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A geometric study of Fibonacci groups

in memoriam Roger C. Lyndon († 1988)

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Abstract. Fibonacci groups are recovered as fundamental groups of certain closed hyperbolic 3-manifolds. This is achieved by constructing compact hyperbolic polyhedra in dimension 3 which are fundamental domains of torsion free lattices in $\mathrm{SL}_2(\mathbb{C})$. Their presentation can be read off by classical methods. This presentation can easily be given standard Fibonacci form.

Introduction

If a group G is only given by a finite presentation, it is usually very difficult or even impossible to decide the simplest questions concerning the structure of G. Many cases are known in the literature where some information has been extracted from a presentation. A well–known example are the Fibonacci groups:

$$F(2,m) = \langle x_1, x_2, \ldots, x_m;\ x_i x_{i+1} = x_{i+2}, i \bmod m\rangle. \tag{1}$$

These were introduced by J. Conway [4]. References for the combinatorial study of the $F(2,m)$ are given in [3].

The first problem which arises is whether $F(2,m)$ is trivial. This can easily be settled by computing the commutator quotient of $F(2,m)$, which turns out to be always finite and non–trivial for $m \neq 1,2$. The next question is considerably more difficult. It asks whether $F(2,m)$ is infinite. This has been the concern of most of the existing literature. The following has been shown:

(P1) $F(2,m)$ is finite only for $m = 1,2,3,4,5,7$.

It is comparatively easy to prove the finiteness of $F(2,m)$ in the above cases. The proofs which show that $F(2,m)$ is infinite in all other cases are of considerable ingenuity. The difficult case $m = 9$ was settled by M. Newman [9].

[1] The first part of the present work was done during a workshop in Korea, organised by the second named author. We gratefully acknowledge excellent hospitality, and financial support from Deutsche Forschungsgemeinschaft, and from Korean Science and Engineering Council.

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A very general method which can be used for problems of the above type is small cancellation theory. This path has been followed by R. C. LYNDON. In an unpublished manuscript, LYNDON has by an application of small cancellation theory proved that $F(2,m)$ is infinite for $m \geq 11$. This manuscript which originally contained some flaws was corrected by D. JOHNSON and a student of his. To apply small cancellation theory, the presentation (1) is not suitable. But it is easy to see that $F(2,m)$ is generated by x_1 and x_2, and can be defined by two relators. These two relators can be manipulated to exhibit a suitable small cancellation condition.

Our approach here is different. We start off from the presentation (1). We try to construct a nice 3–dimensional complex which has $F(2,m)$ as its fundamental group. An obvious suggestion is to take m triangles with edges labelled as in figure 1 and patch them according to the labelling of the edges. We obtain a connected finite 2–complex with fundamental group $F(2,m)$. It is, of course, not at all clear whether this object is the 2–skeleton of a nice 3–complex. In the case that $m \geq 6$ is an even integer we show in chapter 1 that a suitable modification of this idea works. We prove

Theorem A. *Let $m = 2n$ be an even positive integer. Then there is a 3–complex M_n which consist of one 3–cell, $2n$ triangles, $2n$ edges, and one vertex. The complex M_n is a closed, compact, orientable 3–manifold, and it satisfies*

$$\pi_1(M_n) = F(2,2n).$$

As a simple application, we could now establish (P1) for even parameters m. To do this, we just take the list of possible finite fundamental groups of closed, compact 3–manifolds, and prove by again computing commutator quotients that $F(2,2n)$ is none of them for $n \geq 3$. We also obtain all properties satisfied by fundamental groups of 3–manifolds now for $F(2,2n)$. We note

(P2) $F(2,2n)$ is a Noetherian group.

This means that every finitely generated subgroup is finitely presented. To proceed in our study of $F(2,2n)$ we now have to study the 3–manifolds M_n. The theory of W. THURSTON forces us to look for geometric structures on M_n or its universal cover. The first result in this direction is

Proposition B. *M_3 is an affine Riemannian manifold.*

The group–theoretic implication of Proposition B is:

(P3) $F(2,6)$ is a torsion–free finite extension of $\mathbb{Z}^3$.

This result was known. It is in fact easy to see that $F(2,6)$ is isomorphic to a 3–dimensional affine group.

Next we establish

Theorem C. *M_n is a hyperbolic manifold for $n \geq 4$. This means that $F(2,2n)$ is isomorphic to a discontinuous subgroup Γ_n of the group of orientation preparing isometries of hyperbolic 3-space $\mathbb{H}^3$ so that*

$$M_n \cong \Gamma_n \backslash \mathbb{H}^3. \tag{2}$$

We establish this result by describing a tessellation of $\mathbb{H}^3$ by compact polyhedra which are replica of the complex M_n. We note that for $n = 5$, our tessellation is the well–known tessellation of $\mathbb{H}^3$ by regular icosahedra. In all other cases, the tessellation is not regular, but it is isohedral: all the faces on the boundary are regular triangles, and they are all isometric. Hence the classical icosahedral tessellation described by H. S. M. Coxeter belongs to an infinite one–parameter family. As group–theoretic consequences we note:

(P4) $F(2,2n)$ is torsion–free for $n \geq 4$.

(P5) Every abelian subgroup of $F(2,2n)$ is cyclic for $n \geq 4$.

It seems interesting to remark that we also have solved the classical algorithmic problems of M. Dehn for $F(2,2n)$:

(P6) The groups $F(2,2n)$ have solvable word and conjugacy problems .

In fact, together with our tessellation of $\mathbb{H}^3$ comes an explicit embedding of $F(2,2n)$, $n \geq 4$, into $\mathrm{PSL}_2(\mathbb{C})$, which is the group of orientation–preserving isometries of $\mathbb{H}^3$. The algorithms for (P6) can then be found by the usual method from the action of $F(2,2n)$ on $\mathbb{H}^3$.

Since a first version of the present paper was written in preprint form the groups were studied in particular by Hilden, Lozano, Montesinas [12], [13]. They found another way to answer the question which Fibonacci groups are arithmetic. For this reason, we drop our original proof.

It is our pleasure to offer cordial thanks to a number of colleagues for stimulating discussions, to W. Haken and to C. C. Sims, and in particular to F. Grunewald whom we owe much inspiration and who helped us to turn the final manuscript into a form which is more suitable for publication. Our thanks also go to Stephan Helling for some numerical checks on the computer. Actually one of theses checks showed us how to transform a combinatorial tessellation arising from the universal covering of a 3–manifold into a semiregular tessellation of hyperbolic 3–space. Our thanks go to J. Montesinos, Madrid, and J. Howie, CBE, Edinburgh, who showed us how our work relates to some special problems in knot theory, and to Thurston's theory of hyperbolic structures on 3–manifolds.

We dedicate this work to the memory of Roger C. Lyndon as a mathematician of great ingenuity, as a colleague, and as a friend. Although his work on Fibonacci groups does not directly enter into the present work, we owe to his inspiration to see algebraic and even arithmetic problems from a geometric point of view.

The first part of the present work was done during a workshop in Korea, organised by the second named author. We gratefully acknowledge excellent hospitality, and financial support from Deutsche Forschungsgemeinschaft, and from Korean Science and Engineering Council.

1. Fibonacci groups as fundamental groups of certain 3–manifolds

In this section, we shall define a series of closed, compact, orientable 3–manifolds M_n such that

$$\pi_1(M_n) = F(2, 2n).$$

The following configuration was found by the authors in special cases, and then by C. C. SIMS in general.
The configuration is a tessellation of the 2–sphere S^2, consisting of $4n$ triangles, $6n$ edges, and $2n+2$ vertices.

The oriented edges can be labelled in the following manner. The oriented edges fall into $2n$ classes, each class consisting of 3 edges. Oriented edges in the same class carry the same label, say $x_1, x_2, \ldots, x_{2n}$.

Each triangle has a boundary $x_j x_{j+1} x_{j+2}^{-1}$ for some j mod $2n$. For each j mod $2n$, there are precisely two triangles with this boundary.

Notice that for $n = 5$, we obtain a combinatorial icosahedron. We want to construct a 3–manifold from this configuration.

Consider an oriented polyhedron which is bounded by $2m$ triangles. For each triangle, define a mate, such that the boundary consists of m pairs of triangles. For each pair, define an identification of the two triangles, such that the triangle inherits opposite orientations from either side. The resulting 3–dimensional complex K is an orientable pseudomanifold. It is homogeneous except possibly for the vertices, where a neighbourhood is a star bounded by a surface of genus $h \geq 0$.

There is a simple criterion, due to H. SEIFERT and W. THRELFALL, for K to be a manifold, i.e. for $h = 0$.

Proposition 1.1. (See [10], p.208, Satz I) *Let K be an orientable, closed, 3–dimensional pseudomanifold arising from a (simply connected) polyhedron by identifying pairs of faces on the boundary.*

K is a manifold if and only if its Euler characteristic vanishes:

$$\chi(K) = 0.$$

Consider the polyhedron which is the 2–ball with the boundary displayed in figure 1. The faces are labelled by $F_1, F_2, \ldots, F_{2n}$, each F_i occurring precisely twice. For each i mod $2n$, identify the two copies of F_i such that the corresponding oriented edges on the boundary carrying the same label are identified. The identification produces a complex M_n, say, with

$$\begin{array}{lll} \alpha^0 & = 1 & \text{vertex} \\ \alpha^1 & = 2n & \text{edges} \\ \alpha^2 & = 2n & \text{2–cells} \\ \alpha^3 & = 1 & \text{3–cell .} \end{array}$$

Proposition 1.1 applies and yields

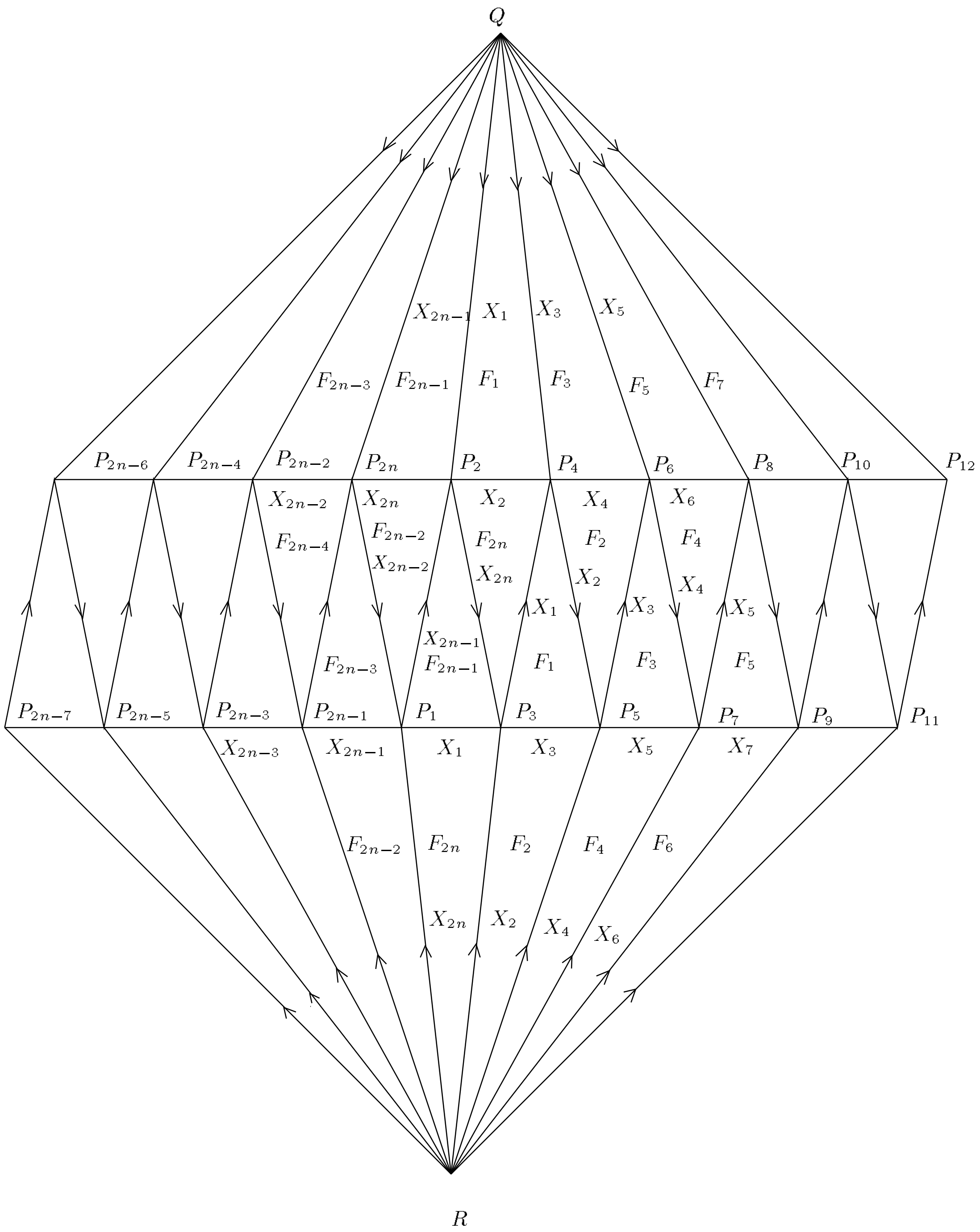
Q
X_{2n-1} X_1 X_3 X_5
F_{2n-3} F_{2n-1} F_1 F_3 F_5 F_7
P_{2n-6} P_{2n-4} P_{2n-2} P_{2n} P_2 P_4 P_6 P_8 P_{10} P_{12}
X_{2n-2} X_{2n} X_2 X_4 X_6
F_{2n-4} F_{2n-2} F_{2n} F_2 F_4
X_{2n-2} X_{2n} X_2 X_4
X_1 X_3 X_5
X_{2n-1}
F_{2n-3} F_{2n-1} F_1 F_3 F_5
P_{2n-7} P_{2n-5} P_{2n-3} P_{2n-1} P_1 P_3 P_5 P_7 P_9 P_{11}
X_{2n-3} X_{2n-1} X_1 X_3 X_5 X_7
F_{2n-2} F_{2n} F_2 F_4 F_6
X_{2n} X_2 X_4 X_6
R

Fig. 1

Theorem 1.2. *The complex M_n constructed above is a closed, compact, orientable 3–manifold. We have*

$$\pi_1(M_n) \cong F(2,2n).$$

The proof of the last statement is obvious, since the $2n$ triangles F_i with boundary $x_i x_{i+1} x_{i+2}^{-1}$ form the 2–skeleton of M_n, and there is only one vertex.

Fundamental groups of closed, compact, orientable 3–manifolds are a very restricted class of finitely generated groups. Hence Theorem 1.2 contains a lot of information about the group–theoretical structure of the abstract group $F(2,2n)$, such as described in the introduction.

For the sake of completeness, we reproduce the commutator quotient group, which now occurs as the first homology group of M_n.

Corollary 1.3.
$$H_1(M_n, \mathbb{Z}) \cong F(2,2n)^{ab}.$$

$$F(2,2n)^{ab} = \begin{cases} \mathbb{Z}_N \times \mathbb{Z}_{5N},\ N = f(n') \cdot g(n') & \textit{for } n = 2n' \\ \mathbb{Z}_N \times \mathbb{Z}_N,\ N = g(n) & \textit{for } n = 2n'+1. \end{cases} \tag{1}$$

Here $f(n), g(n)$ are the Fibonacci–Lucas numbers defined by the equation

$$h(n+2) = h(n) + h(n+1),$$

and the initial values

$$f(0) = 0, f(1) = 1, \ \textit{and } g(0) = 2, g(1) = 1.$$

The order of the commutator quotient group is in the literature, see [3].

After we had completed our work, it was discovered independently by MONTESINOS, LOZANO, HILDEN, and by J. HOWIE that the manifolds M_n are branched coverings of the 3–sphere S^3, branched cyclically over the figure 8 knot, which is the knot 4_1 in the table of ALEXANDER, BRIGGS, REIDEMEISTER.

Theorem 1.4. (MONTESINOS, LOZANO, HILDEN, and J.HOWIE)[2]. *The manifolds M_n are branched coverings of the 3–sphere S^3, branched cyclically over the knot 4_1.*

2. Some elementary hyperbolic geometry

In this section, we shall collect some formulae from 3–dimensional hyperbolic geometry. The proofs are standard, and we can leave them to the reader.

We shall mainly work in the upper halfspace model of 3–dimensional hyperbolic geometry:

$$\mathbb{H}^3 = \{P = (z,r), z \in \mathbb{C}, r \in \mathbb{R}^+\}. \tag{1}$$

[2] private communication, letters of J. MONTESINOS, and of J. HOWIE to the authors, both dated Feb. 28th, 1989

The distance between two points $P=(z,r)$ and $P'=(z',r')$ is given by

$$d(P,P') = \log(\delta + \sqrt{\delta^2 - 1}), \tag{2}$$

$$\delta = \delta(P,P') = \frac{|z-z'|^2 + r^2 + r'^2}{2rr'}. \tag{3}$$

We shall work with the distance function $\delta(P,P')$.

The planes are the northern hemispheres perpendicular to the plane $r=0$, and the lines are the half–circles in $\mathbb{H}^3$ which are perpendicular to the plane $r=0$. The angles are the Euclidean angles.

For our purposes, it is more convenient to describe the points, planes and lines differently.

Consider

$$P = \begin{pmatrix} a & b \\ c & -\overline{a} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}), \tag{4}$$

$$a \in \mathbb{C}; \quad b, c \in \mathbb{R}; \quad c > 0.$$

The matrix $P \in \mathrm{SL}_2(\mathbb{C})$ can also be used to describe the point $P=(z,r) \in \mathbb{H}^3$. The two descriptions are related by

$$\begin{pmatrix} a & b \\ c & -\overline{a} \end{pmatrix} \leftrightarrow (\frac{a}{c}, \frac{1}{c}). \tag{5}$$

A plane and a line can also be described by elements of $\mathrm{SL}_2(\mathbb{C})$.

$$E = \begin{pmatrix} a & i\,b \\ i\,c & \overline{a} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}), \tag{6}$$

$$a \in \mathbb{C}; \quad b, c \in \mathbb{R}.$$

E describes the hyperbolic plane over the circle $E\overline{z} = z$ in the plane $r=0$.

$$G = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}). \tag{7}$$

G describes the hyperbolic line over the fixed points of the matrix G in the plane $r=0$.

Remark . Remember that $\mathbb{H}^3 \cong \mathrm{SL}_2(\mathbb{C})/SU_2$. The group SU_2 is the maximal compact subgroup of $\mathrm{SL}_2(\mathbb{C})$. The description (4) of points is a canonical choice of coset representatives for the coset space $\mathrm{SL}_2(\mathbb{C})/SU_2$.

Proposition 2.1. *The group of isometries $G = \mathrm{PSL}_2(\mathbb{C})$ operates on points P, on lines G, and planes E as follows:*

$$X: \quad \begin{array}{l} P \to X\,P\,\overline{X}^{-1} \\ G \to X\,G\,X^{-1} \\ E \to X\,E\,\overline{X}^{-1} \end{array} \qquad X \in \mathrm{SL}_2(\mathbb{C}). \tag{8}$$

Proposition 2.2. (i) *A point P belongs to a line G iff*

$$GP = P\overline{G}; \tag{9}$$

(ii) *A line G belongs to a plane E iff*

$$GE = -E\overline{G}; \tag{10}$$

(iii) *A point P belongs to a plane E iff*

$$\mathrm{tr}(P\overline{E}) = 0. \tag{11}$$

Proposition 2.3. (i) *The distance function $\delta(P,Q)$ for two points P,Q is given by*

$$\delta(P,Q) = -\frac{1}{2}\,\mathrm{tr}(P\overline{Q}); \tag{12}$$

(ii) *The angle between two intersecting planes E,F is given by*

$$\cos(E,F) = \frac{1}{2}\,\mathrm{tr}(E\overline{F}). \tag{13}$$

Proposition 2.4. *Let P,Q,R be points which are not collinear. Put*

$$x_1 = \delta(P,Q), x_2 = \delta(Q,R), x_3 = \delta(R,P).$$

The (oriented) plane E through the points P,Q,R is given by

$$E = \frac{1}{2\sqrt{1+2x_1x_2x_3 - x_1^2 - x_2^2 - x_3^2}}\,(P\overline{Q}R - R\overline{Q}P). \tag{14}$$

Notice that the quantity

$$\det(P,Q,R) = \begin{vmatrix} 1 & x_1 & x_2 \\ x_1 & 1 & x_3 \\ x_2 & x_3 & 1 \end{vmatrix} = 1 + 2x_1x_2x_3 - x_1^2 - x_2^2 - x_3^2 \tag{15}$$

vanishes iff P,Q,R are collinear.

Proposition 2.5. *A plane E and a line G intersect in a point P if and only if*

$$\mathrm{tr}(F\,G\,\overline{E}\,\overline{G}) < 2 \tag{16}$$

The point P is given by

$$\pm P = \frac{1}{\sqrt{2 - \mathrm{tr}(E\,G\,\overline{E}\,\overline{G})}}\,(GE + E\overline{G}). \tag{17}$$

The line G is perpendicular to the plane E if and only if

$$GE = E\overline{G}. \tag{18}$$

This happens if and only if

$$\operatorname{tr}(E\,G\,\overline{E}\,\overline{G}) = -2, \tag{19}$$

and in this case, the formula (17) *reduces to*

$$P = GE. \tag{20}$$

If $\operatorname{tr}(E\,G\,\overline{E}\,\overline{G}) > 2$*, then*

$$F = \frac{1}{\sqrt{2 - \operatorname{tr}(E\,G\,\overline{E}\,\overline{G})}}(GE + E\overline{G}) \tag{21}$$

is the plane perpendicular to both E *and* G*.*

Proposition 2.6. *The planes* E, F *intersect in a line* G *if and only if*

$$|\operatorname{tr}(E\overline{F})| < 2. \tag{22}$$

The line G *is given by*

$$G = \frac{1}{\sqrt{4 - \operatorname{tr}(E\overline{F})^2}}(E\overline{F} - F\overline{E}). \tag{23}$$

If $|\operatorname{tr}(E\overline{F})| > 2$*, then* (23) *gives the line* G *which is perpendicular to both* E *and* F*.*

Proposition 2.7. *Let* E, F, H *be three planes, any two of which intersect. Put*

$$x_1 = \frac{1}{2}\operatorname{tr}(E\overline{F}), x_2 = \frac{1}{2}\operatorname{tr}(E\overline{H}), x_3 = \frac{1}{2}\operatorname{tr}(F\overline{H}).$$

Then

$$|x_i| < 1, \qquad i = 1, 2, 3. \tag{24}$$

Put

$$\det(E, F, H) = \begin{vmatrix} 1 & x_1 & x_2 \\ x_1 & 1 & x_3 \\ x_2 & x_3 & 1 \end{vmatrix} = 1 + 2x_1x_2x_3 - x_1^2 - x_2^2 - x_3^2. \tag{25}$$

The planes E, F, H *intersect in a point* P *if and only if*

$$\det(E, F, H) > 0. \tag{26}$$

The point P *is given by*

$$\pm P = \frac{1}{2\sqrt{\det(E, F, H)}}\,(E\overline{F}H - H\overline{F}E). \tag{27}$$

If

$$\det(E, F, H) < 0, \tag{28}$$

and if (24) *holds, then there is a uniquely determined plane* K *which is perpendicular to* E, F, H*. It is given by*

$$K = \frac{1}{2\sqrt{\det(E, F, H)}}(E\overline{F}H - H\overline{F}E). \tag{29}$$

Proposition 2.8. *Let x_1, x_2, x_3 be real numbers, and suppose*

$$x_i > 1, \qquad i = 1, 2, 3. \tag{30}$$

A triangle with vertices P_1, P_2, P_3, and edges of length

$$\delta(P_i, P_j) = x_k, \qquad i, j, k = 1, 2, 3,$$

all different, exists if and only if

$$\triangle = \begin{vmatrix} 1 & x_1 & x_2 \\ x_1 & 1 & x_3 \\ x_2 & x_3 & 1 \end{vmatrix} = 1 + 2x_1x_2x_3 - x_1^2 - x_2^2 - x_3^2 > 0. \tag{31}$$

An equivalent condition is this: The quadratic form belonging to the symmetric matrix

$$B = \begin{pmatrix} 1 & x_1 & x_2 \\ x_1 & 1 & x_3 \\ x_2 & x_3 & 1 \end{pmatrix} \tag{32}$$

satisfies the real equivalence

$$B \underset{\mathbb{R}}{\approx} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{33}$$

Proposition 2.9. *Let $x_{i,j}$, $i, j = 1, 2, 3, 4$, $i \neq j$, be real numbers satisfying*

$$x_{i,j} > 1, \qquad x_{i,j} = x_{j,i}. \tag{34}$$

A tetrahedron with vertices P_1, P_2, P_3, P_4, and edges of length $\delta(P_i, P_j) = x_{i,j}$ exists if and only if the symmetric matrix

$$B = \begin{pmatrix} 1 & x_{12} & x_{13} & x_{14} \\ x_{12} & 1 & x_{23} & x_{24} \\ x_{13} & x_{23} & 1 & x_{34} \\ x_{14} & x_{24} & x_{34} & 1 \end{pmatrix} \tag{35}$$

satisfies the equivalence over $\mathbb{R}$:

$$B \underset{\mathbb{R}}{\approx} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{36}$$

Proposition 2.10. *Let $x_{i,j}$, $i, j = 1, 2, 3, 4$, $i \neq j$, be real numbers satisfying*

$$|x_{i,j}| < 1; \qquad x_{i,j} = x_{j,i}. \tag{37}$$

A tetrahedron spanned by four planes E_i, $i = 1, 2, 3, 4$, such that any two planes E_i, E_j intersect in a line under the angle

$$\cos \sphericalangle(E_i, E_j) = x_{i,j} \tag{38}$$

exists if and only if the symmetric matrix

$$C = \begin{pmatrix} -1 & x_{12} & x_{13} & x_{14} \\ x_{12} & -1 & x_{23} & x_{24} \\ x_{13} & x_{23} & -1 & x_{34} \\ x_{14} & x_{24} & x_{34} & -1 \end{pmatrix} \tag{39}$$

satisfies the equivalence over $\mathbb{R}$*:*

$$C \underset{\mathbb{R}}{\approx} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{40}$$

Proposition 2.11. *Consider a tetrahedron spanned by the points* P_i*,* $i = 1,2,3,4$*. Put*

$$x_{i,j} = \delta(P_i, P_j). \tag{41}$$

The angle between the faces meeting in the edge (P_i, P_j) *is given by*

$$\cos(P_i, P_j) = \frac{\begin{vmatrix} 1 & x_{ij} & x_{ik} \\ x_{ij} & 1 & x_{il} \\ x_{ik} & x_{il} & x_{kl} \end{vmatrix}}{\sqrt{\triangle(i,j,k)}\sqrt{\triangle(i,j,l)}}, \tag{42}$$

where

$$\triangle(i,j,k) = 1 + 2x_{ik}x_{jk}x_{ij} - x_{ik}^2 - x_{jk}^2 - x_{ij}^2. \tag{43}$$

Remark 1. Notice that (42) is the 3–dimensional version of the cosine formula in elementary plane hyperbolic geometry.

Remark 2. Notice that we have formulated most of the elementary propositions such that the generalisation to n–dimensional hyperbolic geometry for $n > 3$ is obvious.

Proposition 2.12. *Let* $x_1 = \cos\alpha, x_2 = \cos\beta, x_3 = \cos\gamma$*, where* α, β, γ *are three angles satisfying*

$$0 \le \alpha, \beta, \gamma \le 180°. \tag{44}$$

Then

$$\alpha + \beta + \gamma = 360° \tag{45}$$

if and only if

$$\begin{vmatrix} 1 & x_1 & x_2 \\ x_1 & 1 & x_3 \\ x_2 & x_3 & 1 \end{vmatrix} = 0. \tag{46}$$

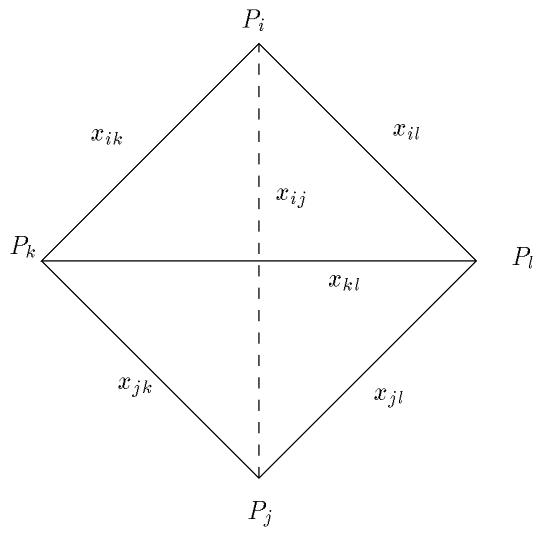

Fig. 2

3. A semiregular tesselation of hyperbolic 3–space

In this section, we shall construct a series of semiregular tessellations of hyperbolic 3–space, for $n \geq 4$. A fundamental domain is a semiregular polyhedron which is a metric realisation of the combinatorial polyhedron described in Section 1.

The polyhedron is bounded by $4n$ regular triangles which are all congruent. It consists of three parts: a pyramid on the top, a box in the middle, and a pyramid underneath.

The pyramid on the top consists of n tetrahedra A_j.
The tetrahedron A_j has vertices Q, Z_1, P_{2j}, P_{2j+2}. The face $Q\,P_{2j}\,P_{2j+2}$ is a regular triangle. The angle at the edge (Q, Z_1) is $\frac{2\pi}{n}$. The angles at the edges (P_{2j}, Z_1) and (P_{2j+2}, Z_1) are right angles.

We introduce some notation.

$$\begin{aligned} x &= \delta(Q, P_{2j}) = \delta(Q, P_{2j+2}) = \delta(P_{2j}, P_{2j+2}) \\ y &= \delta(Z_1, P_{2j}) = \delta(Z_1, P_{2j+2}) \\ z &= \delta(Q, Z_1) \\ \alpha_n &= \alpha = \cos\tfrac{2\pi}{n}. \end{aligned} \tag{1}$$

All tetrahedra A_j are congruent. Hence we shall simply refer to the tetrahedron A.

Proposition 3.1. *In the tetrahedron A, the quantities y, z can be expressed as follows:*

$$yz = x \tag{2}$$

$$y^2 = \frac{x - \alpha}{1 - \alpha}. \tag{3}$$

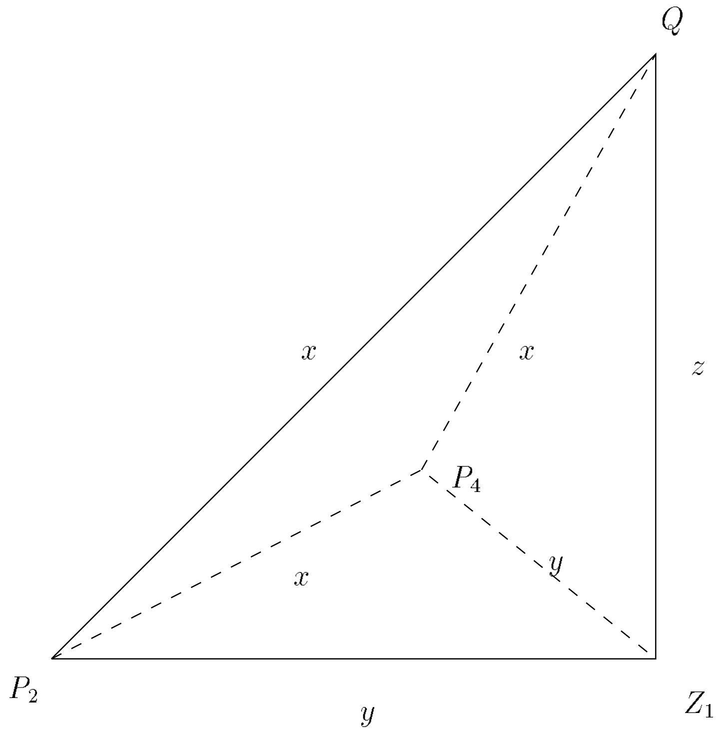

Fig. 3

The angle at the edge (Q,P_2) *or* (Q,P_4) *is given by*

$$\cos(Q,P_2)_A = \cos(Q,P_4)_A = \sqrt{\frac{x - x\alpha - \alpha}{2x+1}}. \tag{4}$$

The angle at the edge (P_2,P_4) *is given by*

$$\cos(P_2,P_4)_A = x\sqrt{\frac{1+\alpha}{(x-\alpha)(2x+1)}}. \tag{5}$$

In the pyramid, the angle at the edge (Q,P_2) *is twice the angle given by* (4)*:*

$$\cos(Q,P_2) = -\frac{2x\alpha + 2\alpha + 1}{2x+1}. \tag{6}$$

Proof. The proof is a straightforward computation, invoking Proposition 2.11. ∎

Join the two tetrahedra A_1 and A_2 along the face $Q\,P_4\,Z_1$. Consider the tetrahedron spanned by Q,P_2,P_6,P_4. Notice that in this tetrahedron, the angle at the edge (Q,P_4) is given by (6). Use Proposition 2.11 to compute the distance between P_2 and P_6, obtaining:

$$\delta(P_2,P_6) = 2x + 2x\alpha - 2\alpha - 1. \tag{7}$$

We consider the box.

We introduce the center Z of the box, as the midpoint between Q and R.

We project from Z to each of the triangles $P_j\, P_{j+1}\, P_{j+2}$ shown in the middle strip of figure 1. We obtain n tetrahedra which we call B_j.

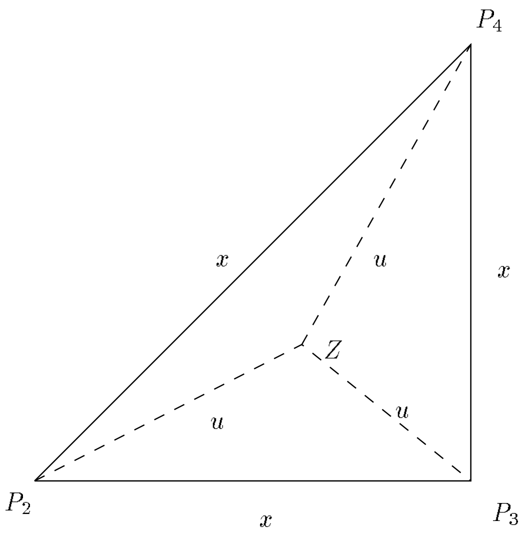

Fig. 4

The tetrahedron B_j has the vertices Z, P_j, P_{j+1}, P_{j+2}. The distances between Z and P_j are all equal.

We introduce the notation

$$u = \delta(Z, P_j) = \delta(Z, P_{j+1}) = \delta(Z, P_{j+2}). \tag{8}$$

The tetrahedra B_j are all congruent, and will be referred to as tetrahedra B.

Proposition 3.2. *In the tetrahedron B, the quantity u can be expressed as follows:*

$$u^2 = \frac{2x - x\alpha - \alpha}{2(1-\alpha)} + \frac{x-1}{2(1-\alpha)}\sqrt{\frac{1+\alpha}{2}}. \tag{9}$$

The angle at the edge (P_4, Z) is given by

$$\cos(P_4, Z)_B = -\frac{1}{2} + \sqrt{\frac{1+\alpha}{2}}. \tag{10}$$

The angle at the edge (P_2, P_4) is given by

$$\cos(P_2, P_4)_B = \frac{u\sqrt{2}}{\sqrt{2x+1}}\sqrt{1 - \sqrt{\frac{1+\alpha}{2}}}. \tag{11}$$

The angle at the edge (P_2, P_3) *of the polyhedron is twice the angle given by* (11)*:*

$$\cos(P_2, P_3) = \frac{x - (x+1)\sqrt{2(1+\alpha)}}{2x+1}. \tag{12}$$

Proof. Use Proposition 2.11 to produce the formulae

$$\cos(P_2, P_4)_B = \frac{u\sqrt{x-1}}{\sqrt{2x+1}\sqrt{2u^2-x-1}} \tag{13}$$

$$\cos(P_4, Z)_B = \frac{u^2 - x}{2u^2 - x - 1}. \tag{14}$$

Join the tetrahedra B_2, B_3, B_4 along their common faces, obtaining a polyhedron with vertices $P_2, P_3, P_4, P_5, P_6, Z$. At the edge (P_4, Z), the angle in this polyhedron is three times the angle given by (14). In the polyhedron, consider the tetrahedron spanned by P_2, P_4, P_6, Z. Apply Proposition 2.11 to this tetrahedron, obtaining

$$\delta(P_2, P_6) = \frac{16u^4x - 12u^4 - 24u^2x^2 + 12u^2 + 4u^2 + 9x^3 - 2x^2}{(2u^2 - x - 1)^2} \tag{15}$$

Comparing the distances (7), (15), we obtain a polynomial equation between u and x:

$$8u^2(1-\alpha) + 8u^2(-2x + x\alpha + \alpha) + 7x^2 - 2x^2\alpha + 2x - 4x\alpha - 1 - 2\alpha = 0. \tag{16}$$

This equation has the solution

$$u^2 = \frac{2x - x\alpha - \alpha}{2(1-\alpha)} + \epsilon\frac{x-1}{2(1-\alpha)}\sqrt{\frac{1+\alpha}{2}}, \ \epsilon = \pm 1. \tag{17}$$

The sign ϵ in (17) is determined as follows. Insert (17) in (13) and compute $\sin(P_2, P_4)_B$, obtaining

$$\sin(P_2, P_4)_B = \sqrt{\frac{x+1}{2(2x+1)}}\sqrt{1 + \epsilon\sqrt{2+2\alpha}}. \tag{18}$$

The last term on the right hand side of (18), which is independent of x, is real only for $\epsilon = +1$.

Insert (17), with $\epsilon = +1$, into (13), (14), obtaining (10), (11), and (12). The proof of Proposition 3.2 is complete. ■

Join the n tetraheda B_j along their common faces. We shall show that the configuration closes, giving a mould with center. We have to fill this mould from above, and from below, in order to obtain the box.

In order to fill the mould, we consider the tetrahedron C_j spanned by the vertices P_{2j}, P_{2j+2}, Z, Z_1. The lengths of the edges are as follows:

$$\begin{aligned} \delta(P_{2j}, Z_1) &= \delta(P_{2j+2}, Z_1) = y \\ \delta(P_{2j}, Z) &= \delta(P_{2j+2}, Z) = u \\ \delta(P_{2j}, P_{2j+2}) &= x; \end{aligned} \tag{19}$$

The angle at the edge (Z, Z_1) is $\frac{2\pi}{n}$.

All tetrahedra C_j with the data (19) are congruent, and will be referred to as tetrahedra C.

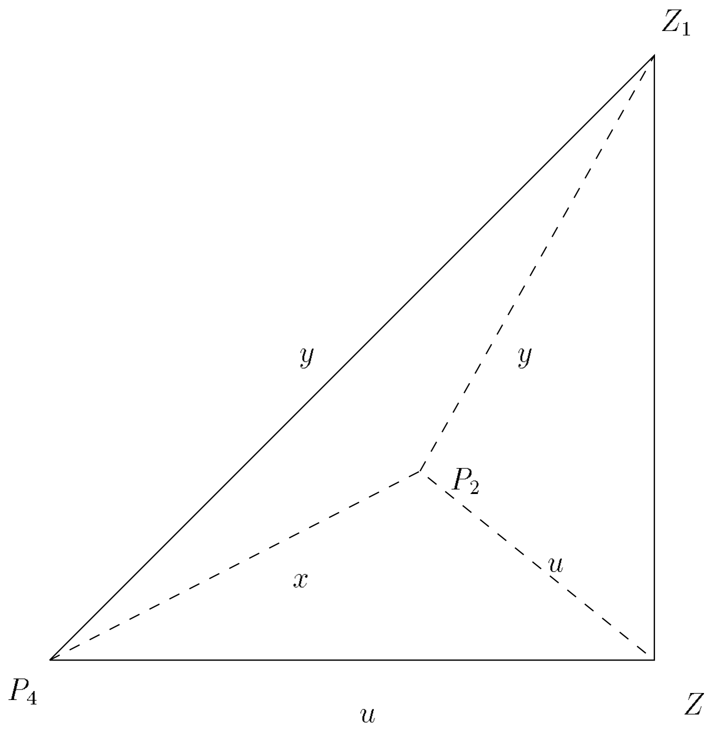

Fig. 5

Proposition 3.3. *In the tetrahedron C, the distance $\delta(Z, Z_1)$ is given by*

$$\delta(Z, Z_1) = \frac{u}{y}. \tag{20}$$

The angles at the edges (P_2, Z_1) and (P_4, Z_1) are right angles:

$$\cos(P_2, Z_1)_C = \cos(P_4, Z_1)_C = 0. \tag{21}$$

The angles at the edges (P_2, Z) and (P_4, Z) are equal, and are given by

$$\cos(P_2, Z)_C = \sqrt{-\frac{1+3\alpha}{2} + (1+\alpha)\sqrt{\frac{1+\alpha}{2}}}. \tag{22}$$

The angle at the edge (P_2, P_4) is given by

$$\cos(P_2, P_4)_C = \frac{u}{y}\sqrt{\frac{1+\alpha}{1-\alpha}(2 - \sqrt{2+2\alpha})}. \tag{23}$$

Proof. The proof is a straightforward application of Proposition 2.11, making use of Propositions 3.1 and 3.2. ∎

So far, we have collected local conditions which are necessary for the existence of the semiregular polyhedron. We now turn to the global situation.

Proposition 3.4. *Assume that the tetrahedra A, B, C exist, for a given value of the parameter x.*

Join the n tetrahedra B_j along common faces, obtaining a configuration which closes, a mould.

Join the n tetrahedra C_j along common faces, along their common edge (Z, Z_1), obtaining a polyhedron which will be referred to as the inverted pyramid.
The inverted pyramid has a top face which is a regular n–gon.
The inverted pyramid fits into the mould from above.
The same way, insert a copy of the inverted pyramid into the mould from below, obtaining a polyhedron referred to as the box.
Join the n tetrahedra A_j along common faces, along their common edge (Q, Z_1), obtaining a pyramid. Put the pyramid on top of the box.
Join a copy of the pyramid to the other side, underneath the box.
Obtain a semiregular polyhedron which is bounded by $4n$ regular triangles. All edges have the same length.
There are precisely three different angles between adjacent faces. Using the labels of edges described in figure 1, these angles occur at each edge x_j. The angles are given by (6), (12), *and*

$$\begin{aligned}(24)\quad \cos(P_2, P_4) = {} & \frac{1}{\sqrt{2}(x-\alpha)(2x+1)}\Big\{\sqrt{2}x^2(1+\alpha) - x(x+1)\sqrt{1+\alpha} \\ & -\sqrt{x^4(1-3\alpha+2\alpha^2) + x^3(1-8\alpha+8\alpha^2) + x^2(-1-6\alpha+12\alpha^2) + x(-1+8\alpha^2) \\ + \alpha + 2\alpha^2 + 2(x^2 - x^2\alpha + x - 2x\alpha - \alpha)(x^2+x)\sqrt{2+2\alpha}}\Big\}.\end{aligned}$$

Proof. Join the n tetrahedra C_j along common faces, along their comon edge (Z_1, Z). The angle condition in (19) guarantees that we obtain an inverted pyramid. The equations (21) guarantee that the inverted pyramid has one top face, which is a regular n–gon.
Join the n tetrahedra B_j along common faces. The configuration may not close at one face, say at P_2, P_1, Z.
We show that the inverted pyramid fits locally into the possibly not closed mould. We consider the edge (P_4, Z). In this edge, five faces meet. The angles are three times the angle (10), coming from the tetrahedra B of the mould, and two times the angles (22), coming from the inverted pyramid. We must show that these angles add up to $360°$.
Abbreviate the right hand sides of (10) and (22) by r, t, respectively.
The angles add up to $360°$ if and only if

$$4r^3 - 3r = 2t^2 - 1, \tag{25}$$

and if the angles lie in the appropriate range. An easy computation shows that (25) holds. Use (10) to conclude that three times the angle (10) is more than $180°$. Use (22) to conclude that twice the angle (22) is less than $180°$. Hence the angles add up to $360°$, and hence the inverted pyramid fits locally into the mould.
Inserting the inverted pyramid into the mould, we conclude that the mould closes in the vertex P_2.

Use the same argument inserting a copy of the inverted pyramid into the mould from below, concluding that the mould also closes in the vertex P_1. Hence it closes at the face P_2, P_1, Z.
The remaining statements of Proposition 3.4 are now obvious. The slightly complicated formula (24) arises by adding up the angles (23), (11), and (5). The proof of Proposition 3.4 is complete. ∎

It remains to show the existence of the tetrahedra A, B, C for some fixed value of the parameter x. We shall come back to this problem in the proof of the next theorem.
We want to construct a semiregular tessellation of hyperbolic 3–space such that the polyhedron discussed in Proposition 3.4 is a fundamental domain. A necessary condition for the existence of the tessellation is that the three angles in Proposition 3.4 add up to $360°$. It was show by B. MASKIT [7] that this angle condition is also sufficient.
Here is our result.

Theorem 3.5. *For $n \geq 4$, there exists a semiregular tessellation of $\mathbb{H}^3$, called the Fibonacci tessellation $F(2,2n)$ with the following properties.*
A fundamental domain for the tessellation is the semiregular polyhedron discussed in Proposition 3.4. The polyhedron is bounded by $4n$ regular triangles. All edges have the same length, given by

$$x = \frac{1}{4(1-\alpha)}\{4 + 2\alpha - 4\alpha^2 + (3-2\alpha)\sqrt{2+2\alpha}\}, \tag{26}$$

where

$$\alpha = \cos(\frac{2\pi}{n}). \tag{27}$$

The three angles are obtained by inserting (26) in (24), (12), (6). The Fibonacci group $F(2,2n)$ operates transitively on copies of the polyhedron as described in Section 1.

Proof. We want to choose the parameter x such that the three angles in Proposition 3.4 add up to $360°$.
A direct application of Proposition 2.12 leads to an awkward computation. So we use a slightly modified argument.
Collecting the angles of the tetrahedra A, B, C which are involved, we can state the angle condition as follows:

$$2.\sphericalangle(Q, P_4)_A + \sphericalangle(P_2, P_4)_A + 3.\sphericalangle(P_2, P_4)_B + \sphericalangle(P_2, P_4)_C = 360°. \tag{28}$$

It is advantageous to add up the contributions from the tetrahedra A, B, C separately. For the reader's convenience, we write out these contributions:

$$
\begin{aligned}
x_1 &= \cos(2.(Q,P_4)_A + (P_2,P_4)_A) \\
&= \frac{-2x^2+(-3+2\alpha)x+2\alpha}{(2x+1)\sqrt{2x+1}\sqrt{x-\alpha}}\sqrt{1+\alpha}, \\
x_2 &= \cos(3.(P_2,P_4)_B) \\
&= -\frac{u}{(2x+1)\sqrt{2x+1}}(1+2(x+1)\sqrt{2+2\alpha})\sqrt{2-\sqrt{2+2\alpha}}, \\
x_3 &= \cos(P_2,P_4)_C = u\sqrt{\frac{1+\alpha}{x-\alpha}(2-\sqrt{2+2\alpha})}.
\end{aligned} \tag{29}
$$

At this point, we do not want to use the range assumptions in Proposition 2.12. Without these assumptions, Proposition 2.6 still gives a necessary condition for (28).
Insert (29) into (2.18), obtaining a polynomial of degree 4 in x:

$$
\begin{aligned}
x^4(-28-44\alpha-8\alpha^2)+x^3(-68-88\alpha+8\alpha^2+8\alpha^3) &= x^2(-\tfrac{105}{2}-39\alpha+50\alpha^2+20\alpha^3 \\
+x(-13+10\alpha+44\alpha^2+16\alpha^3)&-\tfrac{1}{2}+5\alpha+10\alpha^2+4\alpha^3 \\
+\{x^4(20+20\alpha)+x^3(48+38\alpha-16\alpha^2)&+x^2(\tfrac{73}{2}+12\alpha-38\alpha^2) \\
+x(9-10\alpha-28\alpha^2)+\tfrac{1}{2}&-4\alpha-6\alpha^2\}\sqrt{2+2\alpha}=0.
\end{aligned}
$$

This polynomial has twice the factor $x+1$. Dividing out these factors, we obtain:

$$
\begin{aligned}
&x^2(-28-44\alpha-8\alpha^2)+x(-12+24\alpha^2+8\alpha^3)-\tfrac{1}{2}+5\alpha+10\alpha^2+4\alpha^3 \\
&\quad+\{x^2(20+20\alpha)+x(8-2\alpha-16\alpha^2)+\tfrac{1}{2}-4\alpha-6\alpha^2\}\sqrt{2+2\alpha}=0.
\end{aligned} \tag{30}
$$

For $n=5$, the polynomial (30) vanishes identically.
For $n \neq 5$, the equation (30) can be solved in the field $\mathbb{Q}(\sqrt{2+2\alpha}) = \mathbb{Q}(\cos\frac{\pi}{n})$. The solutions are:

$$x = \frac{1}{4(1-\alpha)}\{2\alpha+\sqrt{2+2\alpha}\}, \text{ and} \tag{31}$$

$$x = \frac{1}{4(1-\alpha)}\{4+2\alpha-4\alpha^2+(3-2\alpha)\sqrt{2+2\alpha}\}. \tag{32}$$

For $n=4$, the solution (31) is $x<1$, and hence not a geometric solution.

For $n \geq 6$, one can easily verify that the solution (31) does not satisfy the angle condition (28), and that (32) does satisfy (28), also for $n=4$.

The case $n=5$ must be treated directly, using Proposition 2.12. It turns out that also in this case, (32) is the only solution with $x>1$ which satisfies the angle condition (28).

We shall now prove the existence of the tetrahedra A, B, C for the value x in (32).

For the tetrahedron A, use (2), (3), and Proposition 2.9 to conclude that the tetrahedron exists if

$$x > \frac{\alpha}{1-\alpha}.$$

Conclude from (32) that this condition holds true for $n \geq 4$. Similarly, use (2), (3), (9), (20), and Proposition 2.9 to prove the existence of the tetrahedra B and C.

We have now shown the existence of one tile of the tessellation. Consider the ordered triples of points (P_3, P_4, P_5) and (Q, P_2, P_4). Both triples are the vertices of a regular triangle on the boundary of the polyhedron.

There is precisely one orientation–preserving isometry x_1 which maps one ordered triple onto the other:

$$x_1 : P_3, P_4, P_5 \to Q, P_2, P_4.$$

The same way, we define $2n$ isometries x_j:

$$\begin{gathered} x_{2j-1} : P_{2j+1}, P_{2j+2}, P_{2j+3} \to Q, P_{2j}, P_{2j+2} \\ x_{2j} : P_{2j+2}, P_{2j+3}, P_{2j+4} \to R, P_{2j+1}, P_{2j+3}, \\ j \bmod n. \end{gathered} \tag{33}$$

Each of these isometries maps the fundamental polyhedron onto a neighbouring polyhedron, and for each face, there are two neighbours which meet in the face.

By MASKIT's Theorem, we obtain a tessellation of hyperbolic 3–space. Let Γ be the group of isometries generated by $x_1, \ldots, x_{2n}$:

$$\Gamma = \langle x_1, x_2, \ldots, x_{2n} \rangle. \tag{34}$$

We can easily verify from (33) that

$$x_1 x_2 x_3^{-1}(P_4) = P_4,$$

$$x_1 x_2 x_3^{-1}(Q) = Q.$$

Consider the centers of the polyhedra which meet in the edge (P_4, Q). One can easily verify that at least one of these centers is fixed under $x_1 x_2 x_3^{-1}$. Hence this isometry is the identity:

$$x_1 x_2 x_3^{-1} = 1.$$

The same argument shows that

$$x_j x_{j+1} x_{j+2}^{-1} = 1,\ j \bmod n. \tag{35}$$

Hence the relations of $F(2, 2n)$ hold in Γ.

The tessellation is a covering of the manifold M_n. Because the hyperbolic space is simply connected, the covering is the uniquely determined universal covering.

Let F be the monodromy group belonging to the universal covering. For any two representatives of M_n in the universal covering which meet in a common face, there is precisely one element of F which maps one representative onto the other.

The same holds for the group Γ. Hence we can view Γ as a metric realisation of the monodromy group F. The monodromy group F in turn is isomorphic to the fundamental group $\pi_1(M_n)$. Hence we have established the isomorphism

$$\Gamma \cong \pi_1(M_n), \tag{36}$$

and the tessellation is a metric realisation of the universal covering of M_n.

By Theorem 1.4 we have the isomorphism

$$\Gamma \cong F(2,2n). \tag{37}$$

We have now completed the proof of Theorem 3.5. ∎

Corollary 3.6. *The Fibonacci group $F(2,2n)$ acts effectively as a discontinuous group of transformations on hyperbolic 3–space. A fundamental domain is the semiregular polyhedron described in Theorem* 3.5.

Proof. The Corollary is just a reformulation of Theorem 3.5. ∎

Corollary 3.7. *There exists a one–parameter family of semiregular polyhedra as described in Proposition* 3.4.

Proof. The functions in Proposition 2.9 are continuous functions. The Corollary follows by continuity, when x ranges in a certain interval. ∎

Corollary 3.8. *For large n, the polyhedron in Theorem* 3.5 *looks like a flat disk. In fact, as n goes to infinity, we have*

$$\begin{gathered} x \to \infty \\ u = \delta(Z, P_j) \to \infty \\ \delta(Q, R) \to 1 \\ \sphericalangle(Q, P_2) \to 180^\circ \\ \sphericalangle(P_2, P_3) \to 120^\circ \\ \sphericalangle(P_2, P_4) \to 60^\circ \end{gathered} \tag{38}$$

Proof. The first assertion follows from (26), for $\alpha \to 1$. Inserting (26) in (9), we find

$$u = \frac{1}{4(1-\alpha)}\{2 + (3 - 2\alpha)\sqrt{2 + 2\alpha}\}. \tag{39}$$

This yields the second assertion.

Obtain from (2), (3), (9), (20), (26):

$$\delta(Q, Z) = 2 + \alpha - 2\alpha^2. \tag{40}$$

This yields the third assertion (38).

The formulae for the angles now follow from (6), (12), (24). For the readers convenience, we give the numerical data for the first few polyhedra:

n	x	u	$\delta(Q,Z)$	$\sphericalangle(Q,P_2)$	$\sphericalangle(P_2,P_3)$	$\sphericalangle(P_2,P_4)$
4	2.06	1.56	2.0	101.26	116.28	142.46
5	2.93	2.12	2.12	120.00	120.00	120.00
6	3.73	2.73	2.0	132.63	121.83	105.54
7	4.55	3.43	1.85	141.65	122.66	95.68
8	5.42	4.21	1.71	148.36	122.97	88.66
9	6.35	5.09	1.59	153.50	123.01	83.49
10	7.37	6.06	1.5	157.51	122.91	79.58
11	8.47	7.13	1.43	160.71	122.74	76.55
12	9.67	8.30	1.37	163.29	122.55	74.16

This completes the proof. ∎

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