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Dirac Operators, Conformal Transformations and Aspects of Classical Harmonic Analysis

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Abstract. The main thrust of this paper is to investigate the intimate link between the conformal group and singular integral operators, in particular, but not exclusively, operators of Calderón–Zygmund type, together with associated commutators acting on the L^2 spaces of surfaces. Clifford analysis and Dirac operators are the basic tools used to help to unify these themes. These surfaces lie in euclidean space, the sphere or the hyperbola. We illustrate how these results extend to a general class of submanifolds with arbitrary codimension in euclidean space, the sphere or the hyperbola.

1. Introduction

Clifford analysis has been extensively applied to clarify and solve various problems arising in classical harmonic analysis, see for instance [AuT, GaLQ, LMcQ, LMcS, Mc, Mi, S]. In these papers Dirac operators and Cauchy kernels are used to study properties of singular integral operators acting on the L^p spaces associated to various types of surfaces in R^n. Usually these surfaces are Lipschitz surfaces, or some minimal smoothness criterion is assumed of the surface. In most cases the Clifford algebra is used to show that results known for singular integral operators acting on L^p spaces of minimally smooth curves in the complex plane can be adapted to higher dimensions, see for instance [10, 9]. In this setting Clifford analysis has proved to be a powerful tool in extending complex variable techniques to solving problems in higher dimensions.

Links between Möbius transformations and Dirac operators have been developed in a number of papers, see for instance [5, 8, 19, 21, 22]. All of these papers make use of the Vahlen matrices arising from work in [1, 28] and elsewhere.

In particular, in [8, 21] it is shown that if two surfaces in R^n are conformally equivalent then their L^2 spaces are isometric, and there is a simple association between the operators acting on these L^2 spaces. In [21] it is pointed out that the associated Hardy spaces H^2 of solutions to the Dirac equation for the domains complementing the surfaces are also isometric. This follows from the conformal invariance of the Clifford analysis analogues of the Plemelj operators.

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The main thrust here is to show that by introducing Clifford algebras and, in particular, Vahlen matrices one can easily show that many important results known in classical harmonic analysis on euclidean space possess a conformal covariance. In other words once one extends from euclidean space and includes the Clifford algebra generated from that space many traditional results associated with classical harmonic analysis described in [25] and elsewhere have a rather natural association with the conformal group, and seem to be intimately related to that structure. For instance, in [21] it is shown how certain key results on convolution operators acting on the L^2 spaces of Lipschitz graphs, described in [15, 16], can be carried over to some other surfaces via conformal transformations. In this paper we develop this theme further. We demonstrate that convolution operators of Calderón–Zygmund type acting on the L^2 space of R^{n-1} transform under Möbius transformations acting on R^{n-1} to other operators of Calderón–Zygmund type. We are also able to show that the commutators of such operators with multiplier operators are preserved in the sense that the resulting operators are also the commutators of operators of Calderón–Zygmund type with multiplier operators. This analysis leads us to show that a considerable part of the key results in [15, 16] naturally lend themselves to invariance under the action of the Lie group $V(n-1)$, the group of Vahlen matrices over R^{n-1}. In particular, when viewed as operators the left and right monogenic functions defined on sector domains in [15, 16] possess an extremely simple and elegant conformal invariance which we describe here on the sphere and hyperbola. This makes use of various results and constructions made in [22] for Dirac operators on spheres and hyperbolae.

We also consider the intimate link between the conformal group and the Riesz potentials described in [25] and elsewhere.

We conclude the paper by illustrating that many of the results appearing in the earlier part of the paper also hold for a wide class of submanifolds of euclidean space, the sphere or hyperbola. With this objective in mind we establish Plemelj formulae for such manifolds. Plemelj formulae for general manifolds are described in [7]. Here we explicitly use vector space structures to set up the formulae.

Most of the proofs in this paper follow by similar arguments to those used in [19, 21, 22]. For this reason we primarily indicate how the results can be deduced rather than repeat details from those references.

2. Background on Clifford Algebras

In this section we set up most of the background material on Clifford algebras that we will need for the rest of the paper. We shall use the real Clifford algebra, Cl_n and its complexification $Cl_n(C)$. We will assume that R^n is embedded in Cl_n, and that Cl_n is the 2^n-dimensional algebra with basis

$$1, e_1, \ldots, e_n, \ldots, e_{j_1} \ldots, e_{j_r}, \ldots, e_1 \ldots, e_n,$$

where $e_1, \ldots, e_n$ is an orthonormal basis for R^n, $1 \leq r \leq n$, and the basis vectors $e_1, \ldots, e_n$ satisfy the anti-commutation relationship $e_i e_j + e_j e_i = -2\delta_{i,j}$, where $\delta_{i,j}$ is the Kronecker delta function. From the anti-commutation relationship it follows that each non-zero vector $x \in R^n$ has a multiplicative inverse $x^{-1} = \frac{-x}{\|x\|^2}$.

Up to the minus sign this inverse corresponds to the Kelvin inverse of a vector. It follows that the unit sphere, S^{n-1}, of R^n, is the generator of a group in Cl_n. This group is called the pin group and it is denoted by $\mathrm{Pin}(n)$. We will need to use the anti-automorphism

$$\tilde{} : Cl_n \to Cl_n : (e_{j_1} \ldots, e_{j_r})^{\tilde{}} = e_{j_r} \ldots, e_{j_1}.$$

For each $a \in Cl_n$ we denote $a^{\tilde{}}$ by $\tilde{a}$. When $a \in \mathrm{Pin}(n)$ the action $ax\tilde{a}$ corresponds to an orthogonal transformation on R^n, where x is a variable in R^n. This can be seen by considering the action yxy where $y \in R^n$, and decomposing x into vectors parallel to y and perpendicular to y. By using the anti-commutation relationship it may be observed that this action gives a reflection of x in the direction of y. It may be observed that the set $\{a \in \mathrm{Pin}(n) : a = y_1 \ldots, y_m \text{ with } m \text{ even}\}$ is a subgroup of $\mathrm{Pin}(n)$. This is the spin group, which is denoted by $\mathrm{Spin}(n)$. In [20] and elsewhere it is observed that $\mathrm{Pin}(n)$ is a double covering of the orthogonal group $O(n)$ while $\mathrm{Spin}(n)$ is a double covering of the special orthogonal group $SO(n)$.

The conformal group of the one point compactification, $R^n \cup \{\infty\}$, of R^n is the group of diffeomorphisms generated by orthogonal transformations, translations, dilations and Kelvin inversion. Under the first three types of transformations that we just mentioned, the point at infinity remains fixed, while under Kelvin inversion the origin and the point at infinity are interchanged.

Following [1, 28] we note that each conformal transformation can be expressed as $(ax+b)(cx+d)^{-1}$, where a, b, c and $d \in Cl_n$ and satisfy:
(i) a, b, c and d are all products of vectors from R^n.
(ii) $a\tilde{c}$, $c\tilde{d}$, $d\tilde{b}$, and $b\tilde{a} \in R^n$.
(iii) $a\tilde{d} - b\tilde{c} = \pm 1$.

A matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, with a, b, c and d satisfying (i)–(iii) is called a Vahlen matrix. The set of all such matrices form a group under matrix multiplication. This group is a covering group of the conformal group and it is denoted by $V(n)$. If in condition (iii) we insist that $a\tilde{d} - b\tilde{c} = 1$ then we obtain a subgroup of $V(n)$ which we denote by $V_+(n)$.

For each natural number m we can consider the Clifford algebra Cl_m, and the Vahlen groups $V(m)$ and $V_+(m)$. When $m = n-1$ the Möbius transformation $(ax+b)(cx+d)^{-1}$ preserves upper half space, $R^{n,+} = \{x = x_1e_1 + \ldots + x_ne_n : x_n > 0\}$ and lower half space, $R^{n,-} = \{x = x_1e_1 + \ldots x_ne_n : x_n < 0\}$, whenever $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in V_+(n-1)$. $V_+(n-1)$ also preserves the boundary $R^{n-1} \cup \{\infty\}$ of $R^{n,\pm}$.

When $m = n+1$ the Cayley transformation

$$K_1(x) = (x - e_{n+1})(-e_{n+1}x + 1)^{-1}$$

transforms R^n onto the punctured sphere, $S^n \backslash \{e_{n+1}\}$.

Besides the anti-automorphism $\tilde{}$ we shall also need the anti-automorphism

$$\bar{} : Cl_n \to Cl_n : (e_{j_1} \ldots e_{j_r})^{\bar{}} = (-1)^r e_{j_r} \ldots e_{j_1}.$$

For $a \in Cl_n$ we denote $(a)^{\bar{}}$ by $\overline{a}$. This anti-automorphism is the analogue of conjugation on the complex and quaternionic algebras. In particular, if $a =$

$a_0 + \ldots + a_{1\ldots n}e_1 \ldots e_n$, then the real part of $a\bar{a}$ gives $(a_0^2 + \ldots + a_{1\ldots n}^2)$, the square of the norm $\|a\|$ of a.

Besides the Clifford algebra Cl_m we shall also need the Clifford algebra $Cl_{n,1}$ generated from the Minkowski, or Krein, space $R^{n,1} = \{x_1e_1 + \ldots + x_ne_n + y_{n+1}f_{n+1} : x_1, \ldots, x_n, y_{n+1} \in R\}$. On identifying f_{n+1} with ie_{n+1} one can see that $Cl_{n,1}$ is a real 2^{n+1}-dimensional subalgebra of $Cl_{n+1}(C)$, that the vector f_{n+1} anticommutes with each e_j for $1 \leq j \leq n$, and $f_{n+1}^2 = 1$.

It may be observed that there is a Cayley transformation

$$K_2 : R^n \backslash S^{n-1} \rightarrow H_n \backslash \{-f_{n+1}\} : K_2(x) = (-x + f_{n+1})(f_{n+1}x + 1)^{-1},$$

where H_n comprises the hyperbola $\{y \in R^{n,1} : y^2 = 1\}$. H_n has two components H_n^+ and H_n^-. The hyperbola H_n^+ contains the vector f_{n+1} while the hyperbola H_n^- contains the vector $-f_{n+1}$. Also, $K_2^{-1}(H_n^+) = D(0,1)$, the open unit disc in R^n, while $K_2^{-1}(H_n^- \backslash \{-f_{n+1}\}) = \{x \in R^n : \|x\| > 1\}$.

Further results on Clifford algebras can be found in [20] and elsewhere.

3. Some Clifford Analysis

We begin by developing some background on Clifford analysis. The Dirac operator that we shall use here is the differential operator $D = \sum_{j=1}^n e_j \frac{\partial}{\partial x_j}$. One simple but important feature of this operator is that $D^2 = -\triangle_n$, where $\triangle_n$ is the Laplacian on R^n. For U a domain in R^n and f a Cl_n valued C^1 function defined on U we say that f is left monogenic if $Df = 0$. Similarly if $g : U \rightarrow Cl_n$ is a C^1 function satisfying $gD = 0$ then g is said to be right monogenic. Here $gD = \sum_{j=1}^n \frac{\partial g}{\partial x_j} e_j$. It should be noted that f is left monogenic if and only if $\overline{f}$ is right monogenic.

Many results on monogenic function theory can be found in [6, 14] and in many other texts. In particular, there are the following analogues of Cauchy's theorem and Cauchy's integral formula.

Theorem 3.1. *Suppose that f and g are respectively left and right monogenic functions on U and that S is a compact Lipschitz surface bounding a subregion V of U, and such that $cl(V)$, the closure of V, is contained in U. Then $\int_S g(x)n(x)f(x)d\sigma(x) = 0$, where $n(x)$ is the outward normal vector to S at x, and σ is the Borel measure on S.*

Theorem 3.2. *Suppose that f, V and S are as in the previous theorem, and that $y \in V$. Then $f(y) = \frac{1}{\omega_n} \int_S G(x-y)n(x)f(x)d\sigma(x)$, where ω_n is the surface area of the unit sphere in R^n and $G(x) = \frac{-x}{\|x\|^n}$.*

It should be noted that as the surface S is a Lipschitz surface then, as a consequence of a theorem of Rademacher, the outward normal vector function on S is defined almost everywhere on S and is an L^∞ function. It should also be noted that the function G is both left and right monogenic.

Using the local arguments presented in [13] one also has the following analogues of the Plemelj formulae, see also [15, 16].

Theorem 3.3. *Suppose that S is a Lipschitz surface in R^n and $\eta \in L^2(S)$. Then:*
(i) If S^+ is the component of $R^n \backslash S$ into which the normal vector function of S points, $y(t) \in S^+$ for $t \in [0,1)$, and $y(t)$ approaches $y \in S$ non-tangentially as t tends to 1, then

$$\lim_{t\to 1}\int_S G(x-y(t))n(x)\eta(x)d\sigma(x) = \frac{1}{2}\eta(y) + pv\frac{1}{\omega_n}\int_S G(x-y)n(x)\eta(x)d\sigma(x).$$

(ii) If S^- is the other component of $R^n \backslash S$, $w(t) \in S^-$ for $t \in [0,1)$, and $w(t)$ approaches $w \in S$ non-tangentially as t approaches 1, then

$$\lim_{t\to 1}\int_S G(x-w(t))n(x)\eta(x)d\sigma(x) =$$

$$\frac{-1}{2}\eta(w) + pv\frac{1}{\omega_n}\int_S G(x-w)n(x)\eta(x)d\sigma(x).$$

In Theorem 3.3 we use the L^2 space of S, which is defined to be the set of all Cl_n valued functions, f, defined on S such that $\| \int_S \overline{f}(x)f(x)d\sigma(x)\| < +\infty$.

The bounded operators

$$\frac{1}{2}I \pm C_S : L^2(S) \to L^2(S) : (\frac{1}{2}I \pm C_S)\eta = \frac{1}{2}\eta \pm pv\frac{1}{\omega_n}\int_S G(x-)n(x)\eta(x)d\sigma(x)$$

are projection operators onto the Hardy spaces $H^2(S^\pm)$ of left monogenic functions defined on $S^\pm$ respectively and which extend continuously in the L^2 sense to S. See for instance [15, 16] and elsewhere.

By using the generators of the group $V(n)$ and arguments presented in [22, 26] it may be observed that if $y = \psi(x) = (ax+b)(cx+d)^{-1}$ is a Möbius transformation, and $f(y)$ is a left monogenic function, then $J(\psi,x)f(\psi(x))$ is left monogenic in the variable x, where $J(\psi,x) = \widetilde{(cx+d)}\|cx+d\|^{-n}$. Moreover, in [8, 21] it is observed that if S_1 and S_2 are surfaces and $\psi(S_1) = S_2$ in $R^n \cup \{\infty\}$, then for each pair of functions η, $\nu \in L^2(S_2)$ we have that

$$\int_{S_2}\overline{\eta}(y)\nu(y)d\sigma(y) = \int_{S_1}\overline{\eta}(\psi(x))\nu(\psi(x))\frac{1}{\|cx+d\|^{2n-2}}d\sigma(x).$$

As $\frac{1}{\|cx+d\|^{2n-2}} = \overline{J}(\psi,x)J(\psi,x)$, it follows that the right Cl_n linear transform

$$Is_{2,1} : L^2(S_2) \to L^2(S_1) : Is_{2,1}\eta(y) = J(\psi,x)\eta(\psi(x))$$

is an isometry. It is also a consequence of the previous remarks that $Is_{2,1}H^2(S_2^+) = H^2(S_1^\pm)$ while $Is_{2,1}H^2(S_2^-) = H^2(S_1^\mp)$.

As $G(\psi(x)-\psi(w)) = J(\psi,w)^{-1}G(x-w)\widetilde{J}(\psi,x)^{-1}$ and the volume element $n(y)d\sigma(y)$ on S_2 pulls back to $\widetilde{J}(\psi,x)n(x)J(\psi,x)d\sigma(x)$ on S_1, then

$$J(\psi,x)(\frac{1}{2}I \pm C_{S_2})\eta(\psi(y))) = (\frac{1}{2}I \pm C_{S_1})J(\psi,x)\eta(\psi(x)),$$

where, as usual, $y = \psi(x)$. Consequently the Plemelj projection operators are preserved via conformal transformations, when acting on L^2 spaces of surfaces.

We shall denote the operators $\frac{1}{2}I \pm C_S$ by $C_S^{\pm}$ respectively. Besides the operators $C_S^{\pm}$ we shall need the operators $C_S^{\pm\star}$, where

$$C_S^{\pm\star}(\nu)(w) = \frac{1}{2}\nu(w) \pm n(w)\frac{1}{\omega_n}\int_S G(x-w)\nu(x)d\sigma(x),$$

where $\nu \in L^2(S)$.

From the previous remarks it may be noted that the orthogonal projection operators

$$P_S^+ : L^2(S) \to H^2(S^+),$$

and

$$P_S^- : L^2(S) \to H^2(S^-)$$

are conformally covariant.

Lemma 3.4. *Suppose that S_1 and S_2 are surfaces and that $\psi(S_1) = S_2$ for some Möbius transformation $y = \psi(x) = (ax+b)(cx+d)^{-1}$. Then*

$$J(\psi, x)P_{S_2}^{\pm}\nu(y) = P_{S_1}^{\pm}J(\psi, x)\nu(\psi(x))$$

for each $\nu \in L^2(S_2)$.

The orthogonal projection operators $P_S^{\pm}$ are usually referred to as Szegö projections. Lemma 3.4 demonstrates the conformal covariance of the Szegö projection operators.

Strictly speaking the operators $C_S^{\pm\star}$ do not have the same conformal covariance as the operators $C_S^{\pm}$ and $P_S^{\pm}$. However, for each pair ν, $\mu \in L^2(S_2)$ it may be observed that

$$\langle \mu(y), C_{S_2}^{\pm\star}(\nu)(y)\rangle_{S_2} = \langle J(\psi, x)\mu(\psi(x)), C_{S_1}^{\pm\star}(J(\psi, x)\nu)(\psi(x))\rangle_{S_1},$$

where $\langle \nu, \mu\rangle_S = \int_S \overline{\nu}(x)\mu(x)d\sigma(x)$ for any pair ν, $\mu \in L^2(S)$. Consequently, the bilinear form $\langle \nu, C_S^{\pm\star}\mu\rangle_S$ is covariant under Möbius transformations. It follows that the Kerzman–Stein kernels $C_S^{\pm} - C_S^{\pm\star}$ have a bilinear form covariance given by

$$\langle \nu(y), (C_{S_2}^{\pm} - C_{S_2}^{\pm\star})(\mu)(y)\rangle_{S_2} =$$
$$\langle J(\psi, x)\nu(\psi(x)), (C_{S_1}^{\pm} - C_{S_1}^{\pm\star})(J(\psi, x)\mu)(\psi(x))\rangle_{S_1}.$$

In [3, 23] it is noted that $P_S^{\pm} = C_S^{\pm}P_S^{\pm}$, $P_S^{\pm} = P_S^{\pm\star} = P_S^{\pm\star}C_S^{\pm\star} = P_S^{\pm}C_S^{\pm\star}$, and $C_S^{\pm} = P_S^{\pm}C_S^{\pm}$, where $P_S^{\pm\star}$ is the adjoint of $P_S^{\pm}$. Following [3, 23] it may be observed that $P_S^{\pm} - C_S^{\pm} = P_S^{\pm}(C_S^{\pm\star} - C_S^{\pm})$. Consequently

$$P_S^{\pm}(I + (C_S^{\pm\star} - C_S^{\pm})) = C_S^{\pm}. \tag{1}$$

The conformal covariance of this formula is an immediate consequence of our preceding remarks.

So far we have tacitly assumed that we have been working solely in euclidean space. However, using the Cayley transformations K_1 and K_2 and the results in [22] one can transfer all results so far mentioned in this section to surfaces in the sphere S^n and on the hyperbola H_n. In these cases one uses the Cayley transformations as chart maps to construct vector bundles over S^n and H_n where each fiber is isomorphic as a vector space to Cl_n. We denote the bundle over S^n by (S^n, Cl_n) and the bundle over H_n by (H_n, Cl_n). If U is a domain in either S^n or H_n then we denote the restriction of the appropriate bundle to U by (U, Cl_n). Similarly if S is a surface in S^n or H_n then the restriction of the appropriate bundle to S is denoted by (S, Cl_n). We shall denote the bundle of square integrable sections on (S, Cl_n) simply by $L^2(S)$, and the corresponding bundle of sections of solutions to the Dirac equation over the complementary domains $S^{\pm}$ which extend continuously in the L^2 sense to the boundary, S, by $H^2(S^{\pm})$ respectively. The Dirac operator D_{S^n} over S^n is set up in [22], see also [27]. The Dirac operator D_{H_n} over H_n is set up in [22]. It should be noted that H_n has two components. So one does need to clarify what is meant by the domains $S^{\pm}$ in the context of H_n.

Definition 3.5. An open subset U of H_n is called a domain if there is a domain $V \subset R^n$ such that $K_2(V\backslash(V \cap S^{n-1})) = U$.

It follows that U can have two components.

From [24, 26] it may be observed that the H^2 decomposition of $L^2(S^{n-1})$ is an orthogonal decomposition. It follows that if S is a surface that is conformally equivalent to $S^{n-1} \subset R^n$ then the H^2 decomposition of L^2 is also an orthogonal decomposition. This happens when $S = R^{n-1}$, or when S is a sphere in S^n of codimension 1. It also happens when $S = S_\alpha = H_n \cap (R^n + \alpha f_{n+1})$, where either $\alpha \in (1, \infty)$ or $\alpha \in (-\infty, -1)$. When $S = aS_\alpha\tilde{a}$ the decomposition is also orthogonal, where $a = a_1 \dots a_m$ with $a_j \in H_n$ and $1 \leq j \leq m$ for some arbitrary positive integer m. The orthogonal decomposition also happens when $S = H_{n-1}$, where H_{n-1} is the restriction of H_n to $R^{n-1,1}$. In this last case S has two components. The decomposition is also orthogonal when $S = aH_{n-1}\tilde{a}$ where a as before is equal to $a_1 \dots a_m$ with $a_j \in H_n$ for $1 \leq j \leq m$. Again in these circumstances the surface has two components.

Via Möbius transformations we may now obtain orthogonal bases or orthonormal bases for $L^2(S)$ and $H^2(S^{\pm})$ for each one of these surfaces. They are simply obtained from the orthogonal and orthonormal bases for $L^2(S^{n-1})$ and $H^2(S^{n-1,\pm})$ respectively. Details for the special case where $S = R^{n-1}$ are worked out in [21], the other cases follow similarly.

4. Operators of Calderón–Zygmund Type

Consider a Lipschitz continuous function $\lambda : R^{n-1} \to R$ where R^{n-1} is spanned by $e_1, \dots, e_{n-1}$. Now consider the graph $\Sigma' = \{x + \lambda(x)e_n : x \in R^{n-1}\}$ of λ. Besides the Lipschitz graph Σ' we also shall consider the Lipschitz graph $\Sigma = a\Sigma'\tilde{a}$ for any $a \in \mathrm{Spin}(n)$. Σ is just an orthogonally transformed copy of Σ'. If the Lipschitz continuous function λ has Lipschitz constant $c \in (0, \infty)$, so that $\|\lambda(u) - \lambda(v)\| \leq c\|u - v\|$, then there is a $\theta \in (0, \frac{\pi}{2})$ such that $c = \tan\theta$. Following [15, 16, 17] we may introduce the open cone $N'(\Sigma') = \{x + x_n e_n : x \in R^{n-1}$ and $x_n > \tan \|x\|\}$. We shall denote the open cone $aN'(\Sigma')\tilde{a}$ by $N(\Sigma)$. One important property of $N(\Sigma)$ is that for each $u \in \Sigma$ the cone $u + N(\Sigma)$ has empty intersection with $\Sigma\backslash\{u\}$.

Definition 4.1. An L^2 bounded operator $T_{K,k}: L^2(\Sigma) \to L^2(\Sigma)$ is said to be of Calderón–Zygmund type if

$$T_{K,k}(\phi)(y) = \lim_{\epsilon \to 0}(\int_{\Sigma(x,\epsilon)} K(x,y)n(x)\phi(x)d\sigma(x) + k(\epsilon n(y)\phi(y))$$

for almost all $y \in \Sigma$, where $K(x,y)$ is a smooth $Cl_n(C)$ valued function defined on $(\Sigma \times \Sigma)\backslash\text{diag}(\Sigma \times \Sigma)$ and satisfying

$$\|K(x,y)\| \leq C\|x-y\|^{-n+1},$$

for some $C \in R^+$. Moreover, k is a $Cl_n(C)$ valued L^∞ function defined on $N(\Sigma)$, while $\Sigma(x,\epsilon) = \{y \in \Sigma : \|y-x\| \geq \epsilon\}$.

Definition 4.1 gives a minor modification of various types of singular integral operators considered in [16] and elsewhere. In [16] the function $K(x,y)$ is considered to be equal to be a function $K'(x-y)$. The formulation given here is more amenable under conformal transformation than that given in [16] and elsewhere.

Theorem 4.2. *Suppose that $T_{K,k}: L^2(\Sigma) \to L^2(\Sigma)$ is as in definition 4.1, and that $\Sigma = \psi(\Psi)$, where $y = \psi(u) = (au+b)(cu+d)^{-1}$. Then the operator*

$$S_{H,h}: L^2(\Psi) \to L^2(\Psi)$$

is L^2 bounded, where

$$H(v,u) = J(\psi,v)K(\psi(v),\psi(u))\tilde{J}(\psi,u),$$

$$h(v) = \frac{(cv+d)k(\psi(v))\widetilde{(cv+d)}}{\|cv+d\|^2}$$

and

$$S_{H,h}(\eta)(v) = \lim_{\delta \to 0}\int_{\Psi(v,\delta)} H(v,u)n(u)\eta(u)d\sigma(u) + h(\delta n(v))\eta(v))$$

for almost all $v \in \Psi$, with $\Psi(v,\delta) = \{w \in \Psi : \|w-v\| \geq \delta\}$.

It should be noted that $\|H(v,u)\| \leq C'\|v-u\|^{n-1}$ for some $C' \in R^+$ and that h is an L^∞ function on $\psi^{-1}(N(\Sigma)) = N(\Psi)$.

We shall also call the operator $S_{H,h}$ appearing in Theorem 4.2 an operator of Calderón–Zygmund type. One important reason for considering the function $K(x,y)$ to not necessarily be equal to $K''(x-y)$ for some function K'' is that the difference $x-y$ is not in general preserved under Möbius transformations. An interesting case to consider is when the Lipschitz surface Σ is R^{n-1}. In this case $L^2(R^{n-1})$ is transformed isometrically to itself, and so:

Proposition 4.3. *The family of operators of Calderón–Zygmund type described in Definition 4.1 and acting on $L^2(R^{n-1})$ is transformed to itself under the action of $V_+(n-1)$.*

This gives another reason for not considering $K(x,y)$ to be equal to $K''(x-y)$.

Let us now turn to look at commutators of multiplier operators with operators of Calderón–Zygmund type. We begin with:

Theorem 4.4. *Suppose that $g:\Sigma\to Cl_n(C)$ is such that the commutator*

$$[T_{K,k}, M_g] : L^2(\Sigma)\to L^2(\Sigma)$$

is L^2 bounded for each operator $T_{K,k}$ of Calderón–Zygmund type, where M_g is the multiplier operator $M_g(\mu)=g(x)\mu(x)$. Then for each Möbius transformation $y=\psi(u)=(au+b)(cu+d)^{-1}$ the operator $[T_{K,k},g]$ is conformally pulled back to the L^2 bounded operator

$$[S_{H,h}, M_p] : L^2(\Psi)\to L^2(\Psi),$$

where $S_{H,h}$ is as in Theorem 4.2 and

$$p(v)=\frac{\widetilde{(cv+d)}g(\psi(v))(cv+d)}{\|cv+d\|^2}$$

Theorem 4.4 shows us that commutators of operators of Calderón–Zygmund type with multiplier operators are preserved under conformal transformations. For the special case where the surface Σ is R^{n-1} we have:

Proposition 4.5. *The class of commutators*

$$[T_{K,k}, M_g] : L^2(R^{n-1})\to L^2(R^{n-1}) \tag{2}$$

is preserved under the action of $V_+(n-1)$.

In [11] it is noted that the operator given in (2) is bounded if and only if $g\in BMO(R^{n-1})$. It follows that in this case $p\in BMO(R^{n-1})$.

It should be pointed out that Propositions 4.3 and 4.5 hold equally well if R^{n-1} is replaced by the unit sphere S^{n-1} in R^n, or the unit sphere $K_1(R^{n-1})\cup\{e_{n+1}\}$ in R^{n+1} or the hyperbola $H_{n-1}=K_2(R^{n-1}\backslash S^{n-2})$, where S^{n-2} is the unit sphere in R^{n-1}. In these cases the group $V_+(n-1)$ is replaced by the groups $KV_+(n-1)K^{-1}$, $K_1V_+(n-1)K_1^{-1}$, and $K_2V_+(n-1)K_2^{-1}$ respectively.

By similar observations to those used to derive Theorem 4.4 and the use of induction we may also arrive at:

Proposition 4.6. *Suppose that $T_{K,k}, S_{H,h}, g$ and p are as in Theorem 4.4 and the n-fold commutator*

$$[\ldots[T_{K,k}, M_g],\ldots, M_g] : L^2(\Sigma)\to L^2(\Sigma)$$

is bounded. Then under the Möbius transformation $y=\psi(u)=(au+b)(cu+d)^{-1}$ this operator pulls back to the bounded operator

$$[\ldots[S_{H,h}, M_p],\ldots, M_p] : L^2(\Psi)\to L^2(\Psi).$$

Within Propositions 4.5 and 4.6 we have a description of the conformal covariance of the multiplier operator M_g. This conformal covariance can easily be applied to describe the conformal covariance of the following Hankel operators on conformally equivalent surfaces S_1 and S_2.

Lemma 4.7. *Suppose that $y = \psi(v) = (av+b)(cv+d)^{-1}$ and $\psi(S_1) = S_2$. Suppose also that $g \in L^\infty(S_2)$. Then the Hankel operator*

$$(I - P^{\pm}_{S_2})M_g : L^2(S_2) \to L^2(S_2)$$

pulls back to the operator

$$(I - P^{\pm}_{S_1})M_p : L^2(S_1) \to L^2(S_1),$$

where $p(v) = \frac{\widetilde{(cv+d)}g(\psi(v))(cv+d)}{\|cv+d\|^2} \in L^\infty(S_1)$.

In all the preceding work the surface Ψ is assumed to lie in R^n, S^n or H_n. In order to adequately cope with the situations where Ψ is in H_n we shall assume that $\Sigma \cap S^{n-1}$, seen as a subset of S^{n-1}, is a set of measure zero.

One important reason why in [15, 16, 17] and elsewhere the function $K(x,y)$ is seen to be equal to $K''(x-y)$ for some function $K''(x)$ is because it is assumed that K'' is a right monogenic function on the sector domain $S_\theta = \{x + x_n e_n : x \in R^{n-1}$ and $|x_n| < \|x\| \tan\theta\}$. Moreover, $\|K''(x)\| \le C\|x\|^{-n+1}$ on S_θ.

In [21] it is observed that sector domains are transformed by Möbius transformations on R^{n-1} to sector domains. For each $x \in R^n$ and each Möbius transformation ψ we shall denote the sector domain $\psi^{-1}(S_\theta + x)$ by $Q_\theta(v)$ where $\psi(v) = x$. Each $Q_\theta(v)$ is a domain in R^n, S^n or H_n. So for each $v \in \Psi$ the function $H(u,v) = J(\psi,v)K''(\psi(u) - \psi(v))\widetilde{J}(\psi,u)$ is right monogenic on $Q_\theta(v)$ and $\|H(u,v)\| \le C'\|u-v\|^{-n+1}$ for some $C' \in R^+$. In [16] it is shown that $K'' = K''_+ + K''_-$ where $K''_\pm$ is a right monogenic function on $S^\pm_\theta = S_\theta \cup R^{n,\pm}$ and $\|K''_\pm(x)\| \le C_\pm\|x\|^{-n+1}$. Consequently, $H(u,v) = H_+(u,v) + H_-(u,v)$ where $H_\pm$ is a right monogenic function on $Q^\pm_\theta(v) = \psi^{-1}(S^\pm_\theta + x)$, for each $v = \psi^{-1}(x) \in \Psi$. Moreover, $H_\pm(u,v) = J(\psi,v)K''_\pm(\psi(u) - \psi(v))\widetilde{J}(\psi,u)$, and $\|H_\pm(u,v)\| \le C'_\pm\|u-v\|^{-n+1}$.

In [16, 15] a specific construction is given for the L^∞ function k defined on $N_\theta(\Sigma)$, and associated to the right monogenic function K''. Via this construction k is defined uniquely up to a constant. In [15, 16] it is shown that $k(x) = k_+(x) + k_-(x)$ where $k_\pm \in L^\infty(N_\theta(\Sigma))$. Moreover explicit constructions of these functions are given in [15, 16]. Under the Möbius transformation ψ these functions are pulled back to the functions $h_\pm(v) = \frac{\widetilde{(cv+d)}k_\pm(\psi(v))(cv+d)}{\|cv+d\|^2}$. Furthermore it is easily seen that $h(v) = h_+(v) + h_-(v)$. In [15, 16] conditions are given for the functions $K''_\pm$ to be left monogenic on $\Sigma^\pm_\theta$ as well as right monogenic, under the assumption that K'' is also left monogenic on Σ_θ. Under these assumptions it is shown that $T_{K'',k} = T_{K''_+,k_+} + T_{K''_-,k_-}$ and that $T_{K''_\pm,k_\pm} : H^2(\Sigma^\pm) \to H^2(\Sigma^\pm)$, is a bounded operator and $T_{K''_\pm,k_\pm}(H^2(\Sigma^\mp)) = \{0\}$. Consequently we have

Theorem 4.8. *Suppose that $T_{K'',k}$, $T_{K''_\pm,k_\pm}$ are as in the previous paragraph. Then the operators*

$$T_{K''_\pm,k_\pm} : H^2(\Sigma^\pm) \to H^2(\Sigma^\pm)$$

pull back via the Möbius transformation ψ to the bounded operators

$$S_{H_\pm,h_\pm} : H^2(\Psi^\pm) \to H^2(\Psi^\pm).$$

Moreover, $S_{H_\pm,h_\pm}(H^2(\Psi^\mp)) = \{0\}$.

When $\Psi \subset R^n$, Theorem 4.8 has previously been described in [21].

5. Other Conformally Covariant Operators

For U a domain in R^n, S^n or H_n, we will denote by $S(U)$ the space of $Cl_n(C)$ valued C^∞ functions on U with compact support. In the special cases where $U \subset S^n$ or $U \subset H_n$ then each $\phi \in S(U)$ is a section on $(U, Cl_n(C))$.

For each $\gamma = \alpha + i\beta \in C$ and for each $\phi \in S(U)$ we may at least formally consider the convolutions

$$G_\gamma(\phi)(y) = \int_U (x-y)\|x-y\|^\gamma \phi(x)dx^n$$

and

$$I_\gamma(\phi)(y) = \int_U \|x-y\|^{\gamma+1}\phi(x)dx^n.$$

We shall only consider the cases where these operators are well defined on $S(U)$ and when $cl(U)$, the closure of U, is compact. This happens when the real part of γ belongs to $(-n+1, \infty)$.

Let us denote the image space $G_\gamma(S(U))$ by $P_\gamma(U)$ and the image space $I_\gamma(S(U))$ by $Q_\gamma(U)$.

Theorem 5.1. *Suppose that $y = \psi(v) = (av+b)(cv+d)^{-1}$ is a Möbius transformation. Then*

$$\int_U (x-y)\|x-y\|^\gamma \phi(x)dx^n =$$

$$J_\gamma(\psi, v)^{-1}\int_{\psi^{-1}(U)} (u-v)\|u-v\|^\gamma \tilde{J}_{\gamma+2n}(\psi, u)^{-1}\phi(\psi(u))du^n,$$

and

$$\int_U \|x-y\|^{\gamma+1}\phi(x)dx^n =$$

$$K_\gamma(\psi, v)^{-1}\int_{\psi^{-1}(U)} \|u-v\|^{\gamma+1}K_{\gamma+1+2n}(\psi, u)^{-1}\phi(\psi(u))du^n,$$

where $J_\gamma(\psi, v) = \widetilde{(cv+d)}\|cv+d\|^\gamma$ and $K_\gamma(\psi, v) = \|cv+d\|^{-\gamma-1}$.

Outline of Proof: The result follows from noting that

$$(\psi(u)-\psi(v))\|\psi(u)-\psi(v)\|^\gamma = J_\gamma(\psi, v)^{-1}(u-v)\|u-v\|^\gamma \tilde{J}(\psi, u)^{-1},$$

that

$$\|\psi(u)-\psi(v)\|^{\gamma+1} = K_\gamma(\psi, v)^{-1}\|u-v\|^{\gamma+1}K_\gamma(\psi, u)^{-1},$$

and the Jacobian of the transformation is $\|cv+d\|^{-2n}$.

We shall denote the function space $\{\tilde{J}_{\gamma+2n}(\psi, v)^{-1}\phi(\psi(v)) : \phi \in S(U)\}$ by $S_{\gamma,J}(\psi^{-1}(U))$. We denote the function space $\{K_{\gamma+1+2n}(\psi, v)^{-1}\phi(\psi(v)) : \phi \in S(U)\}$ by $S_{\gamma,K}(\psi^{-1}(U))$. Moreover the space $G_\gamma S_{\gamma,J}(\psi^{-1}(U))$ will be denoted by $R_\gamma(\psi^{-1}(U))$ while the space $I_\gamma S_{\gamma,K}(\psi^{-1}(U))$ will be denoted by $W_\gamma(\psi^{-1}(U))$.

Theorem 5.1 tells us that the multiplier operators $J_\gamma(\psi,)$ and $\tilde{J}_{\gamma+2n}(\psi,)^{-1}$ intertwine the convolution operator G_γ and the multiplier operators $K_\gamma(\psi,)$ and $K_{\gamma+1+2n}(\psi,)^{-1}$ intertwine the convolution operator I_γ. So the transforms

$$J_\gamma(\psi,)G_\gamma : S(U) \to R_\gamma(\psi^{-1}(U)) \tag{3}$$

and

$$G_\gamma \widetilde{J}_{\gamma+2n}(\psi,)^{-1} : S(U) \to R_\gamma(\psi^{-1}(U)) \tag{4}$$

are the same, and the transforms

$$K_\gamma(\psi,)I_\gamma : S(U) \to W_\gamma(\psi^{-1}(U)) \tag{5}$$

and

$$I_\gamma K_{\gamma+1+2n}(\psi,)^{-1} : S(U) \to W_\gamma(\psi^{-1}(U)) \tag{6}$$

are the same.

For $-n < \gamma < 0$ and U a subset of R^n the operators I_γ correspond to the Riesz potentials described in [25] and elsewhere. Following the Fourier analysis arguments given for R^n in [25] it may be observed that for these values of γ the operator I_γ is invertible. From Theorem 5.1 it follows that for these values of γ the operators I_γ acting on the function spaces set up in Theorem 5.1 for the domain $\psi^{-1}(U)$ are invertible. When $\gamma = -n+2$ the Riesz potential on R^n is the fundamental solution to the Laplacian. This operator has $\triangle_n$ as its inverse. When $\gamma = -n+1$ then as observed in [25] the inverse of I_γ is $\triangle_n^{\frac{1}{2}} = \|D\|$. It follows from Theorem 5.1 that each one of these inverse operators have analogues on the appropriate function spaces on the domain $\psi^{-1}(U)$.

From Theorem 5.1 and equations 3–6 it follows that when the operators $G_\gamma : S(U) \to P_\gamma(U)$ and $I_\gamma : S(U) \to Q_\gamma(U)$ are invertible then

$$G_\gamma^{-1} J_\gamma(\psi,)^{-1} : R_\gamma(\psi^{-1}(U)) \to S(U)$$

and

$$\widetilde{J}_{\gamma+2n} G_\gamma^{-1} : R_\gamma(\psi^{-1}(U)) \to S(U)$$

are the same, and the transforms

$$I_\gamma^{-1} K_\gamma(\psi,)^{-1} : W_\gamma(\psi^{-1}(U)) \to S(U)$$

and

$$K_{\gamma+1+2n} I_\gamma(\psi,)^{-1} : W_\gamma(\psi^{-1}(U)) \to S(U)$$

are the same.

As a consequence of the preceding remarks one may introduce analogues of the pseudo-differential operator $\triangle_n^{\frac{1}{2}}$, acting on function spaces $S(U)$ where U is a domain in S^n or H_n.

In [25] and elsewhere it is shown that when $-n+1 < \gamma < 0$ then

$$I_\gamma : L^p(R^{n-1}) \to L^q(R^{n-1})$$

is a bounded operator if and only if $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, where $\alpha = n - \gamma$, and $1 < p < \infty$.

First let us note:

Lemma 5.2. *The function $\phi(y) \in L^p(R^{n-1})$ if and only if*

$$K_{\frac{-2n+2}{p}-1}(\psi, v)\phi(\psi(v)) \in L^p(R^{n-1})$$

for each $\psi \in V_+(n-1)$, and $1 < p < \infty$.

Similarly $\phi(y) \in L^p(R^{n-1})$ if and only if $J_{\frac{-2n+2}{p}}(\psi, v)\phi(\psi(v)) \in L^p(R^{n-1})$ for each $\psi \in V_+(n-1)$ and $1 < p < \infty$.

Consequently one may deduce:

Theorem 5.3. *Suppose that* $1 < p < \infty$ *and* $\frac{1}{p} + \frac{1}{q} = 1$ *then the multiplier operators* $K_{(\frac{1}{p}-1)(2n-2)+1}(\psi,)$ *and* $K_{(\frac{1}{p}+1)(2n-2)+1}(\psi,)$ *intertwine the operator*

$$I_{\frac{2-2n}{p}+1} : L^p(R^{n-1}) \to L^q(R^{n-1})$$

while the multiplier operators $J_{(\frac{1}{p}-1)(2n-2)}(\psi,)$ *and* $J_{(\frac{1}{p}+1)(2n-2)}(\psi,)$ *intertwine the operator*

$$G_{\frac{2-2n}{p}} : L^p(R^{n-1}) \to L^q(R^{n-1})$$

for each $\psi \in V_+(n-1)$.

In fact most of the previous material from this section can be reformulated for arbitrary smooth orientable manifolds in R^n, S^n or H_n. Consider a p-dimensional smooth and orientable manifold M_p in R^n, S^n or H_n, and M_p is such that $cl(M_p)$ is compact. Then we may introduce the space $S(M_p)$ of smooth sections on M_p with compact support. These sections will take their values in the bundle $(M_p, Cl_n(C))$, where $(M_p, Cl_n(C))$ is the appropriate sub-bundle of $(S^n, Cl_n(C))$ when $M_p \subset S^n$. It is a sub-bundle of $(H_n, Cl_n(C))$ when $M_p \subset H_n$ and it is the bundle $M_p \times Cl_n(C)$ when $M_p \subset R^n$.

In these circumstances we may formally introduce the convolution operators $G_{\gamma,M_p}\phi(y) = \int_{M_p}(x-y)\|x-y\|^\gamma\phi(x)d\nu(x)$ and $I_{\gamma,M_p}\phi(y) = \int_{M_p}\|x-y\|^{\gamma+1}\phi(x)d\nu(x)$, where $\phi \in S(M_p)$ and ν is the Borel measure on M_p. These operators are well defined and bounded when the real part of γ belongs to $(-p, \infty)$. We shall denote the image space $G_{\gamma,M_p}S(U)$ by $P_\gamma(M_p)$ and the image space $I_{\gamma,M_p}S(U)$ by $Q_\gamma(M_p)$.

Theorem 5.4. *Suppose that* $y = \psi(v) = (av+b)(cv+d)^{-1}$ *is a Möbius transformation then for each* $\phi \in S(U)$

$$\int_{M_p}(x-y)\|x-y\|^\gamma\phi(x)d\nu(x) =$$

$$J_\gamma(\psi, v)\int_{\psi^{-1}(M_p}(u-v)\|u-v\|^\gamma\widetilde{J}_{\gamma+2p}(\psi, u)^{-1}\phi(\psi(u))d\nu(u)$$

and

$$\int_{M_p}\|x-y\|^{\gamma+1}\phi(x)d\nu(x) =$$

$$K_\gamma(\psi, v)^{-1}\int_{\psi^{-1}(M_p)}\|u-v\|^{\gamma+1}K_{\gamma+2p}(\psi, u)^{-1}\phi(\psi(u))d\nu(u).$$

The result follows from noting that the Jacobian of this transformation is $\|cv+d\|^{-2p}$ and by applying the same arguments used to deduce Theorem 5.1.

Theorem 5.4 tells us that the multiplier operators $J_\gamma(\psi,)$ and $\widetilde{J}_{\gamma+2p}(\psi,)^{-1}$ intertwine the operator G_{γ,M_p} and also its inverse when that exists, and that $K_\gamma(\psi,)$ and $K_{\gamma+1+2p}(\psi,)^{-1}$ intertwine the operator I_{γ,M_p} and also its inverse when that exists.

6. Generalised Plemelj Formulae

When regarded as a principal value integral, the convolution operator G_{-p+1,M_p} is also well defined on $S(M_p)$. We shall consider the case where M_p is compact and is the boundary of a smooth orientable manifold M. We shall also assume that ∂M_p is Liapunov, so that M_p is C^1 and has Hölder continuous first derivative. So the function $n : M_p \to S^{n-1}$ is Hölder continuous, where the function n assigns to each $x \in M_p$ the unit vector $n(x)$ in TM_x normal to $TM_p(x)$ and pointing out from the manifold M, where $TM(x)$ is the tangent space to M at x and $TM_p(x)$ is the tangent space to M_p at x.

Let $L^2(M_p)$ denote the class of L^2 integrable functions defined on M_p and with values in the bundle $(M_p, Cl_n(C))$. It is a simple matter to follow the classical arguments presented in [4] and elsewhere on compact Liapunov surfaces in R^n to deduce

Theorem 6.1. *The convolution operator*

$$G_{-p+1,M_p,M} : L^2(M_p) \to L^2(M_p) :$$

$$\frac{1}{\omega_p} G_{-p+1,M_p,M}\phi(y) = pv\frac{1}{\omega_p}\int_{M_p} \frac{(x-y)}{\|x-y\|^p} n(x)\phi(x)d\nu(x)$$

is an L^2 bounded operator, where ω_p is the surface area of the unit sphere in R^p.

It should be noted that the the normal function $n(x)$ to M_p depends on the choice of M. Consequently the operator $G_{-p+1,M_p,M}$ also depends on M. Also, when $p = n$ the operator $G_{-p+1,M_p,M}$ is the same as the Cauchy kernel G.

Using the Dirac operator over each tangent space TM_x with $x \in M_p$ we obtain the following Plemelj formula, which also follows by the same reasoning as that presented for surfaces in R^n in [4].

Theorem 6.2. *For each $y \in M_p$ and each $y(t) \in M$ such that $y(t)$ approaches y non-tangentially as t approaches 1 then*

$$\lim_{t\to 1} \frac{1}{\omega_p}\int_{M_p} \frac{(x-y)}{\|x-y\|^p} n(x)\phi(x)d\nu(x) =$$

$$\frac{1}{2}\phi(y) + pv\frac{1}{\omega_p}\int_{M_p} \frac{(x-y)}{\|x-y\|^p} n(x)\phi(x)d\nu(x),$$

for almost all $y \in M_p$ and each $\phi \in L^2(M_p)$.

We may denote the Hilbert space $(\frac{1}{2} + G_{-p+1,M_p})L^2(M_p, M)$ by $H^2(M)$.

It is an easy matter to mimic the arguments used earlier here to show that if ψ is a Möbius transformation and for two manifolds $M_{p,1}$ and $M_{p,2}$ with $\psi(M_{p,1}) = M_{p,2}$ then the Hilbert spaces $L^2(M_{p,1})$ and $L^2(M_{p,2})$ are isometric. In particular, each $\phi \in L^2(M_{p,2})$ is transformed to $J_{-p+1}(\psi,)\phi \in L^2(M_{p,1})$. Via this isometry it may be deduced that $H^2(M_{p,1})$ is isometric to $H^2(M_{p,2})$. Also the multiplier operator $J_{-p+1}(\psi,)$ intertwine the operator G_{-p+1} and also the operator $\frac{1}{2} + G_{-p+1}$. Consequently, a significant part of the results mentioned earlier for L^2 spaces of surfaces in R^n, S^n and H_n go through automatically to the setting described in this section. Though we have assumed the manifold $M_{p,2}$ is compact and Liapunov it does not follow that the manifold $M_{p,2}$ is either compact or Liapunov.

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