always non-isomorphic quotient elliptic curves, as we will see in the last section.

2. Proofs

Before we prove the theorem, we need to deduce some auxiliary results.

Lemma 2.1. *Let k be a quadratic number field, $\mathcal{O}_k$ its ring of algebraic integers and $\Gamma \subset k^2$ a subgroup of $\mathbb{Z}$-rank three which is discrete in $\mathbb{C}^2$.*

Then there exists an elliptic curve E (isogenous to $\mathbb{C}/\mathcal{O}_k$) such that $\mathbb{C}^2/\Gamma \simeq \mathbb{C}^* \times E$.

Proof. Since Γ is discrete, k cannot be totally real. Hence $k = \mathbb{Q}[\sqrt{-n}]$ for some $n \in \mathbb{N}$. Let $\tau = \sqrt{-n} \in k$. Let $\Gamma_{\mathbb{Q}} = \Gamma \otimes_{\mathbb{Z}} \mathbb{Q}$. Consider $H = \Gamma_{\mathbb{Q}} \cap \tau(\Gamma_{\mathbb{Q}})$. Since H is the intersection of two $\mathbb{Q}$-hyperplanes in k^2, it is clear that $\dim_{\mathbb{Q}}(H) = 2$. Moreover H is stable under scalar multiplication with elements of k. Hence $H \otimes_{\mathbb{Q}} \mathbb{R}$ is a complex line which we will call $H_{\mathbb{C}}$. The intersection $H \cap \Gamma$ is a lattice of $\mathbb{Z}$-rank 2 in $H_{\mathbb{C}}$. Since H is k-vector space, and since $\mathcal{O}_k$ is a lattice of $\mathbb{Z}$-rank 2, it follows that $H_{\mathbb{C}}/(H \cap \Gamma)$ is isogenous to $\mathbb{C}/\mathcal{O}_k$. On the other hand, $rank_{\mathbb{Z}}(H \cap \Gamma) = 2$ implies that $rank_{\mathbb{Z}}\,\pi(\Gamma) = 1$ where π denotes the natural projection $\mathbb{C}^2 \to \mathbb{C}^2/H_{\mathbb{C}}$. Therefore $\pi(\Gamma) \simeq \mathbb{Z}$ and Γ contains an element γ_0 such that $\pi(\gamma_0)$ generates $\pi(\Gamma)$. It follows that

$$\mathbb{C}^2/\Gamma \simeq (\mathbb{C}\gamma_0/\mathbb{Z}\gamma_0) \times (H_{\mathbb{C}}/(H \cap \Gamma)).$$

This implies the statement. ■

Lemma 2.2. *Let k be a cubic number field. Then $GL_2(\mathbb{Q})$ has exactly two orbits in $\mathbb{P}_1(k)$, namely $\mathbb{P}_1(\mathbb{Q})$ and its complement.*

Proof. We start by the claim: *$PGL_2(\mathbb{Q})$ acts freely on $\Omega = \mathbb{P}_1(k) \setminus \mathbb{P}_1(\mathbb{Q})$.*

Indeed, let $x \in k \setminus \mathbb{Q}$. Then $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is in the isotropy at $[x:1]$ iff

$$x(cx+d) = ax+b \iff cx^2 + (d-a)x - b = 0.$$

Now $x \in k \setminus \mathbb{Q}$ and $\deg(k/\mathbb{Q}) = 3$ imply that $1, x, x^2$ are all $\mathbb{Q}$-linearly independent. Thus the above equation implies that $c = b = 0$ and $a = d$. This yields the claim.

Next we consider the Borel group

$$B = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, d \in \mathbb{Q}^*, b \in \mathbb{Q} \right\}.$$

Such an element in B maps $[x:1]$ to $[ax+b:d] = [\frac{a}{d}x + \frac{b}{d} : 1]$. Fix a generator ("primitive element") τ for the field extension $k/\mathbb{Q}$. Then every element of k be written uniquely as $a + b\tau + c\tau^2$ with $a, b, c \in \mathbb{Q}$. We define a map $\phi : \Omega \to \mathbb{P}_1(\mathbb{Q})$ as follows:

$$[a + b\tau + c\tau^2 : 1] \to [b : c].$$

The fibers of ϕ are precisely the B-orbits in Ω.

We continue with a second claim: *There exists a cubic polynomial $Q \in \mathbb{Q}[X]$ such that for all $\lambda \in \mathbb{Q}$ with $Q(\lambda) \neq 0$ there exists an element $A \in GL_2(\mathbb{Q})$ such that $[\lambda\tau + \tau^2 : 1] = A([\tau : 1])$.*

Let $\tau^3 = p_2\tau^2 + p_1\tau + p_0$ ($p_i \in \mathbb{Q}$)). Choose $c = 1$, $d = -(\lambda + p_2)$, $a = p_1 - \lambda(\lambda + p_2)$ and $b = p_0$.

Then

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = -p_1(\lambda + p_2) + \lambda(\lambda + p_2)^2 - p_0$$

and

$$(\lambda\tau + \tau^2)(c\tau + d) = (a\tau + b).$$

This proves the second claim with $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $Q(\lambda) = -p_1(\lambda + p_2) + \lambda(\lambda + p_2)^2 - p_0$.

Now we can prove the lemma. Let $G = PGL_2(\mathbb{Q})$. By the last claim, the G-orbit through $[\tau : 1]$ contains all of Ω with the possible exception of at most three B-orbits. But G acts freely on Ω and $G/B \simeq \mathbb{P}_1(\mathbb{Q})$ is infinite. Hence a union of finitely many B-orbits cannot be a G-orbit. It follows that G acts transitively on Ω. ∎

Corollary 2.3. *Let k be a cubic number field, and Γ_1, Γ_2 lattices in $\mathbb{C}$ which are contained in k. Then the two elliptic curves $\mathbb{C}/\Gamma_1$ and $\mathbb{C}/\Gamma_2$ are isogenous.*

Proof. This follows because two lattice Γ_1, Γ_2 in $\mathbb{C}$ have isogenous quotient elliptic curves if and only if there is an element $A \in PGL_2(\mathbb{Q})$ such that the associated fractional linear transformation carries Γ_1 to Γ_2. ∎

Corollary 2.4. *Let k be a cubic number field, τ a primitive element for the field extension $k/\mathbb{Q}$ and Γ a lattice in $\mathbb{C}^2$ which is contained in k^2. Let E be an elliptic curve for which there exists a surjective holomorphic Lie group homomorphism $\pi : \mathbb{C}^2/\Gamma \to E$.*

Then E is isogenous to $\mathbb{C}/\langle 1, \tau\rangle_{\mathbb{Z}}$.

Proof. Such a surjective homomorphism π is induced by a linear map $\tilde{\pi} : \mathbb{C}^2 \to \mathbb{C}$. Since Γ has $\mathbb{Z}$-rank 3, there is a non-zero element of Γ in the kernel of $\tilde{\pi}$. Therefore there are $\mu \in \mathbb{C}^*$ and $a_i \in k$ such that $\tilde{\pi}(x_1, x_2) = \mu(a_1 x_1 + a_2 x_2)$. It follows that

$$\frac{1}{\mu}\tilde{\pi}(\Gamma) \subset k.$$

Thus E is isogenous to $\mathbb{C}/\langle 1, \tau\rangle_{\mathbb{Z}}$ by the preceding claim. ■

Here we would like to mention that in the situation of the above corollary in general $\mathbb{C}^2/\Gamma$ is not a direct product:

Example 2.5. Let $\tau = \omega\sqrt[3]{2}$, where ω is a primitive third root of unity, say $\omega = -\frac{1}{2} + \sqrt{\frac{3}{4}}i$. Let $k = \mathbb{Q}[\tau]$. Now $\tau^3 = 2$. Thus k is a cubic number field. Then

$$\Gamma = \left\langle \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}\tau\\\tau^2\end{pmatrix} \right\rangle_{\mathbb{Z}}$$

is discrete $\mathbb{Z}$-module of rank three in $\mathbb{C}^2$ with $\Gamma \subset k^2$. Let $L = \Gamma \otimes \mathbb{R} \cap i\Gamma \otimes \mathbb{R}$. Then L is a complex line in $\mathbb{C}^2$, and $\mathbb{C}^2/\Gamma$ is a direct product of $\mathbb{C}^*$ and an elliptic curve precisely if $\Gamma \cap L$ is cocompact (i.e. of $\mathbb{Z}$-rank two) in L. However,

$$L = \left\langle \begin{pmatrix}\Im(\tau)\\\Im(\tau^2)\end{pmatrix} \right\rangle_{\mathbb{C}} = \left\langle \begin{pmatrix}\sqrt[3]{2}\\-\sqrt[3]{4}\end{pmatrix} \right\rangle_{\mathbb{C}} = \left\langle \begin{pmatrix}1\\-\sqrt[3]{2}\end{pmatrix} \right\rangle_{\mathbb{C}}$$

and $-\sqrt[3]{2} \notin k$, because $k \cap \mathbb{R} = \mathbb{Q}$. This implies $L \cap \Gamma \subset L \cap k^2 = \{0\}$. Therefore $\mathbb{C}^2/\Gamma$ cannot be isogenous to a direct product of $\mathbb{C}^*$ and an elliptic curve.

We are now able to prove the theorem:

Proof. For $R \in \{\mathbb{Q}, \mathbb{R}, \mathbb{C}\}$ the R-module generated by Γ will be denoted by Γ_R. Let $H = \Gamma_{\mathbb{R}} \cap i\Gamma_{\mathbb{R}}$. Then for every $\gamma \in \Gamma \setminus H$ we obtain a surjective Lie group homomorphism onto an elliptic curve as follows: Let L_γ denote the quotient map from $\mathbb{C}^2$ to $Q = \mathbb{C}^2/\mathbb{C}\cdot\gamma$. By construction the image $L_\gamma(\Gamma)$ has rank at most two (because $\mathbb{Z}\cdot\gamma \subset \ker L_\gamma$). On the other hand $\gamma \notin H$ implies $L_\gamma(H) = Q$ and therefore $L_\gamma(\Gamma_{\mathbb{R}}) = Q$. It follows that $L_\gamma(\Gamma)$ is a lattice in Q and $Q/L_\gamma(\Gamma)$ is an elliptic curve.

By choosing a basis for the complex vectorspace $\mathbb{C}^2$ inside $\Gamma \setminus H$ we may assume that

$$\Gamma = \left\langle \begin{pmatrix}1\\0\end{pmatrix} \begin{pmatrix}0\\1\end{pmatrix} \begin{pmatrix}\alpha\\\beta\end{pmatrix} \right\rangle_{\mathbb{Z}}$$

with $\alpha, \beta \in \mathbb{C} \setminus \mathbb{R}$.

Consider

$$\gamma = \gamma_{m,n,p} = n\begin{pmatrix}1\\0\end{pmatrix} + m\begin{pmatrix}0\\1\end{pmatrix} + p\begin{pmatrix}\alpha\\\beta\end{pmatrix}$$

for $(m, n, p) \in \mathbb{Q}^3$. Let I denote the $\mathbb{Q}$ vector subspace of $\mathbb{Q}^3$ of all (m, n, p) for which $\gamma_{m,n,p} \in H$. Evidently $\dim_{\mathbb{Q}}(I) \in \{0, 1, 2\}$.

Let us first deal with the case $\dim_{\mathbb{Q}}(I) = 2$. Then $H \cap \Gamma$ has $\mathbb{Z}$-rank two and $H/(H \cap \Gamma)$ is compact. Consider the linear projection $\pi : \mathbb{C}^2 \to \mathbb{C}^2/H$. Then

$\pi(\Gamma)$ is a subgroup of $(\mathbb{C},+)$ of $\mathbb{Z}$-rank one. Hence $\pi(\Gamma) \simeq \mathbb{Z}$. It follows that $T \simeq \mathbb{C}^* \times H/(H \cap \Gamma)$.

Thus we may assume that $\dim_{\mathbb{Q}}(I) \in \{0,1\}$.

We claim that this implies $\langle 1,\alpha\rangle_{\mathbb{Q}} \neq \langle 1,\beta\rangle_{\mathbb{Q}}$. Indeed, if $\alpha = x\beta + y$ for some $x, y \in \mathbb{Q}$, then

$$\begin{pmatrix} x\beta \\ \beta \end{pmatrix} = \begin{pmatrix} x\beta + y \\ \beta \end{pmatrix} - \begin{pmatrix} y \\ 0 \end{pmatrix} \in \Gamma_{\mathbb{Q}}.$$

Consequently $H = \{(xt, t) : t \in \mathbb{C}\}$ and $rank_{\mathbb{Z}}(H \cap \Gamma) = \dim_{\mathbb{Q}}(I) = 2$.

For $(m,n,p) \in \mathbb{Q}^3 \setminus \{(0,0,0)\}$ we define $E_{m,n,p}$ as the $\mathbb{Q}$-vector space

$$E_{m,n,p} = \langle m + p\beta, n + p\alpha, m\alpha - n\beta\rangle_{\mathbb{Q}}.$$

One verifies easily that for all $(m,n,p) \in \mathbb{Q}^3 \setminus \{(0,0,0)\}$ the matrix

$$\begin{pmatrix} m & 0 & p \\ n & p & 0 \\ 0 & m & -n \end{pmatrix}$$

has rank 2 which implies that $E_{m,n,p}$ is a $\mathbb{Q}$-plane in $V = \langle 1,\alpha,\beta\rangle_{\mathbb{Q}}$.

Thus $(m,n,p) \mapsto E_{m,n,p}$ defines a map from $\mathbb{Q}^3 \setminus \{(0,0,0)\}$ to the Grassmann variety $G_{\mathbb{Q}}$ of $\mathbb{Q}$-planes in V.

If $(m,n,p) \in \mathbb{Q}^3 \setminus I$ we have a projection $L_{m,n,p} : \mathbb{C}^2 \to \mathbb{C}^2/\mathbb{C}\gamma_{m,n,p}$ which in coordinates can be described as

$$(z_1, z_2) \mapsto (m + p\beta)z_1 - (n + p\alpha)z_2.$$

In this case $E_{m,n,p}$ can be identified with the image of $\Gamma_{\mathbb{Q}}$ under the projection $L_{m,n,p}$.

Let E be an arbitrary $\mathbb{Q}$-plane in V. There are two possibilities: Either $1 \in E$ or $1 \notin E$. In the first case $E = \langle 1, m\alpha - n\beta\rangle_{\mathbb{Q}}$ for some $(m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\}$. Then $E = \psi(m,n,0)$. Let us now discuss the second case. Since E is a $\mathbb{Q}$-hyperplane in V, we have that both $E \cap \langle 1,\alpha\rangle_{\mathbb{Q}}$ and $E \cap \langle 1,\beta\rangle_{\mathbb{Q}}$ have $\mathbb{Q}$-dimension one. Therefore E is the direct sum of two $\mathbb{Q}$-lines which arise as intersections of E with $\langle 1,\alpha\rangle_{\mathbb{Q}}$ resp. $\langle 1,\beta\rangle_{\mathbb{Q}}$. Thus

$$E = \langle m + p\alpha, n + p\beta\rangle_{\mathbb{Q}}$$

for some $(m,n,p) \in \mathbb{Q}^3 \setminus \{(0,0,0)\}$. It follows that $E = \psi(m,n,p)$.

In this way we have shown that the map $\psi : \mathbb{Q}^3 \setminus \{0\} \to G_{\mathbb{Q}}$ is surjective.

Observe that I is at most one-dimensional and that $\psi(m,n,p)$ depends only on $[m:n:p]$. Therefore we see: *With at most one exception every element of $G_{\mathbb{Q}}$ is in the image of $\mathbb{Q}^3 \setminus I$ under ψ.* Now let $\mathbb{Z}'$ denote the set of all $k \in \mathbb{Z}$ for which $\langle 1, \beta + k\alpha\rangle_{\mathbb{Q}}$ is contained in the image $\psi(\mathbb{Q}^3 \setminus I)$. Then $\mathbb{Z} \setminus \mathbb{Z}'$ contains at most one element.

Now $\mathbb{C}/\langle 1,\alpha\rangle_{\mathbb{Z}}$ is isogenous to $\mathbb{C}/\langle 1, \beta + k\alpha\rangle_{\mathbb{Z}}$ for all $k \in \mathbb{Z}'$.

Thus for every $k \in \mathbb{Z}'$ there is a matrix

$$A_k = \begin{pmatrix} a_k & b_k \\ c_k & d_k \end{pmatrix} \in GL_2(\mathbb{Q})$$

such that $A_k(\alpha) = \frac{a_k\alpha+b_k}{c_k\alpha+d_k} = \beta + k\alpha$. Then $A_k(\alpha) - A_n(\alpha) = (k-n)\alpha$ for all $k, n \in \mathbb{Z}'$. It follows that

$$(a_k\alpha + b_k)(c_n\alpha + d_n) - (a_n\alpha + b_n)(c_k\alpha + d_k) = (k-n)\alpha(c_k\alpha + d_k)(c_n\alpha + d_n)$$

Thus for all $(k,n) \in \mathbb{Z}' \times \mathbb{Z}'$ there is a $\mathbb{Q}$-polynomial $P_{k,n}$ of degree at most two such that

$$\alpha^3(k-n)c_kc_n = P_{k,n}(\alpha).$$

Therefore either $\deg(\alpha) \leq 3$ or $(k-n)c_kc_n = 0$ for all $k, n \in \mathbb{Z}'$.

However, $A_k(\alpha) = \beta + k\alpha$ combined with $\langle 1, \alpha\rangle_{\mathbb{Q}} \neq \langle 1, \beta\rangle_{\mathbb{Q}}$ implies that $c_k \neq 0$.

Therefore $\deg(\alpha) \leq 3$.

Furthermore $A_k(\alpha) = \beta + k\alpha$ implies $\beta \in \mathbb{Q}(\alpha)$.

For this reason we may deduce that $\Gamma \subset k^2$ for some number field k of degree at most three.

Finally we recall that T splits into a direct product of $\mathbb{C}^*$ and an elliptic curve if $\deg(k/\mathbb{Q}) = 2$ (lemma 2.1). ∎

3. Isogeny vs. isomorphism

Lemma 3.1. *Let z, w be complex numbers with $\Im(z), \Im(w) > 1$.*

Then the elliptic curves $\mathbb{C}/\langle 1, z\rangle_{\mathbb{Z}}$ and $\mathbb{C}/\langle 1, w\rangle_{\mathbb{Z}}$ are biholomorphic if and only if $z - w \in \mathbb{Z}$.

Proof. This follows easily from the well-known fact that

$$F = \left\{ z \in \mathbb{C} : |z| > 1, \Im(z) > 0 \text{ and } |\Re(z)| < \frac{1}{2} \right\}$$

is a fundamental domain for the $PSL_2(\mathbb{Z})$-action on the upper halfplane. ∎

Proposition 3.2. *Let Γ be a discrete subgroup of $\mathbb{C}^2$ of $\mathbb{Z}$-rank three.*

Then there exists non-isomorphic quotient elliptic curves.

Proof. Without loss of generality we may assume that

$$\Gamma = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \alpha \\ \tau \end{pmatrix} \right\rangle_{\mathbb{Z}}$$

with $\Im(\tau) > 0$ and $\alpha \in \mathbb{C}$. For every $m \in \mathbb{Z}$ the group Γ contains $(m, 1)$. The quotient by the complex line through this element is given by

$$(z_1, z_2) \mapsto mz_2 - z_1$$

The image of Γ in $\mathbb{C}$ under this projection is

$$\Lambda_m = \langle -1, m, m\tau - \alpha\rangle_{\mathbb{Z}} = \langle 1, m\tau - \alpha\rangle_{\mathbb{Z}}.$$

By the preceding lemma the quotients $\mathbb{C}/\Lambda_m$ and $\mathbb{C}/\Lambda_n$ are not isomorphic for integers m, n with

$$m > n > \frac{1 + \Im(\alpha)}{\Im(\tau)}$$

because $n > \frac{1+\Im(\alpha)}{\Im(\tau)}$ is equivalent to $\Im(n\tau - \alpha) > 1$ and $m\tau - n\tau$ is never contained in $\mathbb{Z}$. ∎

Example 3.3. Let $\Gamma = \mathbb{Z} \times \mathbb{Z}[i] \subset \mathbb{C} \times \mathbb{C}$. Then $(z_1, z_2) \mapsto z_1 - mz_2$ maps $\mathbb{C}^2/\Gamma$ onto the elliptic curve $\mathbb{C}/\left\langle 1, mi \right\rangle_{\mathbb{Z}}$.

References

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Received June 13, 2003
and in final form January 28, 2004