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Quantization of cohomology in semi-simple Lie algebras

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Abstract. Let $\mathfrak{g}$ be a complex, finite-dimensional Lie algebra, and $\mathbf{M}_o$ a contractable neighborhood of a complex homogeneous space on which $\mathfrak{g}$ acts transitively. The present article investigates quantization of $\mathrm{H}^1(\mathfrak{g}; \mathcal{C}^\omega(\mathbf{M}_o)/\mathbb{C})$ by the condition that the corresponding realization by first-order operators admits a finite-dimensional invariant subspace of functions. The quantization of cohomology phenomenon surfaced during the recent classification of finite-dimensional Lie algebras of first and zero order operators in two complex variables; this classification revealed that quantization holds for all two-dimensional homogeneous spaces.

The present article presents the first known counter-examples to quantization of cohomology; it is shown that quantization can fail even if $\mathfrak{g}$ is semi-simple, and even if the homogeneous space in question is compact.

A explanation for the quantization phenomenon is given in the case of semi-simple $\mathfrak{g}$. It is shown that the set of classes in H^1 that admit finite-dimensional invariant subspaces is a semigroup that lies inside a finitely-generated abelian group. In order for this abelian group be a discrete subset of H^1, i.e. in order for quantization to take place, some extra conditions on the isotropy subalgebra are required. Two different instances of such necessary conditions are presented.

1. Introduction

The present article deals with realizations of finite-dimensional Lie algebras by first-order differential operators on homogeneous spaces. Differential realizations are widely applied to the problems of quantum mechanics as well as being interesting from a purely mathematical point of view.

Realizations that admit a finite-dimensional invariant subspace of functions are particularly useful in the context of algebraically solvable spectral problems [8] [7] [3]. A prototypical example is the following realizations of $\mathfrak{sl}_2$

$$\partial_z, \quad 2z\partial_z - \lambda, \quad z^2\partial_z - \lambda z,$$

where λ is a scalar parameter. Note that if λ is a natural number then then $1, z, \ldots, z^\lambda$ span an invariant subspace. It can be shown [4] that finite-dimensional

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invariant subspaces do not exist for other values of the λ parameter. Thus the existence of a finite-dimensional invariant subspace seems to behave as a kind of a quantization condition.

More generally one may consider realizations of a finite dimensional Lie algebra, $\mathfrak{f}$, by first- and zero-order operators,

$$V(a) + \lambda_1\eta_1(a) + \ldots + \lambda_n\eta_n(a), \quad a \in \mathfrak{f}$$

where V is a vector field (possibly zero), the η's are functions, and the λ_i scalar parameters. Using Sophus Lie's classification of finite-dimensional Lie algebras of vector fields in two complex variables [5], Gonzalez-Lopez, Karman, and Olver were able to describe all such two-dimensional realizations [2]. Calculating case by case, these authors were able to confirm that finite-dimensional modules of functions exist only for certain discrete values of the λ parameters. There is a natural way, described below, to regard the tuple $(\lambda_1, \ldots, \lambda_n)$ as an element of a certain H^1, and so, the authors of [2] named this phenomenon quantization of cohomology and wondered whether or not it continues to hold for higher dimensions.

The purpose of the present article is twofold. First, it will be shown (Example 4.1) that quantization of cohomology does not hold in general. Indeed, there exist counter-examples even when the Lie algebra in question is semi-simple (Example 4.2), and even when the underlying homogeneous space is compact (Example 4.3). The second aim is to describe some necessary conditions that will assure that quantization of cohomology takes place. In regards to this second objective the present paper will only consider complex semi-simple Lie algebras. Indeed, it is possible to show that with this assumption the cohomology classes that correspond to finite-dimensional invariant subspaces are a subset of a finitely generated abelian subgroup of H^1 (Theorem 5.1.) The fact that this abelian subgroup is finitely generated does not guarantee that it is discrete; additional assumptions about the isotropy subalgebra are required for discreteness (Theorems 5.2 and 5.4).

The present approach is based on two ideas. The finite-dimensional irreducible representations of a complex, semi-simple Lie algebras correspond to dominant weights that are integral and positive; this is quantization of sorts. To make use of this observation, one has to relate the abstract representation theory to realizations by differential operators. This will be accomplished via the technique of induced representations, and through the use of Frobenius reciprocity in the context of finite-dimensional representations of Lie algebras.

The rest of the article is organized as follows. Section 2. describes and motivates the notion of quantization of cohomology. Section 3. recasts the problem in terms of induced representations. Section 4. describes several instances where quantization of cohomology fails. Finally, Section 5. investigates quantization of cohomology in the semi-simple case, and gives some necessary conditions on the isotropy algebra that imply quantization.

2. Quantization of cohomology

Let $\mathbf{M}_o$ be an open neighborhood of a complex manifold. A holomorphic first order differential operator defined on $\mathbf{M}_o$ has the form $V + \eta$, where the first term is a vector field, and where the second term is a multiplication operator corresponding to an $\eta \in \mathcal{C}^\omega(\mathbf{M}_\mathrm{o})$. Thus, a local realization of a finite dimensional Lie algebra, $\mathfrak{g}$, by holomorphic, first order differential operators is specified by two items: a realization of $\mathfrak{g}$ by vector fields $V(a), a \in \mathfrak{g}$, and a linear map $\eta : \mathfrak{g} \to \mathcal{C}^\omega(\mathbf{M}_\mathrm{o})$ with the property that resulting collection of operators $\{V(a) + \eta(a),\ a \in \mathfrak{g}\}$ is closed under the operator bracket. This is equivalent to the condition that

$$V(a)(\eta(b)) - V(b)(\eta(a)) = \eta([a,b]), \quad a,b \in \mathfrak{g}.$$

Equivalently, using the language of Lie algebra cohomology one can say that η must be a 1-cocycle of $\mathfrak{g}$ with coefficients in $\mathcal{C}^\omega(\mathbf{M}_\mathrm{o})$.

Given a realization of $\mathfrak{g}$ by differential operators one can obtain many other realizations by conjugating the given operators by an arbitrary multiplication operator e^σ, $\sigma \in \mathcal{C}^\omega(\mathbf{M}_\mathrm{o})$, i.e. by applying a gauge transformation. It therefore makes sense to consider differential realizations up to gauge equivalence. Note that for a first order operator $V + \eta$ and a function σ one has

$$\mathrm{e}^{-\sigma}(V+\eta)\mathrm{e}^{\sigma} = V + \eta + V(\sigma).$$

This means that the effect of a gauge transformation on a realization by first order operators is to add a coboundary term, $\delta\sigma$, to the 1-cocycle η. Thus, for a fixed realization by vector fields, and up to gauge equivalence, realizations of $\mathfrak{g}$ by first order differential operators are classified by $\mathrm{H}^1(\mathfrak{g}; \mathcal{C}^\omega(\mathbf{M}_\mathrm{o}))$.

More generally consider a realization of a finite-dimensional Lie-algebra, $\mathfrak{f}$, by holomorphic first and zero order differential operators. Such operators have the form $V + \eta$, where the vector field term is allowed to be zero. Let $W \subset \mathfrak{f}$ denote the abelian ideal of zero-order elements, i.e. those operators for which $V = 0$, and let $\mathfrak{g}$ denote the quotient $\mathfrak{f}/W$. The projection $V + \eta \mapsto V$ gives a realization of $\mathfrak{g}$ by vector fields. In this situation one can describe η as a linear map from $\mathfrak{g}$ to $\mathcal{C}^\omega(\mathbf{M}_\mathrm{o})/W$, and one can easily verify that closure under the operator bracket is equivalent to the condition that η be a 1-cocycle of $\mathfrak{g}$ with coefficients in $\mathcal{C}^\omega(\mathbf{M}_\mathrm{o})/W$. It is well known that central extensions of $\mathfrak{g}$ by W are classified by $\mathrm{H}^2(\mathfrak{g}; W)$. Let $\mathfrak{g} \ltimes_\eta W$ be a central extension whose class corresponds to the image of $[\eta]$ in the following long exact sequence of cohomology:

$$\ldots \to \mathrm{H}^1(\mathfrak{g}; \mathcal{C}^\omega(\mathbf{M}_\mathrm{o})) \to \mathrm{H}^1(\mathfrak{g}; \mathcal{C}^\omega(\mathbf{M}_\mathrm{o})/W) \to \mathrm{H}^2(\mathfrak{g}; W) \to \ldots \tag{1}$$

Indeed, it is not hard to verify that $\mathfrak{f}$ is isomorphic to such a central extension. Thus, a local realization of a finite dimensional Lie algebra by first and zero order differential operators requires 3 ingredients: a realization of a finite-diemnsional Lie algebra, $\mathfrak{g}$, by vector fields, a finite dimensional $\mathfrak{g}$-module, W, realized as an invariant subspace of $\mathcal{C}^\omega(\mathbf{M}_\mathrm{o})$, and a 1-cocycle, η, of $\mathfrak{g}$, with coefficients in $\mathcal{C}^\omega(\mathbf{M}_\mathrm{o})/W$. The realized Lie algebra will be isomorphic to $\mathfrak{g} \ltimes_\eta W$.

As before, a gauge transformation adds a coboundary term to η. Thus, for a fixed realization of $\mathfrak{g}$ by vector fields, and a fixed $\mathfrak{g}$-module $W \subset \mathcal{C}^\omega(\mathbf{M}_\mathrm{o})$, the set of all possible realizations by first and zero order operators of all possible central extensions of $\mathfrak{g}$ by W is specified up to gauge equivalence by $\mathrm{H}^1(\mathfrak{g}; \mathcal{C}^\omega(\mathbf{M}_\mathrm{o})/W)$.

Let $\Lambda \subset \mathrm{H}^1(\mathfrak{g}; \mathcal{C}^\omega(\mathbf{M}_\mathrm{o})/W)$ denote the set of all cohomology classes, η, such that the corresponding realization of $\mathfrak{g} \ltimes_\eta W$ admits a finite-dimensional invariant subspace of $\mathcal{C}^\omega(\mathbf{M}_\mathrm{o})$. There is an important observation that greatly simplifies the determination of Λ.

Proposition 2.1. (Lemma 2 of [2]) *If W is neither 0, nor the module of constants, then Λ is the empty set.*

The proof of this proposition is straightforward: repeated multiplication by a non-constant function generates an infinite-dimensional vector space of functions. Thus, without loss of generality, one need only consider extensions of $\mathfrak{g}$ by $\mathbb{C}$, the module of constants. The cohomology space of interest is therefore $\mathrm{H}^1(\mathfrak{g}; \mathcal{C}^\omega(\mathbf{M}_\mathrm{o})/\mathbb{C})$.

Definition 2.2. Let $V(a)$, $a \in \mathfrak{g}$ be a realization of $\mathfrak{g}$ by vector fields. If Λ, as defined above, is a discrete subset of $\mathrm{H}^1(\mathfrak{g}; \mathcal{C}^\omega(\mathbf{M}_\mathrm{o})/\mathbb{C})$, then quantization of cohomology will be said to hold for V.

It is not particularly worthwhile to study quantization of cohomology for the full class of vector field realizations. Indeed, the very definition of quantization of cohomology, as stated above, is problematic, because in general $\mathrm{H}^1(\mathfrak{g}; \mathcal{C}^\omega(\mathbf{M}_\mathrm{o})/\mathbb{C})$ is infinite-dimensional, and thus lacking a canonical topology. This means that there isn't even a clear notion of what it means for Λ to be discrete.

For this reason the rest of this paper will be based on the assumption that $\mathbf{M}_\mathrm{o}$ is an open subset of a homogeneous space, and that V is the realization of the corresponding Lie algebra by infinitesimal automorphisms. This assumption improves the situation tremendously. One can show that $\mathrm{H}^1(\mathfrak{g}; \mathcal{C}^\omega(\mathbf{M}_\mathrm{o})/\mathbb{C})$ is finite dimensional (this is a straightforward consequence of Proposition 3.3 below) and endow H^1 with the canonical vector-space topology. In this way the notion of discreteness becomes meaningful.

To this end let $\mathbf{G}$ be a complex Lie group, $\mathbf{H}$ a closed subgroup, and let $\mathfrak{g}$ and $\mathfrak{h}$ denote the corresponding Lie algebras. Let $\mathbf{M} = \mathbf{G}/\mathbf{H}$ and $\pi : \mathbf{G} \to \mathbf{M}$ denote, respectively, the homogeneous space of right cosets, and the canonical projection. For $a \in \mathfrak{g}$ let a^L and a^R denote, respectively, the corresponding left- and right- invariant vector fields on $\mathbf{G}$, and $\mathfrak{g}^\mathrm{L}$ and $\mathfrak{g}^\mathrm{R}$ the collections of all such. To avoid any possible confusion, it should be noted that $\mathfrak{g}^\mathrm{L}$ corresponds to *right* group actions, and $\mathfrak{g}^\mathrm{R}$ to *left* ones. Let $a^\pi = \pi_*(a^\mathrm{L})$, $a \in \mathfrak{g}$ denote the realization of $\mathfrak{g}$ by projected vector fields (i.e. by infinitesimal automorphisms). It will also be assumed that $\mathfrak{h}$ does not contain any ideals of $\mathfrak{g}$. This will ensure that $a \mapsto a^\pi$ is a faithful realization. Let $\mathrm{o} = \pi(e)$ denote the basepoint of $\mathbf{M}$. Henceforth let $\mathbf{M}_\mathrm{o}$ be a contractable neighborhood thereof.

One should also note it is essential that the notion of quantization of cohomology be formulated in terms of $\mathrm{H}^1(\mathfrak{g}; \mathcal{C}^\omega(\mathbf{M}_\mathrm{o})/\mathbb{C})$, rather than merely in terms of $\mathrm{H}^1(\mathfrak{g}; \mathcal{C}^\omega(\mathbf{M}_\mathrm{o}))$. Indeed, consider the following realizations of $\mathfrak{gl}(2,\mathbb{C})$:

$$x\partial_x + \lambda,\ x\partial_y,\ y\partial_x,\ y\partial_y + \lambda, \quad \lambda \in \mathbb{C}.$$

Taking $x = 1$, $y = 0$ as the basepoint, and using Proposition 3.3 one checks that H^1 with coefficients in $\mathcal{C}^\omega(\mathbf{M}_\mathrm{o})$ is one-dimensional, and that λ parameterizes the cohomology classes. Also note that for all values of λ the above operators do not raise the total degree of polynomials, and hence for all λ admit finite-dimensional invariant subspaces. Quantization of cohomology is not invalidated by this example precisely because the cocycle of the above realization is trivial when the cohomology coefficients are taken to be $\mathcal{C}^\omega(\mathbf{M}_\mathrm{o})/\mathbb{C}$ rather than merely $\mathcal{C}^\omega(\mathbf{M}_\mathrm{o})$.

A further simplification occurs when $\mathfrak{g}$ is semi-simple. In this case $\mathrm{H}^2(\mathfrak{g};\mathbb{C}) = 0$, and consequently all central-extensions by $\mathbb{C}$ are trivial. From (1) one has, $\mathrm{H}^1(\mathfrak{g};\mathcal{C}^\omega(\mathbf{M}_\mathrm{o})/\mathbb{C}) = \mathrm{H}^1(\mathfrak{g};\mathcal{C}^\omega(\mathbf{M}_\mathrm{o}))$, and hence all possible realizations of $\mathfrak{g} \ltimes \mathbb{C}$ by first and zero order operators are obtained by adjoining multiplication by constants to realizations of $\mathfrak{g}$ by strictly first-order operators. Therefore, in the semi-simple case it suffices to define Λ as the set of those $[\eta] \in \mathrm{H}^1(\mathfrak{g};\mathcal{C}^\omega(\mathbf{M}_\mathrm{o}))$ such that the collection of operators $a^\pi + \eta(a)$, $a \in \mathfrak{g}$ admits a finite-dimensional invariant subspace of functions, and to say that quantization of cohomology holds if this Λ is a discrete subset of $\mathrm{H}^1(\mathfrak{g};\mathcal{C}^\omega(\mathbf{M}_\mathrm{o}))$.

As mentioned in the introduction, the study of quantization of cohomology began when a classification of all finite-dimensional Lie algebras of first and zero order operators in two complex variables, and of the corresponding finite-dimensional modules of functions was obtained in [2]. A case-by-case examination of this classification revealed the following.

Theorem 2.3. *If $\mathbf{M}$ is a 2-dimensional complex homogeneous space, then quantization of cohomology holds for the corresponding realization by infinitesimal automorphisms.*

As was mentioned in the introduction, in order for quantization of cohomology to hold on higher dimensional homogeneous spaces, additional assumptions are required about the isotropy subalgebra. This issue will be taken up in Section 5..

3. Induced representations

Let $\mathfrak{h} \subset \mathfrak{g}$ be a finite-dimensional Lie algebra, subalgebra pair. Let $\mathbf{G}$ be the simply connected Lie group corresponding to $\mathfrak{g}$, and $\mathbf{G}_e \subset \mathbf{G}$ an open, connected subset that contains the identity element. Let U be a finite-dimensional $\mathfrak{h}$-module, and let $\mathcal{C}^\omega(\mathbf{G}_e, U)^\mathfrak{h}$ denote the vector space of holomorphic, U-valued, $\mathfrak{h}^\mathrm{R}$-equivariant functions on $\mathbf{G}_e$. In other words, $\phi \in \mathcal{C}^\omega(\mathbf{G}_e, U)^\mathfrak{h}$ if and only if

$$(a^\mathrm{R}(\phi))(g) = a \cdot (\phi(g)), \quad a \in \mathfrak{h},\ g \in \mathbf{G}.$$

Since left-invariant and right-invariant vector fields commute, actions by the former give $\mathcal{C}^\omega(\mathbf{G}_e, U)^\mathfrak{h}$ the structure of a $\mathfrak{g}$-module. With this structure, $\mathcal{C}^\omega(\mathbf{G}_e, U)^\mathfrak{h}$ should be thought of as a representation of $\mathfrak{g}$ induced from U. Note that if $\mathbf{G}_e$ is all of $\mathbf{G}$, and if the $\mathfrak{h}$-action on U comes from the action of a closed subgroup $\mathbf{H}$, then the above definition is equivalent to the usual definition of the induced representation as holomorphic sections of the homogeneous vector bundle $\mathbf{G} \times_\mathbf{H} U$ [1].

The obvious drawback of the present definition of induced representation is that it seems to depend on the choice of $\mathbf{G}_e$. However, the "finite-dimensional content" of the induced representation is independent of this choice. This is a consequence of the following version of Frobenius reciprocity. Let W be a finite-dimensional $\mathfrak{g}$-module, and note that W is also naturally a $\mathbf{G}$-module, because $\mathbf{G}$ is assumed to be simply connected.

Proposition 3.1. *The map from* $\mathrm{Hom}_{\mathfrak{g}}(W, \mathcal{C}^\omega(\mathbf{G}_e, U)^{\mathfrak{h}})$ *to* $\mathrm{Hom}_{\mathfrak{h}}(W, U)$ *given by*

$$\phi \mapsto \phi(-)(e), \quad \phi \in \mathrm{Hom}_{\mathfrak{g}}(W, \mathcal{C}^\omega(\mathbf{G}_e, U)^{\mathfrak{h}})$$

is a vector space isomorphism. For $\alpha \in \mathrm{Hom}_{\mathfrak{h}}(W, U)$*, the inverse image,* $\phi \in \mathrm{Hom}_{\mathfrak{g}}(W, \mathcal{C}^\omega(\mathbf{G}_e, U)^{\mathfrak{h}})$ *is given by*

$$\phi(w)(g) = \alpha(g \cdot w), \quad w \in W,\ g \in \mathbf{G}_e.$$

The above definition of induced representation is relevant to the present discussion because, as the following Proposition will show, the $\mathfrak{g}$-action on $\mathcal{C}^\omega(\mathbf{M}_\mathrm{o})$ coming from a realization of $\mathfrak{g}$ by first-order operators is really the same thing as a $\mathfrak{g}$-representation induced from a 1-dimensional $\mathfrak{h}$-module.

Note that the vector space of all 1-dimensional representations of $\mathfrak{h}$ is naturally isomorphic to $\mathrm{H}^1(\mathfrak{h};\mathbb{C})$. For a character, $\hat{\eta}$, of a 1-dimensional representation of $\mathfrak{h}$, let $\mathbb{C}_{\hat{\eta}}$ denote the corresponding 1-dimensional module. For $\hat{\eta} \in \mathrm{H}^1(\mathfrak{h},\mathbb{C})$ it will be convenient to denote the corresponding induced representation, $\mathcal{C}^\omega(\mathbf{G}_e, \mathbb{C}_{\hat{\eta}})^{\mathfrak{h}}$, simply as $\mathcal{C}^\omega(\mathbf{G}_e)^{\hat{\eta}}$; the latter is just the vector space of scalar-valued functions, ϕ, that satisfy $a^{\mathrm{R}}(\phi) = \hat{\eta}(a)\phi,\ a \in \mathfrak{h}$. Given an open, contractable $\mathbf{M}_\mathrm{o} \subset \mathbf{M}$, choose $\mathbf{G}_e \subset \mathbf{G}$ such that there exists a trivialization $\mathbf{G}_e = \mathbf{M}_\mathrm{o} \times \mathbf{H}_e$, where $\mathbf{H}_e$ is an open, contractable subset of $\mathbf{H}$. Choosing $\mathbf{G}_e$ in this way ensures that there will exist a nowhere vanishing $\phi \in \mathcal{C}^\omega(\mathbf{G}_e)^{\hat{\eta}}$. Choose one such ϕ and set

$$\eta(a) = a^{\mathrm{L}}(\phi)/\phi, \quad a \in \mathfrak{g}. \tag{2}$$

Proposition 3.2. *Equation* (2) *defines a 1-cocycle with coefficients in* $\mathcal{C}^\omega(\mathbf{M}_\mathrm{o})$ *whose cohomology class is independent of the choice of* ϕ*. Furthermore, letting* $\mathfrak{g}$ *act on* $\mathcal{C}^\omega(\mathbf{M}_\mathrm{o})$ *by operators* $a^\pi + \eta(a),\ a \in \mathfrak{g}$ *makes the map* $f \mapsto f\phi,\ f \in \mathcal{C}^\omega(\mathbf{M}_\mathrm{o})$ *from* $\mathcal{C}^\omega(\mathbf{M}_\mathrm{o})$ *to* $\mathcal{C}^\omega(\mathbf{G}_e)^{\hat{\eta}}$ *into a* $\mathfrak{g}$*-module isomoprhism .*

As a consequence of this Proposition there is well defined map $\hat{\eta} \mapsto [\eta]$ from $\mathrm{H}^1(\mathfrak{h};\mathbb{C}))$ to $\mathrm{H}^1(\mathfrak{g};\mathcal{C}^\omega(\mathbf{M}_\mathrm{o}))$; this map is easily seen to be linear and injective. The following proposition will assert that it is, in fact, an isomorphism. To exhibit the inverse let $\eta \in \mathrm{Z}^1(\mathfrak{g};\mathcal{C}^\omega(\mathbf{M}_\mathrm{o}))$ be given and set

$$\hat{\eta}(a) = \eta(a)(\mathrm{o}),\ a \in \mathfrak{h}. \tag{3}$$

It's not hard to check that $\hat{\eta}$ annihilates $[\mathfrak{h},\mathfrak{h}]$, and hence can be considered as an element of $\mathrm{H}^1(\mathfrak{h};\mathbb{C})$. Furthermore, if η is a coboundary, then $\hat{\eta} = 0$, and thus one obtains a well-defined linear map from $\mathrm{H}^1(\mathfrak{g};\mathcal{C}^\omega(\mathbf{M}_\mathrm{o}))$ to $\mathrm{H}^1(\mathfrak{h};\mathbb{C})$.

Proposition 3.3. *The linear map* $[\eta] \mapsto \hat{\eta}$*, as given in* (3)*, is an isomorphism of* $\mathrm{H}^1(\mathfrak{g};\mathcal{C}^\omega(\mathbf{M}_\mathrm{o}))$ *and* $\mathrm{H}^1(\mathfrak{h};\mathbb{C})$*. It is the inverse of the map from* $\mathrm{H}^1(\mathfrak{h};\mathbb{C})$ *to* $\mathrm{H}^1(\mathfrak{g};\mathcal{C}^\omega(\mathbf{M}_\mathrm{o}))$ *described by Proposition 3.2*

For more details, as well as an extension of this isomorphism to higher cohomology spaces the reader is referred to [6].

Recall that Λ is the set of all those classes $[\eta] \in \mathrm{H}^1(\mathfrak{g};\mathcal{C}^\omega(\mathbf{M}_\mathrm{o}))$ such that the corresponding realization of $\mathfrak{g}$ by operators $a^\pi + \eta(a), a \in \mathfrak{g}$ admits a finite dimensional invariant subspace of $\mathcal{C}^\omega(\mathbf{M}_\mathrm{o})$. One can use Proposition 3.1 (Frobenius reciprocity) to characterize Λ.

Before stating the theorem, two items of notation used in it have to be explained. First, for an $\mathfrak{h}$-module U, let $\Lambda_\mathfrak{h}(U) \subset \mathrm{H}^1(\mathfrak{h};\mathbb{C})$ denote the set of $\mathfrak{h}$-characters that correspond to 1-dimensional submodules of U. Second, for a $\mathfrak{g}$-module W, let W^* denote the dual vector space of linear forms, with the following $\mathfrak{g}$-action:

$$(a \cdot \alpha)(w) = \alpha(a \cdot w), \quad a \in \mathfrak{g},\ w \in W,\ \alpha \in W^*.$$

This gives W^* the structure of a $\mathfrak{g}$-antimodule, the infinitesimal analogue of a right module of a Lie group.

Theorem 3.4. *Identifying* $\mathrm{H}^1(\mathfrak{h};\mathbb{C})$ *and* $\mathrm{H}^1(\mathfrak{g};\mathcal{C}^\omega(\mathbf{M}_\mathrm{o}))$ *as per Proposition 3.3, one has*

$$\Lambda = \bigcup_W \Lambda_\mathfrak{h}(W^*),$$

where the union is taken over all finite-dimensional, irreducible $\mathfrak{g}$*-modules.*

Proof. Given an $\hat\eta \in \mathrm{H}^1(\mathfrak{h};\mathbb{C})$, choose a corresponding $\eta \in \mathrm{Z}^1(\mathfrak{g};\mathcal{C}^\omega(\mathbf{M}_\mathrm{o}))$ as per (2). Use operators $a^\pi + \eta(a), a \in \mathfrak{g}$ to give $\mathcal{C}^\omega(\mathbf{M}_\mathrm{o})$ the structure of a $\mathfrak{g}$-module, and identify it with $\mathcal{C}^\omega(\mathbf{G}_e)^{\hat\eta}$ as per Proposition 3.2. Let W be a finite-dimensional, irreducible $\mathfrak{g}$-module and note that a non-zero element of $\mathrm{Hom}_\mathfrak{g}(W, \mathcal{C}^\omega(\mathbf{M}_\mathrm{o}))$ is just a faithful realization of W by functions on $\mathbf{M}_\mathrm{o}$. By Frobenius reciprocity (Proposition 3.1), $\mathrm{Hom}_\mathfrak{g}(W, \mathcal{C}^\omega(\mathbf{M}_\mathrm{o}))$ is isomorphic to $\mathrm{Hom}_\mathfrak{h}(W, \mathbb{C}_{\hat\eta})$. But $\mathrm{Hom}_\mathfrak{h}(W, \mathbb{C}_{\hat\eta})$ is just the set of $\alpha \in W^*$ such that $\alpha(a \cdot w) = \hat\eta(a)\alpha(w)$ for all $a \in \mathfrak{h}$, $w \in W$, i.e. $\hat\eta$ is the character of a 1-dimensional $\mathfrak{h}$-submodule of W^* spanned by such an α.

To complete the proof, note that if $\mathcal{C}^\omega(\mathbf{M}_\mathrm{o})$ possesses a finite-dimensional $\mathfrak{g}$-submodule, then it must certainly possess one that is finite dimensional and irreducible. ∎

It is important to note that Theorem 3.4 deals with cohomology whose coefficients are $\mathcal{C}^\omega(\mathbf{M}_\mathrm{o})$ rather than $\mathcal{C}^\omega(\mathbf{M}_\mathrm{o})/\mathbb{C}$, and is not, therefore, directly applicable to the general question of quantization of cohomology. However, as remarked in the preceding section, if one assumes $\mathfrak{g}$ to be semisimple, then $\mathrm{H}^1(\mathfrak{g};\mathcal{C}^\omega(\mathbf{M}_\mathrm{o})/\mathbb{C}) = \mathrm{H}^1(\mathfrak{g};\mathcal{C}^\omega(\mathbf{M}_\mathrm{o}))$, and Theorem 3.4 becomes immediately relevant.

In order to illustrate Frobenius reciprocity and Theorem 3.4 it may be useful to consider the following example. The following three generators give a 2-dimensional realizations of $\mathfrak{sl}(2,\mathbb{C})$ by first order differential operators:

$$a_1 = \partial_x - y^2\partial_y,\ a_2 = 2x\partial_x - 2y\partial_y - \lambda,\ a_3 = x^2\partial_x - \partial_y - \lambda x, \quad \lambda \in \mathbb{C}. \tag{4}$$

Take $x = 0$, $y = 0$ to be the basepoint; it follows that the isotropy algebra, $\mathfrak{h}$, is generated by a_2. Using Proposition 3.3, one sees that H^1 is 1-dimensional, and that λ parameterizes the cohomology classes. Label the 1-dimensional characters of $\mathfrak{h}$ by their value on a_2. Let W_n, $n \in \mathbb{N}$ denote the $(n+1)$-dimensional irreducible $\mathfrak{sl}(2,\mathbb{C})$ module, and note that W_n, as an $\mathfrak{h}$-module, breaks up into $n+1$, 1-dimensional components, $\mathbb{C}_{-n} \oplus \mathbb{C}_{2-n} \oplus \ldots \oplus \mathbb{C}_{n-2} \oplus \mathbb{C}_n$. Therefore by Theorem 3.4, there exists a finite-dimensional invariant subspace of $\mathcal{C}^\omega(\mathbf{M}_\mathrm{o})$ if and only if λ is an integer. Furthermore, by Frobenius reciprocity, a copy of W_n occurs in $\mathcal{C}^\omega(\mathbf{M}_o)$ if and only if $\lambda \in \{-n, 2-n, \ldots, n-2, n\}$. Therefore, the direct sum of all finite-dimensional $\mathfrak{sl}_2$ submodules of $\mathcal{C}^\omega(\mathbf{M}_o)$ is

$$W_{|\lambda|} \oplus W_{|\lambda|+2} \oplus \ldots . \tag{5}$$

It isn't too difficult to determine explicitly what these finite-dimensional invariant subspaces are. If λ is an integer, then each natural number $k \geq -\lambda$ indexes a $\mathfrak{sl}_2$-module spanned by

$$x^i y^j (1-xy)^{-k}, \quad 0 \leq i \leq k+\lambda,\ 0 \leq j \leq k.$$

This module is isomorphic to $W_{k+\lambda} \otimes W_k$. The module with index k is included in the module with index $k+1$, and the totality of these modules form an infinite tower that is isomorphic to the direct sum given in (5).

4. Counter examples

Example 4.1. This example furnishes a 3-dimensional counter-example to quantization of cohomology. Consider the following realization of $\mathfrak{gl}(2,\mathbb{C})$:

$$\begin{aligned} a_1 &= \partial_x - y^2\partial_y - yz\partial_z + by, & a_2 &= x^2\partial_x - \partial_y - rxz\partial_z - ax, \\ a_3 &= 2\partial_x - 2y\partial_y - (r+1)z\partial_z + (b-a), & a_4 &= (r-1)z\partial_z + (a+b), \end{aligned}$$

where x, y, z are local, complex coordinates, $a, b \in \mathbb{C}$ are cocycle parameters, and $r \in \mathbb{C}$ is a realization parameter. Even though the above realization employs two cocycle parameters, H^1 is only 1-dimensional. Here is the reason. With $(0,0,1)$ as the basepoint, the isotropy subalgebra is seen to be spanned by $(r-1)a_3+(r+1)a_4$, and hence, by Proposition 3.3, H^1 is parameterized by $a+br$. In this respect, note that when $a+br=0$, the cocycle is the coboundary of $-b\log(z)$. Notice also that if both a and b are natural numbers, then there exists a finite dimensional module of functions, namely the span of $x^i y^j$, $0 \leq i \leq a$, $0 \leq j \leq b$. Now if r is a negative, irrational number, then $\{a+br : a, b \in \mathbb{N}\}$ is dense on the real part of H^1. Therefore quantization of cohomology fails to hold for such r.

The problem with the present example is that the isotropy subalgebra does not always generate a closed subgroup. Indeed, in terms of the customary representation of $\mathfrak{gl}(2,\mathbb{C})$ by two-by-two matrices, the isotropy generator is given by

$$\begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix}$$

This representation makes it easy to see that if r is irrational, then the corresponding subgroup is not closed. However, as the following example will show, quantization of cohomology can fail even when the isotropy subgroup is closed.

Example 4.2. This example will consider the homogeneous space, $\mathbf{G}/\mathbf{H}$, where $\mathbf{G} = \mathrm{SL}(5,\mathbb{C})$, and where $\mathbf{H}$ and the corresponding Lie algebra generator are shown below:

$$\begin{pmatrix} e^{z} & 0 & 0 & 0 & 0 \\ 0 & e^{rz} & 0 & 0 & 0 \\ 0 & 0 & e^{-(1+r)z} & 0 & 0 \\ 0 & 0 & 0 & 1 & z \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, z \in \mathbb{C} \qquad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & r & 0 & 0 & 0 \\ 0 & 0 & -1-r & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{6}$$

It is evident that $\mathbf{H}$ is a closed, 1-dimensional subgroup of $\mathbf{G}$. As in the preceding example, r is a parameter, and quantization of cohomology will fail if r is an irrational number. It would be too cumbersome to explicitly write down the infinitesimal generators on the homogeneous space (24 generators in 23 variables); it will be best to proceed abstractly.

Again, label the 1-dimensional characters of the isotropy subalgebra according to their value on the generator shown in (6). Consider $\mathbb{C}^5$ with the obvious structure of an $\mathfrak{sl}(5,\mathbb{C})$ module. Restriction of $\mathbb{C}^5$ to the isotropy subalgebra yields 1-dimensional characters of $\mathfrak{h}$ that are labeled by 0, 1, r, and $-1-r$. Hence, the various tensor powers of $\mathbb{C}^5$ will yield characters that are labeled by $a+br-c(1+r)$, where $a,b,c \in \mathbb{N}$. If r is a real, irrational number, then the set of all such characters is clearly dense on the real part of $\mathrm{H}^1(\mathfrak{h};\mathbb{C})$. Therefore, by Theorem 3.4 quantization of cohomology fails when r is irrational.

Example 4.3. The purpose of this example is to show that quantization of cohomology can fail even for a compact homogeneous space. Compact, complex homogeneous spaces were analyzed by H. C. Wang in [9]; the present example uses some tools from that article. Once again take $\mathbf{G} = \mathrm{SL}(5,\mathbb{C})$. For the isotropy subalgebra, $\mathfrak{h}$, take all upper-triangular, nilpotent matrices, as well as the following two generators:

$$a_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & r & 0 & 0 \\ 0 & 0 & 0 & \mathrm{i}-r & 0 \\ 0 & 0 & 0 & 0 & -\mathrm{i} \end{pmatrix}, \qquad a_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{i} & 0 \\ 0 & 0 & 0 & 0 & -\mathrm{i} \end{pmatrix}$$

By Proposition (7.3) of Wang's article, $\mathfrak{h}$ generates a closed subgroup of $\mathbf{G}$. In Proposition (3.1) Wang also shows that $\mathbf{G}/\mathbf{H}$ is compact whenever

$$\dim \mathbf{G} - \dim K(\mathbf{G}) = \dim \mathbf{H} - \dim K(\mathbf{H}),$$

where $K(X)$ denotes a maximal compact connected subgroup of X, and dim denotes the real dimension. For the present case $K(\mathbf{G}) = \mathrm{SU}(5)$ and $\mathbf{H}$ has no non-trivial compact subgroups. Consequently both sides are the above relation evaluate to 24, and therefore $\mathbf{G}/\mathbf{H}$ is compact.

Turning the question of quantization of cohomology, let $\alpha_i \in \mathfrak{h}^*$, $i = 1,2$ denote the linear form that annihilates the nilpotent matrices, and satisfies $\alpha_i(a_j) = \delta_{ij}$. Clearly α_1 and α_2 are linearly independent, and span $\mathrm{H}^1(\mathfrak{h};\mathbb{C})$. Let $e_1, \dots e_5$ denote the canonical column vector basis of $\mathbb{C}^5$. Note that e_1 and $e_1 \wedge e_2 \wedge e_3$ are both eigenvectors of $\mathfrak{h}$ and that α_1, and $r\alpha_1$ are the respective characters. Hence, both α_1 and $r\alpha_1$ are elements of Λ, and therefore when r is a negative irrational number, Λ is not a discrete subset of $\mathrm{H}^1(\mathfrak{h};\mathbb{C})$.

5. The semisimple case

In the present section $\mathfrak{g}$ is assumed to be semisimple, and $\mathbf{G}$ will denote the simply connected Lie group corresponding to $\mathfrak{g}$. As before, $\mathfrak{h}$ is a subalgebra that generates a closed subgroup, $\mathbf{H}$, of $\mathbf{G}$, and $\Lambda \subset \mathrm{H}^1(\mathfrak{g}; \mathcal{C}^\omega(\mathbf{M}_o)/\mathbb{C})$ denotes the set of cohomology classes that admit a finite-dimensional invariant subspace of functions. Using the fact that $\mathrm{H}^1(\mathfrak{g}; \mathcal{C}^\omega(\mathbf{M}_o)/\mathbb{C}) = \mathrm{H}^1(\mathfrak{g}; \mathcal{C}^\omega(\mathbf{M}_o))$ as well as Theorem 3.4, one can identify Λ with the subset of $\mathrm{H}^1(\mathfrak{h}; \mathbb{C})$ consisting of 1-dimensional $\mathfrak{h}$-characters obtained by restriction from finite-dimensional $\mathfrak{g}$-modules. Note that if W_1, W_2 are two $\mathfrak{g}$-modules and $w_1 \in W_1$, $w_2 \in W_2$ are two eigenvectors of $\mathfrak{h}$ with weights $\lambda_1, \lambda_2 \in \mathrm{H}^1(\mathfrak{h}; \mathbb{C})$, then $w_1 \otimes w_2 \in W_1 \otimes W_2$ is also an eigenvector of $\mathfrak{h}$ with weight $\lambda_1 + \lambda_2$. In this way the addition operation on $\mathrm{H}^1(\mathfrak{h}; \mathbb{C})$ endows Λ with the structure of a semigroup. Also of interest is the abelian subgroup of $\mathrm{H}^1(\mathfrak{h}; \mathbb{C})$ additively generated by Λ; it will be denoted by $\Lambda_{\mathfrak{h}}$. The relevance of $\Lambda_{\mathfrak{h}}$ stems from the following fact: if $\Lambda_{\mathfrak{h}}$ is a discrete subset of $\mathrm{H}^1(\mathfrak{h}; \mathbb{C})$, then the same is true of Λ.

Theorem 5.1. $\Lambda_{\mathfrak{h}}$ *is finitely generated.*

Proof. Let $\mathfrak{s}$ denote the radical of $\mathfrak{h}$. Choose a Borel (maximal solvable) subalgebra $\mathfrak{b} \subset \mathfrak{g}$ that contains $\mathfrak{s}$. The choice of $\mathfrak{b}$ singles out a Cartan subalgebra $\mathfrak{c} \subset \mathfrak{b}$, and a set of positive roots, $R^+ \subset \mathfrak{c}^*$, such that $\mathfrak{b} = \mathfrak{c} \oplus \mathfrak{n}$, where $\mathfrak{n}$ is the nilpotent Lie algebra spanned by root vectors corresponding to R^+. Let $\Lambda_{\mathfrak{c}} \subset \mathfrak{c}^*$ denote the weight lattice corresponding to $\mathfrak{c}$, and $\Lambda_{\mathfrak{s}} \subset \mathfrak{s}^*$ the abelian algebra generated by 1-dimensional $\mathfrak{s}$-characters obtained by restriction from finite-dimensional $\mathfrak{g}$-modules. Let $\pi_{\mathfrak{bc}} : \mathfrak{b} \to \mathfrak{c}$ denote the projection induced by the decomposition $\mathfrak{b} = \mathfrak{c} \oplus \mathfrak{n}$, and $\iota_{\mathfrak{sb}} : \mathfrak{s} \to \mathfrak{b}$ the inclusion injection.

The first claim is that $\Lambda_{\mathfrak{s}}$ is contained in $\iota^*_{\mathfrak{sb}}(\pi^*_{\mathfrak{bc}}(\Lambda_{\mathfrak{c}}))$. To that end let W be a finite-dimensional $\mathfrak{g}$-module, and $w \in W$ an eigenvector of $\mathfrak{s}$ with weight $\mu \in \Lambda_{\mathfrak{s}}$. Let $\Lambda_{\mathfrak{c}}(W) \subset \Lambda_{\mathfrak{c}}$ denote the set of $\mathfrak{c}$-weights that occur in the decomposition of W, and write $w = \sum_\lambda w_\lambda$, where the index runs over $\Lambda_{\mathfrak{c}}(W)$, and where each w_λ is an eigenvector of $\mathfrak{c}$ with weight λ. Choose a $\lambda_0 \in \Lambda_{\mathfrak{c}}(W)$ such that $w_{\lambda_0} \neq 0$ and such that it is impossible to write λ_0 as $\lambda' + \alpha$, $\lambda' \in \Lambda_{\mathfrak{c}}(W)$, $\alpha \in R^+$. Let $a \in \mathfrak{b}$ be given and write $a(w) = \sum_\lambda a(w)_\lambda$. From the way λ_0 was chosen one must have $a(w)_{\lambda_0} = \lambda_0(\pi_{\mathfrak{bc}}(a))w_{\lambda_0}$, and consequently $\mu = \iota^*_{\mathfrak{sb}}(\pi^*_{\mathfrak{bc}}(\lambda_0))$. This proves the claim.

Since $\Lambda_{\mathfrak{c}}$ is finitely generated, the above claim implies that $\Lambda_{\mathfrak{s}}$ is finitely generated too. Now $\mathrm{H}^1(\mathfrak{h}; \mathbb{C})$ is the subset of $\mathfrak{h}^*$ that annihilates all commutators of $\mathfrak{h}$. Consequently, one can identify $\mathrm{H}^1(\mathfrak{h}; \mathbb{C})$ with the subspace of those elements of $\mathfrak{s}^*$ that annihilate $\mathfrak{s} \cap [\mathfrak{h}, \mathfrak{h}]$. In this way one can regard $\mathrm{H}^1(\mathfrak{h}; \mathbb{C})$ as a subspace of $\mathrm{H}^1(\mathfrak{s}; \mathbb{C})$. It immediately follows that $\Lambda_{\mathfrak{h}} \subset \Lambda_{\mathfrak{s}}$, and therefore $\Lambda_{\mathfrak{h}}$ is finitely generated as well. ■

Note well that Λ is in general smaller than $\Lambda_{\mathfrak{h}}$. This is true, for instance, in the case of the 1-dimensional realization of $\mathfrak{sl}_2$ shown in the Introduction; there Λ corresponds to all natural number values of the parameter λ, where as $\Lambda_{\mathfrak{h}}$ corresponds to all integer values. Even in those instances, such as the example at the end of Section 3. , when $\Lambda = \Lambda_{\mathfrak{h}}$, the semigroup structure requires more generators than the group structure: at least two generators are required to generate $\mathbb{Z}$ as a semi-group, whereas one generator suffices to generate it as a group. This example, as well as others known to the author, make the following conjecture seem plausible.

Conjecture. Λ is finitely generated as a semigroup.

The proof of Theorem 5.1 relied critically on the fact that a subgroup of a free, finitely generated, abelian group is itself free and finitely generated. The analogous statement is not true for semigroups. Consequently, a proof of the above conjecture will likely require a classification of primitive elements of Λ, i.e. the elements that cannot be obtained as a sum of two others in Λ. As a starting point it seems reasonable to search for a proof of the conjecture for the case where $\mathfrak{h}$ is solvable.

The fact that $\Lambda_{\mathfrak{h}}$ is finitely generated is at the heart of the quantization of cohomology phenomenon. When the generators of $\Lambda_{\mathfrak{h}}$ are linearly independent (as complex vectors), then quantization of cohomology will clearly hold. However, the presence of a linear relation among the generators does not automatically destroy the quantization; in order for that to happen the relation must be based on irrational, real coefficients. It is upon this principle that the counter-examples of Section 4. are constructed. Thus, necessary conditions for quantization of cohomology, are really just various conditions on the isotropy subalgebra that guarantee the absence of irrational relations. Two instances of such necessary conditions are given below. The premise of Theorem 5.2 ensures that the generators of $\Lambda_{\mathfrak{h}}$ are linearly independent, while the premise of Theorem 5.4 produces a $\Lambda_{\mathfrak{h}}$ with exactly two linearly dependent generators, but where the linear dependence is based on a coefficient with non-zero imaginary part.

Recall that an element of a Cartan subalgebra is called rational if the value of all roots on that element is a rational number. A subspace of a Cartan subalgebra will be called rational if it is spanned by rational elements.

Theorem 5.2. *If a Cartan subalgebra of $\mathfrak{h}$ is a rational subspace of a Cartan subalgebra of $\mathfrak{g}$, then quantization of cohomology holds.*

Proof. Choose Cartan subalgebras $\mathfrak{j}$ and $\mathfrak{c}$ of, respectively, $\mathfrak{h}$ and $\mathfrak{g}$, such that $\mathfrak{j}$ is a rational subalgebra of $\mathfrak{c}$. Let l and m denote, respectively, the dimensions of $\mathfrak{c}$ and $\mathfrak{j}$. Since $\mathfrak{j}$ is a rational subspace of $\mathfrak{c}$, one can always find a basis $a_1, \dots a_l$ of $\mathfrak{c}$ such that the first m elements span $\mathfrak{j}$, and such that the dual basis of $\mathfrak{c}^*$, call it $\lambda_1, \dots, \lambda_l$, additively generates the weight lattice of $\mathfrak{c}$. Let $\Lambda_{\mathfrak{j}}$ denote the abelian group additively generated by $\lambda_1, \dots, \lambda_m$. Since the λ_i are linearly independent, $\Lambda_{\mathfrak{j}}$ is a discrete subset of $\mathfrak{j}^*$.

Let $\mathfrak{h} = \mathfrak{j} \oplus \mathfrak{h}_1$ be the Fitting decompositions of $\mathfrak{h}$ relative to $\mathfrak{j}$. Since $\mathfrak{g}$ is semisimple, $\mathrm{ad}_{\mathfrak{g}}(a)$ is diagonalizable for all $a \in \mathfrak{c}$, and hence the same is true of $\mathrm{ad}_{\mathfrak{h}}(a)$ for all $a \in \mathfrak{j}$. Hence, $[\mathfrak{j}, \mathfrak{h}_1] = \mathfrak{h}_1$, and a fortiori, $\mathfrak{h} = \mathfrak{j} + [\mathfrak{h}, \mathfrak{h}]$. Hence, one can identify $\mathrm{H}^1(\mathfrak{h}; \mathbb{C})$ with the subspace of those elements of $\mathfrak{j}^*$ that annihilate $[\mathfrak{h}, \mathfrak{h}]$. With this identification one must have $\Lambda \subset \Lambda_{\mathfrak{j}}$, and since the latter is discrete, so is the former. ∎

The setting for the next theorem about quantization of cohomology will be a compact homogeneous space. First it will be necessary to recall Wang's [9] necessary and sufficient conditions on the isotropy subalgebra $\mathfrak{h}$ in order that $\mathbf{G}/\mathbf{H}$ be compact.

Theorem 5.3. (Wang's Criterion) *Suppose that* $\mathbf{H}$ *is connected. Then,* $\mathbf{G}/\mathbf{H}$ *is compact if and only if* $\mathfrak{h}$ *satisfies the following conditions.*

(a). There exists a Cartan subalgebra, $\mathfrak{c} \subset \mathfrak{g}$*, and an ordering of the* $\mathfrak{c}$*-root vectors such that* $\mathfrak{h}$ *is spanned by* $\mathfrak{j} = \mathfrak{h} \cap \mathfrak{c}$*, all the positive root vectors, and some of the negative root vectors.*

(b). The complex $\mathbf{G}$*-subgroup generated by* $\mathfrak{j}$ *contains a real subgroup isomorphic to* $\mathbb{R}^l$*, where* l *is the rank of* $\mathfrak{g}$*.*

Note that if $\mathfrak{j} = \mathfrak{c}$, then condition (a) implies that $\mathfrak{h}$ is a parabolic subalgebra of $\mathfrak{g}$. To understand the general case, note that the abelian $\mathbf{G}$-subgroup generated by $\mathfrak{c}$ is isomorphic (as a real group) to $\mathbb{R}^l \times \mathbb{T}^l$. Thus, the gist of Wang's analysis is that in order to obtain a compact homogeneous space, one must start with a parabolic subalgebra, and form the isotropy subalgebra by discarding an even number of circle factors from the Cartan subgroup. More precisely, Wang shows that the abelian $\mathbf{G}$-subgroup generated by $\mathfrak{j}$ is isomorphic to $\mathbb{R}^l \times \mathbb{T}^v$, where $l - v = 2p$ is the number of discarded circle factors.

Theorem 5.4. *Let* $\mathbf{H}$ *denote the connected subgroup of* $\mathbf{G}$ *generated by* $\mathfrak{h}$*, and suppose that* $\mathbf{G}/\mathbf{H}$ *is compact. Let* l *and* m *denote the rank of* $\mathfrak{g}$*, and* $\mathfrak{h}$*, respectively. If* $l - m \leq 1$*, then quantization of cohomology holds.*

The proof will require the following Lemma based on Wang's criterion. Let $\mathfrak{c}$ and $\mathfrak{j}$ be as in Theorem 5.3.

Lemma 5.5. *There exists a basis* $a_1, \ldots, a_l$ *of* $\mathfrak{c}$ *and a basis* $b_1, \ldots, b_{p+v}$ *of* $\mathfrak{j}$ *such that elements of the dual basis of* $\mathfrak{c}^*$ *additively generate the weight lattice of* $\mathfrak{c}$*; such that*

$$b_i = \begin{cases} a_i + \sum_{j=p+1}^{2p} k_{ij} a_j, & i \leq p \\ a_{p+i}, & i > p, \end{cases} \tag{7}$$

where $k_{ij} \in \mathbb{C}$*; and such that for every* i *there exists at least one* j *such that* k_{ij} *is not real.*

Proof. Let T denote the maximal torus contained in the subgroup generated by $\mathfrak{j}$; recall that $v = l - 2p$ is its dimension. One can always choose $a_1, \ldots, a_l$ such that the dual basis generates the weight lattice and such that $\mathrm{i}a_{2p+1}, \ldots, \mathrm{i}a_l$ are the real infinitesimal generators of T. Since the complex dimension of $\mathfrak{j}$ is $(l+v)/2 = l - p$, one requires p additional elements to obtain a basis of $\mathfrak{j}$. Choose $\hat{b}_1, \ldots, \hat{b}_p$ in the span of $a_1, \ldots a_{2p}$ such that the $\hat{b}_i$'s together with $a_{2p+1}, \ldots, a_l$ are a basis of $\mathfrak{j}$. Write $\hat{b}_i = \sum_j \hat{k}_{ij} a_j$ where i ranges from 1 to p and j ranges from 1 to $2p$. Row reduce the matrix $\{\hat{k}_{ij}\}$ to obtain the row canonical form, call it $\{k_{ij}\}$, and set $b_i = \sum_j k_{ij} a_j$. Exchanging some of the elements of the list

$a_1, \dots a_{2p}$, if necessary, one can without loss of generality assume that the b_i's have the form shown in (7).

Next, suppose that one of the rows, say row i, of $\{k_{ij}\}$ consists entirely of real numbers. All the elements of the row must be rational numbers, otherwise b_i would not generate a closed subgroup, and consequently neither would $\mathfrak{j}$ nor $\mathfrak{h}$. On the other hand if the row consisted of rational numbers, then $\mathrm{i}b_i$ would generate (in the real sense) a torus, and thereby violate the maximality assumption about T. Therefore a contradiction is obtained, and the lemma is proved. ∎

Proof of Theorem 5.4. Let $\mathfrak{c}$, $\mathfrak{j}$, and their respective bases, $a_1, \dots, a_l$ and $b_1, \dots, b_m$ be as in Theorem 5.3 and Lemma 5.5. If $l = m$ then quantization of cohomology holds by Theorem 5.2. Suppose then that $l = m+1$. This implies that $p = 1$, $b_1 = a_1 + ka_2$, where $k \in \mathbb{C}$ is not real, and $b_i = a_{i+1}$, $i = 2, \dots m$. Recall that the a_i's were chosen so that the dual basis of $\mathfrak{c}^*$, call it $\lambda_1, \dots, \lambda_l$, additively generates the weight lattice of $\mathfrak{c}$. Let $\Lambda_{\mathfrak{j}} \subset \mathfrak{j}^*$ denote the abelian Lie algebra generated by 1-dimensional $\mathfrak{j}$-characters obtained by restriction from finite-dimensional $\mathfrak{g}$-modules. Evidently, $\Lambda_{\mathfrak{j}}$ is generated by $\lambda_1, \lambda_2, \dots, \lambda_l$. The restrictions of $\lambda_3, \dots, \lambda_l$ to $\mathfrak{j}$ are linearly independent, while $(k\lambda_1 - \lambda_2)|_{\mathfrak{j}} = 0$. However, since k is not real, $\{(n_1\lambda_1 + n_2\lambda_2)|_{\mathfrak{j}} : n_1, n_2 \in \mathbb{Z}\}$ is a discrete subset of the 1-dimensional complex vector space spanned by $\lambda_1|_{\mathfrak{j}}$, and therefore $\Lambda_{\mathfrak{j}}$ is a discrete subset of $\mathfrak{j}^*$. From property (a) of Theorem 5.3 one sees that $\mathfrak{h} = \mathfrak{j} + [\mathfrak{h}, \mathfrak{h}]$, and consequently $\mathrm{H}^1(\mathfrak{h}; \mathbb{C})$ can be identified with the vector space of those elements of $\mathfrak{j}^*$ that annihilate $[\mathfrak{h}, \mathfrak{h}]$. With this identification one must have $\Lambda \subset \Lambda_{\mathfrak{j}}$, and since the latter is discrete, so is the former. ∎

Example 4.3 shows that the preceding theorem is sharp in the sense that compactness of $\mathbf{G}/\mathbf{H}$ alone is not sufficient to guarantee quantization of cohomology. Indeed, in that example the Cartan subalgebra of $\mathfrak{h}$ was a codimension 2 subspace of a Cartan subalgebra of $\mathfrak{g}$.

References

[1] Bott, R., *Homogeneous vector bundles*, Ann. Math. **66** (1957), 203–247.

[2] Gonzalez-Lopez, A., N. Kamran, and P. J. Olver, *Quasi-exactly solvable lie algebras of differential operators in two complex variables*, J. Phys. A **24** (1991), 3995–4008.

[3] —, *New quasi-exactly solvable hamiltonians in two dimensions*, Commun. Math. Phys. **159** (1994), 503–537.

[4] Kamran N., and P. J. Olver, *Lie algebras of differential operators and lie-algebraic potentials*, J. Math. Anal. Appl. **145** (1990), 342–356.

[5] Lie, S., „Theorie der Transformationgruppen,“ Band 3, B. G. Teubner, Leipzig, 1893.

[6] Milson, R., *Representations of finite-dimensional Lie algebras by first-order differential operators. Some local results in the transitive case*, J. London Math. Soc. **52** (1995), 285–302.

[7] Shifman, M. A., and A. V. Turbiner, *Quantal problems with partial algebraization of the spectrum*, Commun. Math. Phys **126** (1989), 347–365.

[8] Turbiner, A. V., *Quasi-exactly-solvable problems and* sl(2) *algebra*, Comm. Math. Phys. **118** (1988), 467–474.

[9] Wang, H. C., *Closed manifolds with homogeneous complex structure*, Amer. J. Math., **76** (1954), 1–32.

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