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Haar measure on linear groups over local skew fields

Helge Glöckner[1]

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Abstract. General and special linear groups over local fields, as well as their projective counterparts, are prominent examples of totally disconnected, locally compact groups. In this article, an explicit description of Haar measure on these groups is given by computing the measure on special local bases, consisting of open compact subgroups. These data can be used to compute the measure of any open set. In addition to that, I show that all of the groups above are unimodular, and I give a description of Haar measure on the general linear groups in terms of Haar measure on vector spaces over local fields.

1. Notational Conventions

Throughout this article, K denotes a local field, that is, a non–discrete, totally disconnected, locally compact field, commutative or not. As Weil [11] points out, K can be endowed with an ultrametric absolute value which induces the given topology on K, as follows: For any unit $x \in K^\times$, left multiplication with x yields an automorphism l_x of $(K,+)$. We define $|x|$ to be the module of l_x, that is, if m is a Haar measure on $(K,+)$ and U is a measurable subset of K such that $0 < m(U) < \infty$, we define

$$|x| := \mathrm{mod}_K\,(l_x) := \frac{m(l_x(U))}{m(U)} = \frac{m(xU)}{m(U)}.$$

This definition is being completed by setting $|0| := 0$.
We fix the following notation:

$$R := \{\, x \in K : |x| \le 1\,\}, \qquad P := \{\, x \in K : |x| < 1\,\}.$$

Theorem 1.1. *R is the unique maximal compact subring of K. Its group of units is $R^\times = \{\, x \in K : |x| = 1\,\}$, and P is the unique maximal proper ideal*

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of R. The quotient by this ideal, $k := R/P$, is a finite field. If q denotes its order, then $|K^\times| = \langle q \rangle \leq \mathbb{R}^+$. Furthermore, there is an element $\pi \in R$ such that $P = \pi R = R\pi$; its absolute value is $|\pi| = q^{-1}$.

Proof. See Weil *loc. cit.* Chapter I.–4, Theorem 6. ■

Note that since $|.|$ takes discrete values on $K^\times$, the subsets R and $R^\times$ are *open* in K. Also, since open subgroups of topological groups are closed, P is closed in R, hence compact.
For $\ell \in \mathbb{N} = \{1, 2, \dots\}$ we define $P_\ell := \pi^\ell R = R\pi^\ell = \{x \in K : |x| \leq q^{-\ell}\}$. Plainly these sets constitute a local base at 0 of the topology on K; each P_ℓ is an open and compact two–sided ideal of R.

Theorem 1.2. *Let T be a transversal of R/P such that $\{0, 1\} \subseteq T$. Then, for any $x \in K$, there is a unique sequence $(a_i)_{i\in\mathbb{Z}}$ in T, with support bounded below, such that*

$$x = \sum_{i\in\mathbb{Z}} a_i \pi^i.$$

Proof. See Weil *loc. cit.* Chapter I.–4, Corollary 2 to Theorem 6. ■

For $n \in \mathbb{N}$, $\mathrm{M}(n, K) = K^{n\times n}$ denotes the ring of $n \times n$–matrices with entries in K. We define

$$\begin{aligned} \mathrm{GL}(n,K) &:= \mathrm{M}(n,K)^\times \\ \mathrm{SL}(n,K) &:= \mathrm{GL}(n,K)' \\ \mathrm{PGL}(n,K) &:= \mathrm{GL}(n,K)/Z \\ \mathrm{PSL}(n,K) &:= \mathrm{SL}(n,K)/(Z \cap \mathrm{SL}(n,K)), \end{aligned}$$

where $Z = \mathrm{Z}(K^\times)\mathbf{1}$ denotes the centre of $\mathrm{GL}(n,K)$. The quotient morphisms $\mathrm{GL}(n,K) \to \mathrm{PGL}(n,K)$ and $\mathrm{SL}(n,K) \to \mathrm{PSL}(n,K)$ will be denoted by α and β, respectively. $\mathrm{SL}(n,K)$ is the kernel of the Dieudonné determinant $\det\colon \mathrm{GL}(n,K) \to \bar{K}$, where $\bar{K} = K^\times/(K^\times)'$. We denote the coset of $x \in K^\times$ in $\bar{K}$ by $\bar{x}$. Recall that, for $n \geq 2$, $\mathrm{SL}(n,K)$ is the subgroup of $\mathrm{GL}(n,K)$ generated by the elementary matrices $B_{ij}(\lambda) = \mathbf{1} + \lambda E_{ij}$, where $i \neq j$ and $\lambda \in K$, see Artin [1]. Finally, note that $\mathrm{SL}(1,K) = (K^\times)'$.

2. Special Local Bases at the Identity

With product topology, $\mathrm{M}(n,K)$ is a locally compact topological ring. For $\ell \in \mathbb{N}$, the open compact additive subgroups $P_\ell^{n\times n} = \mathrm{M}(n, P_\ell)$ constitute a local base of the topology at 0. We give $\mathrm{GL}(n,K)$ and $\mathrm{SL}(n,K)$ the topologies induced by $\mathrm{M}(n,K)$, and we give the projective linear groups the respective quotient topologies. For $\ell \in \mathbb{N}$, define

$$\begin{aligned} U_\ell^n &:= \mathbf{1} + P_\ell^{n\times n} \\ V_\ell^n &:= U_\ell^n \cap \mathrm{SL}(n,K) \\ \widetilde{U}_\ell^n &:= \alpha(U_\ell^n) \\ \widetilde{V}_\ell^n &:= \beta(V_\ell^n). \end{aligned}$$

Proposition 2.1. $\mathrm{GL}(n,K)$ *is a locally compact topological group. For* $\ell\in\mathbb{N}$*, the sets* U_ℓ^n *constitute a local base for the topology at the identity element, consisting of open compact subgroups.*

Proof. Any U_ℓ^n is an open compact subset of $\mathrm{M}(n,K)$, which is multiplicatively closed, since $P_\ell^{n\times n} = (\pi^\ell\mathbf{1})\mathrm{M}(n,R)$ is an ideal of $\mathrm{M}(n,R)$: We have $U_\ell^n U_\ell^n = (\mathbf{1}+P_\ell^{n\times n})(\mathbf{1}+P_\ell^{n\times n}) \subseteq \mathbf{1}+P_\ell^{n\times n} = U_\ell^n$. Any matrix $A\in U_\ell^n$ is invertible, with inverse in U_ℓ^n, since we presently show that $A^{-1}=(\mathbf{1}+B)^{-1} = \sum_{i=0}^\infty(-B)^i$, where $B:=A-\mathbf{1}\in P_\ell^{n\times n}$. Once we have established the convergence of this so-called *von Neumann series*, it is clear that its limit is the desired inverse of A. Since K, hence $\mathrm{M}(n,K)$, is complete, we only need to check that the series above is a Cauchy series. To this end, note that $(-B)^i\in P_{i\ell}^{n\times n}$ for any $i\in\mathbb{N}$. Therefore $\sum_{i=\nu}^{\nu'}(-B)^i\in P_{\nu\ell}^{n\times n}$, for all $\nu\le\nu'\in\mathbb{N}$. This implies the Cauchy property. Fixing $\nu=0$ and letting ν' tend to infinity, the previous formula also shows that $\sum_{i=0}^\infty(-B)^i\in\mathbf{1}+P_\ell^{n\times n}=U_\ell^n$.
Since $\mathrm{GL}(n,K)$ contains the open subset U_1^n of $\mathrm{M}(n,K)$ and left multiplication by a unit is a homeomorphism of $\mathrm{M}(n,K)$, we conclude that $\mathrm{GL}(n,K)$ is an open subset of $\mathrm{M}(n,K)$. Now since multiplication is continuous, inversion is continuous if it is continuous at the identity element. But this is guaranteed, since the von Neumann series converges absolutely and uniformly on U_1^n. ∎

Lemma 2.2. *The Dieudonné determinant* $\det\colon \mathrm{GL}(n,K)\to\bar{K}$ *is continuous.*

Proof. Since det is a homomorphism of groups, it suffices to check continuity at the identity. We claim $\det(U_\ell^n)\subseteq(U_\ell^1)^-$. To see this, let $A=(a_{ij})\in U_\ell^n$. Left multiplication with the elementary matrix $B_{n1}(-a_{n1}a_{11}^{-1})\in U_\ell^n$ yields a matrix $A'\in U_\ell^n$ whose $(n,1)$–entry vanishes; here we used that $a_{11}\in U_\ell^1$ is a unit. Continuing in this way, we can replace $a_{n1},\ldots,a_{21}$ by zero, and then we apply the same procedure to the other columns. We conclude that there is a lower triangular matrix $B\in U_\ell^n\cap\mathrm{SL}(n,K)$ and an upper triangular matrix $(c_{ij})=C\in U_\ell^n$ such that $BA=C$. Then $\det(A)=\det(C)=\bar{c}_{11}\cdot\ldots\cdot\bar{c}_{nn}\in(U_\ell^1)^-$. ∎

Lemma 2.3. *The commutator subgroup* $(K^\times)'$ *is closed in* $K^\times$.

Proof. The centre κ of K is a local field, and K has finite dimension over κ, see Weil [11] Chapter I.–4, Proposition 5. Hence K is a central division algebra over κ. Let F be a splitting field for K, that is, a commutative extension field of κ such that there is an isomorphism of F–algebras $g: K\otimes_\kappa F\to\mathrm{M}(n,F)$ for some $n\in\mathbb{N}$. Such a splitting field always exists, and we may assume that $F|\kappa$ is a finite Galois extension, see Cohn [4] Chapter 7.2. Then F is a local field. Given $x\in K$, one can show that $\mathrm{RN}_{K/\kappa}(x):=\det_{F^n}(g(x\otimes 1))\in\kappa$, see Cohn *loc. cit.* Chapter 7.3. The mapping $\mathrm{RN}_{K/\kappa}: K\to\kappa$ is called the *reduced norm* on K. We claim that $\mathrm{RN}_{K/\kappa}$ is continuous. Recall that any finite dimensional vector space over a local field L admits precisely one topology which makes it a topological L–vector space, and that every linear map between finite dimensional L–vector spaces is continuous, see Weil [11] Chapter I.–2, Corollary 1 to Theorem 3. We give $K\otimes_\kappa F$ and $M(n,F)$ the unique F–vector space topologies. Then g is continuous. With the topology above, $K\otimes_\kappa F$ is a topological κ–vector space as well. Hence the κ–linear map $K\to K\otimes_\kappa F$, $x\mapsto x\otimes 1$, is continuous. Finally, the determinant mapping $\mathrm{M}(n,F)\to F$ is continuous since F is a topological

field. Hence $\mathrm{RN}_{K/\kappa}$ is continuous, being a composite of continuous maps.
The reduced norm induces a continuous homomorphism $K^\times \to \kappa^\times$, also denoted by $\mathrm{RN}_{K/\kappa}$; its kernel is closed in $K^\times$. But $\ker \mathrm{RN}_{K/\kappa} = (K^\times)'$, for any local field K, a fact usually expressed by saying that the *reduced Whitehead group* $\mathrm{SK}_1(K) := (\ker \mathrm{RN}_{K/\kappa})/(K^\times)'$ is trivial, see Draxl [6]. ■

Proposition 2.4.

(1) $\mathrm{SL}(n,K)$, $\mathrm{PGL}(n,K)$, *and* $\mathrm{PSL}(n,K)$ *are locally compact groups.*

(2) *For any* $\ell \in \mathbb{N}$, V_ℓ^n *is an open and compact subgroup of* $\mathrm{SL}(n,K)$. $\{V_\ell^n : \ell \in \mathbb{N}\}$ *is a local base of the topology at* $\mathbf{1}$.

(3) *A similar statement holds for the projective linear groups; here, the open and compact subgroups are* $\widetilde{U}_\ell^n$ *and* $\widetilde{V}_\ell^n$, *respectively.*

Proof. By the previous lemmas, the Dieudonné determinant is a continuous homomorphism in a Hausdorff topological group. Hence $\mathrm{SL}(n,K) = \ker \det$ is closed in $\mathrm{GL}(n,K)$. Since $\mathrm{Z}(K^\times)$ is a closed subgroup of $K^\times$, the respective quotient topologies on $\mathrm{PGL}(n,K)$ and $\mathrm{PSL}(n,K)$ are locally compact Hausdorff. The remainder is obvious, since quotient morphisms of topological groups are open and continuous. ■

Remark 2.5. Note that U_ℓ^n and V_ℓ^n are *normal* subgroups of $\mathrm{GL}(n,R)$ and $\mathrm{GL}(n,R) \cap \mathrm{SL}(n,K)$, respectively, for any $\ell \in \mathbb{N}$. This is due to the fact that since any $A \in \mathrm{GL}(n,R)$ is a unit of $\mathrm{M}(n,R)$ and $P_\ell^{n\times n}$ is an ideal of this ring, $AU_\ell^n = A(\mathbf{1} + P_\ell^{n\times n}) = A + P_\ell^{n\times n} = U_\ell^n A$.
Now R is an open and compact subring of K, hence $\mathrm{M}(n,R)$ is an open and compact subring of $\mathrm{M}(n,K)$. If $A \in \mathrm{M}(n,R)\backslash\mathrm{GL}(n,R)$, then $A + P_1^{n\times n} \cap \mathrm{GL}(n,R) = \emptyset$. In fact, if there was some matrix B in this intersection, then $A \in B + P_1^{n\times n} = B\,(\mathbf{1} + P_1^{n\times n})$ would be invertible, a contradiction. Hence $\mathrm{GL}(n,R)$ is a closed subset of $\mathrm{M}(n,R)$, from which we conclude that $\mathrm{GL}(n,R)$ is an open and compact subgroup of $\mathrm{GL}(n,K)$.[2] This in turn implies that $\mathrm{GL}(n,R) \cap \mathrm{SL}(n,K)$ is an open and compact subgroup of $\mathrm{SL}(n,K)$.

3. Some Quotients and their Orders

Definition 3.1. For $\ell \in \mathbb{N}$, $\psi_\ell : R \to R/P_\ell : x \mapsto x + P_\ell$ denotes the quotient morphism. We define $\psi_\ell^n : \mathrm{M}(n,R) \to \mathrm{M}(n,R/P_\ell)$ by the prescription $A = (a_{ij}) \mapsto (\psi_\ell(a_{ij}))$. We also write $x_{[\ell]} := \psi_\ell(x)$, $A_{[\ell]} := \psi_\ell^n(A)$.

Lemma 3.2. *Let* $\ell \in \mathbb{N}$.

(1) *Assume that* $a, b \in R$ *are given, with expansions* $a = \sum_{i=0}^\infty a_i \pi^i$ *and* $b = \sum_{i=0}^\infty b_i \pi^i$, *respectively. Then* $a_{[\ell]} = b_{[\ell]}$ *iff* $a_i = b_i$ *for* $0 \le i \le \ell - 1$.

(2) $(\forall A, B \in \mathrm{M}(n,R))\;\; A + P_\ell^{n\times n} = B + P_\ell^{n\times n} \Leftrightarrow A_{[\ell]} = B_{[\ell]}$.

[2] Serre [10] shows that, in the commutative case, $\mathrm{GL}(n,R)$ is a maximal compact subgroup of $\mathrm{GL}(n,K)$, and that all maximal compact subgroups of $\mathrm{GL}(n,K)$ are conjugate.

(3) ψ_ℓ^n *is a quotient morphism of topological rings, and* $\ker\psi_\ell^n = P_\ell^{n\times n}$.
(4) $(\forall A,B\in \mathrm{GL}(n,R))\ \ AU_\ell^n = BU_\ell^n \Leftrightarrow A_{[\ell]} = B_{[\ell]}$.
(5) $(\forall A,B\in \mathrm{SL}(n,K)\cap\mathrm{GL}(n,R))\ \ AV_\ell^n = BV_\ell^n \Leftrightarrow A_{[\ell]} = B_{[\ell]}$.

Proof. (1) and (2) are obvious.
Ad (3). It is clear that ψ_ℓ^n is a surjective morphism of rings with $\ker\psi_\ell^n = (\ker\psi_\ell)^{n\times n} = P_\ell^{n\times n}$. Since this is an open subset of $\mathrm{M}(n,R)$, we conclude that ψ_ℓ^n is continuous.
Ad (4). $P_\ell^{n\times n}$ is an ideal of $\mathrm{M}(n,R)$, and A is a unit. Therefore $AU_\ell^n = A(\mathbf{1}+P_\ell^{n\times n}) = A + AP_\ell^{n\times n} = A + P_\ell^{n\times n}$. We conclude that $AU_\ell^n = BU_\ell^n \Leftrightarrow A+P_\ell^{n\times n} = B + P_\ell^{n\times n}$. Now apply (2).
Ad (5). Since $V_\ell^n \le U_\ell^n$, we infer from (3) that $AV_\ell^n = BV_\ell^n \Rightarrow A_{[\ell]} = B_{[\ell]}$. Conversely, if $A_{[\ell]} = B_{[\ell]}$, where $A,B\in\mathrm{SL}(n,K)\cap\mathrm{GL}(n,R)$, then there is some $C\in U_\ell^n$ such that $AC = B$, by (4). But this implies $C = A^{-1}B \in U_\ell^n\cap\mathrm{SL}(n,K) = V_\ell^n$. ∎

Lemma 3.3. *Let* $a_{ij}\in\delta_{ij}+P$ *for* $i,j\in J$, *where* $J := \{1,\ldots,n\}^2\backslash\{(1,1)\}$, *and set* $\mathcal{A} := \{a_{11}\in U_1^1 : (a_{ij})\in V_1^n\}$. *Then* $\mathcal{A}$ *is the union of* $y := [V_1^1 : V_\ell^1]$ *equivalence classes modulo* U_ℓ^1, *for every* $\ell\in\mathbb{N}$. *In particular,* $\mathcal{A}$ *is not empty.*

Proof. We may assume $n\ge 2$, since the case $n=1$ is trivial. Let $a_{11}\in U_1^1$ and set $A := (a_{ij})$. Then there is a lower unitriangular matrix $B\in V_1^n$, independent of a_{11}, such that $AB =: (c_{ij})$ is an upper triangular matrix. We can construct B as follows. Multiplication of A on the right by the product of elementary matrices $B_n := B_{n1}(-a_{nn}^{-1}a_{n1})\cdot\ldots\cdot B_{nn-1}(-a_{nn}^{-1}a_{nn-1})$ yields a matrix (a'_{ij}) such that $a'_{ni} = 0$ for $i = 1,\ldots,n-1$. We proceed analogously with columns $n-1,\ldots,2$, obtaining lower triangular matrices $B_{n-1},\ldots,B_2\in V_1^n$, all of whose diagonal entries are 1 and which are independent of a_{11}, such that $AB_n\cdots B_2$ is upper triangular. Now set $B := B_n\ldots B_2$.

Note that $c_{11} = a_{11}+r$ for some $r\in P$, where r is independent of a_{11}. Also, $c_{22},\ldots,c_{nn}$ are independent of a_{11}. Set $s := c_{22}\cdot\ldots\cdot c_{nn}\in U_1^1$. Then $A\in\mathrm{SL}(n,K)$ if and only if $(a_{11}+r)s\in(K^\times)'$, and, indeed, if and only if $(a_{11}+r)s\in(K^\times)'\cap U_1^1 = V_1^1$. Now choose representatives $t_1,\ldots,t_y$ of the cosets of $V_\ell^1 = (K^\times)'\cap U_\ell^1$ in V_1^1. For $k=1,\ldots,y$, set $a_k := t_ks^{-1}-r\in U_1^1$. Then $(a_k+r)s = t_k$, whence $a_k\in\mathcal{A}$. Note that for $a,b\in\mathcal{A}$, $a_{[\ell]} = b_{[\ell]} \Leftrightarrow ((a+r)s)_{[\ell]} = ((b+r)s)_{[\ell]}$; here we use that ψ_ℓ is a ring homomorphism, and that s, hence $s_{[\ell]}$, is a unit. From this the claim follows. ∎

Lemma 3.4. *With notation as in Lemma 3.3, assume* $A' = (a'_{ij})\in V_\ell^n$ *such that* $(a'_{ij})_{[\ell]} = (a_{ij})_{[\ell]}$ *for* $(i,j)\in J$. *Then there is* $a_{11}\in\mathcal{A}$ *such that* $(a_{11})_{[\ell]} = (a'_{11})_{[\ell]}$.

Proof. Let $a_{11}\in U_1^1$ and set $A = (a_{ij})$. Define B, C, s as above, and let B', C', s' be the corresponding expressions for A'. Then $B_{[\ell]} = B'_{[\ell]}$. Also, $s_{[\ell]} = s'_{[\ell]}$, and $(c_{ij})_{[\ell]} = (c'_{ij})_{[\ell]}$ for $(i,j)\in J$. Further, $c_{11} = a_{11}+r$ and $c'_{11} = a'_{11}+r'$ for some $r,r'\in P_\ell$, where $r_{[\ell]} = r'_{[\ell]}$. We have $t := (a'_{11}+r')s'\in(K^\times)'\cap U_1^1$, since $A'\in V_1^n$. Then $(a_{11})_{[\ell]} = (a'_{11})_{[\ell]}$ if and only if $((a_{11}+r)s)_{[\ell]} = t_{[\ell]}$. We may replace a_{11} by $\tilde a_{11} := ts^{-1}-r$ without changing B, C, r, and s. Then $\tilde a_{11}\in\mathcal{A}$, and $(\tilde a_{11})_{[\ell]} = (a'_{11})_{[\ell]}$. ∎

Proposition 3.5. *For any $\ell \in \mathbb{N}$,*

(1) $[U_1^n : U_\ell^n] = q^{n^2(\ell-1)}$

(2) $[\widetilde{U}_1^n : \widetilde{U}_\ell^n] = x_\ell^{-1} q^{n^2(\ell-1)}$

(3) $[V_1^n : V_\ell^n] = y_\ell \, q^{(n^2-1)(\ell-1)}$

(4) $[\widetilde{V}_1^n : \widetilde{V}_\ell^n] = z_{\ell,n}^{-1} \, [V_1^n : V_\ell^n]$.

Here $x_\ell := [U_1^1 \cap \kappa^\times : U_\ell^1 \cap \kappa^\times]$ divides $q^{\ell-1}$, and so do $y_\ell := [V_1^1 : V_\ell^1]$ and $z_{\ell,n} := [U_1^1 \cap W_n : U_\ell^1 \cap W_n]$, where κ denotes the centre of K, and where $W_n := \{x \in \kappa^\times : x^n \in (K^\times)'\}$.

Remark 3.6. Note that if K is commutative, W_n is just the group of n–th roots of unity in K, which is finite. Hence $z_{\ell,n}$ becomes stationary for $\ell \in \mathbb{N}$ sufficiently large. Also note that $y_\ell = 1$ and $x_\ell = q^{\ell-1}$ in the commutative case.

Proof. Ad (1). By Lemma 3.2 (1) and (4), the set of matrices with entries of the form $a_{ij} = \delta_{ij} + \sum_{k=1}^{\ell-1} \alpha_{ijk} \pi^k$, where $\alpha_{ijk} \in T$, is a transversal of U_1^n/U_ℓ^n. But this set has $q^{n^2(\ell-1)}$ elements.

Ad (2). Set $Z := \mathrm{Z}(\mathrm{GL}(n,K)) = \kappa^\times \mathbf{1}$. Since $U_\ell^n Z \cap U_1^n = U_\ell^n (U_1^1 \cap \kappa^\times)\mathbf{1}$ and $[U_\ell^n (U_1^1 \cap \kappa^\times)\mathbf{1} : U_\ell^n] = [(U_1^1 \cap \kappa^\times)\mathbf{1} : (U_1^1 \cap \kappa^\times)\mathbf{1} \cap U_\ell^n] = [U_1^1 \cap \kappa^\times : U_\ell^1 \cap \kappa^\times]$, we have $[\widetilde{U}_1^n : \widetilde{U}_\ell^n] = [U_1^n : U_\ell^n Z \cap U_1^n] = [U_1^n : U_\ell^n] \cdot [U_1^1 \cap \kappa^\times : U_\ell^1 \cap \kappa^\times]^{-1}$, from which (2) follows.

Ad (3). For $n = 1$, we compute $[V_1^1 : V_\ell^1] = [(K^\times)' \cap U_1^1 : (K^\times)' \cap U_\ell^1] = [((K^\times)' \cap U_1^1) U_\ell^1 : U_\ell^1]$, which divides $[U_1^1 : U_\ell^1] = q^{\ell-1}$. Now assume $n \geq 2$. Set $J := \{1, \ldots, n\}^2 \backslash \{(1,1)\}$. We consider the set $\mathcal{R}$ of matrices $A = (a_{ij})$ such that $a_{ij} = \delta_{ij} + \sum_{k=1}^{\ell-1} \alpha_{ijk} \pi^k$ for $(i,j) \in J$, where $\alpha_{ijk} \in T$, and where, for fixed $(a_{ij})_{(i,j)\in J}$, we let a_{11} run through a set of representatives modulo U_ℓ^1 of the possible $(1,1)$–entries, $\mathcal{A}$, as in the proof of Lemma 3.3 (where the representatives were denoted by a_k). We claim that $\mathcal{R}$ is a transversal of V_1^n/V_ℓ^n. To see this, let $B = (b_{ij}) \in V_1^n$, and, for $i,j \in \{1, \ldots, n\}$, let $b_{ij} = \delta_{ij} + \sum_{k=1}^{\infty} \beta_{ijk} \pi^k$ be the expansion of b_{ij}, where $\beta_{ijk} \in T$. Then there is precisely one $A = (a_{ij}) \in \mathcal{R}$ such that $AV_\ell^n = BV_\ell^n$, since, by Lemma 3.2, this condition is equivalent to $a_{ij} = \delta_{ij} + \sum_{k=1}^{\ell-1} \beta_{ijk}$ for $(i,j) \in J$, and $(a_{11})_{[\ell]} = (b_{11})_{[\ell]}$. By Lemma 3.4, we can choose a_{11} as required.
To obtain (4), note that $V_\ell^n Z \cap V_1^n = V_\ell^n (W_n \cap U_1^1)\mathbf{1}$. Now copy the proof of (2). ∎

Lemma 3.7. *Assume that K is commutative. If $\det_\ell \colon \mathrm{M}(n, R/P_\ell) \to R/P_\ell$ and $\det \colon \mathrm{M}(n,R) \to R$ denote the determinant mappings, then*

$$\det_\ell(A_{[\ell]}) = (\det A)_{[\ell]},$$

for any matrix $A \in \mathrm{M}(n,R)$ and any $\ell \in \mathbb{N}$. In particular, $\det U_\ell^n \subseteq 1 + P_\ell = U_\ell^1$.

Proof. By an easy computation. ∎

Since ψ_ℓ^n is a morphism of rings by Lemma 3.2, it maps units to units. If K is commutative, Lemma 3.7 implies that $A_{[\ell]} \in \mathrm{SL}(n, R/P_\ell)$, for every $A \in \mathrm{SL}(n,R)$. Hence, for $\ell \in \mathbb{N}$, we can consider

$$\phi_\ell^n := \psi_\ell^n|_{\mathrm{GL}(n,R)}^{\mathrm{GL}(n,R/P_\ell)}$$

$$\chi_\ell^n := \psi_\ell^n|_{\mathrm{SL}(n,R)}^{\mathrm{SL}(n,R/P_\ell)}.$$

Proposition 3.8. *Assume that K is commutative. Then, for all $\ell \in \mathbb{N}$, the mappings ϕ_ℓ^n and χ_ℓ^n are quotient morphisms of groups, and their kernels are* $\ker \phi_\ell^n = U_\ell^n$ *and* $\ker \chi_\ell^n = V_\ell^n$. *In particular,*

$$[\mathrm{GL}(n,R) : U_1^n] = |\mathrm{GL}(n,R/P_1)| = (q^n - 1) \cdot \ldots \cdot (q^n - q^{n-1})$$

and, for $n \geq 2$,

$$[\mathrm{SL}(n,R) : V_1^n] = |\mathrm{SL}(n,R/P_1)| = (q^n - 1) \cdot \ldots \cdot (q^n - q^{n-2})\, q^{n-1}.$$

Proof. *ϕ_ℓ^n is onto:* For any $A \in \mathrm{GL}(n,R/P_\ell)$, there are $\tilde{A}, \tilde{B} \in \mathrm{M}(n,R)$ such that $\psi_\ell^n(\tilde{A}) = A$ and $\psi_\ell^n(\tilde{B}) = A^{-1}$, since ψ_ℓ^n is onto by Lemma 3.2 (3). Lemma 3.7 shows that $(\det \tilde{A}\tilde{B})_{[\ell]} = \det_\ell \mathbf{1} = 1_{[\ell]}$, from which we conclude that $(\det \tilde{A})(\det \tilde{B}) = \det \tilde{A}\tilde{B} \in 1 + P_\ell \subseteq R^\times$ is a unit. This implies $\tilde{A} \in \mathrm{GL}(n,R)$, and we have proved that ϕ_ℓ^n is onto. Now another application of Lemma 3.2 (3) shows $\ker \phi_\ell^n = (\mathbf{1} + \ker \psi_\ell^n) \cap \mathrm{GL}(n,R) = U_\ell^n$.
χ_ℓ^n is onto: Let $A \in \mathrm{SL}(n,R/P_\ell)$. Then there is some $B \in \mathrm{GL}(n,R)$ such that $\phi_\ell^n(B) = A$, as has just been shown. Now $(\det B)_{[\ell]} = \det_\ell(A) = 1_{[\ell]}$, whence $\det B \in 1 + P_\ell$ and $r := (\det B)^{-1} \in 1 + P_\ell$. Set $C := \mathrm{diag}(r, 1, \ldots, 1)$. Then $BC \in \mathrm{SL}(n,R)$, and we have $\chi_\ell^n(BC) = \phi_\ell^n(BC) = \phi_\ell^n(B)\phi_\ell^n(C) = A$, because $C \in U_\ell^n = \ker \phi_\ell^n$. Since A was arbitrary, χ_ℓ^n is onto. Finally, one computes $\ker \chi_\ell^n = \ker \phi_\ell^n \cap \mathrm{SL}(n,R) = U_\ell^n \cap \mathrm{SL}(n,R) = V_\ell^n$.
The remainder of the proposition follows easily now: simply note that $R/P_1 = R/P = k$ is a finite field with q elements, and that the general and special linear groups over these fields have the orders stated above. ∎

4. Computation of Haar Measure

In this section, we give an explicit description of Haar measure on the linear groups over local fields. To this end, the measure of the open compact subgroups introduced in Section 2 is being computed. Lemma 4.1 and Proposition 4.4 show how the Haar measure of any open compact subset can be determined from these data. In fact, the measure of any open subset can be expressed in terms of the values of Haar measure on the local bases, but in a less explicit way. An alternative description of Haar measure on the *general* linear groups will be given in Section 6.

We recall that if G is a locally compact topological group, a positive measure μ on the σ–algebra of Borel sets of G is called a *Haar measure on G*, if it is finite on all compact sets, regular, left–invariant, and if there is an open subset U of G such that $0 < \mu(U) < \infty$. Then, by inner regularity, $\mu(V) > 0$ for any non–empty open subset V of G, since any compact subset of U is covered by finitely many translates of V. It can be shown that on any locally compact group G, there exists a Haar measure μ, which is unique up to a multiplicative positive constant (see Hewitt and Ross [8], Section 15.8). Note that all of the topological groups discussed in this article satisfy the second countability axiom, since K is metric and has a dense countable subset by Theorem 1.2. Now, in any locally compact, second countable group, any open subset is σ-compact. By Rudin [9], Theorem 2.18, any Borel measure on a locally compact space with this property is regular, provided it is finite on compact sets. Hence, as regards the groups we are interested in, regularity of Haar measure is a consequence of the other axioms.

Lemma 4.1. *Let G be a locally compact group and μ a Haar measure on G. If U and V are open compact subgroups of G such that $V \leq U$, then the index of V in U is finite, and $\mu(V) = [U : V]^{-1}\mu(U)$.*

Proof. Since V is an open subgroup of the compact group U, finitely many cosets of V cover U, that is, the index of V in U is finite. If F is a transversal of U/V, then $U = \bigcup_{x\in F} xV$, where the union is disjoint. Now, by left invariance of Haar measure, $\mu(U) = \sum_{x\in F}\mu(xV) = \sum_{x\in F}\mu(V) = [U : V]\,\mu(V)$. ∎

Theorem 4.2. *Let μ_1, μ_2, μ_3, and μ_4 denote Haar measure on $\mathrm{GL}(n,K)$, $\mathrm{PGL}(n,K)$, $\mathrm{SL}(n,K)$, and $\mathrm{PSL}(n,K)$, respectively. Then, with notation as in Lemma 3.5,*

(1) $\mu_1(U^n_\ell) = q^{-n^2(\ell-1)}\,\mu_1(U^n_1)$

(2) $\mu_2(\widetilde{U}^n_\ell) = x_\ell\, q^{-n^2(\ell-1)}\,\mu_2(\widetilde{U}^n_1)$

(3) $\mu_3(V^n_\ell) = y_\ell^{-1} q^{-(n^2-1)(\ell-1)}\,\mu_3(V^n_1)$

(4) $\mu_4(\widetilde{V}^n_\ell) = z_{\ell,n}\, y_\ell^{-1} q^{-(n^2-1)(\ell-1)}\,\mu_4(\widetilde{V}^n_1)$.

If K is commutative and μ_1, μ_3 are chosen such that $\mu_1(\mathrm{GL}(n,R)) = 1$ and $\mu_3(\mathrm{SL}(n,R)) = 1$, respectively, then

(5) $\mu_1(U^n_\ell) = \gamma\, q^{-n^2(\ell-1)}$, *where* $\gamma = |\mathrm{GL}(n,\mathbb{F}_q)|^{-1}$;

(6) $\mu_3(V^n_\ell) = \delta\, q^{-(n^2-1)(\ell-1)}$, *where* $\delta = |\mathrm{SL}(n,\mathbb{F}_q)|^{-1}$.

Proof. The assertions follow immediately from Proposition 3.5, Proposition 3.8 and Lemma 4.1. ∎

Remark 4.3. If K is commutative, the image of the measure μ_3 under the quotient morphism $\beta\colon \mathrm{SL}(n,K) \to \mathrm{PSL}(n,K)$ also yields a Haar measure, λ say, on $\mathrm{PSL}(n,K)$, defined by $\lambda(\Omega) := \mu_3(\beta^{-1}(\Omega))$ for Borel sets Ω of $\mathrm{PSL}(n,K)$. Since $\ker\beta = K^\times\mathbf{1} \cap \mathrm{SL}(n,K)$ is finite, λ inherits the required properties from μ_3.

Proposition 4.4. *Let G be a totally disconnected, locally compact group satisfying the first countability axiom. Then there is a descending countable local base $W_1 \supseteq W_2 \supseteq \cdots$ of the topology, where W_i is an open and compact subgroup of G, for any $i \in \mathbb{N}$. If W is any open and compact subset of G, then there exists $r \in \mathbb{N}$ such that W is the (disjoint) union of finitely many cosets of W_r.*

Proof. Since G is locally compact and totally disconnected, the open and compact subgroups constitute a local base of the topology. By first countability, a countable subbase can be selected. Replacing its elements by suitable finite intersections, we obtain a local base with the required properties. Now since W is open, there is an $i_x \in \mathbb{N}$ such that $xW_{i_x} \subseteq W$, for any $x \in W$. Due to compactness of W, there is a finite subset F of W such that $W = \bigcup_{x\in F} xW_{i_x}$. Then W is a disjoint union of cosets of W_r, where $r := \max\{i_x : x \in F\}$. ∎

Arbitrary open sets can be decomposed into open and compact ones as well:

Proposition 4.5. *Let G and $\mathcal{W} = \{W_i : i \in \mathbb{N}\}$ be as in Proposition* 4.4, *and assume that G is σ–compact. Then, for any open subset U of G, there are a countable set J and families $(g_j)_{j\in J} \in G^J$, $(V_j)_{j\in J} \in \mathcal{W}^J$ such that $U = \bigcup_{j\in J} g_j V_j$, where the union is disjoint. Hence, if μ is a Haar measure on G, we have $\mu(U) = \sum_{j\in J} \mu(V_j)$.*

Proof. For any $g \in U$, there is an $i_g \in \mathbb{N}$ such that $gW_{i_g} \subseteq U$, being minimal with respect to this property. The set $J := \{gW_{i_g} : g \in U\}$ is countable, since, for any $i \in \mathbb{N}$, the index of the open subgroup W_i in the σ–compact group G is countable. We claim that U is the disjoint union of the sets $V \in J$. For let $h, g \in U$ be given such that $hW_{i_h} \cap gW_{i_g} \neq \emptyset$. We may assume that $i_g \leq i_h$. Then $hW_{i_g} = gW_{i_g} \subseteq U$, whence $i_h \leq i_g$, by minimality. We conclude $i_h = i_g$, hence $hW_{i_h} = gW_{i_g}$. ■

Lemma 4.6. *Let G denote one of the groups $K^\times$, $\mathrm{GL}(n,K)$, $\mathrm{SL}(n,K)$, $\mathrm{PGL}(n,K)$, or $\mathrm{PSL}(n,K)$, where $n \geq 2$, let μ be a Haar measure on G, and choose a non–empty open and compact subset U of G. We know that $|R/P| = q$ is a power of some prime p.*

1. *There is $\gamma \in \mathbb{Q}$, such that for any open and compact subset V of G, there are unique numbers $z \in \mathbb{N}$ and $i \in \mathbb{N}_0$ such that $\gcd(p,z) = 1$ and $\mu(V)\,\mu(U)^{-1} = \gamma z p^{-i}$.*
2. *For any $i_0 \in \mathbb{N}_0$, there exists an open and compact subset V of G, such that $i \geq i_0$, with notations as in* (1).

Proof. The assertions follow immediately from Proposition 2.1, Theorem 4.2, and Proposition 4.4. ■

Corollary 4.7. *With notation as in Lemma* 4.6, *let Γ be the set of all rationals $\mu(V)\mu(U)^{-1}$, where V ranges through the open and compact subsets of G. Let p' be a prime number, and let $\nu_{p'}$ denote the p'–adic valuation on $\mathbb{Q}$. Then $p = p'$ if and only if $\nu_{p'}(\Gamma)$ is not bounded below.* ■

Remark 4.8. The following application demonstrates the usefulness of the ideas presented in this article. For $i \in \{1, 2\}$, consider a local field K_i with valuation ring R_i and valuation ideal P_i. Then $q_i := |R_i/P_i|$ is a power of some prime p_i. Let G_i be a general linear group $\mathrm{GL}(n_i, K_i)$, where $n_i \in \mathbb{N}$, or one of the groups $\mathrm{SL}(n_i, K_i)$, $\mathrm{PGL}(n_i, K_i)$, or $\mathrm{PSL}(n_i, K_i)$, where $n_i \geq 2$. Assume that $\theta : G_1 \to G_2$ is a topological isomorphism. We choose a non–empty open and compact subset U_1 of G_1 and set $U_2 := \theta(U_1)$. For $i \in \{1, 2\}$, we define Γ_i as in Corollary 4.7. Then $\Gamma_1 = \Gamma_2$, since if μ_1 is a Haar measure on G_1, then the image of μ_1 under θ is a Haar measure on G_2. The conclusion of the corollary shows $p_1 = p_2$.[3] An interesting special case is stated in Corollary 4.10.

[3] For $n_1, n_2 \geq 2$, this also follows from general investigations on the isomorphy problem of linear groups, cf. Dieudonné [5], which show that the local fields K_1 and K_2 are algebraically isomorphic or antiisomorphic. This actually implies $q_1 = q_2$ in the commutative case, see Glöckner [7].

Lemma 4.9. *Let F be a commutative field, where $\operatorname{char} F \neq 2$, and $n \in \mathbb{N}$. Then $\Delta_2 := \{\operatorname{diag}(\alpha_1, \ldots, \alpha_n) : (\alpha_i)^2 = 1\}$ is a maximal elementary abelian 2-subgroup of $\mathrm{GL}(n,F)$, and all of these are conjugate. An analogous statement holds for $\mathrm{SL}(n,F)$ and its subgroup $\Delta_2 \cap \mathrm{SL}(n,F)$.*

Proof. Let H be an elementary abelian 2–subgroup of $\mathrm{GL}(n,F)$, and $A \in H$. Then $\operatorname{spec}(A) \subseteq \{1,-1\}$, since $A^2 = \mathbf{1}$. For any $x \in F^n$, we have $x = (x+Ax)/2 + (x-Ax)/2$, where $A(x \pm Ax) = \pm(x \pm Ax)$. Hence F^n is the sum of the eigenspaces of A, that is, A is diagonalizable. Note that since H is abelian, the matrices $A \in H$ are simultaneously diagonalizable: there is $S \in \mathrm{SL}(n,F)$ such that $SHS^{-1} \leq \Delta_2$. If $H \leq \mathrm{SL}(n,F)$, then $SHS^{-1} \leq \Delta_2 \cap \mathrm{SL}(n,F)$. The assertions follow easily from this. ∎

Corollary 4.10. *For any $n_1, n_2 \in \mathbb{N}$ and prime numbers p_1 and p_2, the topological groups $\mathrm{GL}(n_1, \mathbb{Q}_{p_1})$ and $\mathrm{GL}(n_2, \mathbb{Q}_{p_2})$ are isomorphic if and only if $n_1 = n_2$ and $p_1 = p_2$. An analogous statement holds for the special linear groups, provided $n_1, n_2 \geq 2$. (Here $\mathbb{Q}_{p_i}$ denotes the field of p_i-adic numbers).*

Proof. Assume that $\theta : \mathrm{GL}(n_1, \mathbb{Q}_{p_1}) \to \mathrm{GL}(n_2, \mathbb{Q}_{p_2})$ is an isomorphism. By Lemma 4.9, there exists a maximal elementary abelian 2–subgroup H of $\mathrm{GL}(n_1, \mathbb{Q}_{p_1})$, of order 2^{n_1}. Then $\theta(H)$ is a maximal elementary abelian 2–subgroup of $\mathrm{GL}(n_2, \mathbb{Q}_{p_2})$, whose order is 2^{n_2}. This implies $n_1 = n_2$. Now since $R_i = \mathbb{Z}_{p_i}$ is the valuation ring of $\mathbb{Q}_{p_i}$, and $P_i = p_i \mathbb{Z}_{p_i}$ its valuation ideal, we infer $|R_i/P_i| = |\mathbb{F}_{p_i}| = p_i$, for $i \in \{1,2\}$. Hence $p_1 = p_2$, by Remark 4.8. ∎

5. Computation of the modular functions

Let G be a locally compact topological group, μ a Haar measure on G and ϕ an (algebraical and topological) automorphism of G. We obtain another Haar measure, ν, on G by defining $\nu(\Omega) := \mu(\phi(\Omega))$ for Borel sets Ω of G. By uniqueness, there is a positive real number $\operatorname{mod}_G(\phi)$, the *module of* ϕ, such that $\nu = \operatorname{mod}_G(\phi)\mu$. For $g \in G$, we set $\operatorname{mod}(g) := \operatorname{mod}_G(I_g)$, where $I_g : G \to G$ denotes the inner automorphism $x \mapsto g^{-1}xg$ of G. The mapping $\operatorname{mod}: G \to \mathbb{R}^+$ is a morphism of topological groups; it is called the *modular function of* G. If $\operatorname{mod} \equiv 1$, G is called *unimodular* (cf. Hewitt and Ross [8], Chapter 15).

Theorem 5.1. *All of the groups $\mathrm{GL}(n,K)$, $\mathrm{SL}(n,K)$, $\mathrm{PGL}(n,K)$, $\mathrm{PSL}(n,K)$ are unimodular.*

Proof. Let G denote one of the groups above. Since $\operatorname{mod}: G \to \mathbb{R}^+$ is a homomorphism into an abelian group, we conclude that $G' \leq \ker \operatorname{mod}$, where G' denotes the derived group.

Now, for $n \geq 2$, we have $\mathrm{SL}(n,K)' = \mathrm{SL}(n,K)$, hence $\mathrm{PSL}(n,K)' = \mathrm{PSL}(n,K)$, which shows that these groups are unimodular. Note that $\mathrm{SL}(1,K) = (K^\times)'$ is a closed subset of $R^\times$, hence compact. Also, $\mathrm{PSL}(1,K)$ is compact, and we conclude that these groups are unimodular.

As regards $\mathrm{GL}(n,K)$, note that $\mathrm{GL}(n,K)' = \mathrm{SL}(n,K) \leq \ker \operatorname{mod}$. For $x \in K^\times$, we define $A_x := \operatorname{diag}(x, 1, \ldots, 1)$. Since $\mathrm{SL}(n,K) \cup \{A_x : x \in K^\times\}$ generates

$\mathrm{GL}(n,K)$, it suffices to show that $\mathrm{mod}\,(A_x)=1$, for all $x\in K^\times$. Hence let $x\in K^\times$ and $A:=A_x$, where we may assume $|x|=q^{-\ell}\le 1$, that is, $\ell\ge 0$ (otherwise compute $\mathrm{mod}\,(A_x^{-1})=\mathrm{mod}\,(A_{x^{-1}})$). For any $B=(b_{ij})\in \mathrm{GL}(n,K)$, we have

$$A^{-1}BA=\begin{pmatrix} x^{-1}b_{11}x & x^{-1}b_{12} & \cdots & x^{-1}b_{1n}\\ b_{21}x & b_{22} & \cdots & b_{2n}\\ \vdots & \vdots & & \vdots\\ b_{n1}x & b_{n2} & \cdots & b_{nn}\end{pmatrix},$$

from which we infer that $A^{-1}U^n_{\ell+1}A$ is the set of all matrices $B=(b_{ij})$ such that $b_{11}\in 1+P_{\ell+1}$, $b_{ij}\in\delta_{ij}+P_{\ell+1}$, $b_{i1}\in P_{2\ell+1}$, and $b_{1j}\in P_1$, where $i,j\in\{2,\ldots,n\}$. Hence

$$U^n_{2\ell+1}\le A^{-1}U^n_{\ell+1}A\le U^n_1\le \mathrm{GL}(n,R).$$

We wish to determine the index $[A^{-1}U^n_{\ell+1}A:U^n_{2\ell+1}]$. By Lemma 3.2 (1) and (4), this can be achieved by counting the possible choices for the first $2\ell+1$ coefficients occuring in the power series expansions of the entries b_{ij} of matrices $(b_{ij})\in A^{-1}U^n_{\ell+1}A$. The first row yields $q^\ell(q^{2\ell})^{n-1}$ possible choices, the remaining coefficients of the first column allow for only one possible choice, and the remainder of the matrix yields another $(q^\ell)^{(n-1)^2}$ choices, whence $[A^{-1}U^n_{\ell+1}A:U^n_{2\ell+1}]=q^{\ell+2\ell(n-1)+\ell(n-1)^2}=q^{\ell n^2}$ holds. Now, if μ denotes Haar measure on G, we obtain $\mu(A^{-1}U^n_{\ell+1}A)=q^{\ell n^2}\mu(U^n_{2\ell+1})=q^{\ell n^2}q^{-2\ell n^2}\mu(U^n_1)=\mu(U^n_{\ell+1})$, which implies $\mathrm{mod}(A)=1$.

Now let λ denote Haar measure on $\mathrm{PGL}(n,K)$. We have $\mathrm{PGL}(n,K)'=\alpha(\mathrm{GL}(n,K)')=\alpha(\mathrm{SL}(n,K))$, and by the preceding argument, it suffices to show $\mathrm{mod}(\alpha(A))=1$ for $A=A_x$ as above. Proceeding as in the proof of Proposition 3.5 (2), we obtain

$$\begin{aligned}[\alpha(A)^{-1}\widetilde{U}^n_{\ell+1}\alpha(A):\widetilde{U}^n_{2\ell+1}] &= [A^{-1}U^n_{\ell+1}A:(U^n_{2\ell+1}Z)\cap A^{-1}U^n_{\ell+1}A]\\ &= [A^{-1}U^n_{\ell+1}A:U^n_{2\ell+1}(U^1_{\ell+1}\cap\kappa^\times)\mathbf{1}]\\ &= [A^{-1}U^n_{\ell+1}A:U^n_{2\ell+1}]\cdot[U^n_{2\ell+1}(U^1_{\ell+1}\cap\kappa^\times)\mathbf{1}:U^n_{2\ell+1}]^{-1}\\ &= [U^n_{\ell+1}:U^n_{2\ell+1}]\cdot[U^n_{2\ell+1}(U^1_{\ell+1}\cap\kappa^\times)\mathbf{1}:U^n_{2\ell+1}]^{-1}\\ &= [\widetilde{U}^n_{\ell+1}:\widetilde{U}^n_{2\ell+1}],\end{aligned}$$

whence $\lambda\big(\alpha(A)^{-1}\widetilde{U}^n_{\ell+1}\alpha(A)\big)=\lambda(\widetilde{U}^n_{\ell+1})$, that is, $\mathrm{mod}(\alpha(A))=1$. ■

6. An alternative description of Haar measure on $\mathrm{GL}(n,K)$

It is well–known that if K is a commutative local field, a Haar measure μ on $\mathrm{GL}(n,K)$ is given by $d\mu=\rho\,d\lambda$, where $\rho(A):=|\det(A)|^{-n}$ for $A\in\mathrm{GL}(n,K)$, and where λ denotes the restriction of Haar measure on $(K^n,+)$ to the Borel sets of $\mathrm{GL}(n,K)$, see Bourbaki [3], Chap. VII–3. We wish to drop the hypothesis that K be commutative.

Lemma 6.1. *Let K be a local field, commutative or not, $|.| = \mathrm{mod}_K$ the absolute value on K described in Section 1, and $\det : \mathrm{GL}(r,K) \to \bar{K}$ the Dieudonné determinant. Since $\mathbb{R}^+$ is abelian, the restriction of mod_K to $K^\times$ factors through $\bar{K} = K^\times/(K^\times)'$, via a homomorphism $f : \bar{K} \to \mathbb{R}^+$.*
Claim: $\mathrm{mod}_{K^r}(A) = f(\det(A))$, *for any* $A \in \mathrm{GL}(r,K)$.

Proof. Any elementary matrix A is in the commutator subgroup $\mathrm{SL}(r,K) = \mathrm{GL}(r,K)'$, hence in the kernel of $\mathrm{mod}_{K^r} : \mathrm{Aut}(K^r) \to \mathbb{R}^+$. This implies $\mathrm{mod}_{K^r}(A) = 1 = f(\det(A))$. Now if $D = \mathrm{diag}(1,\dots,1,a)$ and W is an open subset of K of finite positive measure, then $\mathrm{mod}_{K^r}(D)\,\lambda(W^r) = \lambda(DW^r) = \mathrm{mod}_K(a)\,\lambda(W^r)$, using that Haar measure λ on K^r is the r–fold product of Haar measure on K. We infer $\mathrm{mod}_{K^r}(D) = \mathrm{mod}_K(a) = f(\det(D))$. Since the matrices above generate $\mathrm{GL}(r,K)$, the claim follows. ∎

Theorem 6.2. *With notations as in* 6.1*, define $\rho(A) := (f(\det A))^{-n}$ for $A \in \mathrm{GL}(n,K)$. Let λ' denote Haar measure on $K^{n\times n}$. Then $d\mu = \rho\, d\lambda$ is a Haar measure on $\mathrm{GL}(n,K)$, where λ denotes the restriction of λ' to the Borel sets of $\mathrm{GL}(n,K)$.*

Proof. Since $\mu(U_1^n) = \lambda(U_1^n) = \lambda(\mathbf{1} + P_1^{n\times n}) = \lambda'(P_1^{n\times n}) > 0$, we only need to check that μ is left invariant. Let $V := K^{n\times n}$. Then

$$\begin{aligned} l : \mathrm{GL}(n,K) &\to \mathrm{GL}(n,V), \\ C &\mapsto (l_C\colon A \mapsto CA) \end{aligned}$$

defines a morphism of topological groups. Note that $l_C = C \oplus \cdots \oplus C$, since the action of l_C on each column is multiplication by C. Hence $\det_V(l_C) = (\det_{K^n}(C))^n$, and Lemma 6.1 shows that $\mathrm{mod}_V(l_C) = f(\det(l_C)) = (f(\det(C)))^n$. Now let Ω be a Borel set of $\mathrm{GL}(n,K)$ and $B \in \mathrm{GL}(n,K)$. Then

$$\begin{aligned} \mu(B\Omega) &= \int \mathbf{1}_{B\Omega}(A)\cdot (f(\det A))^{-n}\, d\lambda(A) \\ &= \int (\mathbf{1}_\Omega \circ l_{B^{-1}})(A)\cdot \mathrm{mod}_V^{-1}(l_A)\, d\lambda(A) \\ &= \int (\mathbf{1}_\Omega \circ l_{B^{-1}})(A)\cdot \mathrm{mod}_V^{-1}(l_{BB^{-1}A})\, d\lambda(A) \\ &= \int (\mathbf{1}_\Omega \cdot (\mathrm{mod}_V^{-1} \circ l \circ l_B)) \circ l_{B^{-1}}\, d\lambda \\ &= \int \mathbf{1}_\Omega \cdot (\mathrm{mod}_V^{-1} \circ l \circ l_B)\, dl_{B^{-1}}\lambda, \end{aligned}$$

by transformation of integrals, cf. Bauer [2], Satz 19.1. Here $l_{B^{-1}}\lambda$ denotes the image of λ under the mapping $l_{B^{-1}}$, defined by $(l_{B^{-1}}\lambda)(\omega) := \lambda((l_{B^{-1}})^{-1}(\omega))$ for Borel sets $\omega \subseteq \mathrm{GL}(n,K)$, see Bauer *loc. cit.* Definition 7.6. Now we have $\lambda((l_{B^{-1}})^{-1}(\omega)) = \lambda(l_B(\omega)) = \mathrm{mod}_V(l_B)\,\lambda(\omega)$, hence $l_{B^{-1}}\lambda = \mathrm{mod}_V(l_B)\cdot\lambda$. Since $\mathrm{mod}_V\colon \mathrm{Aut}(V) \to \mathbb{R}^+$ is a homomorphism, one computes

$$\begin{aligned} (\mathrm{mod}_V \circ l \circ l_B)(A) &= \mathrm{mod}_V(l_{l_B(A)}) = \mathrm{mod}_V(l_{BA}) \\ &= \mathrm{mod}_V(l_B l_A) = (\mathrm{mod}_V(l_B)\cdot(\mathrm{mod}_V \circ l))(A). \end{aligned}$$

With these replacements, $\mu(B\Omega) = \int \mathbf{1}_\Omega \cdot (\mathrm{mod}_V(l_B))^{-1} \cdot (\mathrm{mod}_V^{-1} \circ l) \cdot \mathrm{mod}_V(l_B)\, d\lambda = \int \mathbf{1}_\Omega \cdot (\mathrm{mod}_V^{-1} \circ l)\, d\lambda = \mu(\Omega)$. Since Ω was arbitrary, μ is left invariant, and we have proved that μ is a Haar measure on $\mathrm{GL}(n, K)$. ■

Remark 6.3. In particular, Theorem 6.2 shows that a Haar measure on the open compact subgroup $\mathrm{GL}(n, R)$ of $\mathrm{GL}(n, K)$ can be obtained by restricting the Haar measure on $(K^{n\times n}, +)$ to the Borel sets of $\mathrm{GL}(n, R)$. Similar phenomena occur in every standard group, see Serre [10] Part II, Chapter IV, Exercise 5.
A Haar measure λ on the additive group $K^{n\times n}$ can be described explicitely. We consider the open and compact subgroups $P_\ell^{n\times n}$ introduced in Section 2, which constitute a local base of the topology. As in the proof of Proposition 3.5, one computes $[P_1^{n\times n} : P_\ell^{n\times n}] = q^{n^2(\ell-1)}$. This implies $\lambda(P_\ell^{n\times n}) = q^{-n^2(\ell-1)}\lambda(P_1^{n\times n})$.

References

[1] Artin, E., "Geometric Algebra," Interscience Publishers, New York, 1957.

[2] Bauer, H., „Maß– und Integrationstheorie", Walter de Gruyter, Berlin, 1992.

[3] Bourbaki, N., "Intégration, Chap. 7, 8," Hermann Paris, 1963.

[4] Cohn, P. M., "Algebra 3," John Wiley & Sons, New York, 1991.

[5] Dieudonné, J. A., "La Géométrie des Groupes Classiques," Springer-Verlag, Berlin etc., 1971.

[6] Draxl, P. K., "Skew fields," Cambridge University Press, 1983.

[7] Glöckner, H., „Zum Isomorphieproblem linearer Gruppen über lokalen Körpern", Diplomarbeit, TH Darmstadt, 1995.

[8] Hewitt, E., and K. A. Ross, "Abstract Harmonic Analysis I," Springer-Verlag, Berlin etc., 1979.

[9] Rudin, W., "Real and Complex Analysis," MacGraw–Hill, New York etc., 1987.

[10] Serre, J. P., "Lie Algebras and Lie Groups," Springer-Verlag, Berlin etc., 1992.

[11] Weil, A., "Basic Number Theory," Springer-Verlag, Berlin etc., 1967.

Helge Glöckner
Fachbereich Mathematik, AG 5
Technische Hochschule Darmstadt
Schloßgartenstraße 7
D-64289 Darmstadt
gloeckner@mathematik.th-darmstadt.de

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