Journal of Lie Theory
Volume **8** (1998) 129–0

Symmetry algebras and normal forms of third order ordinary differential equations

Annett Schmucker, Günter Czichowski

Communicated by H. Boseck

Abstract. Third order ordinary differential equations are classified according to the dimension and the structure of their symmetry algebra. We establish canonical forms of the generators and representatives for all third order ordinary differential equations possessing 4, 5, 6 or 7 independent symmetries.

1. Introduction

The concept of symmetries of differential equations is due to Lie. He investigated groups of point transformations in order to solve differential equations systematically. The order of an ordinary differential equation (o.d.e.) $q^{(n)} = f(t, q, \ldots, q^{(n-1)})$ that admits a symmetry can be reduced by transforming this symmetry in ∂_q and then substituting Q for $\dot{q}$. The point symmetries of an o.d.e. generate a Lie algebra of vector fields in two coordinates.

If we are given a classification of o.d.e.-s according to their symmetry algebra and the solutions of the representatives of all classes, solving an o.d.e. will be reduced to the determination of the symmetry algebra structure and the transformation of the o.d.e. into the appropriate normal form. This procedure might be successful for normal forms which do not contain a parameter function.

In this work o.d.e.-s of order three are investigated. Lie [3] proved that a third order o.d.e. admits at most seven independent symmetries. In a paper of 1988 [4] Mahomed and Leach treat third order o.d.e.-s with a three dimensional symmetry algebra. Their results and methods are used in Section 2. where we obtain canonical realizations of four dimensional real Lie algebras in terms of vector fields in two coordinates and the associated normal forms of invariant third order o.d.e.-s.

In Section 3. we determine the full symmetry algebra of these representatives. Since any real Lie algebra of dimension 5, 6 or 7 contains a four dimensional subalgebra, the procedure mentioned above leads to a complete list of normal forms for third order o.d.e.-s having 4, 5, 6 or 7 independent symmetries.

ISSN 0949–5932 / \$2.50

2. Equations with four symmetries

Since the symmetry algebra is a Lie algebra, we will first consider the algebraic description of real four dimensional Lie algebras. A classification of these Lie algebras including a list of all subalgebras was given by Patera and Winternitz in [8]. It turns out that any real Lie algebra of dimension four contains a three dimensional subalgebra.

Furthermore, it can be proven that any real Lie algebra of a dimension higher than three has a three dimensional subalgebra. Therefore, the treatment of three dimensional symmetry algebras is fundamental for finding o.d.e.-s with more than three symmetries (see [4] and [9]).

Linear o.d.e.-s play a particular part and will be treated in a special way. There is a general criteria for linearizability [5]:

Theorem 2.1. *An o.d.e. of order n $(n \geq 3)$ is linearizable by means of a point transformation iff its symmetry algebra has an n-dimensional abelian subalgebra. The symmetry algebra of a linear o.d.e. of order n $(n \geq 3)$ has dimension $n+1$, $n+2$ or $n+4$.*

Consequently, a third order o.d.e. that admits three independent, commuting symmetries can be transformed into $\dddot{q}= a(t)\dot{q}$. Moreover, this normal form has at least the four symmetries generated by ∂_q, $t\partial_q$, $h(t)\partial_q$ and $q\partial_q$ where $h(t)$ is a nonlinear solution of $a(t) = \frac{h'''(t)}{h''(t)}$. [4]

Hence it is sufficient to consider only those Lie algebras L of dimension four that do not contain a three dimensional commutative subalgebra (see table 1).

In this case we choose a three dimensional subalgebra S of L (see the third column of table 1), which can be represented in different canonical ways in terms of vector fields (see table 2). A representation of the fourth generator G of L is obtained invoking the commutator relations with G. Perhaps further coordinate transformations will be necessary. Applying the symmetry criteria with G on a normal form of an o.d.e. of third order which is invariant with respect to S, one arrives at a normal form of a third order o.d.e. which is invariant under the action of L.

The procedure will be illustrated by calculating as an example $L = A_{4,7}$. The Lie algebra $A_{4,7}$ (see table 1) contains a three dimensional subalgebra S which is generated by G_1, G_2 and G_3 and isomorphic to $A_{3,1}$. There are two canonical representations of $A_{3,1}$ (see table 2).

1. Let $G_1 = \partial_q$, $G_2 = \partial_t$, $G_3 = \alpha\partial_t + (t+\beta)\partial_q$ and $G_4 = \xi(t,q)\partial_t + \eta(t,q)\partial_q$. Since the commutator relations $[G_1, G_4] = 2G_1$ and $[G_2, G_4] = G_2$ must be satisfied, it is immediate that $\xi = t + \alpha$ and $\eta = 2q + \beta$ where α and β are constants. Hence the equation $t + \beta = 0$ resulting from the commutator $[G_3, G_4] = G_2 + G_3$ cannot be satisfied.

2. Let $G_1 = \partial_q$, $G_2 = \alpha\partial_t + (-t+\beta)\partial_q$, $G_3 = \partial_t$ and $G_4 = \xi(t,q)\partial_t + \eta(t,q)\partial_q$. The commutator relations $[G_1, G_4] = 2G_1$ and $[G_3, G_4] = G_2 + G_3$ lead to the partial differential equations $\xi_q = 0$, $\xi_t = \alpha + 1$, $\eta_q = 2$ and $\eta_t = -t + \beta$. From $[G_2, G_4] = G_2$ follows $\alpha = 0$ and $\xi = t - \beta$. Obviously it is sufficient to apply the symmetry criteria with the generator $t\partial_t + (-\frac{1}{2}t^2 + 2q)\partial_q$ on

type	nontrivial commutators		subalgebra
$2A_2$	$[G_1, G_2] = G_2$	$[G_3, G_4] = G_4$	$< G_1, G_2, G_4 >$
$A_{3,8} \oplus A_1$	$[G_1, G_2] = G_1$ $[G_3, G_1] = 2G_2$	$[G_3, G_2] = G_3$	$< G_1, G_2, G_3 >$
$A_{3,9} \oplus A_1$	$[G_1, G_2] = G_3$ $[G_3, G_1] = G_2$	$[G_2, G_3] = G_1$	$< G_1, G_2, G_3 >$
$A_{4,7}$	$[G_1, G_4] = 2G_1$ $[G_3, G_4] = G_2 + G_3$	$[G_2, G_4] = G_2$ $[G_2, G_3] = G_1$	$< G_1, G_2, G_3 >$
$A_{4,8}$	$[G_2, G_4] = G_2$ $[G_2, G_3] = G_1$	$[G_3, G_4] = -G_3$	$< G_1, G_2, G_3 >$
$A_{4,9}^b$ $(0 < \|b\| < 1)$	$[G_1, G_4] = (1 + b)G_1$ $[G_3, G_4] = bG_3$	$[G_2, G_4] = G_2$ $[G_2, G_3] = G_1$	$< G_1, G_2, G_3 >$
$A_{4,9}^1$	$[G_1, G_4] = 2G_1$ $[G_3, G_4] = G_3$	$[G_2, G_4] = G_2$ $[G_2, G_3] = G_1$	$< G_1, G_2, G_3 >$
$A_{4,9}^0$	$[G_1, G_4] = G_1$ $[G_2, G_3] = G_1$	$[G_2, G_4] = G_2$	$< G_1, G_2, G_3 >$
$A_{4,10}$	$[G_2, G_4] = -G_3$ $[G_2, G_3] = G_1$	$[G_3, G_4] = G_2$	$< G_1, G_2, G_3 >$
$A_{4,11}^a$ $(0 < a)$	$[G_2, G_4] = aG_2 - G_3$ $[G_3, G_4] = G_2 + aG_3$	$[G_1, G_4] = 2aG_1$ $[G_2, G_3] = G_1$	$< G_1, G_2, G_3 >$
$A_{4,12}$	$[G_1, G_4] = -G_2$ $[G_1, G_3] = G_1$	$[G_2, G_4] = G_1$ $[G_2, G_3] = G_2$	$< G_1, G_2, G_3 >$

Table 1: Real four dimensional Lie algebras without commutative three dimensional subalgebra

type	commutators	canonical realizations G_1, G_2, G_3
$A_1 \oplus A_2$	$[G_1, G_3] = G_1$	1. $\partial_t, \partial_q, t\partial_t + a\partial_t + b\partial_q$ 2. $\partial_q, t\partial_q, t\partial_t + q\partial_q$
$A_{3,1}$	$[G_2, G_3] = G_1$	$\partial_q, \partial_t, a\partial_t + (t + b)\partial_q$ $\partial_q, a\partial_t + (-t + b)\partial_q, \partial_t$
$A_{3,3}$	$[G_1, G_3] = G_1$ $[G_2, G_3] = G_2$	1. $\partial_t, \partial_q, (t + a)\partial_t + (q + b)\partial_q$ 2. $\partial_q, t\partial_q, q\partial_q$
$A_{3,8}$ $(sl(2, \mathbb{R}))$	$[G_1, G_2] = G_1$ $[G_2, G_3] = G_3$ $[G_3, G_1] = -2G_2$	1. $\partial_q, q\partial_q, q^2\partial_q$ 2. $\partial_q, t\partial_t + q\partial_q, 2tq\partial_t + q^2\partial_q$ 3. $\partial_q, t\partial_t + q\partial_q, 2tq\partial_t + (q^2 - t^2)\partial_q$ 4. $\partial_q, t\partial_t + q\partial_q, 2tq\partial_t + (q^2 + t^2)\partial_q$
$A_{3,9}$ $(so(3, \mathbb{R}))$	$[G_1, G_2] = G_3$ $[G_2, G_3] = G_1$ $[G_3, G_1] = G_2$	$(1 + t^2)\partial_t + tq\partial_q, tq\partial_t(1 + q^2)\partial_q, q\partial_t + t\partial_q$

Table 2: Representations of real three dimensional Lie algebras in terms of vector fields of two coordinates

type	normal forms of invariant o.d.e.-s
$A_1 \oplus A_2$	1. $\dddot{q}= \ddot{q}^{\frac{3}{2}}\varphi(\dot{q}\ddot{q}^{-2})$ 2. $\dddot{q}= \ddot{q}^2\varphi(t\ddot{q})$
$A_{3,1}$	$\dddot{q}= \varphi(\ddot{q})$
$A_{3,3}$	1. $\dddot{q}= \ddot{q}^2\varphi(\dot{q})$ 2. $\dddot{q}= \ddot{q}\varphi(t)$
$A_{3,8}$	1. $\dddot{q}= \frac{3}{2}\dot{q}^{-1}\ddot{q}^2 + \dot{q}\varphi(t)$ 2. $\dddot{q}= t^{-2}\dot{q}^4\varphi((t\ddot{q} + \frac{1}{2}\dot{q})\dot{q}^{-3}) + 3\dot{q}^{-1}\ddot{q}^2$ 3. $\dddot{q}= t^{-2}(-1+\dot{q}^2)^2\varphi((t\ddot{q} - \dot{q}(1-\dot{q}^2))(-1+\dot{q}^2)^{-\frac{3}{2}})$ $+3\dot{q}(-1+\dot{q}^2)^{-1}\ddot{q}^2$ 4. $\dddot{q}= t^{-2}(1+\dot{q}^2)^2\varphi((t\ddot{q} - \dot{q}(1+\dot{q}^2))(1+\dot{q}^2)^{-\frac{3}{2}})$ $+3\dot{q}(1+\dot{q}^2)^{-1}\ddot{q}^2$
$A_{3,9}$	$\dddot{q}= (1+q^2)^{-\frac{5}{2}}(1+q^2+\dot{q}^2)$ $\times\varphi((q+\ddot{q})^{-\frac{1}{3}}(1+q^2)^{-\frac{1}{2}}(1+q^2+\dot{q}^2)^{\frac{1}{2}})$ $+3\dot{q}(q+\ddot{q})((1+q^2+\dot{q}^2)^{-1} - q(1+q^2)^{-1}) - \dot{q}$

Table 3: Normal forms of third order o.d.e.-s with symmetry algebra $A_1 \oplus A_2$, $A_{3,1}$, $A_{3,3}$, $A_{3,8}$, or $A_{3,9}$

$\dddot{q}= \varphi(\ddot{q})$ which is the normal form of a third order o.d.e. with symmetry algebra $A_{3,1}$. This results in $\dot{\varphi}(\ddot{q}) = \varphi(\ddot{q})$.

We obtain $\dddot{q}= Ce^{\ddot{q}}$ where C is an arbitrary constant as the normal form of a third order o.d.e. with symmetry algebra $A_{4,7}$. It should be noted that invariance under $A^1_{4,9}$ implies $\dddot{q}= 0$. The Lie algebras $A_{3,9} \oplus A_1$, $A_{4,10}$ and $A^a_{4,11}$ are not representable in terms of vector fields of two coordinates.

3. Further symmetries

All normal forms of third order o.d.e.-s derived in the previous section except the linear o.d.e. $\dddot{q}= a(t)\ddot{q}$ contain an arbitrary constant as parameter but no arbitrary function of some argument. Therefore it is easy to determine the full symmetry algebra of these representatives.

We apply the symmetry criteria with the general generator $\xi(t,q)\partial_t + \eta(t,q)\partial_q$ and obtain a system of linear partial differential equations for ξ and η called the system of determining equations by comparing coefficients of functionally independent functions of $\dot{q}, \ldots, q^{(n-1)}$ [1]. The dimension of the solution space is equal to the number of independent symmetries [2]. The Lie algebra structure can be determined by the calculation of the commutators of independent symmetries.

The symmetry algebra L of $\dddot{q}= C\dot{q}^{-1}\ddot{q}^2$, which is one normal form implied by $2A_2$, is calculated in detail as an example.
L is already known for $C = 0$ where $L = \langle \partial_q, t\partial_q, t^2\partial_q, q\partial_q, t^2\partial_t + 2tq\partial_q, t\partial_t, \partial_t \rangle$.
If $C = \frac{3}{2}$ the determining system consists of $\xi_q = 0$, $\xi_{ttt} = 0$, $\eta_t = 0$ and $\eta_{qqq} = 0$. Obviously, $L = \langle t^2\partial_t, t\partial_t, \partial_t, q^2\partial_q, q\partial_q, \partial_q \rangle$.
If $C = 3$ the determining system consists of $\eta_t = 0$, $\xi_{tt} = 0$, $\eta_{qqq} = 0$, $\xi_{qqq} = 0$ and $\xi_{tq} = \eta_{qq}$. One can easily verify that $\xi = \alpha tq + \beta t + \gamma q^2 + \delta q + \varepsilon$ and $\eta = \alpha q^2 + \mu q + \nu$, i.e. $\dim L = 7$ and $\dddot{q}= C\dot{q}^{-1}\ddot{q}^2$ can be transformed into $\dddot{Q}(T) = 0$ by means of

$Q = t$ and $T = q$.
Provided that $C \neq 0, \frac{3}{2}, 3$ we obtain the determining system $\eta_t = 0$, $\xi_{tt} = 0$, $\eta_{qq} = 0$ and $\xi_q = 0$. Evidently $L = \langle \partial_t, t\partial_t, \partial_q, q\partial_q \rangle \cong 2A_2$.

It remains the case $\dddot{q} = a(t)\ddot{q}$. Linear o.d.e.-s of third order, that we can assume to be given in this special form, admit four, five or seven independent symmetries. For which functions $a(t)$ is the symmetry algebra of $\dddot{q} = a(t)\ddot{q}$ five dimensional?

Theorem 3.1. *If a linear third order o.d.e. admits at least five independent symmetries, it can be transformed into one of the following forms:* $\dddot{q} = C\ddot{q}$, $\dddot{q} = (C - 3t)(1+t^2)^{-1}\ddot{q}$ *or* $\dddot{q} = Ct^{-1}\ddot{q}$.

Proof. Any linear third order o.d.e. can be given as $\dddot{q} = a(t)\ddot{q}$. It follows from the defining equations that every symmetry of this o.d.e is of the form $b(t)\partial_t + (d(t) + c(t)q)\partial_q$. Furthermore, any o.d.e. $\dddot{q} = a(t)\ddot{q}$ has three commuting symmetries $\partial_q = G_1$, $t\partial_q = G_2$, $h(t)\partial_q = G_3$, where $h(t)$ is a non-linear solution of $h'''(t) = a(t)h''(t)$, and an induced fourth symmetry $q\partial_q$.

Let $G = b(t)\partial_t + (d(t) + c(t)q)\partial_q$ a fifth independent symmetry. The commutator $[q\partial_q, G] = c(t)q\partial_q - (d(t) + c(t)q)\partial_q = -d(t)\partial_q$ is an element of the symmetry algebra. Since the coefficient function $d(t)$ has to satisfy an o.d.e. of third order, namely $d'''(t) = a(t)d''(t)$, it is contained in the linear span of G_1, G_2 and G_3. So we can assume that a further symmetry G commutes with $q\partial_q$ and can be written as $b(t)\partial_t + c(t)q\partial_q$.

Similarly, $[\partial_q, G] = c(t)\partial_q$, $[t\partial_q, G] = (tc(t) - b(t))\partial_q$ and $[h(t)\partial_q, G] = (h(t)c(t) - b(t)h'(t))\partial_q$ are in the linear span of G_1, G_2 and G_3. Hence, $\langle G_1, G_2, G_3 \rangle = I$ is an ideal of the full symmetry algebra, and adG and $\mathrm{ad(q\partial_q)}$ are linear operators on I. Notice that $\mathrm{ad(q\partial_q)} = -\mathrm{Id}$ independently of the basis of I. For the dimension of I is odd, adG possesses a real eigenvalue λ.

We distinguish several cases. Coordinate transformations $Q = f(t)q$ and $T = g(t)$ that map another basis of $\langle G_1, G_2, G_3 \rangle$ of $3A_1$ to $\{\partial_q, t\partial_q, h(t)\partial_q\}$ and transform $q\partial_q$ and $b(t)\partial_t + c(t)q\partial_q$ into $Q\partial_Q$ and $B(T)\partial_T + C(T)Q\partial_Q$, respectively, will lead to the normal forms listed in the theorem.

1. If λ is a multiple eigenvalue whose eigenspace is of dimension one, the Jordan matrix with respect to a basis $\{G_1, G_2, G_3\}$ is $\begin{pmatrix} \lambda & 1 & * \\ 0 & \lambda & * \\ 0 & 0 & * \end{pmatrix}$. Transforming G_1, G_2, G_3 to ∂_q, $t\partial_q$, $h(t)\partial_q$, respectively, the commutator relations $[G_1, G] = \lambda G_1$ and $[G_2, G] = G_1 + \lambda G_2$ lead to $c(t) \equiv \lambda$ and $b(t) \equiv -1$. Invariance under ∂_t implies $\dddot{q} = C\ddot{q}$.

2. If λ is an multiple eigenvalue whose eigenspace is at least of dimension two, the Jordan matrix has the form $\begin{pmatrix} \lambda & 0 & * \\ 0 & \lambda & * \\ 0 & 0 & * \end{pmatrix}$ with respect to a basis $\{G_1, G_2, G_3\}$. Proceeding as above the commutator relations containing G_1, G_2, G lead to $c(t) \equiv \lambda$ and $b(t) \equiv 0$. This implies that G is a real multiple

of $q\partial_q$. So this case can not occur, if the symmetry algebra is at least five dimensional.

3. If adG possesses another real eigenvalue $\mu \neq \lambda$, the Jordan matrix is of the form $\begin{pmatrix} \lambda & 0 & * \\ 0 & \mu & * \\ 0 & 0 & * \end{pmatrix}$ with respect to a basis $\{G_1, G_2, G_3\}$. As above we derive $c(t) \equiv \lambda$ and $b(t) = (\lambda - \mu)t$. Invariance under $t\partial_t$ implies $\dddot{q} = Ct^{-1}\ddot{q}$.

4. If adG possesses a complex eigenvalue $\tau + i\sigma$, $\sigma \neq 0$, we are given an invariant subspace of I. Furthermore we can assume $\tau = 0$ and $\sigma = 1$. Then the Jordan matrix has the form $\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \lambda \end{pmatrix}$ with respect to a basis $\{G_1, G_2, G_3\}$. The symmetries G_1, G_2, $q\partial_q$, G generate the Lie algebra $A_{4,12}$. Transforming G_1, G_2, G_3 into ∂_q, $t\partial_q$, $h(t)\partial_q$, respectively, and using the results of Section 2. it becomes clear that $\dddot{q} = (C - 3t)(1 + t^2)^{-1}\ddot{q}$. From $a(t) = \frac{h'''(t)}{h''(t)}$ follows $h(t) = \iint e^{C\arctan t}(1 + t^2)^{-\frac{3}{2}} dt dt$.

■

The three normal forms obtained in theorem 3.1 have already been examined for further symmetries in the first part of this section because they are also representatives of third order o.d.e.-s with four symmetries. It turns out that all third order o.d.e.-s with seven independent symmetries can be transformed into $\dddot{q} = 0$.

Finally we show that any real Lie algebra L with $5 \leq \dim L \leq 7$ has a four dimensional subalgebra. This subalgebra can be transformed to one of the representatives derived in Section 2.. So we obtained a complete list of representatives of third order o.d.e.-s with 4, 5, 6 or 7 independent symmetries.

Theorem 3.2. *Any real Lie algebra L with $5 \leq \dim L \leq 7$ contains a four dimensional subalgebra.*

Proof. Consider the Levi-decomposition. If L is solvable, nothing remains to prove. Bear in mind that there are only three real simple Lie algebras of dimension less than 8; these are $sl(2, \mathbb{R})$, $so(3, \mathbb{R})$, $sl(2, \mathbb{C})_{\mathbb{R}}$. Reasoning with dimensions is sufficient except for a real five dimensional Lie algebra whose Levi-decomposition is non-trivial. In that case we invoke that the semi-simple part must be isomorphic to $sl(2, \mathbb{R})$ [11]. ■

4. Conclusions

The results of the Sections 2. and 3. are summarised in the tables 4, 5, 6.

Some interesting facts can be derived from the classification. Let L be the symmetry algebra of a third order o.d.e.

It holds $\dim L = 7$ if and only if the o.d.e. can be transformed into $\dddot{q} = 0$.

$\dddot{q} = C\dot{q}^{-1}\ddot{q}^2$ $(C \neq 0, \frac{3}{2}, 3)$	$L = \langle \partial_t, \partial_q, t\partial_t, q\partial_q \rangle$
$\dddot{q} = Ct^{-2}\dot{q}^{-\frac{1}{2}}(t\ddot{q} + \frac{1}{2}\dot{q})^{\frac{3}{2}} + 3\dot{q}^{-1}\ddot{q}^2$ $(C \neq 0)$	$L = \langle t\partial_t, \partial_q, q\partial_q, 2tq\partial_t + q^2\partial_q \rangle$
$\dddot{q} = Ce^{\ddot{q}}$ $(C \neq 0)$	$L = \langle \partial_t, \partial_q, t\partial_q, t\partial_t + (-\frac{1}{2}t^2 + 2q)\partial_q \rangle$
$\dddot{q} = C\ddot{q}^{\gamma}$ $(C \neq 0,\ \gamma \neq 0, 1)$	$L = \langle \partial_t, \partial_q, t\partial_q, t\partial_t + (1 + \frac{\gamma - 2}{\gamma - 1})q\partial_q \rangle$
$\dddot{q} = \frac{C + 3\dot{q}}{1 + \dot{q}^2}\ddot{q}^2$ $(C \neq 0)$	$L = \langle \partial_t, \partial_q, t\partial_t + q\partial_q, q\partial_t - t\partial_q \rangle$
$\dddot{q} = a(t)\ddot{q}$ $(a(t) \neq C, Ct^{-1}, \frac{C - 3t}{1 + t^2})$	$L = \langle \partial_q, t\partial_q, h(t)\partial_q, q\partial_q \rangle$ $h(t) = \iint e^{\int a(t)dt} dt dt$

Table 4: Normal forms of third order o.d.e.-s with four independent symmetries

$\dddot{q} = 0$	$L = \langle \partial_q, t\partial_q, t^2\partial_q, q\partial_q, t^2\partial_t + 2tq\partial_q, t\partial_t, \partial_t \rangle$
$\dddot{q} = \ddot{q}$	$L = \langle \partial_q, t\partial_q, e^t\partial_q, q\partial_q, \partial_t \rangle$
$\dddot{q} = t^{-1}\ddot{q}$	$L = \langle \partial_q, t\partial_q, t^3\partial_q, q\partial_q, t\partial_t \rangle$
$\dddot{q} = \frac{C - 3t}{1 + t^2}\ddot{q}$	$L = \langle \partial_q, t\partial_q, h(t)\partial_q, q\partial_q, (1 + t^2)\partial_t + tq\partial_q \rangle$ $h(t) = \iint (1 + t^2)^{-\frac{3}{2}} e^{C\mathrm{arctan}t} dt dt$

Table 5: Normal forms of third order o.d.e.-s with five or seven independent symmetries

$\dddot{q} = \frac{3}{2}\dot{q}^{-1}\ddot{q}^2$	$L = \langle \partial_q, q\partial_q, q^2\partial_q, \partial_t, t\partial_t, t^2\partial_t \rangle$
$\dddot{q} = 3\frac{\dot{q}}{1 + \dot{q}^2}\ddot{q}^2$	$L = \langle (t^2 - q^2)\partial_t + 2tq\partial_q, 2tq\partial_t + (q^2 - t^2)\partial_q,$ $t\partial_t + q\partial_q, q\partial_t - t\partial_q, \partial_t, \partial_q \rangle$

Table 6: Normal forms of third order o.d.e.-s with six independent symmetries

If $\dim L = 5$, then the o.d.e. is linearizable.
If $\dim L = 6$, then L is semi-simple.

In contrast to $(n \geq 4)$, where the existence of the Lie subalgebra $(n-1)A_1 \oplus_S A_1$ of the symmetry algebra implies linearizability of an o.d.e. of order n, there are nonlinearizable third order o.d.e.-s with symmetry algebra $2A_1 \oplus_S A_1$, e.g. $\dddot{q} = \ddot{q}^2$ that admits the symmetries ∂_t, ∂_q and $t\partial_t + q\partial_q$ which generate the Lie algebra $2A_1 \oplus_S A_1$.

It is worth mentioning that there are nonlinearizable third order o.d.e.-s with a six dimensional symmetry algebra. That means they have more symmetries than some linear third order o.d.e.-s.

In contrast to second order o.d.e.-s, there exist o.d.e.-s with m independent symmetries, where m can be any integer $(1 \leq m \leq 7)$.

References

[1] Berth, M., Invarianten von Differentialgleichungen und ihre Berechnung mit Mitteln der Computeralgebra, Diplomarbeit, Universität Greifswald, 1995.

[2] Czichowski, G. and M. Thiede, *Gröbner Bases, Standardforms of Differential Equations and Symmetry Computation*, Seminar Sophus Lie-Darmstadt Erlangen Greifswald Leipzig, **3** (1992), 223–233.

[3] Lie, S., „Vorlesungen über kontinuierliche Gruppen", Teubner, Leipzig, 1893.

[4] Mahomed, F. M. and P. G. L. Leach, *Normal forms for third order equations*, in: "Proceedings of the Workshop on Finite Dimensional Integrable Nonlinear Dynamical Systems," World Scientific Pub., 1988, 178–189.

[5] —, *Differential Equations*, Journal of Mathematical Analysis and Applications **151** (1990), 80–107.

[6] —, *Lie Algebras associated with scalar second-order ordinary differential equations*, J. Math. Phys. **30** (1989), 2770.

[7] Olver, P. J., "Applications of Lie Groups to Differential Equations," Graduate Texts in Mathematics **107**, Springer-Verlag, 1986.

[8] Patera, J. and P. Winternitz, *Subalgebras of real three and four dimensional Lie algebras*, J. Math. Phys. **18** (1977), 1449–1455.

[9] Regener, K., Symmetrien von gewöhnlichen Differentialgleichungen 2.Ordnung, Diplomarbeit, Universität Greifswald, 1987.

[10] Schmucker, A., Symmetrien und Normalformen von gewöhnlichen Differentialgleichungen höherer Ordnung, Diplomarbeit, Universität Greifswald, 1996.

[11] Turkowski, P., *Low-dimensional real Lie algebras*, J. Math. Phys. **29** (1988), 2139–2144.

Institut für Mathematik
und Informatik
Universität Greifswald
Jahnstraße 15
D-17487 Greifswald
czicho@rz.uni-greifswald.de

Received July 23, 1997