Journal of Lie Theory
Volume **7** (1997) 1–28

Spherical distributions on harmonic extensions of pseudo-H-type groups

Paolo Ciatti

Communicated by G. Mauceri

Abstract. In the articles "Ciatti, P., Scalar products on Clifford modules and pseudo-H-type Lie algebras, Ann. Mat. Pura Appl., to appear" and "Ciatti, P., Solvable extensions of pseudo-H-type algebras, Boll. Un. Mat. It., to appear," a class of solvable pseudo-Riemannian harmonic manifolds was constructed. Now spherical distributions on such manifolds are investigated. A notion of radiality for distributions is introduced with the aid of a technique due to J. Faraut (Faraut, J., Distributions sphérique sur les espaces hyperboliques, J. Math. Pures Appl., (1979), 369–444). The spherical distributions are the radial eigendistributions of the Laplace-Beltrami operator. They span a space which, depending on the signature of the metric, may have dimension one or two.

0. Introduction

A pseudo-Riemannian manifold M with Laplace-Beltrami operator Δ is said to be harmonic if for all functions $f(x)$ on M which depend only on the geodesic distance $d(x, x_0)$ from a fixed point x_0, $\Delta f(x)$ also depends only on $d(x, x_0)$, (see [14, 15]).

In 1944 A. Lichnérowicz showed that, when the dimension is less than or equal to 4, the harmonic Riemannian spaces are symmetric spaces of rank one. In 1950 Lichnérowicz conjectured that the same holds true for all dimensions. Recently Z. Szabó has showed that the conjecture is true for compact manifolds with finite fundamental group, (see [15]).

In 1992 E. Damek and F. Ricci have exhibited counterexamples to the conjecture of Lichnérowicz, (see [7], see also [1]). These counterexamples consist of one-dimensional solvable extensions of the H-type groups defined in [11].

The more general setting of pseudo-riemannian harmonic spaces is more complicated. In fact, on the one hand in 1944 A. Lichnérowicz and A. G. Walker showed that every harmonic space with a Lorentz metric (i.e., a metric with signature $(\dim M-1, 1)$ or $(1, \dim M-1)$) is symmetric, actually it has constant

ISSN 0949–5932 / \$2.50

curvature. On the other hand T. J. Willmore has proved that the so called recurrent spaces are non-symmetric harmonic spaces with metric of signature (p,q) with $p,q \geq 2$, (see for instance [14]). In [6] we provide a new class of pseudo-Riemannian solvable harmonic spaces, applying the construction of Damek and Ricci to the class of pseudo-H-type Lie groups, which have been previously defined in [5]. We summarize biefly here this construction:

Let $\mathfrak{n}$ be a two-step nilpotent real Lie algebra endowed with a scalar product $\langle \cdot\,,\,\cdot \rangle$. Assume that the sum of the center $\mathfrak{z}$ of $\mathfrak{n}$ with $\mathfrak{v} = \mathfrak{z}^{\perp}$ is direct and equal to $\mathfrak{n}$.

Define for all $Z \in \mathfrak{z}$ a map $J_Z : \mathfrak{v} \to \mathfrak{v}$ by

$$\langle J_Z X, Y \rangle = \langle Z, [X,Y] \rangle$$

for $X, Y \in \mathfrak{v}$.

Definition 0.1. We will call $\mathfrak{n}$ *a pseudo-H-type Lie algebra* if

$$\langle J_Z X, J_Z X \rangle = -\langle Z, Z \rangle \langle X, X \rangle$$

for all $Z \in \mathfrak{z}$ and for all $X \in \mathfrak{v}$. In particular we will say that $\mathfrak{n}$ is a (p,q)*-H-type algebra* if $\langle \cdot\,,\,\cdot \rangle$ has signature (p,q) (see [5]).

Let $n = \dim \mathfrak{z} = p+q$. We will often denote by $\bar{n}$ the integer part of $\frac{n}{2}$. We assume $n \geq 1$, excluding the trivial case. It follows that the dimension of $\mathfrak{v}$ is even, so we set dimension of $\mathfrak{v}$ $= 2m$ [5]. Since we know from [5] that m must be even for $n > 1$, we do not consider the case $p = 1$, $q = 0$ and m odd. Hence, we assume that m is even.

We also assume $p \geq 1$, excluding from what follows the case of Euclidean H-type algebras which have been introduced in 1980 by A. Kaplan (see [11]) and have been extensively studed in the literature (see for istance [12], [3], [4], [7]). It necessarily follows from the last assumption that the signature of the restriction of $\langle \cdot\,,\,\cdot \rangle$ to $\mathfrak{v}$ is (m,m) (see [5]).

As in [6] we construct the Damek-Ricci one dimensional solvable extension of $\mathfrak{n}$, taking a derivation H of $\mathfrak{n}$ such that

$$H|_{\mathfrak{v}} = \frac{1}{2} I_{\mathfrak{v}} \qquad \text{and} \qquad H|_{\mathfrak{z}} = I_{\mathfrak{z}},$$

where $I_{\mathfrak{v}}$ and $I_{\mathfrak{z}}$ are respectively the identity endomorphisms of $\mathfrak{v}$ and $\mathfrak{z}$, setting $\mathfrak{a} = \mathbb{R}H$, and defining $\mathfrak{s}$ to be the semi-direct sum of $\mathfrak{n}$ and $\mathfrak{a}$.

We denote by S the connected, simply connected, Lie group with Lie algebra $\mathfrak{s}$, and by A the multiplicative group $(\mathbb{R}^+, .)$ with Lie algebra $\mathfrak{a}$. In particular any connected rank one symmetric space belongs to this class of manifolds which provide a general framework in which rank one symmetric spaces can be analyzed in a unified way ([3], [4]).

The map

$$(X, Z, a) \mapsto \exp(X+Z)\exp(\log aH)$$

from $\mathfrak{v} \times \mathfrak{z} \times \mathbb{R}^+$ onto S defines a global chart.

We endow S with the left invariant pseudo-Riemannian metric g induced by the following scalar product on $\mathfrak{s}$

$$\langle (X, Z, tH), (X', Z', t'H)\rangle = \langle (X, Z), (X', Z')\rangle_{\mathfrak{n}} - tt'.$$

There exists a geodesic arc connecting the identity $e = (0, 0, 1)$ of S to (X, Z, a) if and only if the function $R : S \mapsto \mathbb{R}$ defined by

$$\frac{R(X, Z, a) = \left(1 + a - \frac{1}{4}\langle X, X\rangle\right)^2 - \langle Z, Z\rangle}{4a.} \tag{0.1}$$

is non-negative in (X, Z, a). If this is the case the geodesic distance, $d(X, Z, a)$, of (X, Z, a) from e is a function of $R(X, Z, a)$ (see [6]).

The pseudo-Riemannian manifold (S, g) is harmonic (for the definition see [1, 2, 14, 15]). This means that if f is defined in a neighborhood of e and depends only on $d(X, Z, a)$, then also Δf is a function of $d(X, Z, a)$, where

$$\Delta = \operatorname{div} \operatorname{grad}$$

is the Laplace-Beltrami operator on S. More precisely, if Φ is a C^2 function of $R(X, Z, a)$

$$\Delta\Phi(R) = R\,(1 - R)\,\Phi''(R) + \left(\frac{n}{2} + \frac{1}{2} - (1 + n + m)\,R\right)\,\Phi'(R), \tag{0.2}$$

(see [6]). The left Haar measure on S, $dm_L(X, Z, a)$, is

$$dm_L(X, Z, a) = a^{-1-m-n}\, d^{2m}X\, d^nZ\, da, \tag{0.3}$$

where $d^{2m}X$, d^nZ, and da are Lebesgue measures on $\mathfrak{v}$, $\mathfrak{z}$, and A respectively.

We will denote by $\mathcal{D}(\mathbb{R})$ the space of C^∞-functions with compact support on the real line and by $\mathcal{D}(S)$ the space of C^∞-functions on S with compact support. Let $\eta_1, \ldots, \eta_k$ be a set of functions on the real line, and let $\mathcal{F}$ be any space of functions on $\mathbb{R}$, we will write

$$f \in \mathcal{F} + \eta_1 \times \mathcal{F} + \cdots + \eta_k \times \mathcal{F},$$

if there exist $f_0, f_1, \ldots, f_k \in \mathcal{F}$ such that $f = f_0 + \eta_1\, f_1 + \ldots + +\eta_k\, f_k$.

The spherical distributions on a pseudo-Riemannian symmetric space of rank-one $X = G/H$ have been defined by J. Faraut [9], as the H-invariant distributions on X which are eigendistributions of the Laplace-Beltrami operator. Since in our case S is not a quotient of groups, we replace the H-invariance of distributions with the condition of being constant on the level sets of the function R in (0.1). Decomposition of the Haar measure on S with respect to the coordinate R leads to the notion of an averaging operator M, obtained by integrating functions in $\mathcal{D}(S)$ on the level sets of R. We call $\mathcal{H}_{(p,q,m))}$ the image of $\mathcal{D}(S)$ under M.

Then $\mathcal{H}_{(p,q,m))}$ is a space of functions on the real line. In the first section we prove that $\mathcal{H}_{(p,q,m)}$ can be described as follows

$$\mathcal{H}_{(p,q,m)} = \mathcal{D}(\mathbb{R}) + \eta^0 \times \mathcal{D}(\mathbb{R}) + \eta^1 \times \mathcal{D}(\mathbb{R}),$$

where the functions η^0 and η^1 are defined in the following table

$p \pmod 2$	$q \pmod 2$	η^0	η^1
0	0	$R_+^{\frac{n-1}{2}}$	$(R-1)_+^{\frac{n-1}{2}+m}$
1	0	$R^{\frac{n-1}{2}} \log\lvert R\rvert$	$(R-1)^{\frac{n-1}{2}+m} \log\lvert R-1\rvert$
0	1	$R_+^{\frac{n-1}{2}}$	$(R-1)_+^{\frac{n-1}{2}+m}$
1	1	$R_-^{\frac{n-1}{2}}$	$(R-1)_-^{\frac{n-1}{2}+m}$

where $n = p+q$ is the dimension of $\mathfrak{z}$ and m is the dimension of $\mathfrak{v}$, and t_+^k and t_-^k are the functions defined by

$$t_+^k = t^k\, Y(t), \quad t_-^k = t^k\, Y(-t), \quad Y(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}.$$

In the second section, following the approach of A.Tengstrand and J. Faraut, we endow $\mathcal{H}$ with a topology with respect to which M becomes a continuous mapping. Then we define radial distributions those distributions on S which are image of $\mathcal{H}'_{(p,q,m)}$ under M'. Finally, in the third section, we determine, among radial distributions, those which are eigendistributions of the Laplace-Beltrami operator. These results extend the work of J. Faraut and M. Kosters for the symmetric case [9, 13].

The determination of the spherical distributions on this class of harmonic spaces is the first step toward the Plancherel formula which we will obtain in a forcoming paper.

1. The space $\mathcal{H}_{(p,q,m)}$

We shall prove that we can associate to any function ϕ in $\mathcal{D}(S)$ a function $M\phi$ with compact support on the real line such that, given a function f in $C(\mathbb{R})$,

$$\int_S f\left(R(X,Z,a)\right)\, \phi(X,Z,a)\, dm_L(X,Z,a) = \int_{\mathbb{R}} f(t)\, M\phi(t)\, dt. \tag{1.1}$$

The function $M\phi$ is smooth except at most in 0 and 1 which are the critical values of R. The critical value $R=1$ corresponds to the isolated critical point $(0,0,1)$. The signature of the Hessian matrix of the function R at this point is $(p+m, q+m)$. The critical value $R=0$ is not isolated, it corresponds to a non degenerate critical submanifold Σ of dimension $2m$. The signature of the Hessian matrix to the subspace in the tangent space normal to Σ has signature (p,q).

Theorem 1.1. *Let $\phi \in \mathcal{D}(S)$. Then $M\phi \in \mathcal{H}_{(p,q,m)}$.*

The proof of the theorem requires the preliminaries and the lemmas which follow.

From the proof of Lemma 4.3 of A. Tengstrand [16] we extract the following technical lemma which will be used repeatedly in course of the section:

Lemma 1.2. *Let $f \in \mathcal{D}(\mathbb{R}^2)$, then the function Nf defined by*

$$Nf(t) = \int_{|t|}^{+\infty} f(s,t)\ (s+t)^{\frac{p-1}{2}}\ (s-t)^{\frac{q-1}{2}}\ ds,$$

belongs to the space $\mathcal{D}(\mathbb{R}) + \lambda \times \mathcal{D}(\mathbb{R})$, with λ given by

(I)	$\lambda(t) = t_+^{\frac{p+q}{2}-1}$	*if p and q are both even,*
(II)	$\lambda(t) = t^{\frac{p+q}{2}-1} \log\lvert t\rvert$	*if p and q are both odd,*
(III)	$\lambda(t) = t_+^{\frac{p+q}{2}-1}$	*if p is odd and q is even,*
(IV)	$\lambda(t) = t_-^{\frac{p+q}{2}2-1}$	*if p is even and q isodd.*

Furthermore N is a surjective map from $\mathcal{D}(S)$ onto $\mathcal{D}(\mathbb{R}) + \lambda \times \mathcal{D}(\mathbb{R})$.

We will also need the following lemma of Tengstrand [16], Lemma 3.1:

Lemma 1.3. *Let f be a function on $\mathbb{R}$ with compact support which is C^∞ except at a point t_0, and let λ be as in Lemma 1.2; then the following are equivalent:*

1. *The function f belongs to the space $f \in \mathcal{D}(\mathbb{R}) + \lambda(t-t_0) \times \mathcal{D}(\mathbb{R})$;*
2. *There exists a sequence $\left\{\varepsilon_j^{(0)}\right\}$ such that for all $N \in \mathbb{N}$, the function $f(t) - \lambda(t-t_0) \sum_{j=0}^{N} \varepsilon_j^{(0)} (t-t_0)^j$ is of class C^r in some neighborhood of t_0 for all non-negative integers $r < N + \frac{p+q}{2}$.* ∎

Lemma 1.4. *Let $f\left(x^2,R\right)$ be a function in $\mathcal{D}\left(\mathbb{R}^2\right)$, even in x. If k is a positive integer then:*

1. $$I_1(R) = \int_0^{+\infty} \left(x^2-R\right)_+^{k-1} f\left(x^2,R\right)\,dx \in \mathcal{D}(\mathbb{R}) + R_+^{k-\frac{1}{2}} \times \mathcal{D}(\mathbb{R}),$$

2. $$I_2(R) = \int_0^{+\infty} \left(x^2-R\right)_+^{k-\frac{1}{2}} f\left(x^2,R\right)\,dx \in \mathcal{D}(\mathbb{R}) + R^k \log|R| \times \mathcal{D}(\mathbb{R}),$$

3. $$I_3(R) = \int_0^{+\infty} \left(x^2-R\right)_-^{k-\frac{1}{2}} f\left(x^2,R\right)\,dx \in R_+^k \times \mathcal{D}(\mathbb{R}),$$

4. $$I_4(R) = \int_0^{+\infty} \left(x^2-R\right)^{k-1} \log\left|x^2-R\right|\, f\left(x^2,R\right)\,dx$$
$$\in \mathcal{D}(\mathbb{R}) + R_-^{k-\frac{1}{2}} \times \mathcal{D}(\mathbb{R}).$$

Proof. (1) We write

$$I_1(R)=\int\limits_0^{+\infty}\left(x^2-R\right)^{k-1}f\left(x^2,R\right)\,dx-\mathrm{Y}(R)\int\limits_0^{\sqrt{R}}\left(x^2-R\right)^{k-1}f\left(x^2,R\right)\,dx.$$

The first term in the sum is a C^∞-function of R with compact support. For the second term, setting $x=w\sqrt{R}$ we find

$$\begin{aligned}&\mathrm{Y}(R)\int\limits_0^{\sqrt{R}}\left(x^2-R\right)^{k-1}f\left(x^2,R\right)\,dx\\&=R_+^{k-\frac{1}{2}}\int\limits_0^{1}\left(w^2-1\right)^{k-1}f\left(R\,w^2,R\right)\,dw\in R_+^{k-\frac{1}{2}}\times\mathcal{D}(\mathbb{R}).\end{aligned}$$

Hence,

$$I_1(R)\in\mathcal{D}(\mathbb{R})+R_+^{k-\frac{1}{2}}\times\mathcal{D}(\mathbb{R}).$$

(2) In this case we find

$$\begin{aligned}I_2(R)&=\int\limits_0^{+\infty}\left(x^2-R\right)_+^{\frac{k-1}{2}}f\left(x^2,R\right)\,dx\\&=\begin{cases}\int\limits_0^{+\infty}\left(x^2-R\right)^{\frac{k-1}{2}}f\left(x^2,R\right)\,dx, & \text{if } R<0,\\ \int\limits_{\sqrt{R}}^{+\infty}\left(x^2-R\right)^{\frac{k-1}{2}}f\left(x^2,R\right)\,dx, & \text{if } R\geq 0.\end{cases}\end{aligned}$$

We aim to study $I_2(R)$ when $|R|$ is small since $I_2\in C^\infty$ for $R\neq 0$ and vanishes when $|R|$ is big. So we assume $|R|<\frac{1}{2}$ and write

$$I_2\left(R\right)=I_{21}(R)+I_{22}(R),$$

where

$$I_{21}(R)=\int\limits_0^{1}\left(x^2-R\right)_+^{k-\frac{1}{2}}f\left(x^2,R\right)\,dx,$$

and

$$I_{22}(R)=\int\limits_1^{+\infty}\left(x^2-R\right)^{k-\frac{1}{2}}f\left(x^2,R\right)\,dx.$$

Clearly I_{22} is a C^∞-function of R for $|R|<\frac{1}{2}$. To deal with I_{21} we take the Taylor expansion of f centered at 0 in the variable x^2,

$$f\left(x^2,R\right)=\sum_{j=0}^{l}\frac{1}{j!}\,\partial_1^j\,f(0,R)\,x^{2j}+\mathcal{R}_{(l)}(x^2,R),$$

where $\mathcal{R}_{(l)}$ is the remainder which satisfies

$$\lim_{u\to 0}\frac{\mathcal{R}_{(l)}(u,R)}{u^l}=0,\quad \lim_{u\to 0}\frac{\mathcal{R}_{(l)}^{(j)}(u,R)}{u^{l-j}}=0, \tag{1.2}$$

uniformly in R for all j in $\{0,\dots,l\}$. We obtain

$$I_{21}(R)=\sum_{j=0}^{l}\frac{1}{j!}\,\partial_1^j f(0,R)\int_0^1 x^{2j}\,\left(x^2-R\right)_+^{k-\frac12}\,dx$$
$$+\int_0^1\left(x^2-R\right)_+^{k-\frac12}\,\mathcal{R}_{(l)}(x^2,R)\,dx.$$

By the Lebesgue Dominated Convergence Theorem, formula (1.2) implies that the integral with the remainder term is of class C^{k+l}. In fact, when $R<0$ replacing x^2 by t we obtain

$$\int_0^1 (t-R)^{k-\frac12}\,\frac{\mathcal{R}_{(l)}(t,R)}{2\sqrt{t}}\,dt.$$

By Leibniz's rule we find

$$\left(\frac{d}{dR}\right)^r\int_0^1 (t-R)^{k-\frac12}\,\frac{\mathcal{R}_{(l)}(t,R)}{2\sqrt{t}}\,dt$$
$$=\sum_{j=0}^{r}(-1)^j\binom{r}{j}\int_0^1 (t-R)^{k-j-\frac12}\,\frac{\mathcal{R}_{(l)}^{(r-j)}(t,R)}{2\sqrt{t}}\,dt.$$

Since the remainder is a C^∞-function of R this procedure is allowed for $r<l+k$, because for $r\geq k+l$ the integral does not tend to a finite limit as $R\to 0$. A similar argument works for $R\geq 0$. It is a matter of computation to show that

$$\int_0^1\left(x^2-R\right)_+^{k-\frac12}\,x^{2j}\,dx=g_j(R)+(-1)^{k+1}\,\frac{(2\,k-1)!!\,(2\,j-1)!!}{2^{k+j+1}\,(k+j)!}\,R^{k+j}\,\log|R|, \tag{1.3}$$

where g_j is C^∞ in a neighborhood of 0. Hence, expanding $\partial_1^j f(0,R)$ in powers of R about 0 we find a sequence $\{c_i\}$ such that

$$I_{21}(R)-R^k\,\log|R|\sum_{j=0}^{l}c_j\,R^j\in C^{k+l}.$$

It follows from Lemma 1.3 that

$$I_2\in\mathcal{D}(\mathbb{R})+R^k\,\log|R|\times\mathcal{D}(\mathbb{R}).$$

(3) We remark that $I_3(R) = 0$ if $R < 0$. Thus we assume $R \geq 0$ and set $x = u\sqrt{R}$ obtaining

$$I_3(R) = R_+^k \int_0^1 \left(1-u^2\right)^{k-\frac{1}{2}} f\left(R\,u^2, R\right)\, du \in R_+^k \times \mathcal{D}(\mathbb{R}).$$

(4) We assume $|R| < \frac{1}{2}$ and write

$$I_4(R) = I_{41}(R) + I_{42}(R),$$

where

$$I_{41}(R) = \int_0^1 \left(x^2 - R\right)^{k-1} \log\left|x^2 - R\right| \, f(x^2, R)\, dx,$$

and

$$I_{42}(R) = \int_1^{+\infty} \left(x^2 - R\right)^{k-1} \log\left|x^2 - R\right| \, f(x^2, R)\, dx.$$

We immediately see that I_{42} is C^∞ with compact support in the domain considered. To deal with I_{41}, just as we did for I_{21} we take the Taylor expansion of f about 0 in the variable x^2,

$$\begin{aligned} I_{41}(R) = &\sum_{j=0}^{l} \frac{1}{j!} \partial_1^j f(0,R) \int_0^1 x^{2j} \left(x^2 - R\right)^{k-1} \log\left|x^2 - R\right| \, dx \\ &+ \int_0^1 \left(x^2 - R\right)^{k-1} \log\left|x^2 - R\right| \, \mathcal{R}_{(l)}(x^2, R)\, dx, \end{aligned} \tag{1.4}$$

where $\mathcal{R}_{(l)}$ is the remainder.

From Lebesgue dominated convergence theorem it follows immediately that the last term is a function of class C^{k-1+l} with compact support. The behavior as $R \to 0$ of the first term in (1.4) depends on the side from which R approaches zero. In fact, there exists two C^∞-functions g_j and h_j such that when $R \leq 0$ the integrals are given by

$$\int_0^1 x^{2\,j} \left(x^2 - R\right)^{k-1} \log\left(x^2 - R\right) \, dx = g_j(R) + h_j(R)\,|R|^{k+j-\frac{1}{2}}, \tag{1.5.a}$$

and when $R > 0$ the integrals are given by

$$\int_0^1 x^{2\,j} \left(x^2 - R\right)^{k-1} \log\left|x^2 - R\right| \, dx = g_j(R). \tag{1.5.b}$$

Expanding $\partial_1^j f(0,R)$ in power of R about 0 in (1.4), and using (1.5), we see that there exists a sequence $\{d_j\}$ such that for all non-negative integers l

$$I_{41} - R_-^{k-\frac{1}{2}} \sum_{j=0}^{l} d_j\, R^j$$

is of class C^{k-1+l} in a neighborhood of 0. Therefore from Lemma 1.3 it follows that

$$I_4(R) \in \mathcal{D}(\mathbb{R}) + R_-^{k-\frac{1}{2}} \times \mathcal{D}(\mathbb{R}).$$

■

Lemma 1.5. *Let $f \in \mathcal{D}\left(\mathbb{R}^2\right)$, and j and k non-negative integers, then:*

1. $$\int_0^{+\infty} t^k\, (t+R)_\pm^{\frac{j}{2}}\, f\,(t,R)\, dt \in \mathcal{D}(\mathbb{R}) + R_\pm^{\frac{j}{2+k+1}} \times \mathcal{D}(\mathbb{R}),$$

2. $$\int_0^{+\infty} t^k\, (t+R)^{\frac{j}{2}} \log|t+R|\, f\,(t,R)\, dt \in \mathcal{D}(\mathbb{R}) + R^{\frac{j}{2}+k+1} \log|R| \times \mathcal{D}(\mathbb{R}).$$

We omit the proof which is similar to that of Lemma 1.4, but considerably easier.
Proof of Theorem 1.1. First of all with some transformation we reduce the integral on the left hand side of (1.1) to a simpler form.

Let $Z_1, \ldots, Z_n$ be an orthonormal basis of $\mathfrak{z}$, i.e. such that

$$\langle Z_\mu, Z_\nu \rangle = \epsilon_\mu(p,q)\, \delta_{\mu\nu} \quad \mu,\nu = 1,\ldots,n,$$

where $\delta_{\mu\nu}$ is the Kronecker delta, and

$$\epsilon_\mu(p,q) = \begin{cases} 1, & \text{for } \mu = 1,\ldots,p, \\ -1, & \text{for } \mu = p+1,\ldots,p+q. \end{cases}$$

Let also $X_1, \ldots, X_{2m}$ be an orthonormal basis of $\mathfrak{v}$ such that

$$\langle X_i, X_j \rangle = \epsilon_i(m,m)\, \delta_{ij} \quad i,j = 1,\ldots,2m.$$

If (X,Z,a) is a generic element of S we can write

$$X = \sum_{i=1}^{2m} x_i\, X_i \quad \text{and} \quad Z = \sum_{\mu=1}^{n} z_\mu\, Z_\mu.$$

We introduce bi-spherical coordinates on $\mathfrak{z}$ and $\mathfrak{v}$ as follows. On $\mathfrak{z}$ we set

$$\sum_{\mu=1}^{p} z_\mu\, Z_\mu = (r+s)^{\frac{1}{2}}\, \omega_p \quad \text{and} \quad \sum_{\mu=p+1}^{p+q} z_\mu\, Z_\mu = (r-s)^{\frac{1}{2}}\, \omega_q,$$

where ω_p belongs to the $(p-1)$-dimensional sphere, S_{p-1}, and $\omega_q \in S_{q-1}$. On $\mathfrak{v}$ we set

$$\sum_{i=1}^{m} x_i X_i = (u+v)^{\frac{1}{2}} \omega_m \quad \text{and} \quad \sum_{i=m+1}^{2m} x_i X_i = (u-v)^{\frac{1}{2}} \omega'_m,$$

where $\omega_m, \omega'_m \in S_{m-1}$. Therefore,

$$\langle Z, Z\rangle = r+s-(r-s) = 2s, \quad \text{and} \quad \langle X, X\rangle = u+v-(u-v) = 2v.$$

We observe that the function R defined in (0.1), depends only on (v,s,a),

$$R(v,s,a) = \frac{\left(1+a-\frac{1}{2}v\right)^2 - 2s}{4a}.$$

The integral on the left hand side of (1.1) in the new coordinates becomes

$$\int_0^{+\infty} da \int_{-\infty}^{+\infty} dv \int_{|v|}^{+\infty} du \int_{-\infty}^{+\infty} ds \int_{|s|}^{+\infty} dr\, f\left(R(v,s,a)\right)\, \Omega\phi(u,v,r,s,a)$$
$$\times\, a^{-Q-1} (r+s)^{\frac{p-1}{2}} (r-s)^{\frac{q-1}{2}} \left(u^2-v^2\right)^{\frac{m-1}{2}} \qquad (1.6)$$

with

$$\Omega\phi(u,v,r,s,a) =$$
$$\int_S \phi\left((r+s)^{\frac{1}{2}}\omega_p, (r-s)^{\frac{1}{2}}\omega_q, (u+v)^{\frac{1}{2}}\omega_m, (u-v)^{\frac{1}{2}}\omega'_m\right) d\omega_p d\omega_q d\omega_m d\omega'_m,$$
$$S = S_{p-1}\times S_{q-1}\times S_{m-1}\times S_{m-1},$$

where $d\omega_k$ is the surface element on the sphere S_{k-1}. It follows immediately from Lemma 4.1 of [16] that $\Omega\phi(u,v,r,s,a)$ is a C^∞-function with compact support of all its variables.

Applying Lemma 1.2 (I) to the variables (u,v), the integral in (1.6) becomes

$$\int_0^{+\infty} da \int_{-\infty}^{+\infty} dv \int_{-\infty}^{+\infty} ds \int_{|s|}^{+\infty} dr\, f\left(R(v,s,a)\right)$$
$$\times \left(\alpha(v,r,s,a) + v_+^{m-1}\, \beta(v,r,s,a)\right) (r+s)^{\frac{p-1}{2}} (r-s)^{\frac{q-1}{2}} \qquad (1.7.0)$$

when m is even, and

$$\int_0^{+\infty} da \int_{-\infty}^{+\infty} dv \int_{-\infty}^{+\infty} ds \int_{|s|}^{+\infty} dr\, f\left(R(v,s,a)\right)$$
$$\times \left(\alpha(v,r,s,a) + v^{m-1}\, \log|v|\, \beta(v,r,s,a)\right) (r+s)^{\frac{p-1}{2}} (r-s)^{\frac{q-1}{2}} \qquad (1.7.1)$$

when m is odd. We have included the factor a^{-1-Q} in the functions α and β, noticing that for what concerns the variable a, the supports of α and β are compact in $(0,+\infty)$. Moreover, we observe that $\alpha,\beta \in \mathcal{D}(\mathbb{R}^3\times\mathbb{R}^+)$.

However, only the formula (1.7.0) occurs when $n>1$. In fact, we know from [5] that m must be even for $n>1$. As anticipated in the introduction we do not consider the case $p=1$, $q=0$ and m odd, we can therefore assume that m is even. Replacing the coordinate s with $R=R(v,s,a)$ and calling

$$\tilde{s}(v,R,a)=\frac{1}{2}\left(1+a-\frac{v}{2}\right)^2-2\,a\,R,$$

the integral (1.7.0) becomes

$$\int_{-\infty}^{+\infty} dR\, f(R)\int_0^{+\infty} da\int_{-\infty}^{+\infty} dv\int_{|\tilde{s}(v,R,a)|}^{+\infty} dr\,\left(\tilde{\alpha}(v,R,r,a)+v_+^{m-1}\,\tilde{\beta}(v,R,r,a)\right)$$
$$\times\left(\tilde{s}(v,R,a)+r\right)^{\frac{p-1}{2}}\left(r-\tilde{s}(v,R,a)\right)^{\frac{q-1}{2}},$$

where $\tilde{\alpha}(v,R,r,a)=\alpha\left(v,\tilde{s}(v,R,a),r,a\right)$ is a C^∞-function with compact support in $\mathbb{R}^3\times\mathbb{R}^+$ and the same for $\tilde{\beta}(v,R,r,a)$. Comparing the last formula with (1.1) we obtain

$$M\phi(R)=\int_0^{+\infty} da\int_{-\infty}^{+\infty} dv\int_{|\tilde{s}(v,R,a)|}^{+\infty} dr\,\left(\tilde{\alpha}(v,R,r,a)+v_+^{m-1}\,\tilde{\beta}(v,R,r,a)\right)$$
$$\times\left(r+\tilde{s}(v,R,a)\right)^{\frac{p-1}{2}}\left(r-\tilde{s}(v,R,a)\right)^{\frac{q-1}{2}}. \qquad (1.8)$$

It is clear that $\tilde{s}(v,R,a)>0$ when $R<0$. When $R\geq 0$ we see that $\tilde{s}(v,R,a)\geq 0$ if and only if

$$v\leq 4\left(1+a-2\sqrt{aR}\right)\quad\text{or}\quad v\geq 4\left(1+a+2\sqrt{aR}\right).$$

We set

$$v=2\,(y+a+1),$$

from which it follows

$$\frac{\hat{s}(y,R,a)=\tilde{s}\left(v(y,R,a),R,a\right)=1}{2\,\left(y^2-4aR\right).}$$

In the new variables $M\phi$ can be written as

$$M\phi\,(R)=J_1(R)+J_2(R),$$

with

$$J_1(R)=\int_0^{+\infty} da\int_{-\infty}^{+\infty} dy\int_{\frac{|y^2-4\,a\,R|}{2}}^{+\infty} dr\,\hat{\alpha}(y,R,r,a)$$
$$\times\left(r+\frac{y^2-4\,a\,R}{2}\right)^{\frac{p-1}{2}}\left(r-\frac{y^2-4\,a\,R}{2}\right)^{\frac{q-1}{2}},$$

and

$$J_2(R) = \int_0^{+\infty} da \int_{-\infty}^{+\infty} dy \int_{\frac{|y^2-4\,a\,R|}{2}}^{+\infty} dr\ (y+a+1)_+^{m-1}\ \hat\beta(y,R,r,a)$$
$$\times \left(r + \frac{y^2-4\,a\,R}{2}\right)^{\frac{p-1}{2}} \left(\frac{r-y^2-4\,a\,R}{2}\right)^{\frac{q-1}{2}}.$$

Since the transformation is linear $\hat\alpha$ and $\hat\beta$ are still C^∞-functions with compact support. Considering J_1 we split the function $\hat\alpha$ into the sum of its even and odd parts,

$$\hat\alpha(y,R,r,a) = \frac{1}{2}\,\hat\alpha_0(y^2,R,r,a) + \frac{1}{2}\,y\,\hat\alpha_1(y^2,R,r,a),$$

and we replace r with $a\,t$, and y with $2\sqrt{a}\,x$, obtaining

$$J_1(R) = \int_0^{+\infty} da \int_0^{+\infty} dx \int_{2\,|x^2-R|}^{+\infty} dt\,\tilde\alpha\left(x^2,R,t,a\right)\ \left(t+2\ \left(x^2-R\right)\right)^{\frac{p-1}{2}}$$
$$\times \left(t-2\ \left(x^2-R\right)\right)^{\frac{q-1}{2}}.$$

Since the transformation is C^∞ and proper on the support of the function $\hat\alpha_0$, the function $\tilde\alpha$ is still C^∞ with compact support.

Finally, executing the integration in a we obtain

$$J_1(R) = \int_0^{+\infty} dx \int_{2\,|x^2-R|}^{+\infty} dt\,\mathcal{A}\left(x^2,R,t\right)\ \left(t+2\ \left(x^2-R\right)\right)^{\frac{p-1}{2}}$$
$$\times \left(t-2\ \left(x^2-R\right)\right)^{\frac{q-1}{2}}. \tag{1.9}$$

where

$$\mathcal{A}\left(x^2,R,t\right) = \int_0^{+\infty} \tilde\alpha\left(x^2,R,t,a\right)\ da$$

is a C^∞-function with compact support.

The computations which follow depend on the parity of the two integers (p,q), we denote by $\bar n$ the integer part of half the dimension $n=p+q$ of $\mathfrak{z}$.

(1) When p and q are both even we obtain from (1.9) by Lemma 1.2 (I)

$$J_1(R) = \int_0^{+\infty} \left(A_1\left(x^2,R\right) + A_2\left(x^2,R\right)\ \left(x^2-R\right)_+^{\bar n-1}\right)\ dx, \tag{1.10}$$

Since A_1 and A_2 are C^∞-functions with compact support from Lemma 1.4 it follows that

$$J_1(R) \in \mathcal{D}(\mathbb{R}) + R_+^{\bar n-\frac{1}{2}} \times \mathcal{D}(\mathbb{R}).$$

(2) When p and q are both odd we obtain from (1.9) by Lemma 1.2 (IV)

$$J_1(R) = \int_0^{+\infty} \left(A_1\left(x^2, R\right) + A_2\left(x^2, R\right)\left(x^2 - R\right)^{\bar{n}-1} \log\left|x^2 - R\right| \right) dx.$$

Here also A_1 and A_2 are C^∞-functions with compact support, therefore Lemma 1.4 implies that

$$J_1(R) \in \mathcal{D}(\mathbb{R}) + R_-^{\bar{n}-\frac{1}{2}} \times \mathcal{D}(\mathbb{R}).$$

(3) In the same way when p is even and q is odd we obtain

$$J_1(R) \in \mathcal{D}(\mathbb{R}) + R_+^{\bar{n}} \times \mathcal{D}(\mathbb{R}).$$

(4) Finally, for p odd and q even we obtain

$$J_1(R) \in \mathcal{D}(\mathbb{R}) + R^{\bar{n}} \log|R| \times \mathcal{D}(\mathbb{R}).$$

We now turn our attention to J_2. To study J_2 near 0, we write

$$J_2(R) = J_{21}(R) - J_{22}(R),$$

where

$$J_{21}(R) = \int_{-\infty}^{+\infty} dy \int_0^{+\infty} da \int_{\frac{|y^2-4aR|}{2}}^{+\infty} dr\ (y+a+1)^{m-1}$$
$$\times \left(r + \frac{y^2-4aR}{2}\right)^{\frac{p-1}{2}} \left(r - \frac{y^2-4aR}{2}\right)^{\frac{q-1}{2}} \hat{\beta}\left(y, R, r, a\right),$$

and

$$J_{22}(R) = (-1)^{m-1} \int_{-\infty}^{+\infty} dy \int_0^{+\infty} da \int_{\frac{|y^2-4aR|}{2}}^{+\infty} dr\ (y+a+1)_-^{m-1}$$
$$\times \left(r + \frac{y^2-4aR}{2}\right)^{\frac{p-1}{2}} \left(r - \frac{y^2-4aR}{2}\right)^{\frac{q-1}{2}} \hat{\beta}\left(y, R, r, a\right),$$

When R is near 0 the considerations developed for J_1 hold for J_{21} if one replaces the function $\hat{\alpha}(y, a, r, R)$ with $(y+a+1)^{m-1}\,\hat{\beta}\left(y, R, r, a\right)$. It follows that J_2 produces at $R = 0$ a singularity of the same type as J_1. We also notice that in 0 the function J_{22} is C^∞ since on the domain of integration the argument of the integral is C^∞ with compact support.

We study now the behavior of J_2 near $R=1$. We replace first y with $2\sqrt{R}\,x$ and r with $2\,R\,t$

$$J_2(R)=\int_{-\infty}^{+\infty}dx\int_0^{+\infty}da\int_{|x^2-a|}^{+\infty}dt\,\left(2\sqrt{R}\,x+a+1\right)_+^{m-1}\times\left(t+x^2-a\right)^{\frac{p-1}{2}}\left(t-x^2+a\right)^{\frac{q-1}{2}}\tilde{\beta}\left(x,R,t,a\right),$$

where $\tilde{\beta}$ is C^∞ with compact support for $R\geq\frac{1}{2}$. Recalling that $\tilde{\beta}\left(x,R,t,a\right)=0$ for $a\leq 0$, we extend the integral in a to $(-\infty,+\infty)$. We replace a with $\xi=2\sqrt{R}\,x+a+1$, and then x with $z=x+\sqrt{R}$; finally, observing that only the even part of $\bar{\beta}$ with respect to z gives contribution to the integral, we obtain

$$\begin{aligned}J_2(R)&=\int_{-\infty}^{+\infty}dz\int_{-\infty}^{+\infty}d\xi\,\xi_+^{m-1}\int_{2\,|z^2+1-R-\xi|}^{+\infty}dt\,\left(t+z^2-R+1-\xi\right)^{\frac{p-1}{2}}\\&\qquad\times\left(t-z^2+R-1+\xi\right)^{\frac{q-1}{2}}\bar{\beta}\left(z,R,t,\xi\right)\\&=\int_0^{+\infty}dz\int_0^{+\infty}d\xi\,\xi^{m-1}\int_{2\,|z^2+1-R-\xi|}^{+\infty}dt\,\left(t+z^2-R+1-\xi\right)^{\frac{p-1}{2}}\\&\qquad\times\left(t-z^2+R-1+\xi\right)^{\frac{q-1}{2}}\bar{\beta}_0\left(z^2,R,t,\xi\right)\end{aligned}\tag{1.11}$$

Since these transformations are proper maps and C^∞ on the domain of the integral the function $\bar{\beta}_0$ is C^∞ with compact support in $[0,+\infty)\times\mathbb{R}^2\times\left(\frac{1}{2},+\infty\right)$.

We noticed above that, at the point zero, J_2 produces the same type of singularity as does J_1, therefore we pass to the study of J_2 about 1.

(1) When p and q are both even (1.11) gives by Lemma 1.2 (I)

$$J_2(R)=\int_0^{+\infty}dz\int_0^{+\infty}d\xi\,\xi^{m-1}\times\left(\mathcal{B}_1\left(z^2,R,\xi\right)+\mathcal{B}_2\left(z^2,R,\xi\right)\left(z^2-R+1-\xi\right)_+^{\bar{n}-\frac{1}{2}}\right),$$

where $\mathcal{B}_1$ and $\mathcal{B}_2$ are C^∞-functions with compact support. Interchanging now the order of integration we see that the internal integral becomes one of the same type as I_1. So from Lemma 1.4 (1) it follows that

$$\int_0^{+\infty}\left(\mathcal{B}_1\left(z^2,R,\xi\right)+\mathcal{B}_2\left(z^2,R,\xi\right)\left(z^2-R+1-\xi\right)_+^{\bar{n}-\frac{1}{2}}\right)dz=B_1(R,\xi)+B_2(R,\xi)\left(R-1+\xi\right)_+^{\bar{n}-\frac{1}{2}},$$

where B_1 and B_2 are C^∞-functions with compact support. Hence,

$$J_2(R) = \int_0^{+\infty} \xi^{m-1} \left(B_1(R,\xi) + B_2(R,\xi)\ (R-1+\xi)_+^{\bar{n}-\frac{1}{2}} \right) d\xi, \tag{1.12}$$

and by Lemma 1.5 (1) we find

$$J_2(R) \in \mathcal{D}(\mathbb{R}) + R_+^{\bar{n}-\frac{1}{2}} \times \mathcal{D}(\mathbb{R}) + (R-1)_+^{\bar{n}+m-\frac{1}{2}} \times \mathcal{D}(\mathbb{R}).$$

The same argument applies to the other cases giving:

(2) For p and q both odd

$$J_1(R) \in \mathcal{D}(\mathbb{R}) + R_-^{\bar{n}-\frac{1}{2}} \times \mathcal{D}(\mathbb{R}) + (R-1)_-^{\bar{n}+m-\frac{1}{2}} \times \mathcal{D}(\mathbb{R}).$$

(3) For p even and q odd

$$J_2(R) \in \mathcal{D}(\mathbb{R}) + R_+^{\bar{n}} \times \mathcal{D}(\mathbb{R}) + (R-1)_+^{\bar{n}+m} \times \mathcal{D}(\mathbb{R}).$$

(4) For p odd and q even

$$J_2(R) \in \mathcal{D}(\mathbb{R}) + R^{\bar{n}} \log|R| \times \mathcal{D}(\mathbb{R}) + (R-1)^{\bar{n}+m} \log|R-1| \times \mathcal{D}(\mathbb{R}).$$

Collecting together the results concerning J_1 and J_2 we get the theorem. ∎

We conclude the section with the following theorem:

Theorem 1.6. *The map $M : \mathcal{D}(S) \to \mathcal{H}_{(p,q,m)}$ defined by (1.1) is linear and onto.*

Proof. We have showed that if $\phi \in \mathcal{D}(S)$ then $M\phi \in \mathcal{H}_{(p,q,m)}$. It is also clear that M is linear, so it remains only to show that it is onto. To fix the ideas we will assume that p and q are even; the other cases can be handled in a similar way. Since

$$\mathcal{H}_{(0,0)} = \left\{ f = f_0 + \eta^0 f_1 + \eta^1 f_2 \mid f_0, f_1, f_2 \in \mathcal{D}(\mathbb{R}) \right\},$$

given $f_0, f_1, f_2 \in \mathcal{D}(\mathbb{R})$, to prove the surjectivity of M it is enough to exhibit $\phi_0, \phi_1, \phi_2 \in \mathcal{D}(S)$ such that

$$M\phi_0 = f_0, \quad M\phi_1 = g_0 + \eta^0 f_1, \quad M\phi_2 = h_0 + \eta^0 h_1 + \eta^1 f_2,$$

where $g_0, h_0, h_1 \in \mathcal{D}(\mathbb{R})$. We will show only that there exist ϕ_0 and ϕ_1 with the required properties, the proof in the remaining case is similar.

We first find, in each case, A_1, A_2 such that the integral in (1.10) gives the desired result. For f_0, let $U \in \mathcal{D}(\mathbb{R})$ be such that

$$\int_0^{+\infty} U\left(x^2\right) dx = 1.$$

Choosing

$$A_1\left(x^2,R\right)=U\left(x^2\right)\,f_0(R)\quad\text{and}\quad A_2=0,$$

we find

$$f_0(R)=\int\limits_0^{+\infty}A_1\left(x^2,R\right)\,dx.$$

For f_1, we take $g\in\mathcal{D}(\mathbb{R})$ such that

$$\Lambda_g(R)=2\int\limits_0^1 g\left(R\,u^2\right)\,\left(u^2-1\right)^{\bar n-1}\,dx\neq 0\quad\text{if } R\in\operatorname{supp}f_1\,.$$

We define

$$F_1(R)=-\frac{f_1(R)}{\Lambda_g(R)},$$

clearly $\operatorname{supp}F_1=\operatorname{supp}f_1$ and $F_1\in C^\infty$. We set

$$A_2\left(x^2,R\right)=g\left(x^2\right)\,F_1(R).$$

We find with the usual computations

$$\int\limits_0^{+\infty}\left(x^2-R\right)_+^{\bar n-1}\,A_2\left(x^2,R\right)\,dx=F_1(R)\int\limits_0^{+\infty}\left(x^2-R\right)_+^{\bar n-1}\,g\left(x^2\right)\,dx$$

$$=F_1(R)\int\limits_0^{+\infty}\left(x^2-R\right)^{\bar n-1}\,g\left(x^2\right)\,dx-F_1(R)\,Y(R)\int\limits_0^{\sqrt{R}}\left(x^2-R\right)^{\bar n-1}\,g\left(x^2\right)\,dx$$

$$=g_0(R)-F_1(R)\,R_+^{\bar n-\frac12}\int\limits_0^1\left(u^2-1\right)^{\bar n-1}\,g\left(R\,u^2\right)du=g_0(R)+R_+^{\frac{\bar n-1}{2}}\,f_1(R),$$

with $g_0\in\mathcal{D}(\mathbb{R})$.

Once we have determined A_1 and A_2, we apply Lemma 1.2 to find $\mathcal{A}\left(x^2,R,t\right)$ in $\mathcal{D}\left(\mathbb{R}^3\right)$ such that

$$\int\limits_0^{+\infty}dx\int\limits_{2\,|x^2-R|}^{+\infty}dt\,\mathcal{A}\left(x^2,R,t\right)\,\left(t+2\,\left(x^2-R\right)\right)^{\frac{p-1}{2}}\,\left(t-2\,\left(x^2-R\right)\right)^{\frac{q-1}{2}}$$

$$=\int\limits_0^{+\infty}\left(A_1\left(x^2,R\right)+A_2\left(x^2,R\right)\,\left(x^2-R\right)_+^{\bar n-1}\right)\,dx.$$

Now let $H\in\mathcal{D}\left((0,+\infty)\right)$ be such that

$$\int_0^{+\infty}H(a)\,da=1.$$

Setting

$$\hat{\alpha}\,(y,R,t,a)=\frac{1}{2}\,\mathcal{A}\left(\frac{y^2}{4\,a},R,t\right)\;H(a),$$

we find

$$\int_0^{+\infty} da \int_{-\infty}^{+\infty} dy \int_{\frac{|y^2-4\,a\,R|}{2}}^{+\infty} dr\;\hat{\alpha}(y,R,r,a)\;\left(r+\frac{y^2-4\,a\,R}{2}\right)^{\frac{p-1}{2}}\;\left(r-\frac{y^2-4\,a\,R}{2}\right)^{\frac{q-1}{2}}$$

$$=\int_0^{+\infty} dx \int_{2\,|x^2-R|}^{+\infty} dt\;\mathcal{A}\left(x^2,R,t\right)\;\left(t+2\;\left(x^2-R\right)\right)^{\frac{p-1}{2}}\;\left(t-2\;\left(x^2-R\right)\right)^{\frac{q-1}{2}}.$$

Composing the function $\hat{\alpha}$ with the transformations

$$\tilde{s}(v,R,a)=\frac{1}{2}\;\left(1+a-\frac{v}{2}\right)^2-2\,a\,R,\quad\text{and}\quad r=a\,t,$$

we obtain a function $\alpha(v,r,s,a)$ which lies in $\mathcal{D}\left(\mathbb{R}^3\times\mathbb{R}^+\right)$.

Given α, it follows from Lemma 1.2 that there exists $\psi\in\mathcal{D}\left(\mathbb{R}^4\times\mathbb{R}^+\right)$ such that for any $f\in\mathcal{D}(\mathbb{R})$

$$\int_0^{+\infty} da \int_{-\infty}^{+\infty} dv \int_{|v|}^{+\infty} du \int_{-\infty}^{+\infty} ds \int_{|s|}^{+\infty} dr\, f\left(R(v,s,a)\right)\;\psi(u,v,r,s,a)$$
$$\times\; a^{-Q-1}\,(r+s)^{\frac{p-1}{2}}\,(r-s)^{\frac{q-1}{2}}\left(u^2-v^2\right)^{\frac{m-1}{2}}$$

$$=\int_0^{+\infty} da \int_{-\infty}^{+\infty} dv \int_{-\infty}^{+\infty} ds \int_{|s|}^{+\infty} dr\, f\left(R(v,s,a)\right)\;\alpha(v,r,s,a)\,(r+s)^{\frac{p-1}{2}}\,(r-s)^{\frac{q-1}{2}}.$$

Finally, from Lemma 4.1 of [16] we immediately see that the map $\Omega:\mathcal{D}(S)\to\mathcal{D}\left(\mathbb{R}^4\times\mathbb{R}^+\right)$ defined for $\phi\in\mathcal{D}(S)$ by $\Omega\phi(u,v,r,s,a)=$

$$\int_S \phi\left((r+s)^{\frac{1}{2}}\omega_p,(r-s)^{\frac{1}{2}}\omega_q,(u+v)^{\frac{1}{2}}\omega_m,(u-v)^{\frac{1}{2}}\omega_m'\right)d\omega_p d\omega_q d\omega_m d\omega_m',$$

$S=S_{p-1}\times S_{q-1}\times S_{m-1}\times S_{m-1}$, is onto, and this completes the proof. ∎

2. Radial distributions on S

In this section and in the next we will briefly write $\mathcal{H}$ for $\mathcal{H}_{(p,q,m)}$. We fix once for all two functions χ_0 and χ_1 in $\mathcal{D}(\mathbb{R})$ which are 1 in a neighborhood of 0 and 1 respectively and have disjoint supports. If f is a function in $\mathcal{H}$ there are $f_0,f_1,f_2\in\mathcal{D}(\mathbb{R})$ such that $f=f_0+\eta^0\,f_1+\eta^1\,f_2$. We write

$$f=f_0+\eta^0\,f_1+\eta^1\,f_2=f_0+\chi^0\,\eta_0\,f_1+\left(1-\chi^0\right)\,\eta_0\,f_1+\chi^1\,\eta_1\,f_1+\left(1-\chi^1\right)\,\eta_1\,f_2,$$

and set

$$\tilde{f}_0 = f_0 + \left(1-\chi^0\right)\,\eta_0\,f_1 + \left(1-\chi^1\right)\,\eta_1\,f_2,$$

$$\tilde{\eta}_0 = \chi^0\,\eta_0 \quad \tilde{\eta}_1 = \chi^1\,\eta_1.$$

Hence, we decompose a generic function f in $\mathcal{H}$ as

$$f = \tilde{f}_0 + \tilde{\eta}^0\,f_1 + \tilde{\eta}^1\,f_2, \tag{2.1}$$

where $\tilde{f}_0, f_1, f_2 \in \mathcal{D}(\mathcal{R})$, and the singular parts $\chi^0\,\eta_0\,f_1$, $\chi^1\,\eta_1\,f_2$ have disjoint supports about 0 and 1.

With this decomposition at hand we define the following linear functional on $\mathcal{H}$

$$\langle\theta^0_k, f\rangle = (-1)^k\,\tilde{f}_0^{(k)}(0), \quad \langle\theta^1_k, f\rangle = (-1)^k\,\tilde{f}_0^{(k)}(1), \tag{2.2.a}$$

$$\langle\varepsilon_k, f\rangle = (-1)^k\,f_1^{(k)}(0), \quad \langle\varepsilon^1_k, f\rangle = (-1)^k\,f_2^{(k)}(1), \tag{2.2.b}$$

where $k = 0, 1, \ldots$, and $f^{(k)}$ denotes the k-th derivative of f. Even though the decomposition (2.1) is not unique, it is easy to see that the above sequences are uniquely determined by f (see [16] (3.3)).

We define the topology on $\mathcal{H}$ as follows. For any $r > 0$, we endow the space

$$\mathcal{H}^r = \{f \in \mathcal{H} : \mathrm{supp} f \subset (-r, 1+r)\},$$

with the following semi-norms:

1. $$\|f\|_{N,j} = \sup_t \left| \left(\frac{d}{dt}\right)^j \left(f(t) - \tilde{\eta}^0(t) \sum_{k=0}^{N-\bar{n}} \langle\varepsilon^0_k, f\rangle\, t^k \right.\right.$$ $$\left.\left. -\tilde{\eta}^1(t) \sum_{k=0}^{N-\bar{n}} \langle\varepsilon^1_k, f\rangle\,(t-1)^k \right)\right|, \text{ for all } N \in \mathbb{N} \text{ and all } j \leq N;$$

2. $$\|f\|_{(k)} = \left|\langle\varepsilon^0_{(k)}, f\rangle\right|, \quad \|f\|^1_{(k)} = \left|\langle\varepsilon^1_{(k)}, f\rangle\right|, \text{ for all } k \in \mathbb{N}.$$

These semi-norms define a Fréchet space topology on $\mathcal{H}^r$. We provide the space $\mathcal{H}$ with the strict inductive limit topology. From these definitions we immediately obtain the following:

Lemma 2.1. 1. *$\mathcal{D}(\mathbb{R})$ is a topological subspace of $\mathcal{H}$, that is the injection $\iota : \mathcal{D}(\mathbb{R}) \to \mathcal{H}$ is a homeomorphism onto its image.*

2. *The maps*

$$\upsilon^{(0)} : f \mapsto \eta^0\,f, \quad \textit{and} \quad \upsilon^{(1)} : f \mapsto \eta^1\,f,$$

are continuous from $\mathcal{D}(\mathbb{R})$ into $\mathcal{H}$.

The topological dual $\mathcal{H}'$ of $\mathcal{H}$ is described by the following proposition which can be immediately deduced from Lemma 3.3 of [16].

Proposition 2.2. *Given T in $\mathcal{H}'$, there exist a unique $\tilde{T} \in \mathcal{D}'(\mathbb{R})$ and two sets of constants $\{c_k : k = 0, \ldots, r\}$, $\left\{c_k^1 : k = 0, \ldots, s\right\}$ such that for $f \in \mathcal{H}$*

$$\langle T, f \rangle = \left\langle \tilde{T}, f - \tilde{\eta}^0\, P^0_{l-\bar{n}} - \tilde{\eta}^1\, P^1_{l-\bar{n}-m} \right\rangle + \sum_{k=0}^{r} c_k^0 \,\langle \varepsilon_k^0, f \rangle + \sum_{k=0}^{s} c_k^1 \,\langle \varepsilon_k^1, f \rangle, \quad (2.3)$$

where:

1. $\tilde{T} \in \mathcal{D}'(\mathbb{R})$ *has order* l *on the union of the supports of* $\chi^{(0)}$ *and* $\chi^{(1)}$,
2. *for* $j \in \mathbb{N}$

$$P_j^0(t) = \begin{cases} 0 & \text{if } j < 0, \\ \sum_{k=0}^{j} \langle \varepsilon_k^0, f \rangle\, t^k & \text{if } j \geq 0, \end{cases}$$

$$P_j^1(t) = \begin{cases} 0 & \text{if } j < 0, \\ \sum_{k=0}^{j} \langle \varepsilon_k^1, f \rangle\, (t-1)^k & \text{if } j \geq 0. \end{cases}$$

Conversely, given $\tilde{T} \in \mathcal{D}'(\mathbb{R})$ and two sets of constants $\{c_k : k = 0, \ldots, r\}$ and $\left\{c_k^1 : k = 0, \ldots, s\right\}$, the linear functional defined by (2.3), with l equal to the order of $\tilde{T}$ on the union of the supports of $\tilde{\eta}^{(0)}$ and $\tilde{\eta}^{(1)}$, is in $\mathcal{H}'$.

Remark 2.3. The linear functionals defined by (2.2.a) and (2.2.b) belong to $\mathcal{H}'$.

From Proposition 2.2 and Remark 2.3 we immediately obtain the following

Corollary 2.4. *Any $F \in \mathcal{H}'$ with support in $\{0\} \cup \{1\}$ is a finite linear combination of the functionals $\theta_k^0, \theta_k^1, \varepsilon_k^0, \varepsilon_k^1$.*

Remark 2.5. It follows from Lemma 2.1 that the restriction to $\mathcal{D}(\mathbb{R})$ of an element of $\mathcal{H}'$ is a distribution, i.e. if $S \in \mathcal{H}'$ then $S \circ \iota \in \mathcal{D}'(\mathbb{R})$.

Definition 2.6. Let $F \in \mathcal{H}'$ be given by (2.7), the *order* of F is equal to $\max(l, r, s)$, with the convention that the zero functional has order -1.

The transpose map $M' : \mathcal{H}' \mapsto \mathcal{D}'(S)$ of M is defined as usual by

$$\langle M'T, \phi \rangle = \langle T, M\phi \rangle,$$

with $T \in \mathcal{H}'$ and $\phi \in \mathcal{D}(S)$.

As it has been showed by Faraut in [9] and Kosters in [13] in the case of a rank-one symmetric space $X = G/H$, a distribution $F \in \mathcal{D}'(X)$ is H-invariant if and only if it is the M'-image of some $T \in \mathcal{H}'$. More generally we give the following

Definition 2.7. A distribution F on S is called *radial* if there exists $T \in \mathcal{H}'$ such that

$$F = M'T. \quad (2.4)$$

3. Spherical distributions on S

Definition 3.1. We say that a distribution F on S is *spherical* if (a) it is radial, and (b) it is an eigendistribution of the Laplace-Beltrami operator, i.e.

$$\Delta F = \lambda F,$$

with $\lambda \in \mathbb{C}$.

Remark 3.2. The index κ will refer to the critical values 0 and 1 of the function R and will correspondingly take the values 0 and 1. We denote by I_0 the open interval $(-\infty, 1)$, and by I_1 the open interval $(0, +\infty)$. We also set

$$\mu_0 = \frac{n-1}{2} \quad \text{and} \quad \mu_1 = \frac{n-1}{2} + m. \tag{3.1}$$

The operator M' discussed in the previous sections intertwines the Laplace-Beltrami operator Δ on S with the second order ordinary differential operator

$$L = t\,(1-t)\,\frac{d^2}{dt^2} + \left(\frac{n}{2} + \frac{1}{2} - (1+n+m)\,t\right)\frac{d}{dt}. \tag{3.2}$$

Therefore, the radial solutions of $\Delta F = \lambda F$ are of the form $M'T$, where $T \in \mathcal{H}'$ (with the notations of Section 2) is a solution of

$$LT = \lambda T. \tag{3.3}$$

The operator $L - \lambda$ is hypergeometric. It has three regular singular points (see [14] sec. 10.3): $0, 1, +\infty$. The characteristic exponents in 0 and 1 are given respectively by

$$(0, -\mu_0) \quad \text{and} \quad (0, -\mu_1)\,.$$

The Frobenius theory of ordinary differential equations with singular coefficients shows that the classical solutions of the equation (3.3) behave differently near the singular points according to whether the characteristic exponents are integers or not. So we distinguish between two cases, which depend on the parity of $n = \dim \mathfrak{z}$:

(1) n even. On the open half line $I_0 = (-\infty, 1)$ the solution of the equation $(L-\lambda)\,u = 0$ are the linear combinations of

$$\Phi_0(t), \qquad \Psi_0(t) = |t|^{\frac{1}{2}-\frac{n}{2}}\,\overline{\Psi}_0(t)$$

where Φ_0 and $\overline{\Psi}_0$ are analytic on $(-\infty, 1)$ and can be chosen so that $\Phi_0(0) = \overline{\Psi}_0(0) = 1$. Similarly, on the open half line $I_1 = (0, +\infty)$ the space of classical solutions of (3.3) is generated by two functions

$$\Phi_1(t), \qquad \Psi_1(t) = |t-1|^{\frac{1}{2}-\frac{n}{2}-m}\,\overline{\Psi}_1(t),$$

where Φ_1 and $\overline{\Psi}_1$ are analytic on $(0,+\infty)$ and such that $\Phi_1(1)=\overline{\Psi}_1(1)=1$.
(2) n odd. On $(-\infty,1)$ the space of functions which are solution of (3.3) has a basis given by

$$\Phi_0(t), \qquad \Psi_0(t)=A_0\,\Phi_0(t)\,\log|t|+t^{\frac{1}{2}-\frac{n}{2}}\,\overline{\Psi}_0(t),$$

where A_0 is a constant, and Φ_0 and $\overline{\Psi}_0$ are as in (1). In the case $n=1$ the constant A_0 is equal to 1.

On $(0,+\infty)$ the space of classical solutions of (3.3) is generated by

$$\Phi_1(t), \qquad \Psi_1(t)=A_1\,\Phi_1(t)\,\log|t-1|+(t-1)^{\frac{1}{2}-\frac{n}{2}-m}\,\overline{\Psi}_1(t),$$

where Φ_1 and $\overline{\Psi}_1$ are as before and A_1 is a constant.

In looking for the solutions of (3.3) in $\mathcal{H}'$ it is useful to deal only with one singularity at a time, so we introduce the following open subsets of S

$$S_0=\left\{(X,Z,a)\in S \;\middle|\; R(X,Z,a)<1\right\},$$

and

$$S_1=\left\{(X,Z,a)\in S \;\middle|\; R(X,Z,a)>0\right\}.$$

Correspondingly we consider

$$\mathcal{H}_0=\left\{M\phi \;\middle|\; \phi\in\mathcal{D}(S_0)\right\},$$

and

$$\mathcal{H}_1=\left\{M\phi \;\middle|\; \phi\in\mathcal{D}(S_1)\right\}.$$

Consequently the support of the functions in $\mathcal{H}_0$ is contained in $(-\infty,1)$ and the support of functions in $\mathcal{H}_1$ is contained in $(0,+\infty)$.

Let $\mathrm{Pf}_{\varepsilon\downarrow 0}$ denote the Hadamard finite part of the integral (see for instance [10], p. 70). The solutions of (3.3) in the required space are described in terms of the following linear functionals.

1. For $f\in\mathcal{H}_0$, we define

$$\langle S_{(0,+)},f\rangle=\mathrm{Pf}_{\varepsilon\downarrow 0}\int_{\varepsilon}^{1}\Phi_0(t)\,f(t)\,dt, \quad \langle S_{(0,-)},f\rangle=\mathrm{Pf}_{\varepsilon\downarrow 0}\int_{-\infty}^{-\varepsilon}\Phi_0(t)\,f(t)\,dt, \tag{3.4.a}$$

$$\langle T_{(0,+)},f\rangle=\mathrm{Pf}_{\varepsilon\downarrow 0}\int_{\varepsilon}^{1}\Psi_0(t)\,f(t)\,dt, \quad \langle T_{(0,-)},f\rangle=\mathrm{Pf}_{\varepsilon\downarrow 0}\int_{-\infty}^{-\varepsilon}\Psi_0(t)\,f(t)\,dt.$$

2. For $f\in\mathcal{H}_1$, we define

$$\langle S_{(1,+)},f\rangle=\mathrm{Pf}_{\varepsilon\downarrow 0}\int_{1+\varepsilon}^{+\infty}\Phi_1(t)\,f(t)\,dt, \quad \langle S_{(1,-)},f\rangle=\mathrm{Pf}_{\varepsilon\downarrow 0}\int_{0}^{1-\varepsilon}\Phi_1(t)\,f(t)\,dt, \tag{3.4.b}$$

$$\langle T_{(1,+)},f\rangle=\mathrm{Pf}_{\varepsilon\downarrow 0}\int_{1+\varepsilon}^{+\infty}\Psi_1(t)\,f(t)\,dt, \quad \langle T_{(1,-)},f\rangle=\mathrm{Pf}_{\varepsilon\downarrow 0}\int_{0}^{1-\varepsilon}\Psi_1(t)\,f(t)\,dt.$$

3. When n is odd we also define for $f \in \mathcal{H}_0$

$$\langle U_0, f\rangle = \mathrm{Pf}_{\varepsilon\downarrow 0}\left\{\varepsilon\, \Psi_0(\varepsilon)\, f(\varepsilon)\right\}, \tag{3.4.c}$$

and for $f \in \mathcal{H}_1$

$$\langle U_1, f\rangle = \mathrm{Pf}_{\varepsilon\downarrow 0,\varepsilon'\downarrow 1}\left\{(\varepsilon' - 1)\, \Psi_1(\varepsilon)\, f(\varepsilon)\right\}, \tag{3.4.d}$$

Remark 3.3. Clearly $S_{(\kappa,+)}$, $S_{(\kappa,-)}$, $T_{(\kappa,+)}$, $T_{(\kappa,-)}$, and U_κ are in $\mathcal{H}'_\kappa$ for $\kappa = 0, 1$.

Remark 3.4. We notice that if n is odd and $f \in \mathcal{D}(\mathbb{R})$

$$\langle U_0, f\rangle = \mathrm{Pf}_{\varepsilon\downarrow 0}\left\{\varepsilon^{1-\bar n}\, \overline{\Psi}_0(\varepsilon)\, f(\varepsilon)\right\} =$$

$$\frac{1}{(\bar n - 1)!}\left(\frac{d}{dt}\right)^{\bar n - 1}\left(\overline{\Psi}_0(t)\, f(t)\right)_{t=0} = \sum_{k=0}^{\bar n - 1} u_{(0,k)}\, \langle \theta_k, f\rangle,$$

with

$$u_{(0,k)} = \frac{(-1)^k}{k!\,(\bar n - k)!}\left(\frac{d}{dt}\right)^{(\bar n - 1 - k)} \overline{\Psi}_0\Big|_{t=0}.$$

Analogously we find

$$\langle U_{(1)}, f\rangle = \frac{1}{(\bar n + m - 1)!}\left(\frac{d}{dt}\right)^{\bar n + m - 1}\left(\overline{\Psi}_1(t)\, f(t)\right)_{t=1}.$$

We also notice that

$$\langle U_\kappa, \eta^\kappa f\rangle = 0.$$

In the rest of the section we will prove the following theorem:

Theorem 3.5. *The subspace of solutions of* (3.3) *is generated:*

1. *by* $\left\{S_{(\kappa,+)} + S_{(\kappa,-)}, T_{(\kappa,-)}\right\}$ *when* $p, q \equiv 0 \pmod 2$.
2. *by* $\left\{S_{(\kappa,+)} + S_{(\kappa,-)}, T_{(\kappa,+)} + T_{(\kappa,-)}\right\}$ *when* $p \equiv 1 \pmod 2$ *and* $q \equiv 0 \pmod 2$.
3. *by* $\left\{S_{(\kappa,+)}, S_{(\kappa,-)}\right\}$ *when* $p \equiv 0 \pmod 2$ *and* $q \equiv 1 \pmod 2$ *and* $n = 1$, *and by* $\left\{A_\kappa\, S_{(\kappa,+)} + U_\kappa, S_{(\kappa,+)} + S_{(\kappa,-)}\right\}$ *when* $p \equiv 0 \pmod 2$ *and* $q \equiv 1 \pmod 2$ *and* $n > 1$.
4. *by* $\left\{S_{(\kappa,+)} + S_{(\kappa,-)}, T_{(\kappa,+)}\right\}$ *when* $p, q \equiv 0 \pmod 2$.

The formal adjoint of L, denoted by L^*, is defined for a C^2 function f by

$$L^* f = \frac{d^2}{dt^2}\left(t\,(1-t)\, f(t)\right) - \frac{d}{dt}\left(\left(\frac{n}{2} + \frac{1}{2} - (1 + n + m)\, t\right) f(t)\right).$$

Solving (3.3) in the space $\mathcal{H}_\kappa{}'$ is equivalent to looking for elements D of $\mathcal{H}_\kappa{}'$ which satisfy

$$0 = \langle (L - \lambda)\, D, f\rangle = \langle D, (L^* - \lambda)\, f\rangle \quad \text{for all } f \in \mathcal{H}^\kappa. \tag{3.5}$$

By Lemma 2.2 $\mathcal{D}(I_\kappa)$ and $\eta^\kappa \times \mathcal{D}(I_\kappa)$ with $\kappa = 0, 1$ are topological subspaces of $\mathcal{H}_\kappa$. It follows that $D \in \mathcal{H}'_\kappa$ satisfies (3.5), if and only if the kernel of D contains the spaces $(L^* - \lambda)\, \mathcal{D}(I^\kappa)$ and $(L^* - \lambda)\, (\eta^\kappa \times \mathcal{D}(I^\kappa))$. To find the functionals in $\mathcal{H}'_\kappa$ with this property we need the action of D on $(L^* - \lambda)\, \mathcal{D}(I_\kappa)$ and on $(L^* - \lambda)\, (\eta^\kappa \times \mathcal{D}(I_\kappa))$ with $\kappa = 0, 1$. These actions are described by the following lemma, the proof can be found in Appendix A.3 of [9].

Lemma 3.6. 1. *When* $p, q \equiv 0 \pmod 2$ *we have*

$$(L-\lambda)\ S_{\pm}^{\kappa} = \pm\mu_{\kappa}\theta^{\kappa}, \qquad (L-\lambda)\ T_{+}^{\kappa} = -\mu_{\kappa}\varepsilon^{\kappa}, \qquad (L-\lambda)\ T_{-}^{\kappa} = 0.$$

2. *When* $p \equiv 1 \pmod 2$ *and* $q \equiv 0 \pmod 2$ *we have*

$$(L-\lambda)\ S_{\pm}^{0} = \begin{cases} \mp\varepsilon^{0} & \text{if } n=1, \\ \pm\mu_0\,\theta^{0} & \text{if } n>1, \end{cases} \qquad (l-\lambda)\ s_{\pm}^{1} = \pm\mu_1\theta^{1},$$

$$(L-\lambda)\ T_{\pm}^{0} = \begin{cases} \pm\left(\theta^{0}-\varepsilon^{0}\right) & \text{if } n=1, \\ \pm\left(F^{0}-\varepsilon^{0}\right) & \text{if } n>1, \end{cases}$$

$$(L-\lambda)\ T_{\pm}^{1} = \pm\left(F^{1}-\varepsilon^{1}\right), \qquad (L-\lambda)\ U_{\kappa} = \mu_{\kappa}\left(\varepsilon^{\kappa}-A_{\kappa}\,\theta^{\kappa}\right).$$

3. *When* $p \equiv 0 \pmod 2$ *and* $q \equiv 1 \pmod 2$ *we have*

$$(L-\lambda)\ S_{\pm}^{\kappa} = \pm\mu_{\kappa}\,\theta^{\kappa}, \qquad (L-\lambda)\ T_{+}^{0} = \begin{cases} \theta^{0}+\varepsilon^{0} & \text{if } n=1, \\ F^{0}-\mu_0\,\varepsilon^{0} & \text{if } n>1, \end{cases}$$

$$(L-\lambda)\ T_{+}^{1} = F^{1}-\mu_1\,\varepsilon^{1}, \qquad (L-\lambda)\ T_{-}^{0} = \begin{cases} -\theta^{0} & \text{if } n=1, \\ -F^{0} & \text{if } n>1, \end{cases}$$

$$(L-\lambda)\ T_{-}^{1} = -F^{1}, \qquad (L-\lambda)\ U_{\kappa} = -\mu_{\kappa}\,A_{\kappa}\,\theta^{\kappa}.$$

4. *When* $p, q \equiv 1 \pmod 2$ *we have*

$$(L-\lambda)\,S_{\pm}^{\kappa} = \pm\mu_{\kappa}\theta^{\kappa}, \quad (L-\lambda)\,T_{+}^{\kappa} = 0, \quad (L-\lambda)\,T_{-}^{\kappa} = \mu_{\kappa}\varepsilon^{\kappa}.$$

Here μ_{κ} *is given by* (3.1) *and* $F^{\kappa} = \mu_{\kappa}\,\theta^{\kappa}+$ *lower order terms for* $n>1$ *and odd.* ∎

Since the operator $L-\lambda$ is elliptic on $(-\infty,0)\cup(0,1)\cup(1,+\infty)$ every distribution which satisfies (3.5) actually is a C^{∞}-function on each of these intervals and can be written as a linear combination of the corresponding Φ_{κ} and Ψ_{κ}. Hence, any solution D of (3.3) in $\mathcal{D}'(I^{\kappa})$ has the form

$$D = c_1\,S_{(\kappa,+)} + c_2\,S_{(\kappa,-)} + c_3\,T_{(\kappa,-)} + c_4\,T_{(\kappa,+)} + E^{\kappa},$$

where c_1, c_2, c_3, c_4 are complex constants, and $E^{(\kappa)}$ is a distribution with support in κ, i.e.

$$E^{(\kappa)} = a_0\,\delta_{\kappa} + a_1\,\delta_{\kappa}^{(1)} + \ldots + a_k\,\delta_{\kappa}^{(k)},$$

where δ_{κ} is the Dirac mass in κ, k is a non-negative integer, and $a_0, a_1, \ldots, a_k \in \mathbb{R}$. It follows from Lemma 2.2 that the solution of (3.3) in ${\mathcal{H}_{\kappa}}'$ have the form

$$D = c_1\,S_{+}^{(\kappa)} + c_2\,S_{-}^{(\kappa)} + c_3\,T_{+}^{(\kappa)} + c_4\,T_{-}^{(\kappa)} + \overline{E}^{(\kappa)}, \tag{3.6}$$

where, by Corollary 2.4,

$$\overline{E}^{\kappa} = a_0\,\theta^{\kappa} + a_1\,\theta^{\kappa}{}_1 + \ldots + a_k\,\theta^{\kappa}{}_k + b_0\,\varepsilon^{\kappa} + b_1\,\varepsilon^{\kappa}{}_1 + \ldots + b_l\,\varepsilon^{\kappa}{}_l,$$

with $b_0, b_1, \ldots, b_l \in \mathbb{R}$.

Lemma 3.7. *In $\mathcal{H}_\kappa{}'$ the following relations hold*

$$L\theta^\kappa{}_{(j)} = \frac{n-2\,j-3}{2}\,\theta^\kappa{}_{(j+1)} + + \sum_{i=0}^{j} a_j\,\theta^\kappa{}_{(j)}, \tag{3.7.a}$$

and

$$L\varepsilon^\kappa{}_{(j)} = (j+1)\,\left(j+\frac{n+1}{2}\right)\,\varepsilon^\kappa{}_{(j+1)} + \sum_{i=0}^{j} b_j\,\varepsilon^\kappa{}_{(j)}. \tag{3.7.b}$$

where $a_1,\ldots,a_j,b_1,\ldots,b_j$ are constants.

Proof. We only prove (3.7.a). We have

$$\langle L\theta^\kappa{}_{(j)}, f\rangle = (-1)^j\,\langle\theta^\kappa, (L^* f)^{(j)}\rangle =$$

$$(-1)^j\left\langle\theta^\kappa, \left(t\,(1-t)\,f'' + \left((n+m-3)\,t+\frac{3-n}{2}\right)\,f' + (n+m-1)\,f\right)^{(j)}\right\rangle$$

$$= (-1)^j\left\langle\theta^\kappa, t\,(1-t)\,f^{(j+2)} + \left(j\,(1-2\,t) + (n+m-3)\,t + \frac{3-n}{2}\right)\,f^{(j+1)}\right.$$

$$\left. + \left(-j\,(j-1) + j\,(n+m-3) + n+m-1\right)\,f^{(j)}\right\rangle$$

$$= (-1)^j\,\left\langle\theta^\kappa, \left(j+\frac{3-n}{2}\right)\,f^{(j+1)} + \left((j+1)\,(n+m) - j^2 - 2\,j - 1\right)\,f^{(j)}\right\rangle$$

$$= \left\langle -\left(j+\frac{3-n}{2}\right)\,\theta^\kappa{}_{(j+1)} + (j+1)\,(n+m-j-1)\,\theta^\kappa{}_{(j)}, f\right\rangle.$$

Hence,

$$L\theta^\kappa{}_{(j)} = \frac{n-2\,j-3}{2}\,\theta^\kappa{}_{(j+1)} + (j+1)\,(n+m-j-1)\,\theta^\kappa{}_{(j)}. \qquad \blacksquare$$

Let D be given by (3.6). According to the classification made in Lemma 3.6 we have the following cases:

1. When $p \equiv 0 \pmod 2$ and $q \equiv 0 \pmod 2$

$$(L-\lambda)\,D = \mu_\kappa\,(c_1-c_2)\,\theta^\kappa - c_3\,\mu_\kappa\,\varepsilon^\kappa + (L-\lambda)\,\overline{E}^\kappa. \tag{3.8.a}$$

2. When $p \equiv 0 \pmod 2$ and $q \equiv 1 \pmod 2$

$$(L-\lambda)\,D = \begin{cases} (c_3-c_4)\,\theta^\kappa + c_3\,\varepsilon^\kappa + (L-\lambda)\,\overline{E}^\kappa, & \text{if } n=1,\\ \mu_\kappa\,(c_1-c_2)\,\theta^\kappa + (c_3-c_4)\,F_\kappa - \mu_\kappa\,c_3\,\varepsilon^\kappa & \\ +(L-\lambda)\,\overline{E}^\kappa, & \text{if } n>1. \end{cases} \tag{3.8.b}$$

3. When $p \equiv 1 \pmod 2$ and $q \equiv 0 \pmod 2$

$$(L-\lambda)\,D = \begin{cases} (c_3-c_4)\,\theta^\kappa + (c_3-c_4-c_1+c_2)\,\varepsilon^\kappa + (L-\lambda)\,\overline{E}^\kappa, & \text{if } n=1,\\ \mu_\kappa\,(c_1-c_2)\,\theta^\kappa + (c_3-c_4)\,(F_\kappa-\varepsilon^\kappa) + (L-\lambda)\,\overline{E}^\kappa, & \text{if } n>1. \end{cases} \tag{3.8.c}$$

4. When $p \equiv 1 \pmod 2$ and $q \equiv 1 \pmod 2$

$$(L-\lambda)\,D = \mu_\kappa\,(c_1-c_2)\,\theta^\kappa + c_4\,\mu_\kappa\,\varepsilon^\kappa + (L-\lambda)\,\overline{E}^\kappa. \tag{3.8.d}$$

In any case we see that if D is a solution of (3.3) then $(L-\lambda)\,E$ has support in the set $\{\kappa\}$ and order less than or equal to the minimum between 0 and the order of F, that is at most 0 in the case when n is even, and at most $\frac{n-1}{2}$ when n is odd.

From Lemma 3.7 it follows that

$$(L-\lambda)\overline{E}^\kappa = \frac{n-2\,k-3}{2}\,a_k\,\theta^\kappa{}_{(k+1)} + (l+1)\left(l+\frac{n+1}{2}\right)\,b_l\,\varepsilon^\kappa{}_{(l+1)}$$

$+$ lower order terms. (3.9)
This relation means that $(L-\lambda)\,\overline{E}^\kappa$ has order equal to $\max(k+1,l+1)$ for any k not equal to $\frac{n-3}{2}$.

Lemma 3.8. *When $p,q \equiv 0 \pmod 2$ the space of solutions of (3.3) consists of linear combinations of $\{S_{(\kappa,+)}+S_{(\kappa,-)},T_{(\kappa,-)}\}$.*

Proof. Since n is even we see from (3.9) that $(L-\lambda)\,E$ has order greater than or equal to 1. Whence, from (3.8.a) D is solution of (3.3) if only if $\overline{E}^\kappa = 0$, $c_1 = c_2$ and $c_3 = 0$. ∎

Analogously we get:

Lemma 3.9. *When $p,q \equiv 1 \pmod 2$ the space of solutions of (3.3) consists of linear combinations of $\{S_{(\kappa,+)}+S_{(\kappa,-)},T_{(\kappa,+)}\}$.* ∎

The situation when n is odd is a bit more complicated:

Lemma 3.10. *The solutions of (3.3) when $p \equiv 0 \pmod 2$ and $q \equiv 1 \pmod 2$ are linear combinations of:*

1. $\{S_{(\kappa,+)},S_{(\kappa,-)}\}$ *if* $n=1$,
2. $\{A_\kappa\,S_{(\kappa,+)}+U_\kappa,S_{(\kappa,+)}+S_{(\kappa,-)}\}$ *if* $n>1$.

Proof. (1) We see from (3.8.b) that the order of $(L-\lambda)\,E$ is 0, hence (3.9) implies $E=0$. Therefore, from (3.8.b) $c_3=c_4=0$ and $c_1=c_2$.

(2) When n is odd and greater than 1 Lemma 3.6 (2) shows that $F=\pm(L-\lambda)\,T_\pm$ is of order $\frac{n-1}{2}$. On the other hand, we know from (3.9) that when $\overline{E}^\kappa$ has order k then $(L-\lambda)\,\overline{E}^\kappa$ has order $k+1$ unless k is equal to $\frac{n-3}{2}$. If this is the case then $(L-\lambda)\,\overline{E}^\kappa$ has order $\frac{n-3}{2}$. In any case $(L-\lambda)\,\overline{E}^\kappa$ cannot be of order $\frac{n-1}{2}$. Thus $(L-\lambda)\,\overline{E}^\kappa$ has order 0, and c_3 equals c_4. We have two possibilities still: $\overline{E}^\kappa=0$, or $\overline{E}^\kappa$ has order $\frac{n-3}{2}$.

The functional U_κ defined by (3.4.c) has order $\frac{n-3}{2}$. Let c be the non-zero constant such that $\overline{E}^\kappa - c\,U_\kappa$ is of order strictly less than $\frac{n-3}{2}$. We see from the paragraph above and Lemma 3.6(2) that $(L-\lambda)\big(\overline{E}^\kappa - cU_\kappa\big)$ has order zero. Therefore $\overline{E}^\kappa - c\,U_\kappa = 0$. Using this relation and $c_3=c_4$ in (3.8.b) we get $c_3=c_4=0$ and $c\,A_0 = c_1-c_2$. ∎

Similarly we obtain:

Lemma 3.11. *The solutions of* (3.3) *when* $p \equiv 1 \pmod 2$ *and* $q \equiv 0 \pmod 2$ *are linear combinations of* $\{S_{(\kappa,+)} + S_{(\kappa,-)}, T_{(\kappa,+)} + T_{(\kappa,-)}\}$.

The linear functionals acting on $\mathcal{H}^0$ and $\mathcal{H}^1$ defined by (3.4) give rise to radial distributions with support in S^0 and S^1 respectively. Correspondingly, Theorem 3.5 gives a list of radial eigendistributions of the Laplace-Beltrami operator supported in S^0 and S^1. In order to obtain the radial eigendistributions of Δ in $\mathcal{D}'(S)$ we extend the linear functionals $S_{(0,+)}$, $S_{(1,-)}$, $T_{(0,+)}$, and $T_{(1,-)}$ on $\mathcal{H}^0$ to linear functionals on the whole $\mathcal{H}$ by setting, for $f \in \mathcal{H}$,

$$\begin{aligned} \langle \overline{S}_{(0,+)}, f\rangle &= \mathrm{Pf}_{\varepsilon\downarrow 0} \int_{\varepsilon}^{1-\varepsilon} \Phi_0(t) f(t)\, dt, \\ \langle \overline{T}_{(0,+)}, f\rangle &= \mathrm{Pf}_{\varepsilon\downarrow 0} \int_{\varepsilon}^{1-\varepsilon} \Psi_0(t)\, f(t)\, dt, \end{aligned} \tag{3.10.a}$$

$$\begin{aligned} \langle \overline{S}_{(1,-)}, f\rangle &= \mathrm{Pf}_{\varepsilon\downarrow 0} \int_{\varepsilon}^{1-\varepsilon} \Phi_1(t) f(t)\, dt, \\ \langle \overline{T}_{(1,-)}, f\rangle &= \mathrm{Pf}_{\varepsilon\downarrow 0} \int_{\varepsilon}^{1-\varepsilon} \Psi_1(t) f(t)\, dt. \end{aligned} \tag{3.10.b}$$

On the open interval $(0,1)$ both the sets of functions Φ_0, Ψ_0 and Φ_1, Ψ_1 generate the space of the solutions of (3.3). Therefore, there are constants $\alpha, \beta, \gamma, \delta$ and $\overline{\alpha}, \overline{\beta}, \bar{\gamma}, \bar{\delta}$ such that

$$\begin{gathered} \Phi_1 = \alpha\, \Phi_0 + \beta\, \Psi_0, \quad \Psi_1 = \gamma\, \Phi_0 + \delta\, \Psi_0, \quad \text{and} \\ \Phi_0 = \tilde{\alpha}\, \Phi_1 + \tilde{\beta}\, \Psi_1, \quad \Psi_0 = \tilde{\gamma}\, \Phi_1 + \tilde{\delta}\, \Psi_1. \end{gathered}$$

Hence, we obtain the following lemma, whose proof is similar to that of Lemma 3.6.

Lemma 3.12. 1. *When* $p, q \equiv 0 \pmod 2$ *we have*

$$\begin{aligned} (L-\lambda)\, \overline{S}^0_+ &= \mu_0\, \theta^0 - \tilde{\alpha}\, \mu_1\, \theta^1 \\ (L-\lambda)\, \tilde{T}^0_+ &= -\mu_0\, \varepsilon^0 - \tilde{\gamma}\, \mu_1\, \theta^1. \end{aligned}$$

2. *When* $p \equiv 0 \pmod 2$ *and* $q \equiv 1 \pmod 2$ *we have*

$$\begin{aligned} (L-\lambda)\, \overline{S}^1_- &= \begin{cases} -\alpha\, \varepsilon^0 + \beta\, \left(\theta^0 - \varepsilon^0\right) - \varepsilon^1 & \text{if } n = 1, \\ \alpha\, \mu_0\, \theta^0 + \beta\, \left(F^0 - \varepsilon^0\right) - \mu_1\, \theta^1 & \text{if } n > 1, \end{cases} \\ (L-\lambda)\, T^1_- &= \begin{cases} -\gamma\, \varepsilon^0 + \delta\, \left(\theta^0 - \varepsilon^0\right) - \theta^1 + \varepsilon^1 & \text{if } n = 1, \\ \gamma\, \mu_0\, \theta^0 + \delta\, \left(F^0 - \varepsilon^0\right) - F^1 + \varepsilon^1 & \text{if } n > 1. \end{cases} \end{aligned}$$

3. *When* $p \equiv 1 \pmod 2$ *and* $q \equiv 0 \pmod 2$ *we have*

$$(L-\lambda)\,\overline{S}^1_- = \alpha\,\mu_0\,\theta^0 + \beta\,\left(F^0 - \mu_0\,\varepsilon^0\right) - \mu_1\,\theta^1,$$
$$(L-\lambda)\,T^1_- = \gamma\,\mu_0\,\theta^0 + \delta\,\left(F^0 - \mu_0\,\varepsilon^0\right) - \mu_1\,F^1.$$

4. *When* $p, q \equiv 1 \pmod 2$ *we have*

$$(L-\lambda)\,\overline{S}^1_- = -\mu_1\theta^1 + \alpha\mu_0\theta^0$$

$$(L-\lambda)\,\overline{T}^1_- = \mu_1\varepsilon^1 + \gamma\mu_0\theta^0. \qquad \blacksquare$$

We denote by $\mathcal{H}'_{(p,q,m;\lambda)}$ the space $\left\{F \in \mathcal{H}'_{(p,q,m)} : L\,F = \lambda\,F\right\}$. From Lemma 3.6 and Lemma 3.12 we obtain the following theorem:

Theorem 3.13. 1. *When* $p, q \equiv 0 \pmod 2$ *for any* $\lambda \in \mathbb{C}$ $\left\{S_{(0,-)} + \overline{S}_{(0,+)} + \tilde{\alpha} S_{(1,+)}, T_{(0,-)}\right\}$ *is a basis of* $\mathcal{H}'_{(p,q,m;\lambda)}$.

2. *When* $p \equiv 1 \pmod 2$ *and* $q \equiv 0 \pmod 2$ *for any* $\lambda \in \mathbb{C}$ $\left\{\alpha S_{(0,-)} + \beta T_{(0,-)} + \overline{S}_{(1,-)} + S_{(1,+)}, \gamma S_{(0,-)} + \delta T_{(0,-)} + \overline{T}_{(1,-)} + T_{(1,+)}\right\}$ *is a basis of* $\mathcal{H}'_{(p,q,m;\lambda)}$.

3. *When* $p \equiv 0 \pmod 2$ *and* $q \equiv 1 \pmod 2$ *for any* $\lambda \in \mathbb{C}$ *not equal to* $k\,(n+m-k)$, *the set* $\left\{A_1\,S_{(1,+)} + U_{(1)}\right\}$ *is a basis of* $\mathcal{H}'_{(p,q,m;\lambda)}$. *If* $\lambda = k\,(n+m-k)$ *with* $k = 1, 2, 3, \ldots$ *a basis of* $\mathcal{H}'_{(p,q,m;\lambda)}$ *is given by* $\alpha\,S_{(0,-)} + \beta\,T_{(0,-)} + \overline{S}_{(1,-)} + S_{(1,+)}$.

4. *When* $p, q \equiv 1 \pmod 2$ *for any* $\lambda \in \mathbb{C}$ $\left\{\alpha S_{(0,-)} + \overline{S}_{(1,-)} + S_{(1,+)}, T_{(1,+)}\right\}$ *is a basis of* $\mathcal{H}'_{(p,q,m;\lambda)}$.

Proof. The only point which needs some explanation is (3). From Lemma 3.6 (3) we see that $A_1\,S_{(1,+)} + U_{(1)}$ is an eigenvector of L for any λ. On the other hand from Lemma 3.11 we see that

$$(L-\lambda)\,\left(\alpha\,S_{(0,-)} + \beta\,T_{(0,-)} + \overline{S}_{(1,-)} + S_{(1,+)}\right) = -\beta\,\mu_0\,\varepsilon^0.$$

It follows that $\alpha\,S_{(0,-)} + \beta\,T_{(0,-)} + \overline{S}_{(1,-)} + S_{(1,+)}$ is eigenvector of L if and only if Φ_1 is regular in 0, and from the general theory of the hypergeometric equation (see [8] pag. 68) we know that this happens precisely when $\lambda = k\,(n+m-k)$ with $k = 1, 2, 3, \ldots \in \mathbb{N}$. $\blacksquare$

References

[1] Berndt, J., F. Tricerri, and L. Vanhecke, "Generalized Heisenberg Groups and Damek-Ricci Harmonic Spaces," Springer-Verlag Berlin etc., 1995.

[2] Besse, A., "Manifold all of whose geodesics are closed," Springer-Verlag Berlin etc., 1978.

[3] Cowling, M., A. Dooley, A. Korányi, F. Ricci, *H-type groups and Iwasawa decompositions*, Adv. Math. **87**, (1991), 1–41.

[4] —, *An approach to symmetric spaces of rank one via groups of Heisenberg type*, J. Geom. Anal., to appear.

[Cti] Ciatti, P., *Scalar products on Clifford modules and pseudo-H-type Lie algebras*, Annali di Matematica pura ed Applicata, to appear.

[6] —, *Solvable extensions of pseudo-H-type algebras* Bollettino UMI, to appear.

[7] Damek,E., F. Ricci, *Harmonic analysis on solvable extensions of H-type groups*, J. Geom. Anal. **2** (1992), 213–248.

[8] Erdélyi, A., W. Magnus, F. Oberhettinger, F. Tricomi, "Bateman Manuscript Project," MacGraw-Hill , 1953.

[9] Faraut, J., *Distributions sphérique sur les espaces hyperboliques*, J. Math. Pures Appl. **58** (1979), 369–444.

[10] Hörmander, L., "The Analysis of Linear Partial Differential Operators I," Springer-Verlag, 1983.

[11] Kaplan, A., *Fundamental solutions for a class of hypoelliptic PDE generated by composition of quadratic forms*, Trans. Amer. Math. Soc. **258** (1980), 147–153.

[12] Korányi, A., *Geometric properties of Heisenberg-type groups*, Adv. Math. **56** (1985), 28–38.

[13] Kosters, M. T., "Sperical distribution on rank one symmetric spaces," Dissertation, University of Leiden, 1983.

[14] Ruse, H. S., A. G. Walker, T. J. Willmore, "Harmonic Spaces," Edizioni Cremonese, 1961.

[15] Szabó, Z., *The Lichnerowicz conjecture on harmonic manifolds*, J. Diff. Geom. **31** (1990), 1–28.

[16] Tengstrand, A., *Distributions invariant under an orthogonal group of arbitrary signature* Math. Scand. **8** (1960), 201–218.

Dipartimento di Matematica,
Politecnico di Torino,
Corso Duca degli Abruzzi, 24,
10129, Torino
ciatti@polito.it

Received February 2, 1996
and in final form November 18, 1996