Journal of Lie Theory
Volume **6** (1996) 207–213

A Note on Central Extensions of Lie Groups

Karl-Hermann Neeb

Communicated by K. H. Hofmann

Abstract. In this note we show that the existence of a central extension of a Lie group G for a prescribed central extension of its Lie algebra can be completely characterized by the exactness of a certain set of 1-forms on G which are obtained by contracting the left-invariant 2-form Ω defined by the 2-cocycle of the Lie algebra extension with right-invariant vector fields. This criterion simplifies a criterion derived by Tuynman and Wiegerinck in the sense that they required in addition that the group of periods of Ω is discrete.

Introduction

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ and A a connected abelian Lie group with Lie algbera $\mathfrak{a}$. A *central extension* of G by A is a short exact sequence of Lie groups

$$(*) \qquad \{\mathbf{1}\} \longrightarrow A \longrightarrow H \longrightarrow G \longrightarrow \{\mathbf{1}\}.$$

It is clear that each such sequence induces a short exact sequence

$$(**) \qquad \{\mathbf{0}\} \longrightarrow \mathfrak{a} \longrightarrow \mathfrak{h} \longrightarrow \mathfrak{g} \longrightarrow \{\mathbf{0}\}$$

on the level of Lie algebras, but in general a central extension of Lie algebras does not integrate to a central extension on the group level.

In [TW87] Tuynman and Wiegerinck have shown that for $\dim A = \mathbf{1}$, the existence of the central extension on the group level can be characterized by two conditions. To explain these conditions, we write $\mathfrak{h}$ as $\mathfrak{g} \times \mathbb{R}$ with the bracket given by

$$[(X,t),(X',t')] = \big([X,X'], \omega([X,X']\big),$$

where $\omega \in \Lambda^2(\mathfrak{g}^*)$ is a cocycle defining the Lie algebra extension of $\mathfrak{g}$ by $\mathfrak{a}$. Let Ω denote the corresponding left-invariant 2-form on G defined by

$$\Omega(g)(d\lambda_g(\mathbf{1})v, d\lambda_g(\mathbf{1})w) := \omega(v,w)$$

ISSN 0949–5932 / \$2.50

for $v, w \in \mathfrak{g} \cong T_{\mathbf{1}}(G)$, where $\lambda_g(x) = gx$ denotes the left-translations on G. Let further

$$\operatorname{Per}\Omega := \left\{ \int_\gamma \Omega \colon \gamma \text{ 2-cycle on } G \right\} \subseteq \mathbb{R}$$

denote the *group of periods* of Ω. Since the mapping $\gamma \to \int_\gamma \Omega$ is a homomorphism of abelian groups, it is clear that $\operatorname{Per}\Omega$ is a subgroup of $\mathbb{R}$.

To formulate the aforementioned criterion, we need one more definition. For $g \in G$ we write $\rho_g \colon x \mapsto xg$ for the right translations on G and for $X \in \mathfrak{g}$ we write X_r for the right-invariant vector field on G given by $X_r(g) = d\rho_g(\mathbf{1}).X$, $g \in G$.

Now the criterion proved in [TW87, Th. 5.4] states that for a Lie algebra extension $(**)$ a central extension $(*)$ exists if and only if

(C1) for each $X \in \mathfrak{g}$ the 1-form $i(X_r).\Omega$ on G is exact, and

(C2) $\operatorname{Per}\Omega$ is a discrete subgroup of $\mathbb{R}$.

In this note we show that (C2) is superfluous, i.e., that the existence of H as in $(*)$ is equivalent to (C2). In view of the results of Tuynman and Wiegerinck, this shows in particular that (C1) implies (C2). The basic tool in our proof is Lie's Third Theorem which ensures the existence of a simply connected Lie group for a given finite dimensional Lie algebra. We note that a general treatment of extensions of connected Lie groups by abelian groups can be found in [Ho51].

I. A Criterion for the Existence of a Central Extendsion

We consider the situation described in the introduction where

$$\{\mathbf{0}\} \longrightarrow \mathfrak{a} \longrightarrow \mathfrak{h} \longrightarrow \mathfrak{g} \longrightarrow \{\mathbf{0}\}$$

is a central extension of the Lie algebra $\mathfrak{g}$ by the abelian Lie algebra $\mathfrak{a}$. We are asking for a criterion which describes the condition under which this central extension integrates to a central extension of G by A.

Let $\widetilde{A}$, $\widetilde{G}$ and $\widetilde{H}$ denote the simply connected Lie groups with Lie algebras $\mathfrak{a}$, $\mathfrak{g}$ and $\mathfrak{h}$. Further we write $q_A \colon \widetilde{A} \to A$ and $q_G \colon \widetilde{G} \to G$ for the corresponding covering homomorphisms.

Lemma I.1. *The natural homomorphisms $\widetilde{A} \to \widetilde{H} \to \widetilde{G}$ define a central extension of $\widetilde{G}$ by $\widetilde{A}$.*

Proof. Since $\widetilde{H}$ is simply connected, the connected normal subgroup $\exp_{\widetilde{H}} \mathfrak{a}$ which is the image of $\widetilde{A}$ under the natural map $i_{\widetilde{A}} \colon \widetilde{A} \to \widetilde{H}$ is closed and simply connected ([Ho65, p.135]). Therefore $i_{\widetilde{A}}$ is an embedding and in particular injective.

Moreover, $\widetilde{H}/\widetilde{A}$ is also simply connected ([Ho65, p.135]), so that the fact that its Lie algebra is $\mathfrak{h}/\mathfrak{a} \cong \mathfrak{g}$ implies that the kernel of the map $\widetilde{H} \to \widetilde{G}$ coincides with $\widetilde{A}$. ∎

In the preceding lemma we have used Lie's Third Theorem which ensures the existence of at least a central extension on the level of simply connected groups. Now we have to modify this "first approximation" appropriately to find a solution of our problem. The first crucial observation is that the kernel of the adjoint representation $\mathrm{Ad}_{\widetilde{H}}\colon \widetilde{H} \to \mathrm{Aut}(\mathfrak{h})$ coincides with the center of $\widetilde{H}$, hence contains $\widetilde{A}$ and thus factors to a representation $\mathrm{Ad}_{\widetilde{G},\mathfrak{h}}\colon \widetilde{G} \to \mathrm{Aut}(\mathfrak{h})$. Further we recall that $\ker q_G \subseteq \widetilde{G}$ can be identified with the fundamental group $\pi_1(G)$. Now we can state our group theoretical condition for the existence of the central extension:

Theorem I.2. *If G and A are given, then a central extension $\mathfrak{a} \to \mathfrak{h} \to \mathfrak{g}$ integrates to a central extension $A \to H \to G$ if and only if the fundamental group $\pi_1(G)$ acts trivially on $\mathfrak{h}$, i.e., $\pi_1(G) \subseteq \ker \mathrm{Ad}_{\widetilde{G},\mathfrak{h}}$.*

Proof. Suppost first that the central extension $A \to H \to G$ exists. Then the adjoint representation $\mathrm{Ad}_H\colon H \to \mathrm{Aut}(\mathfrak{h})$ factors to a representation $\mathrm{Ad}_{G,\mathfrak{h}}$ of G on $\mathfrak{h}$ satisfying $\mathrm{Ad}_{G,\mathfrak{h}} \circ q_G = \mathrm{Ad}_{\widetilde{G},\mathfrak{h}}$. It follows in particular that $\pi_1(G) = \ker q_G \subseteq \ker \mathrm{Ad}_{\widetilde{G},\mathfrak{h}}$. This proves the necessity of the stated condition.

Now we assume that $\pi_1(G)$ acts trivially on $\mathfrak{h}$. We identify $\widetilde{A}$ with a subgroup of $\widetilde{H}$ (Lemma I.1) and write $\beta\colon \widetilde{H} \to \widetilde{G}$ for the canonical homomorphism. Let $C := \beta^{-1}\big(\pi_1(G)\big)$ and note that our assumption implies that

$$C = \ker(q_G \circ \beta) \subseteq \ker \mathrm{Ad}_{\widetilde{H}} = Z(\widetilde{H}).$$

It follows in particular that C is a closed abelian subgroup of $\widetilde{H}$. Moreover, $\mathfrak{g} \cong \mathfrak{h}/\mathfrak{a}$ implies that the identity component C_0 of C coincides with $\widetilde{A}$. Now we have an exact sequence

$$\{\mathbf{1}\} \longrightarrow \widetilde{A} \cong C_0 \longrightarrow C \longrightarrow \pi_1(G) \longrightarrow \{\mathbf{1}\},$$

and since the group $\widetilde{A}$ which is isomorphic to the additive group $\mathfrak{a}$ is in particular divisible, the identity map $\widetilde{A} \to \widetilde{A}$ extends to a homomorphism $C \to \widetilde{A}$ of abelian groups which is continuous because it is continuous on the identity component. We conclude that $C \cong C_0 \times \pi_1(G)$. In this sense we identify $\pi_1(G)$ in a non-canonical fashion with a discrete subgroup of $Z(\widetilde{H})$.

Now let $D := \pi_1(A) \times \pi_1(G)$ and note that this is a discrete subgroup of C, hence a discrete central subgroup of $\widetilde{H}$. We put $H := \widetilde{H}/D$ and write $q_H\colon \widetilde{H} \to H$ for the quotient map. Then $\ker q_H \cap \widetilde{A} = \pi_1(A)$, so that $\widetilde{A} \to \widetilde{H}$ factors to an embedding $A \cong \widetilde{A}/\pi_1(A) \to H$. Moreover, $D \subseteq \ker(q_G \circ \beta)$, so that there exists a morphism of Lie groups $\gamma\colon H \to G$ with $\gamma \circ q_H = q_G \circ \beta$. Then $\mathfrak{g} \cong \mathfrak{h}/\mathfrak{a}$ implies that γ is surjective, and $\gamma(A) = \gamma\big(q_H(\widetilde{A})\big) = q_G\big(\beta(\widetilde{A})\big) = \{\mathbf{1}\}$. Conversely, $x = q_H(y) \in \ker \gamma$ implies that $y \in \ker(q_G \circ \beta) = C$, hence that $x \in q_H(C) = q_H(\widetilde{A}) = A$. This proves that $\ker \gamma = A$, i.e., that $A \to H \to G$ is a central extension. ∎

Corollary I.3. *The existence of the central extension $A \to H \to G$ for a given Lie algebra extension $\mathfrak{a} \to \mathfrak{h} \to \mathfrak{g}$ does not depend on the global structure of the group A.* ∎

Example I.4. A typical example of a Lie algebra extension which does not integrate to an extension of the corresponding group is the following. Let $\mathfrak{g} = \mathbb{R}^2$ with the cocycle $\omega(x,y) = x_1y_2 - x_2y_1$ defining the central extension $\mathbb{R} \to \mathfrak{h} \to \mathfrak{g}$, where $\mathfrak{h}$ is the three dimensional Heisenberg algebra.

If we consider the group $G := \mathbb{R}^2/\mathbb{Z}^2$, a two-dimensional torus, then the fact that $\mathbb{Z}^2$ acts non-trivially via the adjoint action on $\mathfrak{h}$ implies that there exists no central extension

$$A \longrightarrow H \longrightarrow G.$$

This example also shows clearly that the difficulties for the existence of global central extensions are caused by the fact that for a central subgroup A of a Lie group H the center of H/A might be bigger than the image $Z(H)/A$ of the center of H. Thus for any discrete central subgroup $D \subseteq H/A$ which is not contained in $Z(H)/A$ the central extension $\mathfrak{a} \to \mathfrak{h} \to \mathfrak{h}/\mathfrak{a}$ does not integrate to a central extension of the type

$$A_1 \longrightarrow H_1 \longrightarrow (H/A)/D. \qquad \blacksquare$$

II.2. An Interpretation in Terms of Coadjoint Orbits

To see how Theorem I.2 can be interpreted in the light of the criterion of Tuynman and Wiegerinck, we assume from now on that $\dim \mathfrak{a} = 1$. We consider the Lie algebra $\mathfrak{h} = \mathfrak{g} \times \mathbb{R}$ with the bracket

$$[(X,t),(X',t')] = \big([X,X'], \omega([X,X']\big),$$

where $\omega \in \Lambda^2(\mathfrak{g}^*)$ is a cocycle defining the central extension.

Let $\lambda := (0,1) \in \mathfrak{h}^*$ denote the linear functional given by $\lambda(X,t) = t$. We recall that the adjoint representation $\mathrm{Ad}_{\widetilde{H}}$ of $\widetilde{H}$ also defines an action on the dual $\mathfrak{h}^*$ of $\mathfrak{h}$ which is given by $\mathrm{Ad}^*_{\widetilde{H}}(h) := \mathrm{Ad}_{\widetilde{H}}(h^{-1})^*$ and called the *coadjoint action.* We also note that this action clearly factors to a representation $\mathrm{Ad}^*_{\widetilde{G},\mathfrak{h}}$ of $\widetilde{G}$ on $\mathfrak{h}^*$. We write $\mathcal{O}_f$ for the coadjoint orbit of the element $f \in \mathfrak{h}^*$.

The following lemma shows that the action of $\widetilde{G}$ on the orbit $\mathcal{O}_\lambda$ already displays the obstruction for the existence of the central extension.

Lemma II.1. *The following conditions are equivalent:*

(1) $\pi_1(G)$ *acts trivially on* $\mathfrak{g}$.
(2) $\pi_1(G)$ *acts trivially on* $\mathfrak{g}^*$.
(3) $\pi_1(G)$ *acts trivially on* $\mathcal{O}_\lambda$.
(4) *The functional* λ *is fixed by* $\pi_1(G)$.

Proof. The equivalence of (1) and (2) is trivial, and also that (2) implies (3) implies (4).

To see that (4) implies (2), we note that the dual $\mathfrak{g}^*$ of $\mathfrak{g} \cong \mathfrak{h}/\mathfrak{a}$ can be identified with the subspace $\mathfrak{a}^\perp = \{(\alpha, 0) \in \mathfrak{h}^* : \alpha \in \mathfrak{g}^*\}$ in a natural way.

It follows in particular that the corresponding natural embedding $\mathfrak{g}^* \to \mathfrak{h}^*$ is equivariant with respect to the action of the group $\widetilde{G}$ on $\mathfrak{g}^*$ and $\mathfrak{h}^*$. So it is clear that $\pi_1(G)$ acts trivially on the hyperplane $\mathfrak{g}^* \subseteq \mathfrak{h}^*$. Therefore the assumption that $\pi_1(G)$ fixes the element $\lambda = (0,1)$ implies that $\pi_1(G)$ acts trivially on $\mathfrak{g}^*$. ∎

Now we will show that (3) in the preceding lemma can directly be shown to be equivalent to condition (C1) from the introduction. To see this, let $\eta\colon \widetilde{G} \to \mathcal{O}_\lambda$ denote the orbit map and Ω_λ the canonical symplectic form on $\mathcal{O}_\lambda$ which satisfies

$$\Omega_\lambda(\lambda)(\lambda \circ \operatorname{ad} X, \lambda \circ \operatorname{ad} Y) = \lambda([X,Y])$$

for $X, Y \in \mathfrak{h}$.

Lemma II.2. $\eta^*\Omega_\lambda = q_G^*\Omega$.

Proof. The mapping $\eta\colon \widetilde{G} \to \mathcal{O}_\lambda$ is equivariant with respect to the left action of $\widetilde{G}$ on itself. This shows that both 2-forms are left-invariant, and thus we only have to check at the idetity that they coincide. For the right hand side we have

$$(q_G^*\Omega)(\mathbf{1})(X,Y) = \omega(X,Y). \tag{2.1}$$

To calculate the left hand side we note that

$$d\eta(\mathbf{1})(X) = \frac{d}{dt}\Big|_{t=0} \exp(tX).\lambda = \frac{d}{dt}\Big|_{t=0} \lambda \circ e^{-t\operatorname{ad}_{\mathfrak{h}}(X,0)} = -\lambda \circ \operatorname{ad}_{\mathfrak{h}}(X,0).$$

Thus

$$\begin{aligned}(\eta^*\Omega_\lambda)(\mathbf{1})(X,Y) &= \Omega_\lambda(\lambda)\big(\lambda \circ \operatorname{ad}_{\mathfrak{h}}(X,0), \lambda \circ \operatorname{ad}_{\mathfrak{h}}(Y,0)\big) = \lambda([(X,0),(Y,0)]) \\ &= \lambda\big([X,Y], \omega(X,Y)\big) = \omega(X,Y).\end{aligned}$$

In view of (2.1), this proves the lemma. ∎

Proposition II.3. *The group $\pi_1(G)$ acts trivially on $\mathcal{O}_\lambda$ if and only if the condition* (C2) *is satisfied.*

Proof. Let us first assume that $\pi_1(G)$ acts trivially on $\mathcal{O}_\lambda$ and pick $X \in \mathfrak{g}$. We have to show that the 1-form $i(X_r).\Omega$ on G is exact. For that purpose we first note that the orbit map $\eta\colon \widetilde{G} \to \mathcal{O}_\lambda$ factors to an orbit map $\eta_G\colon G \to \mathcal{O}_\lambda$. We consider the function

$$f_X\colon G \to \mathbb{R}, \quad g \mapsto \langle (X,0), \eta_G(g)\rangle$$

and note that $H_X\colon \mathcal{O}_\lambda \to \mathbb{R}, \alpha \mapsto \alpha(X,0)$ is a linear functional with $f_X = \eta_G^* H_X$.

For $\alpha \in \mathcal{O}_\lambda \subseteq \mathfrak{h}^*$ and $Y \in \mathfrak{h}$ we have

$$\begin{aligned}dH_X(\alpha)(\alpha \circ \operatorname{ad} Y) &= \frac{d}{dt}\Big|_{t=0} \langle \alpha \circ e^{t\operatorname{ad} Y}, (X,0)\rangle = \alpha([Y,(X,0)]) \\ &= \Omega_\lambda(\alpha)(\alpha \circ \operatorname{ad} Y, \alpha \circ \operatorname{ad}(X,0)) = -\big(i_{X_{\mathfrak{h}}}\Omega_\lambda\big)(\alpha)(\alpha \circ Y),\end{aligned}$$

where $X_{\mathfrak{h}}$ is the vector field on $\mathcal{O}_\lambda$ given by $X_{\mathfrak{h}}(\alpha) = \alpha \circ \mathrm{ad}(X,0)$. Next we observe that $(\eta_G)_*.X_r = X_{\mathfrak{h}}$ follows from $\eta_G(\exp tXg) = \eta_G(g) \circ e^{-t\,\mathrm{ad}(X,0)}$ for all $g \in G$. Now we can calculate

$$i(X_r)\Omega = i(X_r)\eta_G^*\Omega_\lambda = \eta_G^*\big(i(X_{\mathfrak{h}})\Omega_\lambda\big) = -\eta_G^* dH_X = -d(\eta_G^* H_X) = -df_X.$$

This shows that (C2) is satisfied.

Suppose, conversely, that the condition (C2) is satisfied and that f_X is a function on G with $df_X = -i(X_r)\Omega$. Then the function $\widetilde{f}_X := f_X \circ q_G$ on $\widetilde{G}$ satisfies $d\widetilde{f}_X = -i(X_r)(q_G^*\Omega)$. On the other hand the calculations from above applied to $\widetilde{G}$ instead of G imply that

$$i(X_r)(q_G^*\Omega) = -d(H_X \circ \eta).$$

We conclude that the function $H_X \circ \eta - \widetilde{f}_X$ is constant, which implies in particular that $H_X \circ \eta$ is constant on the cosets of the subgroup $\pi_1(G)$.

Since the function $\alpha \mapsto \alpha(0,1)$ is constant on the orbit $\mathcal{O}_\lambda \subseteq \mathfrak{h}^*$ because $(0,1) \in \mathfrak{h}$ is central, the functions H_X, $X \in \mathfrak{g}$ separate the points on $\mathcal{O}_\lambda$. Therefore the fact that all these functions are $\pi_1(G)$-invariant implies that $\pi_1(G)$ acts trivially on $\mathcal{O}_\lambda$. This completes the proof. ∎

We collect our observations in the following theorem.

Theorem II.4. *Let $\mathbb{R} \to \mathfrak{h} \to \mathfrak{g}$ be a central extension of the Lie algebra $\mathfrak{g}$ and G a connected group with Lie algebra $\mathfrak{g}$. Let further $\omega \in \Lambda^2(\mathfrak{g}^*)$ denote a cocycle defining the Lie algebra extension by identifying $\mathfrak{h}$ with $\mathfrak{g} \times \mathbb{R}$ with the bracket $[(X,t),(X',t')] = \big([X,X'], \omega(X,X')\big)$, Ω on G the corresponding left-invariant 2-form, and $\lambda = (0,1) \in \mathfrak{h}^*$. Then the following are equivalent:*

1. *There exists a central extension $A \to H \to G$ corresponding to the Lie algebra extension $\mathfrak{a} \cong \mathbb{R} \to \mathfrak{h} \to \mathfrak{g}$.*
2. *$\pi_1(G)$ acts trivially on the Lie algebra $\mathfrak{h}$.*
3. *$\pi_1(G)$ acts trivially on the coadjoint orbit $\mathcal{O}_\lambda \subseteq \mathfrak{h}^*$.*
4. *$\pi_1(G)$ fixes the functional $\lambda \in \mathfrak{h}^*$.*
5. *The 1-forms $i(X_r).\Omega$, $X \in \mathfrak{g}$ on G are exact.* ∎

Remark II.5. In view of the criterion of Tuynman and Wiegerinck, it follows that the condition (C1) implies the condition (C2) that the group of periods $\mathrm{Per}\,\Omega \subseteq \mathbb{R}$ is discrete. ∎

References

[Ho51] Hochschild, G., *Group Extensions of Lie Groups*, Annals of Math. **54:1** (1951), 96–109.

[Ho65] —, "The Structure of Lie Groups," Holden Day, San Francisco, 1965.

[TW87] Tuynman, G. M. and W. A. J. J. Wiegerinck, *Central extensions and physics*, J. Geom. Physics **4:2**(1987), 207–258.

Karl-Hermann Neeb
Mathematisches Institut
Universität Erlangen-Nürnberg
Bismarckstr. $1\frac{1}{2}$
D-91054 Erlangen
Germany

Received May 30, 1996