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Order and Domains of Attraction of Control Sets in Flag Manifolds

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Abstract. Let G be a real semi-simple noncompact Lie group and $S \subset G$ a subsemigroup with $\mathrm{int}\, S \neq \emptyset$. This article relates the Bruhat-Chevalley order in the Weyl group W of G to the ordering of the control sets for S in the flag manifolds of G by showing that the one-to-one correspondence between the control sets and the elements of a double coset $W(S)\backslash W/W_\Theta$ of W reverses the orders. This fact is used to show that the domain of attraction of a control set is a union of Schubert cells.

Key words: semigroups, semi-simple groups, flag manifolds, control sets, Bruhat-Chevalley order.
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1. Introduction

For the action of a semigroup of maps in some state space the transitivity structure is described by the control sets, which are lasting regions of the evolution of the semigroup, and by the transience between the control sets. In this article we consider a noncompact semi-simple Lie group G and look at the transitivity properties of the action of a semigroup $S \subset G$ in the flag manifolds of G. A basic assumption is that the semigroup has nonvoid interior in G. The control sets for these actions were studied in [9], which provides us with the following picture: Let W be the Weyl group of G. Then there is a parabolic subgroup $W(S)$ of W attached to S such that the control sets for the S-action in the maximal flag manifold B of G are in one-to-one correspondence with the cosets in $W(S)\backslash W$. This fact extends to any other flag manifold B_Θ establishing a bijection between the control sets in B_Θ and the double cosets in $W(S)\backslash W/W_\Theta$ where W_Θ is the parabolic subgroup of W associated with B_Θ.

These results bring to the study of the semigroups in G, and their control sets in the flag manifolds, the combinatorics of the Weyl group. The purpose of

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this paper is to relate the Weyl group also to the transience behavior between the control sets. This is made as follows: There is a natural partial order $\leq$ between the control sets, which is defined by putting $D_1 \leq D_2$ if it is possible to steer the points of D_1 into D_2. Clearly, this order describes the transience between the control sets. On the other hand, there is the well known Bruhat-Chevalley partial order in the Weyl group W. These two orders are related by the above bijections. In fact, in Theorem 4.1 below we show that the order between the control sets in the maximal flag manifold is obtained by reversing the Bruhat-Chevalley order in W and projecting onto $W(S)\backslash W$ so that the bijection is order reversing. This fact is extended to the other flag manifolds in Proposition 7.1 where it is proved that the control sets in a flag manifold B_Θ are ordered according to the reverse of the order induced by the Weyl-Chevalley order on the double coset $W(S)\backslash W/W_\Theta$.

Intimately related to the order of the control sets is the concept of domain of attraction of a control set D, which is defined to be the subset of points in the state space that can be steered by the semigroup into D. It is not hard to prove that $D_1 \leq D_2$ if and only if D_1 is contained in the domain of attraction of D_2 (see Proposition 2.1 below). In the Weyl group side this fact parallels the classical Chevalley-Borel-Tits Theorem which characterizes the Schubert cells from the Bruhat-Chevalley order and the Bruhat cells. In fact, we use this theorem in Section 6. to show that the domain of attraction of a control set in the maximal flag manifold is a union of Schubert cells. This is achieved in two steps. First we construct a Schubert cell by starting from a point in B and exhausting successively subsets with fibers of projections from B onto smaller flag manifolds. This construction resembles the Bott-Samelson [1] construction of the desingularization of a Schubert cell. Once we have this characterization of a Schubert cell we apply it to show that the same exhausting procedure, starting now from the minimal control set, yields the domain of attraction of a control set. In this procedure the choice of the smaller flag manifolds is determined by the element of the Weyl group associated with the control set.

The results proved in this paper show that the order of the control sets reduces to the order of the Weyl group. The combinatorics of the Bruhat-Chevalley ordering is extensively studied in the literature. We refer to [4], [5], [6], and references therein, for a description of this order in the different Weyl groups. This description provides the order of the control sets for the semigroups.

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2. Control sets

Let G be a Lie group, $S \subset G$ a semigroup with $\operatorname{int} S \neq \emptyset$ and G/H a compact homogeneous space with $H \subset G$ a closed subgroup. Recall that a subset $D \subset G/H$ is said to be a *control set* for the S-action in G/H provided it satisfies

1. $D \subset \operatorname{cl}(Sx)$ for every $x \in D$,
2. $\operatorname{int} D \neq \emptyset$,

3. the subset $D_0 = \{x \in D : x \in (\operatorname{int} S)\, x\}$ is not empty, and

4. D is maximal with these properties.

The subset D_0 appearing in the third condition is open and dense in D (see [9]). We refer to it as the *set of transitivity* or the *core* of D.

The *domain of attraction* $\mathcal{A}(D)$ of a control set D is the subset of those $x \in G/H$ such that there exists $g \in S$ with $gx \in D$.

The control sets for S on G/H are ordered by putting $D_1 \leq D_2$ if there are $x \in D_1$ and $g \in S$ such that $gx \in D_2$. Equivalently, $D_1 \leq D_2$ if $D_1 \cap \mathcal{A}(D_2) \neq \emptyset$. The following statement clarifies the relation between the domain of attraction and the order of the control sets.

Proposition 2.1. *The domain of attraction $\mathcal{A}(D)$ of the control set D is open and if $x \in \mathcal{A}(D)$ then there exists $g \in \operatorname{int} S$ such that $gx \in D_0$. Moreover, for the control sets D_1 and D_2 the following statements are equivalent:*

1. $D_1 \leq D_2$.

2. *There exists $x \in (D_1)_0$ and $g \in \operatorname{int} S$ such that $gx \in (D_2)_0$.*

3. *For any $y \in (D_1)_0$ and $z \in (D_2)_0$ there exists $g \in \operatorname{int} S$ such that $gy = z$.*

4. $D_1 \subset \mathcal{A}(D_2)$.

Proof. Take $x \in \mathcal{A}(D)$ and $h \in S$ such that $hx \in D$. Then $S(hx)$ contains a dense subset of D so that there exists $g \in S$ with $gx \in D_0$. Since any point in D_0 is fixed by some element in $\operatorname{int} S$ we can choose $g \in \operatorname{int} S$ as claimed. This implies that the open set $g^{-1}(D_0)$ contains x and is contained in $\mathcal{A}(D)$. Hence $\mathcal{A}(D)$ is open.

As to the equivalent statements, suppose that $D_1 \leq D_2$. Then $(D_1)_0 \cap \mathcal{A}(D_2) \neq \emptyset$ because $(D_1)_0$ is dense in D_1 and $\mathcal{A}(D_2)$ is open. Hence (2) follows from the statement about $\mathcal{A}(D)$. Now, take x and g as in (2). If $y \in (D_1)_0$ and $z \in (D_2)_0$ then there are $h_1, h_2 \in S$ such that $h_1 y = x$ and $h_2 gx = z$ (see [9, Prop. 2.2]). So that (3) follows. Item (3) means that $(D_1)_0 \subset \mathcal{A}(D_2)$. Since for any $x \in D_1$ there is $g \in S$ such that $gx \in (D_1)_0$ we have that $D_1 \subset \mathcal{A}(D_2)$ as well. Finally, $D_1 \leq D_2$ if $D_1 \subset \mathcal{A}(D_2)$ as follows from the definitions. ■

In the sequel we write $y \rightsquigarrow z$ if $z \in (\operatorname{int} S)\, y$. From the third equivalent statement in this proposition we have that $y \rightsquigarrow z$ for any $y \in (D_1)_0$ and $z \in (D_2)_0$ if $D_1 \leq D_2$.

The order of the control sets for S^{-1} is given by reversing the order of the control sets for S. In fact, there is a mapping $D \mapsto D^-$ which associates to a control set D for S the control set D^- for S^{-1} which is related to D by $D_0^- = D_0$. From the third equivalent property in the above lemma we have that $D_1^- \leq D_2^-$ if and only if $D_1 \geq D_2$.

Note that, since we are assuming that G/H is compact, a control set D is *invariant* (i.e., $Sx \subset D$ for all $x \in D$) if and only if D is maximal with respect to the order. Also, a control set D is minimal if and only if D^- is S^{-1}-invariant. We refer to [3], [9] for further results about control sets.

3. Flag manifolds

In what follows we are interested in semigroup actions on G/H with G a connected and noncompact semi-simple Lie group and H a parabolic subgroup. We assume throughout that G has finite center. For these groups we use the following standard notation and terminology.

Let $\mathfrak{g}$ be the Lie algebra of G. Take a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ with $\mathfrak{k}$ the compactly embedded subalgebra and denote by θ the corresponding Cartan involution. Let $\mathfrak{a}$ be a maximal abelian subalgebra contained in $\mathfrak{s}$ and denote by Π the set of roots of the pair $(\mathfrak{g}, \mathfrak{a})$. Fix a simple system of roots $\Sigma \subset \Pi$. Denote by Π^+ the set of positive roots and by $\mathfrak{a}^+$ the Weyl chamber

$$\mathfrak{a}^+ = \{H \in \mathfrak{a} : \alpha(H) > 0 \text{ for all } \alpha \in \Sigma\}.$$

Let

$$\mathfrak{n} = \sum_{\alpha \in \Pi^+} \mathfrak{g}_\alpha$$

be the direct sum of the root spaces corresponding to the positive roots and

$$\mathfrak{n}^- = \theta(\mathfrak{n}) = \sum_{\alpha \in \Pi^+} \mathfrak{g}_{-\alpha}$$

the opposed subalgebra.

The notations K, N and N^- are used to indicate the connected subgroups whose Lie algebras are $\mathfrak{k}$, $\mathfrak{n}$ and $\mathfrak{n}^-$ respectively.

Let W be the Weyl group of G. It is constructed either as the subgroup of reflections generated by the roots of $(\mathfrak{g}, \mathfrak{a})$ or as the quotient M^*/M where M^* and M are respectively the normalizer and the centralizer of $\mathfrak{a}$ in K.

A minimal parabolic subalgebra of $\mathfrak{g}$ is given by

$$\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$$

where $\mathfrak{m}$, the Lie algebra of M, is the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$.

Let P be the minimal parabolic subgroup with Lie algebra $\mathfrak{p}$ and put $B = G/P$ for the maximal flag manifold. We denote by b_0 the base point in G/P. This flag manifold fibers over the other boundaries of G, which are built from subsets of Σ as follows: Given $\Theta \subset \Sigma$ let $\langle\Theta\rangle$ be the subset of positive roots generated by Θ and denote by $\mathfrak{n}_\Theta^-$ the subalgebra spanned by the root spaces $\mathfrak{g}_{-\alpha}$, $\alpha \in \langle\Theta\rangle$. Then

$$\mathfrak{p}_\Theta = \mathfrak{n}_\Theta^- \oplus \mathfrak{p}.$$

The normalizer P_Θ of $\mathfrak{p}_\Theta$ in G is a parabolic subgroup which contains P. The corresponding flag manifold $B_\Theta = G/P_\Theta$ is the base space for the natural fibration $\pi_\Theta : B \to B_\Theta$ whose fiber is P_Θ/P. This fiber is a flag manifold of a semi-simple subgroup $M_\Theta \subset G$ whose rank is the order of Θ (see [11]). In particular, the group M_Θ is of rank one if Θ is singleton. The Weyl group of M_Θ is the subgroup W_Θ generated by the reflections with respect to the simple roots in Θ.

A conjugate $\mathrm{Ad}(g)H$, $g \in G$, $H \in \mathfrak{a}^+$ is said to be *split-regular* in $\mathfrak{g}$. Similarly, a split-regular element in G is a exponential $h = \exp(H)$ with $H \in \mathfrak{g}$

split-regular. A split-regular $H \in \mathfrak{g}$ belongs to a unique Weyl chamber in $\mathfrak{g}$ (a conjugate of $\mathfrak{a}^+$). If $h_0 \in A^+ = \exp \mathfrak{a}^+$ then h_0 has a finite number of fixed points in B, namely, the base point $b_0 (= P)$ and its orbit under the subgroup $M^* \subset K$, the normalizer of $\mathfrak{a}$ in K. The action of M^* in this orbit factors through M so that the fixed points are given by wb_0, $w \in W = M^*/M$. The same way, the fixed points in B of a split-regular $h = gh_0g^{-1}$ with $g \in G$ and $h_0 \in A^+$, are the points gwb_0. In what follows we say that gwb_0 is the *fixed point of type* w for h and denote it by $\phi(h,w)$. We note that $\phi(h,1)$ is the only attractor for the action of h in B, in the sense that it is a hyperbolic fixed point with an open stable manifold. On the other hand, h has just one repeller, i.e., an attractor for h^{-1}, which is $\phi(h,w_0)$ where w_0 is the *principal involution* of W, that is, the only element of W which satisfies $w_0\Sigma = -\Sigma$.

Any $h \in A$ leaves invariant the fiber $\pi_\Theta^{-1}\pi_\Theta(b_0)$ for every $\Theta \subset \Sigma$, and if h is regular then its fixed points in this fiber are wb_0 with $w \in W_\Theta$, that is, the orbit of b_0 under the subgroup W_Θ. In particular, if Θ is singleton there are just two fixed points for the action of h in this fiber, one of them being an attractor and the other one a repeller.

Regarding still the notations, we shall say that a root $\alpha \in \Pi$ is *indivisible* if $\alpha/2$ is not a root. The set of indivisible roots is denoted by Δ with a superscript $\Delta^\pm$ to indicate the positive or negative ones.

We consider now a semigroup $S \subset G$ with $\operatorname{int} S \neq \emptyset$. In [9] the control sets for the action of S on the flag manifolds were described by means of the Weyl group W. For this description it is assumed that G has finite center. It is proved that $\operatorname{int} S$ contains enough split-regular elements so that we have a mapping

$$w \longmapsto D(w) \tag{1}$$

which associates to $w \in W$ a control set $D(w)$ in such a way that the core $D(w)_0$ is the set of the fixed points of type w for the split-regular elements in $\operatorname{int} S$. There is just one invariant control set $D(1)$ whose core is the set of attractors for the split-regular elements in $\operatorname{int} S$. Similarly the repellers of the split-regular elements in $\operatorname{int} S$ form the core of a unique minimal control set $D(w_0)$ where w_0 is the principal involution of W.

The level sets of (1) are described by the subset

$$W(S) = \{w \in W : D(w) = D(1)\}.$$

This subset is a parabolic subgroup of W, and for $w_1, w_2 \in W$, $D(w_1) = D(w_2)$ if and only if $w_1w_2^{-1} \in W(S)$ (see [9, Prop. 4.2]). Hence the control sets on the maximal boundary B are in one-to-one correspondence with the cosets in $W(S)\backslash W$.

These facts also apply to the inverted semigroup $S^{-1} = \{g^{-1} : g \in S\}$ and we have a mapping

$$w \longmapsto D^-(w)$$

into the control sets of S^{-1}. The control set $D^-(w)$ contains the fixed points of type w for the split-regular elements in $\operatorname{int} S^{-1}$. Hence, if $h \in \operatorname{int} S$ is split-regular then $\phi(h^{-1},w) \in D^-(w)_0$. If w_0 is the principal involution of W then

$h^{-1} = w_0 h_0 w_0^{-1}$ for some h_0 in the same chamber as h. Therefore, the fixed point of type w for h^{-1} is $\phi(h, w_0 w)$. Since this fixed point belongs to the core of $D(w_0 w)$ we have the following equality between the cores of these control sets.

Proposition 3.1. $D^-(w)_0 = D(w_0 w)_0$.

4. Order of the control sets

Recall the Bruhat-Chevalley order of the Weyl group [2, 4]: Keeping fixed a simple system of roots, take for $w \in W$ a reduced expression $w = s_1 \cdots s_n$ as a product of reflections with respect to the simple roots. Then $w_1 \leq w$ if and only if there are integers $1 \leq i_1 < \cdots < i_j \leq n$ such that $w_1 = s_{i_1} \cdots s_{i_j}$ is a reduced expression for w_1.

In general the order on W depends on the choice of the simple system of roots Σ, that is, on the set of generators of W. Note however that the order obtained from $-\Sigma$ coincides with the order coming from Σ because both simple systems of roots define the same set of generators of W.

A useful elementary fact about the order in W is that $w_1 \leq w$ if and only if $w_1^{-1} \leq w^{-1}$. This follows directly from the definition and the remark that $w^{-1} = s_n \cdots s_1$ is a reduced expression for w^{-1} if the decomposition $w = s_1 \cdots s_n$ is reduced.

From [2, Thm. 3.13] we have that $w \leq w_1$ if and only if $wb_0 \in \mathrm{cl}(Nw_1 b_0)$ where b_0 is the base point in G/P.

We prove next that the order of the control sets in G/P for a semigroup $S \subset G$ with nonvoid interior is precisely the reverse of the Bruhat-Chevalley ordering of the Weyl group. We have

Theorem 4.1. *For $u \in W$ let $D(u)$ denote the control set in G/P given by (1). Let $w_1, w_2 \in W$. Then the following statements are equivalent:*

1. *$D(w_1) \leq D(w_2)$.*
2. *There exists $w \in W$ such that $w_1 \geq w$ and $w \in W(S)w_2$, that is, $D(w) = D(w_2)$.*

Proof. We choose the simple system of roots such that the corresponding Weyl chamber A^+ intersects $\mathrm{int}\, S$. The ordering of the control sets will be provided by the action of a split-regular $h \in A^+ \cap \mathrm{int}\, S$. The h-fixed points in G/P are wb_0, $w \in W$. Recall that $wb_0 \in D(w)_0$ for all $w \in W$.

Suppose that $w_1 \geq w$ and $w_2 \in W(S)w$. Then $wb_0 \in \mathrm{cl}(Nw_1 b_0)$. Since $wb_0 \in D(w)_0$ we have that $Nw_1 b_0 \cap D(w)_0 \neq \emptyset$. Hence there exists $n \in N$ such that $nw_1 b_0 \in D(w)_0$. But $h^{-k} n w_1 b_0 \to w_1 b_0$ as $k \to \infty$, which ensures the existence of an integer $k > 0$ such that $h^{-k} n w_1 b_0 \in D(w_1)_0$. Therefore there exists $g \in \mathrm{int}\, S^{-1}$ and $x \in D(w)_0$ such that $gx \in D(w_1)_0$ so that Proposition 2.1 implies that $D(w_1) \leq D(w) = D(w_2)$.

Conversely assume that $D(w_1) \leq D(w_2)$ and take a reduced expression

$$w_1 = s_1 \cdots s_n$$

where s_i is the reflection with respect to the simple root α_i, $i = 1, \ldots, n$. Let $P_i = P_{\{\alpha_i\}}$ be the parabolic subgroup defined by $\Theta = \{\alpha_i\}$ and denote by $\pi_i : G/P \to G/P_i$ the projection from the maximal flag manifold.

We have from Proposition 2.1 that

$$w_1 b_0 \quad w_2 b_0.$$

If $g \in \operatorname{int} S$ is such that $g w_1 b_0 = w_2 b_0$ then each fibration π_i is equivariant under g so that g interchanges fibers. In particular, g maps the fiber of $w_1 b_0$ onto the fiber of $w_2 b_0$. We shall exploit these fibrations to show that $b_0 \quad w_2 s_{i_k} \cdots s_{i_1} b_0$ for some integers $1 \leq i_1 < \cdots < i_k \leq n$.

Consider the reflection s_n and the corresponding fibration $\pi_n : G/P \to G/P_n$. The fixed points of h in the fiber of $w_1 b_0$ are $w_1 b_0$ itself and $w_1 s_n b_0$. To see this note that w_1^{-1} maps the fiber of $w_1 b_0$ onto the fiber of b_0. Also, w_1^{-1} interchanges the fixed points of h and those of $h_1 = w_1^{-1} h w_1$. Since h_1 is split-regular its fixed points in the fiber of b_0 are b_0 itself and $s_n b_0$. Hence $w_1 b_0$ and $w_1 s_n b_0$ are the fixed points of h in the fiber of $w_1 b_0$ as claimed.

By the same reason $w_2 b_0$ and $w_2 s_n b_0$ are the h-fixed points in the fiber through $w_2 b_0$.

Now, if $g \in \operatorname{int} S$ maps $w_1 b_0$ into $w_2 b_0$ then $g w_1 s_n b_0$ belongs to the same fiber as $w_2 b_0$. One of the two fixed points $w_2 b_0$ or $w_2 s_n b_0$ is the attractor for the action of h inside this fiber. Therefore as $k \to \infty$ we have that $h^k g w_1 s_n b_0$ converges either to $w_2 b_0$ or to $w_2 s_n b_0$. This implies that $D(w_1 s_n) \leq D(w_2)$ or $D(w_1 s_n) \leq D(w_2 s_n)$ and hence we have one of the two possibilities

$$w_1 s_n b_0 \quad w_2 b_0 \text{ or } w_2 s_n b_0.$$

Now we can repeat this argument with $w_1 s_n b_0$ in place of $w_1 b_0$ and with the fibration $\pi_{n-1} : G/P \to G/P_{n-1}$ instead of π_n. The fixed points in the π_{n-1}-fiber of $w_1 s_n b_0$ are $w_1 s_n b_0$ itself and $w_1 s_n s_{n-1} b_0$. Hence according to the above possibilities we get that $w_1 s_n s_{n-1} b_0 \quad w_2 b_0$ or $w_2 s_{n-1} b_0$ if $w_1 s_n b_0 \quad w_2 b_0$. Otherwise, we have $w_1 s_n s_{n-1} b_0 \quad w_2 s_n b_0$ or $w_2 s_n s_{n-1} b_0$ if $w_1 s_n b_0 \quad w_2 s_n b_0$. Hence at least one of the following cases happens

$$w_1 s_n s_{n-1} b_0 \quad w_2 b_0 \text{ or } w_2 s_n b_0 \text{ or } w_2 s_{n-1} b_0 \text{ or } w_2 s_n s_{n-1} b_0.$$

Continuing this way we arrive, after n steps, that there are integers $1 \leq i_1 < \cdots < i_k \leq n$ such that

$$w_1 s_n \cdots s_1 b_0 = b_0 \quad w_2 s_{i_k} \cdots s_{i_1} b_0.$$

This implies that $w_2 s_{i_k} \cdots s_{i_1} b_0$ belongs to the invariant control set $D(1)$. Therefore

$$w_2 s_{i_k} \cdots s_{i_1} \in W(S)$$

so that

$$w_2 \in W(S) s_{i_1} \cdots s_{i_k}.$$

Since $w_1 \geq s_{i_1} \cdots s_{i_k}$ the second condition in the statement is satisfied with $w = s_{i_1} \cdots s_{i_k}$. ∎

We conclude this section with the following fact about the Bruhat-Chevalley order, which will be needed later. Although this fact is known in the literature (see [8], Example 3, p.119) we give here a proof via semigroups, as a consequence of the above theorem.

Corollary 4.2. *Let w_0 be the principal involution of W, and take $w_1, w_2 \in W$. Then $w_1 \leq w_2$ if and only if $w_0 w_1 \geq w_0 w_2$.*

Proof. Let S be a semigroup such that $W(S) = \{1\}$. The existence of such a semigroup is implicit in the theory of [9]: If S is the compression semigroup of some subset C contained in an open Bruhat cell $N^- b_0$ then $W(S) = 1$. By the theorem $w_1 \leq w_2$ if and only if $D(w_1) \geq D(w_2)$. On the other hand, we have from Proposition 3.1 that $D^-(w_0 w_1)_0 = D(w_1)_0$ and $D^-(w_0 w_2)_0 = D(w_2)_0$. Hence in the order of the control sets for S^{-1} we have

$$D^-(w_0 w_1) \leq D^-(w_0 w_2).$$

Applying the theorem again we get $w_0 w_1 \geq w_0 w_2$. ■

5. Schubert cells

A Schubert cell in the maximal flag manifold B is the closure of an orbit of the subgroup N or any of its conjugates. As is well known, the N-orbits on B, the so called Bruhat cells, are Nwb_0 with w running through the Weyl group W. The purpose of this section is to provide a geometric description of the Schubert cells in B in terms of smaller flag manifolds.

In the discussion to follow we keep fixed a simple system of roots Σ. For a finite sequence (with possible repetitions) $\alpha_1, \ldots, \alpha_n$ of simple roots we let $s_1, \ldots, s_n$ be the reflections with respect to these roots. Also, we denote by $P_i = P_{\{\alpha_i\}}$ the parabolic subgroup defined by $\Theta = \{\alpha_i\}$. The corresponding flag manifold is denoted by $B_i = G/P_i$. Associated with B_i there is the canonical fibration $\pi_i : B \to B_i$. For $i = 1, \ldots, n$ we let γ_i stand for the operation of exhausting a subset of B with the fibers of π_i, that is, if $X \subset B$ then

$$\gamma_i(X) = \pi_i^{-1}\pi_i(X)$$

We recall that π_i is equivariant under $g \in G$. This implies that γ_i is also equivariant under g, i.e., $g\gamma_i(X) = \gamma_i(gX)$ for any subset $X \subset B$.

In the sequel we shall prove that a Schubert cell is a subset of the type $\gamma_1 \cdots \gamma_n(b)$ for some sequence of simple roots and $b \in B$.

For $w \in W$ put $N^w = wNw^{-1}$ and let $\mathfrak{n}^w = \mathrm{Ad}(w)\mathfrak{n}$ be its Lie algebra. Every Schubert cell is the image under some $g \in G$ of $\mathrm{cl}(N^w b_0)$ for a $w \in W$. So we start by looking at these cells.

Let $\alpha \in \Sigma$ be a simple root such that $-\alpha \in w(\Delta^+) \cap \Delta^-$ and consider the parabolic subgroup $P_\alpha = P_{\{\alpha\}}$. Denote by B_α the flag manifold G/P_α and let $\pi : B \to B_\alpha$ be the canonical fibration. Also, let N_α^- stand for the nilpotent group generated by $\exp(\mathfrak{g}_{-\alpha} + \mathfrak{g}_{-2\alpha})$. The orbit $P_\alpha b_0$ is the fiber $\pi^{-1}\pi(b_0)$ while $N_\alpha^- b_0$

is dense in this fiber. The fact that $-\alpha \in w(\Delta^+) \cap \Delta^-$ implies that $\mathfrak{g}_{-\alpha} + \mathfrak{g}_{-2\alpha}$ is contained in $\mathfrak{n}^w$. Hence $N_\alpha^- b_0 \subset N^w b_0$. Therefore we have that

$$\gamma(b_0) \subset \mathrm{cl}(N^w b_0). \tag{2}$$

where for a subset $X \subset B$, $\gamma(X) = \pi^{-1}\pi(X)$. From this inclusion we get the following statement which is the main step in the proof of Theorem 5.3 below.

Proposition 5.1. *As above, take $\alpha \in \Sigma$ such that $-\alpha \in w(\Delta^+) \cap \Delta^-$. Then*

$$\mathrm{cl}(N^w b_0) = \gamma(\mathrm{cl}(N^w s b_0))$$

where s is the reflection with respect to the simple root α.

Proof. The fiber $\gamma(b_0)$ contains sb_0 so that $\pi(b_0) = \pi(sb_0)$. This implies that the orbits $N^w\pi(b_0)$ and $N^w\pi(sb_0)$ in B_α coincide. Hence $\pi(N^w b_0) = \pi(N^w s b_0)$ because π is equivariant. Taking closures and using the fact that π is a continuous and closed map we conclude that

$$\pi(\mathrm{cl}(N^w b_0)) = \pi(\mathrm{cl}(N^w s b_0)). \tag{3}$$

Therefore $\mathrm{cl}(N^w b_0) \subset \gamma(\mathrm{cl}(N^w s b_0))$.

In order to show the reverse inclusion, take $x \in \gamma(\mathrm{cl}(N^w s b_0))$. Then (3) ensures that $\pi(x) \in \mathrm{cl}(N^w \pi(b_0))$. Hence there exists a sequence $n_k \in N^w$ such that $n_k\pi(b_0) \to \pi(x)$. By trivializing the bundle $\pi : B \to B_\alpha$ around $\pi(x)$ it is easy to see that there exists a sequence $z_k \in B$ such that $\pi(z_k) = n_k\pi(b_0)$ and $z_k \to x$. Let $y_k = n_k^{-1} z_k$. Then $\pi(y_k) = \pi(b_0)$ for all k so that we have from the inclusion in (2) that $y_k \in \mathrm{cl}(N^w b_0)$ and hence $z_k = n_k y_k \in \mathrm{cl}(N^w b_0)$. Since $z_k \to x$ we have that $x \in \mathrm{cl}(N^w b_0)$ concluding the proof. ■

From this proposition we can proceed by induction to get the desired description of the Schubert cells. Take a reduced expression

$$w = s_1 \cdots s_n$$

with s_i the reflection with respect to the simple, and hence indivisible root $\alpha_i \in \Sigma$. It is well known (see e.g. [10, Thm. 4.15.10]) that the indivisible positive roots that are mapped into negative roots by w are

$$s_n \cdots s_2\alpha_1, \ldots, s_n\alpha_{n-1}, \alpha_n.$$

Applying w to these roots we get that

$$w(\Delta^+) \cap \Delta^- = \{-\alpha_1, -s_1\alpha_2, \ldots, -s_1 \cdots s_{n-1}\alpha_n\}.$$

For the inductive step it is needed the following lemma.

Lemma 5.2. *Let $w = s_1 \cdots s_n$ be a reduced product of reflections with respect to simple roots in Σ. For $k = 1, \ldots, n-1$ put $t_k = s_1 \cdots s_k$ and consider the simple system of roots $\Sigma_k = t_k\Sigma$. Denote by $\Delta_k^\pm$ the positive (respectively negative) indivisible roots defined by Σ_k.*

Then $t_k s_{k+1} t_k^{-1}$ is the reflection with respect to the simple root $t_k\alpha_{k+1} \in \Sigma_k$ and $-t_k\alpha_{k+1} \in w(\Delta^+) \cap \Delta_k^-$.

Proof. For a root β denote by r_β the reflection it defines. Then $r_{u\beta} = ur_\beta u^{-1}$ for any $u \in W$. Hence $t_k s_{k+1} t_k^{-1}$ is the reflection with respect to the simple root $t_k\alpha_{k+1} \in \Sigma_k$. We have that $-t_k\alpha_{k+1} \in \Delta_k^-$ and if $k = n-1$ then $-t_k\alpha_{k+1} = w\alpha_n$ so that $-t_k\alpha_{k+1} \in w(\Delta^+)$. On the other hand, if $k < n-1$ then

$$w^{-1}(-t_k\alpha_{k+1}) = s_n \cdots s_{k+2}\alpha_{k+1}$$

and, as mentioned above, this is one of the positive roots in Δ^+ which is mapped into Δ^-. Therefore $-t_k\alpha_{k+1} \in w(\Delta^+) \cap \Delta_k^-$ as required. ∎

With this background at hand we can give a geometric description of the Schubert cell $\mathrm{cl}(N^w b_0)$.

Theorem 5.3. *Let $w = s_1 \cdots s_n$ be a reduced expression as a product of reflections with respect to the simple roots in Σ. Then for any $k = 1, \ldots, n$, we have*

$$\mathrm{cl}(N^w b_0) = \gamma_1 \cdots \gamma_k (\mathrm{cl}(N^w s_1 \cdots s_k b_0)). \tag{4}$$

In particular $\mathrm{cl}(N^w b_0) = \gamma_1 \cdots \gamma_n(b_w)$ where $b_w = wb_0$ is the only N^w-fixed point in B.

Proof. By induction on k. For $k = 1$, (4) reduces to the formula in Proposition 5.1 with $s = s_1$ and $\alpha = \alpha_1$.

Suppose then that (4) holds for $1 < k \leq n-1$. Let $t_k = s_1 \cdots s_k$ and put $\Sigma_k = t_k\Sigma$, $\Delta_k^\pm = t_k\Delta^\pm$. Then $t_k\alpha_{k+1} \in \Sigma_k$ and the above lemma ensures that $-t_k\alpha_{k+1} \in w(\Delta^+) \cap \Delta_k^-$. This implies that if $u_k = t_k^{-1}w$ then $-\alpha_{k+1} \in u_k(\Delta^+) \cap \Delta^-$. Hence we can apply Proposition 5.1 with $\alpha = \alpha_{k+1}$ and u_k in place of w. In this case $s = s_{k+1}$ and $\gamma = \gamma_{k+1}$ so we have

$$\mathrm{cl}(N^{u_k} b_0) = \gamma_{k+1}\, \mathrm{cl}(N^{u_k} s_{k+1} b_0). \tag{5}$$

Applying t_k to the left hand side gives

$$t_k\, \mathrm{cl}(N^{u_k} b_0) = t_k\, \mathrm{cl}\left(N^{u_k} t_k^{-1} t_k b_0\right) = \mathrm{cl}(N^w t_k b_0).$$

On the other hand, the image under t_k of the right hand side of (5) is

$$t_k\gamma_{k+1}\, \mathrm{cl}(N^{u_k} s_{k+1} b_0) = \gamma_{k+1}\, \mathrm{cl}(N^w t_k s_{k+1} b_0) = \gamma_{k+1}\, \mathrm{cl}(N^w t_{k+1} b_0).$$

Therefore $\mathrm{cl}(N^w t_k b_0) = \gamma_{k+1}\, \mathrm{cl}(N^w t_{k+1} b_0)$ and

$$\gamma_1 \cdots \gamma_k\, \mathrm{cl}(N^w t_k b_0) = \gamma_1 \cdots \gamma_k \gamma_{k+1}\, \mathrm{cl}(N^w t_{k+1} b_0)$$

completing the induction step. ∎

Remark: The above description of a Schubert cell is comparable to the classical result known as Bott-Samelson desingularization which states that the cell $\mathrm{cl}(Nwb_0)$ is the image of the map $\phi : P_1 \times \cdots \times P_n \to B$ defined by

$$\phi(g_1, \ldots, g_n) = g_1 \cdots g_n b_0$$

(see [1, 6, 7]). Here $w = s_1 \cdots s_n$ and P_i is as above the parabolic subgroup associated with s_i. In the light of this result Theorem 4.1 provides an indirect proof that successive applications of the parabolic subgroups P_i yields successive exhaustions of the corresponding fibers.

The other Schubert cells are easily obtained from $\mathrm{cl}\,(N^w b_0)$. In the sequel we are particularly interested in the cells $\mathrm{cl}\,(N^- w b_0)$, $w \in W$, where N^- is the group opposed to N. For these cells we have

Corollary 5.4. *Let w_0 be the principal involution and suppose that $w_0 w = s_n \cdots s_1$ is a reduced expression. Then*

$$\mathrm{cl}\left(N^- w b_0\right) = \gamma_1 \cdots \gamma_n \left(w_0 b_0\right)$$

where $\gamma_i = \pi_i^{-1}\pi_i$ comes from the reflection s_i.

Proof. Since $N^- = w_0 N w_0^{-1}$ we have that $N^- w b_0 = w N^{w^{-1} w_0} b_0$ so that

$$\mathrm{cl}\left(N^- w b_0\right) = w \gamma_1 \cdots \gamma_n \left(w^{-1} w_0 b_0\right)$$

and the corollary follows from the equivariance of γ_i. ∎

6. Domain of Attraction of a Control Set

In this section we apply the previous results about the Schubert cells and the order of control sets to show that the domain of attraction of a control set is a union of Schubert cells.

For this we need the following consequences of Theorem 4.1: From Borel and Tits [2, Thm. 3.13] we know that a Schubert cell is given by

$$\mathrm{cl}\,(N w b_0) = \bigcup_{w_1 \leq w} N w_1 b_0. \tag{6}$$

In this formula N and b_0 are linked by the fact that b_0 is the only N-fixed point. If we take N^- instead of N then $N^- = w_0 N w_0^{-1}$ where w_0 is the principal involution of W. This subgroup is linked to $b^- = w_0 b_0$. For N^- and b^- formula (6) reads

$$\mathrm{cl}\left(N^- w b^-\right) = \bigcup_{w_1 \leq w} N^- w_1 b^- \tag{7}$$

(recall that the orders in W defined by Σ and $-\Sigma$ coincide). We need now a formula similar to (7) where in the left hand side b^- is replaced by b_0.

Proposition 6.1. *The Schubert cell $\mathrm{cl}\,(N^- w b_0)$ is given by*

$$\mathrm{cl}\left(N^- w b_0\right) = \bigcup_{s \geq w} N^- s b_0.$$

Proof. We have

$$\mathrm{cl}\left(N^{-}wb_0\right) = \mathrm{cl}\left(N^{-}\left(ww_0\right)w_0b_0\right) = \mathrm{cl}\left(N^{-}\left(ww_0\right)b^{-}\right).$$

Applying the Borel-Tits formula (7) to the right hand side of this equality we get

$$\mathrm{cl}\left(N^{-}wb_0\right) = \bigcup_{w_1 \leq ww_0} N^{-}w_1b^{-} = \bigcup_{w_1 \leq ww_0} N^{-}\left(w_1w_0\right)b_0.$$

So that by putting $s = w_1w_0$ we have

$$\mathrm{cl}\left(N^{-}wb_0\right) = \bigcup\{N^{-}sb_0 : sw_0 \leq ww_0\}.$$

Now $sw_0 \leq ww_0$ if and only if $w_0s^{-1} \leq w_0w^{-1}$ and by Corollary 4.2 this happens if and only if $s^{-1} \geq w^{-1}$. So that the above union runs through s such that $s \geq w$ as claimed. ∎

As a consequence of this result we can prove that certain Schubert cells are contained in the domain of attraction of a control set. This fact will be needed below in the proof of the main result of this section.

Proposition 6.2. *If $A^{+} \cap \operatorname{int} S \neq \emptyset$ then $\mathrm{cl}\left(N^{-}wb_0\right) \subset \mathcal{A}\left(D\left(w\right)\right)$ for every $w \in W$.*

Proof. From the above proposition we have

$$\mathrm{cl}\left(N^{-}wb_0\right) = \bigcup_{s \geq w} N^{-}sb_0.$$

In this formula each sb_0 belongs $\mathcal{A}\left(D\left(w\right)\right)$. In fact, $sb_0 \in D\left(s\right)_0$. But if $s \geq w$ then $D\left(s\right) \leq D\left(w\right)$ so that $D\left(s\right) \subset \mathcal{A}\left(D\left(w\right)\right)$. On the other hand, if $n \in N^{-}$ and $h \in A^{+} \cap \operatorname{int} S$ then $h^{k}nsb_0 \to sb_0$, as $k \to \infty$, and since sb_0 is in the core of a control set this implies that nsb_0 sb_0. Hence nsb_0 wb_0 so that $nsb_0 \in \mathcal{A}\left(D\left(w\right)\right)$. ∎

We can now prove our main result about domains of attraction.

Theorem 6.3. *Let $C^{*} = D\left(w_0\right)$ be the minimal control set. Then for all $w \in W$ the domain of attraction $\mathcal{A}\left(D\left(w\right)\right)$ of $D\left(w\right)$ is given by*

$$\mathcal{A}\left(D\left(w\right)\right) = \gamma_1 \cdots \gamma_n\left(C^{*}\right). \tag{8}$$

Here the sequence $\gamma_1, \ldots, \gamma_n$ comes from a reduced expression

$$w_0w = s_n \cdots s_1$$

where w_0 is the principal involution of W.

Proof. The inclusion $\gamma_1 \cdots \gamma_n (C^*) \subset \mathcal{A}(D(w))$ is a consequence of the previous characterization of the Schubert cells and the above proposition. In fact, it is easily checked by induction that $\gamma_1 \cdots \gamma_n (C^*) = \bigcup_{b \in C^*} \gamma_1 \cdots \gamma_n (b)$ so it is required to show that

$$\gamma_1 \cdots \gamma_n (b) \subset \mathcal{A}(D(w)) \tag{9}$$

for all $b \in C^*$. Actually it is enough to prove this for $b \in C_0^*$ because if $b \in C^*$ then there exists $g \in S$ such that $gb \in C_0^*$. Since $g\gamma_1 \cdots \gamma_n (b) = \gamma_1 \cdots \gamma_n (gb)$ we have (9) from the inclusion for gb. Now, if $b \in C_0^*$ then there exists a split-regular $h \in \operatorname{int} S$ such that b is the repeller of h in B. So that if we take our basic objects such that $h \in A^+$ then from Corollary 5.4 we have that the left hand side of (9) is the Schubert cell $\operatorname{cl}(N^- w b_0)$, which is contained in $\mathcal{A}(D(w))$ by the above proposition.

The proof that $\mathcal{A}(D(w))$ is contained in the right hand side of (8) is by induction on n, the length $\ell(w_0 w)$ of $w_0 w$. If $n = 0$ then $w_0 w = 1$, $w = w_0$ and $D(w) = C^*$. So that (8) holds because $\mathcal{A}(C^*) = C^*$.

If $\ell(w_0 w) = n$, let $w_0 w = s_n \cdots s_2 s_1$ be a reduced expression and define

$$w_1 = w_0 s_n \cdots s_2.$$

Then $\ell(w_0 w_1) = \ell(w_0 w) - 1$ so we have by the induction hypothesis that

$$\mathcal{A}(D(w_1)) = \gamma_2 \cdots \gamma_n (C^*). \tag{10}$$

Let $\pi_1 : B \to B_1$ be the projection corresponding to s_1. By construction we have that $\pi_1(b_0) = \pi_1(s_1 b_0)$. Applying w_1 to this equality we get that $\pi_1(w_1 b_0) = \pi_1(w b_0)$. This implies that there exists a control set, say E, in B_1 such that

$$\pi_1(D(w)_0) = \pi_1(D(w_1)_0) = E_0$$

(see [9, Prop. 5.1]). Now, take $y \in \mathcal{A}(D(w))$ and $g \in S$ such that $gy \in D(w)_0$. Then $g\pi_1(y) \in E_0$ and since $\pi_1^{-1}\pi_1(gy)$ intersects $D(w_1)_0$ and g interchanges fibers, there exists z in the same fiber as y such that $gz \in D(w_1)_0$. By definition $z \in \mathcal{A}(D(w_1))$ so (10) implies that

$$z \in \gamma_2 \cdots \gamma_n (C^*).$$

Hence $\pi_1(y) = \pi_1(z) \in \pi_1 \gamma_2 \cdots \gamma_n (C^*)$, that is,

$$y \in \gamma_1 \gamma_2 \cdots \gamma_n (C^*)$$

concluding the proof. ∎

7. The other flag manifolds

As before we denote by B_Θ the flag manifold defined by $\Theta \subset \Sigma$ and by $\pi_\Theta : B \to B_\Theta$ the natural projection from the maximal flag manifold B. If $D \subset B$ is a control set then $\pi_\Theta(D_0)$ is the core of a control set in B_Θ and reciprocally if E is a control set in B_Θ then there exists $w \in W$ such that $\pi_\Theta\left(D\left(w\right)_0\right) = E_0$ (see [9, Prop. 5.1]). This gives a well defined map

$$w \longmapsto E\left(w\right) \tag{11}$$

onto the control sets in B_Θ. Since we have further that for $w_1, w_2 \in W$, $\pi_\Theta\left(D\left(w_1\right)_0\right) = \pi_\Theta\left(D\left(w_2\right)_0\right)$ if and only if there exists $w \in W\left(S\right)w_2$ such that $\pi_\Theta\left(wb_0\right) = \pi_\Theta\left(w_2 b_0\right)$ this map factors through $W\left(S\right)\backslash W/W_\Theta$ establishing a one-to-one correspondence between these double cosets and the control sets in B_Θ. From this correspondence we can detect the order and domains of attraction of the control sets in B_Θ.

The Bruhat-Chevalley order in W induces an order in W/W_Θ by putting $w_1 W_\Theta \leq w_2 W_\Theta$ if one of the following equivalent conditions hold:

1. There exists $a \in w_1 W_\Theta$ such that $a \leq w_2$.
2. There exists $b \in w_2 W_\Theta$ such that $w_1 \leq b$.
3. There are $a \in w_1 W_\Theta$ and $b \in w_2 W_\Theta$ such that $a \leq b$.

It is not hard to check, either from the definition of the order or from its relation with the order of the control sets, that these are indeed equivalent conditions. Actually, it can be shown that in each coset wW_Θ there is just one element of minimal length, whose order provide the order between the cosets. In [5] the elements of minimal length in the cosets were written down explicitly for the classical diagrams.

Clearly a similar order is defined in the quotient $W_\Theta\backslash W$. Since $W\left(S\right)$ is a parabolic subgroup of W we have in particular that $W\left(S\right)\backslash W$ is ordered. Note that by Theorem 4.1 the control sets in B are ordered according to this order in $W\left(S\right)\backslash W$. We can now factor this order once again and get the order of the control sets in B_Θ.

Proposition 7.1. *For $u \in W$ let $E\left(u\right)$ be the control set in B_Θ as given by (11). Take $w_1, w_2 \in W$. Then $E\left(w_1\right) \leq E\left(w_2\right)$ if and only if there exists $w \in w_2 W_\Theta$ such that $D\left(w_1\right) \leq D\left(w\right)$.*

Proof. If w is as in the statement then there are $g \in \operatorname{int} S$, $x \in D\left(w_1\right)_0$ and $y \in D\left(w\right)_0$ such that $gx = y$. By the definition in (11) it follows that $\pi_\Theta\left(x\right) \in E\left(w_1\right)_0$, $\pi_\Theta\left(y\right) \in E\left(w\right)_0$ and $E\left(w_2\right) = E\left(w\right)$. Therefore $g\pi_\Theta\left(x\right) \in E\left(w_2\right)_0$ showing that $E\left(w_1\right) \leq E\left(w_2\right)$.

For the converse fix a split-regular $h \in \operatorname{int} S$ and let b_0 be its attractor. Then $ub_0 \in D\left(u\right)$ for all $u \in W$. Hence for $i = 1, 2$, $\pi_\Theta\left(w_i b_0\right)$ belongs to $E\left(w_i\right)_0$. Therefore there exists $g \in \operatorname{int} S$ which maps the fiber through $w_1 b_0$ onto the fiber through $w_2 b_0$. Since h has an attractor in the latter fiber, it follows that there exists $w \in w_2 W_\Theta$ such that $D\left(w_1\right) \leq D\left(w\right)$ concluding the proof. ■

Corollary 7.2. *For $w_1, w_2 \in W$ the following assertions are equivalent*

1. $E(w_1) \leq E(w_2)$.
2. *There exists* $w \leq w_1$ *such that* $w \in W(S)\, w_2 W_\Theta$, *that is,* $E(w) = E(w_2)$.

Proof. From the above proposition we have that $E(w_1) \leq E(w_2)$ if and only if there is $w_3 \in w_2 W_\Theta$ such that $D(w_1) \leq D(w_3)$. On the other hand, by Theorem 4.1 this inequality holds if and only if there is $w \in W(S)\, w_3$ such that $w_1 \geq w$. These two equivalences together imply the corollary. ■

This corollary implies at once that the order in W factors through the double coset $W(S) \backslash W / W_\Theta$ in such a way that the bijection with the control sets is order reversing.

We turn now to the domains of attraction of the control sets in B_Θ. These are projections of the domains of attraction of the control sets in B. In fact,

Proposition 7.3. *As above let* $E(w)$ *be a control set in* B_Θ. *Then*

$$\mathcal{A}(E(w)) = \pi_\Theta(\mathcal{A}(D(w))).$$

Proof. If $x \in \mathcal{A}(D(w))$ then there exists $g \in \operatorname{int} S$ such that $gx \in D(w)_0$. Since $\pi_\Theta(D(w)_0) = E(w)_0$, equivariance of π_Θ implies that $g\pi_\Theta(x) \in E(w)_0$ so that $\pi_\Theta(x) \in \mathcal{A}(E(w))$ showing that $\pi_\Theta(\mathcal{A}(D(w))) \subset \mathcal{A}(E(w))$.

For the other inclusion take $y \in \mathcal{A}(E(w))$. Then there exists $g \in \operatorname{int} S$ such that $gy = \pi_\Theta(wb_0)$ because $\pi_\Theta(wb_0) \in E(w)_0$. Hence g maps the fiber $\pi_\Theta^{-1}(y)$ onto the fiber through wb_0. Therefore there exists $x \in \pi_\Theta^{-1}(y)$ such that $gx = wb_0$. We have that $x \in \mathcal{A}(D(w))$ and $\pi_\Theta(x) = y$ showing that $\mathcal{A}(E(w)) \subset \pi_\Theta(\mathcal{A}(D(w)))$. ■

As a consequence of this proposition we have that the domains of attraction of the control sets in B_Θ are also unions of Schubert cells. In fact, the Schubert cells in B_Θ are projections of the Schubert cells in B. Therefore we have from Theorem 6.3 that

$$\mathcal{A}(E(w)) = \pi_\Theta \gamma_1 \cdots \gamma_n (C^*)$$

is the union of the Schubert cells in B_Θ determined by the split-regular elements in $\operatorname{int} S$.

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