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On solvmanifolds and a conjecture of Benson and Gordon from the hamiltonian viewpoint

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Abstract. In this work we prove a theorem which shows that under some mild restrictions on a solvmanifold G/Γ the existence of a Kähler structure on it forces G to be metabelian and, hence this result is only 'one-step' removed from the original Benson-Gordon conjecture. Applications and examples are discussed. The proof develops the 'hamiltonian' idea of D. McDuff which appeared in her proof of the same conjecture for nilmanifolds [22] as well as ideas of G. Lupton and J. Oprea contained in [20].

1. Introduction

In [4] the authors formulated the following conjecture.

Conjecture. *Any compact quotient of a simply connected completely solvable Lie group by a lattice admits a Kähler structure if and only if it is diffeomorphic to a complex torus.*

Note that the authors of [4] call these manifolds *solvmanifolds*. In the sequel we will also follow this terminology although in general the class of compact homogeneous spaces G/H of simply connected solvable Lie groups is larger [31]. For example, the latter quotients may be not parallelizable (cf. [1]) while quotients by *lattices* are obviously parallelizable (cf. also [28]). Recall that a Lie group G is called *completely solvable* if its Lie algebra $\mathfrak{g}$ satisfies the property that each operator $ad\,V : \mathfrak{g} \to \mathfrak{g}$, $V \in \mathfrak{g}$ has only real eigenvalues.

The conjecture of Benson and Gordon was a result of their analysis of the existence problem for Kähler structures on compact nil- and solvmanifolds (see [4], [5], [28]). The conjecture is true for nilmanifolds and has many proofs. There are several proofs based on *rational homotopy theory* [17], [19], [28] and a nice geometric proof given by McDuff [22]. Also, this fact was known to Hano [16]. The ideas of rational homotopy theory led various authors to partial results giving evidence for the conjecture [15], [28], [24], [27], [13]. The strongest results

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were the proof of the conjecture in dimension 4 [24] and the result of [4] about the algebraic structure of the group G provided that G/Γ admits Kähler structures.

Theorem of Benson and Gordon. *If G is completely solvable and G/Γ is a solvmanifold that admits a Kähler structure, then*

1) *there is an abelian complement $\mathfrak{a}$ in $\mathfrak{g}$ of the derived algebra $\mathfrak{n} = [\mathfrak{g}, \mathfrak{g}]$;*
2) *$\mathfrak{a}$ and $\mathfrak{n}$ are even dimensional;*
3) *the center of $\mathfrak{g}$ intersects $\mathfrak{n}$ trivially;*
4) *the Kähler form is cohomologous to a left invariant symplectic form $\omega = \omega_0 + \omega_1$, where $\mathfrak{n} = \mathrm{Ker}(\omega_0)$ and $\mathfrak{a} = \mathrm{Ker}(\omega_1)$;*
5) *both ω_0 and ω_1 are closed but non exact in $\mathfrak{g}$ (and also in $\mathfrak{n}$ and $\mathfrak{a}$);*
6) *the adjoint action of $\mathfrak{a}$ on $\mathfrak{n}$ is by infinitesimal symplectic automorphisms of $(\mathfrak{n}, \omega)$.*

Thus G is a semidirect product $G = A \times_\varphi N$, where A is a connected abelian subgroup and N is the nilpotent commutator subgroup. Moreover, N admits a left invariant symplectic structure, and the action of A on N is by symplectic automorphisms.

Note that these results are in the framework of a general problem of constructing *symplectic manifolds with no Kähler structures* [28, 10, 13, 14]. Using solvmanifolds, several authors constructed such examples (cf. [15, 26, 27, 28]). Note also that we *don't assume* that complex structure, or Kähler form are G-invariant, since any kind of invariance of the structure forces G/Γ to be a torus and this fact is known even for larger classes of spaces, e.g. for *complex homogeneous symplectic manifolds* [13] or for any compact complex homogeneous manifold admitting a Kähler structure (which may be not invariant, this is the *Borel–Remmert theorem* [6]). In this framework, we can mention also a 'Hamiltonian' approach to the problem [13, 18, 20] as well as Dorfmeister's and Guan's results on pseudo-Kählerian homogeneous compact manifolds [8]. Compact solvmanifolds appeared naturally in various works related to topological obstructions to the existence of complex structures on manifolds (see, e.g. [10, 30]).

The purpose of this note is twofold. First, we prove a theorem which shows that under rather mild restrictions on G/Γ the theorem of Benson and Gordon [4] can be essentially strengthened: our result appears to be only 'one-step' removed from the original conjecture. Indeed, while the conjecture forces the group G to be *abelian*, we are able to show that the existence of a Kähler structure implies that G must be *metabelian*:

$$G = \mathbb{R}^s \times_\varphi \mathbb{R}^t$$

(i.e. a semidirect product of two abelian Lie groups). This result allows for eliminating many Lie groups G when attacking the conjecture. For example, it follows that in *dimension 6 Kähler structures may exist only on solvmanifolds of the form* $(\mathbb{R}^2 \times_\varphi \mathbb{R}^4)/\Gamma$ (it is worth comparing with [15], where the authors explicitly construct symplectic non-Kählerian solvmanifold of the type mentioned above). The second explicit aim of this note is the developing of the 'hamiltonian' approach to the problem inspired by the proof of D. McDuff [22] in the nilmanifold case as well as the papers of D. Guan [13], A.T. Huckleberry [18] and G. Lupton–J. Oprea [20]. The proof of our theorem is, in fact, an extension of the 'hamiltonian' technique of McDuff to the solvmanifold case. It is based on an interesting observation that the *non-Lefschetz condition* 'propagates' from the fibers of the *Mostow bundle* to the total space of it. This observation seems to be of much more general nature, but the authors have no definit results in this direction and further analysis looks promising. However, the extension of the McDuff technique to the solvmanifold case requires more refined techniques of working with solvable Lie groups and solvmanifolds (cf. Section 3). In is quite natural, since the problem of existence of lattices in *solvable* Lie groups is much more delicate then the corresponding problem for nilpotent Lie groups.

Our proof is not directly related to rational homotopy theory, however, the result itself deserves thinking over from the homotopic point of view, since solvmanifolds are 'one-step' removed from *nilpotent spaces* and a version of the minimal model theory can be still developed (this attempt was discussed in [28], see also [15]).

We complete this section with several conventions, notations and facts which will be used in the sequel. First, note that understanding of this paper requires some information about lattices in Lie groups (it is collected, e.g. in Raghunathan's book [25]).

By definition, a finitely graded commutative algebra

$$H = \oplus_{k=0}^{2n} H^k$$

is called *Lefschetz*, or satisfies the *hard Lefschetz condition* if there exists an element $\omega \in H^2$ such that all linear maps $L_{\omega^s} : H^{n-s} \to H^{n+s}, L_{\omega^s}(\theta) = \omega^s\theta$ are isomorphisms of vector spaces H^{n-s} and H^{n+s}. It is known that the hard Lefschetz condition holds for the de Rham cohomology algebra of any compact Kähler manifold [11] and therefore the violation of this condition eliminates Kähler structures on the given manifold (this fact is used by many authors as a tool of constructing symplectic manifolds with no Kähler structure).

If G/Γ is a compact solvmanifold, the nilradical $\widetilde{N}$ of G satisfies the property that $\widetilde{N}\Gamma$ is a closed subgroup in G and the latter fact allows for constructing the following bundle which is called in the sequel the *Mostow bundle*:

$$\widetilde{N}\Gamma/\Gamma \to G/\Gamma \to G/\widetilde{N}\Gamma$$

(see [21]).

Two groups Γ and Γ' are called *commensurable* if the indices $[\Gamma : \Gamma\cap\Gamma']$ and $[\Gamma' : \Gamma\cap\Gamma']$ are finite.

In Section 3 we need the structure of a $\mathbb{Q}$-defined *algebraic* group on the simply connected nilpotent Lie group N containing a lattice. The 'digest' of algebraic groups in our context can be also found in [25] and [31]. As far as topological notions are concerned, we need, in fact, only the Poincaré duality [7]. Symplectic structures and hamiltonian symplectic actions are described, e.g. in [2, 20].

The first example of symplectic non-Kählerian compact manifold was constructed by Thurston [29] and this manifold was also considered by Kodaira. This manifold is defined as a *nilmanifold* of the form $N_3/\Gamma_3 \times \mathbb{R}/\mathbb{Z}$ where N_3 is the 3-dimensional Heisenberg group and Γ_3 is the group of all unitriangular 3×3-matrices with integer entries. In the sequel we will call this manifold *the Kodaira–Thurston manifold*. We will show in Section 3 that the Kodaira–Thurston manifold is 'strongly non-Kählerian', i.e. it cannot even be a fiber of the Mostow bundle with Kählerian total space.

Now, it is important to discuss one subtle difference between the results of this paper and [18]. The author is grateful to the referee for pointing out the necessity of expressing this difference explicitly. It was proved in [18] that any homogeneous space G/Γ determined by a discrete co-compact subgroup Γ and endowed with a G*-ivariant* symplectic form is a torus (see Section 3). However, the spaces considered in this article *do not satisfy* this property, although a "close" property is satisfied: each such manifold admits a symplectic form, whose pullback to G is G-invariant. To stress this, we use the McDuff terminology [22] and introduce the following definition:

Definition 1. Let G/Γ be a homogeneous space. A differential form ω on G/Γ is *homogeneous*, if it lifts to a left-invariant symplectic form on G.

2. Propagation of the non-Lefschetz condition

In the sequel we will need several facts concerning nilpotent torsionfree finitely generated groups (these and only these groups are lattices in nilpotent Lie groups). First recall that there exists a *refinement of the upper central series* for any nilpotent group Γ with no torsion and with finite number of generators:

$$\Gamma \supset Z_2\Gamma \supset Z_3\Gamma \supset \ldots \supset Z_n\Gamma \supset \{1\}$$

with each $Z_i\Gamma/Z_{i+1}\Gamma \cong \mathbb{Z}$. The length of this series is invariant and is called the *rank* of Γ. So, for Γ above, $\mathrm{rank}(\Gamma) = n$. Note that we write the indices of the refined series in a non-standard way (the usual one for the upper central series is $\{1\} \subset Z_1\Gamma \subset Z_2\Gamma \subset \ldots$). This description implies that any $u \in \Gamma$ has a decomposition $u = u_1^{x_1} \cdots u_n^{x_n}$, where

$$\langle u_n \rangle = Z_n\Gamma, \ldots, \langle u_i \rangle = Z_i\Gamma/Z_{i+1}\Gamma. \tag{1}$$

Of course, $u_1, \ldots, u_n$ are generators of Γ and this set is called the *Malcev basis* for Γ. Using this basis, the multiplication in Γ takes the form

$$u_1^{x_1} \cdots u_n^{x_n} u_1^{y_1} \cdots u_n^{y_n} = u_1^{\rho_1(x,y)} \cdots u_n^{\rho_n(x,y)}$$

where

$$\rho_i(x,y) = x_i + y_i + \tau(x_1, \ldots, x_{i-1}, y_1, \ldots, y_{i-1})$$

In the sequel we will use the following observation expressed by the formula below:

$$Z_n\Gamma \subset Z(\Gamma) \subset [\Gamma, \Gamma], \quad Z_n\Gamma = \langle u_n \rangle \cong \mathbb{Z}. \tag{2}$$

Recall once more that, by the Benson–Gordon theorem, *if G/Γ is a completely solvable solvmanifold such that its cohomology algebra satisfies the hard Lefschetz condition, then G is necessarily a semidirect product*

$$G = A \times_\varphi N$$

where A is abelian and N is a commutator subgroup.

Using this notation, we can formulate the main result.

Theorem 1. *Let $S = G/\Gamma$ be a compact completely solvable solvmanifold and let N denote the commutator subgroup $N = [G, G]$. Assume that the following conditions hold:*

1) *$\Gamma \cap N$ is a lattice in N,*
2) *the refined upper central series for the nilpotent group $\Gamma_N = \Gamma \cap N$ is characteristic (i.e. is preserved by all automorphisms of Γ_N).*

Then, if S is a manifold of Lefschetz type (in particular, if S carries Kähler structures), the group G is necessarily a semidirect product of two abelian groups:

$$G = \mathbb{R}^k \times_\varphi \mathbb{R}^s$$

that is, a metabelian Lie group.

Remark 1. Note that the refined upper central series is characteristic for example, for all Γ_N for which the standard upper central series coincides with the refined upper central series, since the standard series is characteristic (it is well known). For instance, it holds for $\Gamma_N = U_n(\mathbb{Z})$, where $U_n(\mathbb{Z})$ denotes the group of all upper unitriangular matrices with integral entries.

Corollary 1. *Theorem 1 is valid under the following weaker assumption: there exists a non-trivial element $z \in [\Gamma_N, \Gamma_N]$ such that $\varphi(a)(z) = z$ for all $a \in \Gamma_A = A \cap \Gamma$. In particular, it is valid if $\dim Z(N) = 1$.*

Proof. Since S is completely solvable, we can always assume that it is equipped with a homogeneous symplectic form, say, ω (it follows from the *Nomizu–Hattori theorem*, see [28, 15], or the invariance argument at the end of the proof of Proposition 4 in [22], which is valid not only for nilmanifolds, but for completely solvable solvmanifolds). Moreover,

$$\omega = \omega_0 + \omega_1$$

with both ω_0 and ω_1 closed and ω_1 representing a homogeneous symplectic form on the nilmanifold $N/N \cap \Gamma$. Note that *symplecticness* of ω_1 as a homogeneous differential form on $N/(N \cap \Gamma)$ follows from (5) of the Benson–Gordon Theorem, which was cited in the Introduction. It was mentioned there that ω_1 is closed and non-exact not only as an element of the Chevalley–Eilenberg complex $(\Lambda\mathfrak{g}^*, \delta)$, but *also as an element in the Chevalley–Eilenberg complex of* $\mathfrak{n}$. Note that by our assumption $N \cap \Gamma$ is a lattice in N. Now, one can easily notice that the structure of a semidirect product on G is inherited by Γ:

$$\Gamma = \Gamma_A \times_\varphi \Gamma_N.$$

Indeed, the restriction of the natural projection $G \to G/N$ onto Γ has the kernel Γ_N while the image can be identified with $\Gamma N/\Gamma = \Gamma/(\Gamma \cap N)$, which yields the exact sequence

$$\{1\} \to \Gamma_N \to \Gamma \to \Gamma/(N \cap \Gamma) \to \{1\}$$

which is splittable because the initial sequence is.

The conditions of Theorem 1 allow for considering the following analogue of the Mostow bundle

$$F \to S \to T,$$

where F denotes the fiber of the commutator fibration, that is, $F = N\Gamma/\Gamma \cong N/(N \cap \Gamma)$, while $T = G/N\Gamma$ is the base.

Indeed, it is not very difficult to prove (and it is done in [31]) that $N \cap \Gamma$ is a lattice if and only if $N\Gamma$ is closed in G (see [31, Theorem 4.5]). Note that in this work subgroups $H \subset G$ such that $H\Gamma$ are closed in G are called Γ*-closed.*

We claim, that under the assumptions of Theorem 1 there exists a free S^1-action on S. We begin with the proof of the latter assertion. Note that, since $N/(N \cap \Gamma)$ is a compact nilmanifold, we have the following two possibilities:

1) N is abelian and, therefore, G is of the prescribed form;
2) N is *non-abelian* and, by the well-known result of McDuff [22], there exists a free circle action on F, which is symplectic with respect to the homogeneous symplectic form ω_1.

Recall this construction. It is proved, first, that $Z(N)$ has a lattice which can be taken as a center of $\Gamma \cap N$. In the sequel we will denote the latter intersection by $\Gamma_N = N \cap \Gamma$, so we claim that

$$Z(\Gamma_N) \subset Z(N)$$

and that, in fact, $Z(\Gamma_N)$ is a discrete and co-compact subgroup in $Z(N)$. This assertion is proved by the following argument (cf. [31]). Following Malcev introduce the natural structure of a $\mathbb{Q}$-defined algebraic group on N *uniquely* determined by Γ_N. Then, Γ_N is commensurable with the subgroup of $\mathbb{Z}$-points $N(\mathbb{Z})$. But the center $Z(N)$ is also a $\mathbb{Q}$-defined algebraic subgroup in N, which implies that $Z(N)(\mathbb{Z})$ is a lattice in $Z(N)$ by the Malcev theorem. Again, since $Z(\Gamma_N)$ is commensurable with $Z(N)(\mathbb{Z})$, we get the required assertion (the relation of commensurability inherits the property of being a lattice, [31]).

Now, since $Z(\Gamma_N)$ is a lattice in the *abelian* Lie group $Z(N)$, we get an isomorphism

$$Z(N)/Z(\Gamma_N) \cong S^1 \times ... \times S^1$$

which allows us to generate an S^1-action on N/Γ_N as follows. Since $\exp : \mathfrak{z}(\mathfrak{n}) \to Z(N)$ is a diffeomorphism, one can find $X_\gamma = \exp^{-1}(\gamma)$, $X_\gamma \in \mathfrak{z}(\mathfrak{n})$ for any $\gamma \in Z(\Gamma_N)$. Let $\pi : Z(N) \to Z(N)/Z(\Gamma_N)$ be the natural projection and denote by

$$[\exp tX_\gamma] = \pi(\exp tX_\gamma) \subset S^1 \times ... \times S^1, \quad t \in \mathbb{R}.$$

We claim, first, that the equality

$$[\exp tX_\gamma] \cdot n\Gamma_N = \exp tX_\gamma \cdot n\Gamma$$

correctly defines an S^1-action on F (this is the action defined by D. McDuff in [22] in the nilmanifold case). Indeed, if we take $\exp tX_\gamma \cdot \gamma'$, $\gamma' \in Z(\Gamma_N) \subset Z(N)$, we will obviously get

$$\exp tX_\gamma \cdot \gamma' \cdot n\Gamma_N = \exp tX_\gamma \cdot n\Gamma_N,$$

since $\gamma' \in Z(N)$.

Now, *we claim that this action can be naturally extended to the whole S, of course, under the assumptions of Theorem 1.* Namely, recall that $\Gamma = \Gamma_A \times_\varphi \Gamma_N$ and the multiplication in the semidirect product is given by the action of automorphisms of the form $\varphi(a_\gamma) \in \mathrm{Aut}(\Gamma_N)$:

$$(a_\gamma, n_\gamma) \cdot (a'_\gamma, n'_\gamma) = (a_\gamma \cdot a'_\gamma, n_\gamma \varphi(a_\gamma)(n'_\gamma))$$

In particular, if we take $(e, u_n) \in Z_n(\Gamma_N)$, we will obtain $z = (e, u_n) \in Z(\Gamma_N)$. Now, consider the following two possibilities:

1) $\varphi(a_\gamma)(u_n) = u_n$ for all $a_\gamma \in \Gamma_A$,
2) $\varphi(a_\gamma)(u_n) = u_n^{-1}$ for some a_γ.

Note that *no other possibility may occur*, since $Z_n(\Gamma_N) \cong \mathbb{Z}$ and the infinite cyclic group has only two automorphisms. Here we use also the assumption of the Theorem that the refined upper series is *characteristic*, which means that $Z_n(\Gamma_N)$ is preserved by any automorphism.

Consider the first possibility. Obviously, the multiplication rule in the semidirect product shows that

$$z = (e, u_n) \in Z(\Gamma) \tag{3}$$

Indeed,

$$(e, u_n) \cdot (a, n) = (a, u_n \cdot n)$$

$$(a, n) \cdot (e, u_n) = (a, n \cdot \varphi(a)(\cdot u_n)) = (a, n \cdot u_n) = (a, u_n \cdot n).$$

Thus, the element z lies simultaneously in the centers of both Γ_N and Γ. Note, however, that the McDuff construction requires *only the existence of a nontrivial element in the center of the lattice.* Therefore, applying the McDuff construction to the case G/Γ we can generate the free S^1-action on S by z *which extends the given S^1-action on F*. Finally, we get a free and obviously symplectic (because of homogeneity of $\omega_0 + \omega_1$) S^1-action on S.

Let ξ be the fundamental vector field corresponding to this action. Now, use the homology and cohomology sequences corresponding to this bundle:

$$H_*(F) \to H_*(S) \to H_*(T)$$

$$H^*(T) \to H^*(S) \to H^*(F)$$

In the sequel, we denote the embedding of F to S by j and the projection of S onto T by p and the corresponding maps on the cohomology and homology level by j^*, p^* (respectively, j_*, p_*). Note that F is a submanifold in S and that the initial S^1-action induces the fundamental vector field ξ^F on F. Recall from [22] that since ξ^F is a fundamental vector field of an S^1-action, the homology class $[\xi^F] \in H_1(F)$ is well-defined. Now, use the *non-commutativity* of N. Recall from formula (2) that in this case we can choose the generating element z lying in the commutator subgroup $[\Gamma_N, \Gamma_N]$ which obviously implies

$$[\xi^F] = 0$$

(since $H_1(F) = \Gamma_N/[\Gamma_N, \Gamma_N]$, this is the argument of McDuff applied to F). Since

$$j_*[\xi^F] = [\xi], \quad [\xi] \in H_1(S)$$

we get $[\xi] = 0$. Note that $[\xi]$ is Poincaré dual to the cohomology class $[i(\xi)\omega^n] \in H^{2n-1}(S)$. Since, by assumption, $H^*(S)$ is a Lefschetz algebra

$$[i(\xi)\omega^n] = 0 \quad \text{if and only if} \quad [i(\xi)\omega] = 0.$$

Note that

$$j^*(i(\xi)\omega) = i(\xi^F)j^*\omega = i(\xi^F)\omega_1.$$

We get

$$j^*[i(\xi)\omega] = [i(\xi^F)\omega_1] = 0$$

since we have already shown that $[i(\xi)\omega^n] = 0$. Finally, we see that if N is non-abelian, $F = N/(N \cap \Gamma)$ admits an S^1-action which satisfies the following conditions

1) $i(\xi^F)\omega_1$ is exact (or the action of S^1 is hamiltonian),
2) ξ^F has no zeros.

Since these two conditions are incompatible (it is well known, cf. [22, Prop. 3], or [23]), we obtain a contradiction which follows from our assumption of the non-commutativity of N.

Thus, it remains to consider the second possibility of the $\varphi(a_\gamma)$-action on the infinite cyclic group $Z_n(\Gamma_N) = \langle u_n \rangle$. Here, however, we can easily reduce the proof to the previous case as follows. Since there are only two automorphisms of infinite cyclic group, the subgroup

$$\Gamma' = \Gamma'_A \times_\varphi \Gamma_N = \{(a,n) | \quad \varphi(a)(u_n) = u_n\}$$

has index 2 in Γ. Hence, we obtain the double covering $G/\Gamma' \to G/\Gamma$ such that G/Γ' satisfies the condition (1) which we have already settled. But the Lefschetz condition is cohomological (over the reals) and does not change up to a finite covering. ■

Proof of Corollary 1

The possibility of weakening assumption (ii) is obvious, since the equality $\varphi(a)(z) = z$ is the only condition which is required for accomplishing the proof. Note that if $\dim Z(N) = 1$, we get $\varphi(a)(Z(\Gamma_N)) \subset Z(\Gamma_N)$, but $Z(\Gamma_N) \cong \mathbb{Z}$ and we accomplish the argument repeating the proof of the Theorem. ∎

3. Discussion of examples

Thus, the non-Lefschetz condition 'propagates' to the total space of the Mostow fibration under an additional assumption $\varphi(a)(z) = z$ for at least one $z \in Z(\Gamma_N)$ and in the latter case it forces G/Γ to be of very particular type. The purpose of this section is to discuss the question whether this condition is very restrictive or not. Note, first, that in general this condition need not be satisfied, as Example 1 in [4] shows. However, one can also notice that this example is of the form G/Γ with $G = \mathbb{R}^2 \times_\varphi \mathbb{R}^2$ and this means that *there are no additional restrictions regarding the action of $\varphi(a)$ on $\mathbb{R}^2$ except the obvious one $\varphi(a) \in SL_2(\mathbb{Z})$.* As one may expect this additional conditions appear when the *class of nilpotency* grows.

Example 1. Kodaira–Thurston manifold and 6-dimensional solvmanifolds

Proposition 1. *Let G/Γ be any solvmanifold such that $G = A \times_\varphi N$ and $\Gamma = \Gamma_A \times_\varphi \Gamma_N$. Assume that:*

1) $N = [G,G]$
2) N contains a 3-dimensional Heisenberg group N_3 such that $\varphi(a)(N_3) \subset N_3$,
3) $\dim(N_3 \cap Z(N)) = 1$,
4) $N_3 \cap \Gamma$ is a lattice in N_3

Then there exists an element $z \in Z(\Gamma_N)$ such that $\varphi(a)(z) = z$ for all $a \in \Gamma_A$.

Corollary 2.

1) The Kodaira–Thurston manifold cannot be a fiber of the Mostow fibration of any Kähler compact solvmanifold of completely solvable Lie group.
2) The assertion of the Theorem is valid for any 4- and 6-dimensional solvmanifold.

Proof. Since $\varphi(a)(\Gamma_N) \subset \Gamma_N$, and $\varphi(a)(N_3) \subset N_3$ we get

$$\varphi(a)(N_3 \cap \Gamma) = \varphi(a)(N_3 \cap \Gamma_N) \subset N_3 \cap \Gamma_N$$

which means that $\varphi(a)$ is an automorphism of the lattice $\Gamma_N \cap N_3$. Since

$$\varphi(a)(Z(\Gamma_N)) \subset Z(\Gamma_N)$$

we see that $\varphi(a)$ *preserves an infinite cyclic group* $N_3 \cap Z(\Gamma_N) \cong \mathbb{Z}$ and the proof follows. ∎

Proof of Corollary 2

Note, first, that the Kodaira–Thurston manifold is a nilmanifold of the form

$$(N_3 \times \mathbb{R})/(\Gamma_3 \times \mathbb{Z})$$

where Γ_3 is a group of upper triangular unipotent matrices with integral entries. Note that any automorphism $\varphi(a)$ lifted from the automorphism of $\Gamma_3 \times \mathbb{Z}$ *preserves* N_3 (it can be verified on the Lie algebra level using the standard base of $\mathfrak{n}_3$, say, e_1, e_2, e_3 with $[e_1, e_2] = e_3$). Since $\varphi(a)$ was lifted from an automorphism of the lattice $\Gamma = \Gamma_3 \times \mathbb{Z}$, we get $\varphi(a)(\Gamma_3) = \varphi(a)(N_3 \cap \Gamma) \subset \Gamma \cap N_3 = \Gamma_3$. The latter implies that $\varphi(a)$ preserves the center of the latter group which is also an *infinite cyclic group.* The proof of (1) is completed.

To prove the second part of the corollary it is enough to notice that *four-dimensional* real Lie algebras are exhausted by the list below [3]:

$$\text{abelian}, \quad \mathfrak{n}_3 \times \mathbb{R}, \quad \mathfrak{n}_4$$

where

$$\mathfrak{n}_4 = \langle x_1, x_2, x_3, x_4 \rangle$$

$$[x_1, x_2] = x_3, \quad [x_1, x_3] = x_4, \quad [x_1, x_4] = 0.$$

Now, one can see that in the first case the proof follows from the first part of the corollary and in the second case it follows directly from Theorem 1, since $\dim \mathfrak{z}(\mathfrak{n}_4) = 1$. Indeed, if $\dim G/\Gamma = 4$ we get even a stronger result [24] and in the 6-dimensional case we have only one possibly non-toral decomposition for dimensions, namely, $2 + 4$. From the table below we see that either the fiber is the Kodaira–Thurston manifold, or it has the form N_4/Γ_4 where N_4 corresponds to $\mathfrak{n}_4$ defined above. Both cases, however, are eliminated by the hard Lefschetz condition. ∎

Example 2. Unipotent groups of Chevalley type

Proposition 2.

1) *Assume that $N = [G, G]$ satisfies the following property: the group $\mathrm{Aut}(N)$ is solvable. Then the manifold G/Γ cannot carry Kähler structures.*
2) *In particular, let $N = [G, G]$ satisfy the property that the subgroup of $\mathbb{Q}$-points N_Q of N considered as a $\mathbb{Q}$-defined algebraic group is a unipotent group of Chevalley type but not of the type A_2. Then G/Γ cannot carry Kähler structures.*

Proof. Recall the known criterion for the existence of lattices in simply connected nilpotent Lie groups. By definition, a *$\mathbb{Q}$-structure on a Lie algebra* $\mathfrak{n}$ is a basis in $\mathfrak{n}$, say, $e_1, ..., e_n$ such that all structure constants c_{ij}^k of $\mathfrak{n}$ are *rational.* Recall [25, Theorem 2.12] that N *contains a lattice if and only if* $\mathfrak{n}$ *admits a $\mathbb{Q}$-structure.* Note that the *exact meaning* of this is described as follows.

Let there be given a $\mathbb{Q}$-structure determined by the basis $e_1, ..., e_n$ and let $\mathfrak{n}_Q$ denote the vector space (and Lie algebra) spanned by $e_1, ..., e_n$ over $\mathbb{Q}$:

$$\mathfrak{n}_Q = \langle e_1, ..., e_n \rangle_Q \subset \mathfrak{n}.$$

Take the lattice $L = \mathbb{Z}e_1 \oplus ... \oplus \mathbb{Z}e_n$ in the real vector space $\mathfrak{n}$. Then, by [25, Theorem 2.12] $\Gamma = \langle \exp(L) \rangle \subset N$ is a lattice in N (here $\langle ... \rangle$ denotes the subgroup generated by $\exp(L)$). Conversely, if $\Gamma \subset N$ is a lattice in N, one obtains that a $\mathbb{Z}$-submodule $L = \langle \exp^{-1}(\Gamma) \rangle$ generated by the set $\exp^{-1}(\Gamma)$ in $\mathfrak{n}$ is a lattice in the real vector space $\mathfrak{n}$ and for *each basis* $e'_1, ..., e'_n$ such that $e'_i \in L, i = 1, ..., n$ the corresponding structure constants $'c^k_{ij}$ are rational.

Now, let $\varphi : \Gamma \to \Gamma$ be any automorphism. From the construction above we see that $\varphi_* : \mathfrak{n} \to \mathfrak{n}$ preserves $\exp^{-1}(\Gamma)$ and, therefore,

$$\varphi_*(L) \subset L.$$

Let

$$\mathfrak{n}^L_Q = \langle L \rangle_Q = \langle e'_1, ..., e'_n \rangle_Q \subset \mathfrak{n}$$

be the Lie subalgebra defined over the rationals in the same way as $\mathfrak{n}_Q$. We see that $\varphi_*(\mathfrak{n}^L_Q) \subset \mathfrak{n}^L_Q$, which follows from the previous two formulas. Let $\mathrm{Aut}^Q(\mathfrak{n})$ be the group of all automorphisms of $\mathfrak{n}$ preserving $\mathfrak{n}^L_Q$. By the assumptions of the proposition the latter group is also solvable and, hence, there exists a basis, say, $e''_1, ..., e''_n$ such that *all automorphisms* in $\mathrm{Aut}^Q(\mathfrak{n})$ are triangular. Since $\mathfrak{n}^L_Q$ is a $\mathbb{Q}$-defined Lie algebra *over the rationals*, we can choose without loss of generality $e''_n \in \mathfrak{z}(\mathfrak{n}^L_Q) \cap [\mathfrak{n}^L_Q, \mathfrak{n}^L_Q]$ (since the center and the commutator subalgebra are characteristic, the same is valid for their intersection). However, since all matrices are triangular, e''_n must be preserved (up to a scalar $\alpha \in \mathbb{Q}$). However, because of the *first part of the criterion*, there exists a (possibly new) lattice

$$\Gamma' = \langle \exp(\mathbb{Z}e''_1 \oplus ... \oplus \mathbb{Z}e''_n) \rangle$$

in N. Of course, Γ' is commensurable with Γ and satisfies the property $\exp(e''_n) \in \Gamma' \cap Z(N) = Z(\Gamma')$. Thus, without loss of generality, we can change N/Γ by N/Γ'. However, the lattice Γ' possesses an element in the center ($z = \exp(e''_n)$) which is preserved under the action of *any* automorphism $\varphi(a), a \in \Gamma'_A$ (as in the Corollary). Again, one should only notice that since z lies in an infinite cyclic group $\varphi(a)(z) = z$, or $\varphi(a)(z) = z^{-1}$.

Note that this argument does not work for abelian fibers: the corresponding group of automorphisms becomes too big and it may be not possible to choose them all triangular.

Now, it is known [9] that any unipotent group of Chevalley type which is not contained in the simple algebraic group A_2 has solvable group of *algebraic* automorphisms. However, $\mathrm{Aut}_Q(N)$ consists of algebraic automorphisms (it is obvious, since the structure of an algebraic group on N is given by the exponential mapping such that exp becomes an isomorphism of algebraic varieties (cf. [31, p. 45]). The proof is completed. ∎

Remark 2. We end this article with the remark due to the referee and [18]. This remark shows what is the difficulty in developing a "hamiltonian" approach to the problem. The following proposition is contained in [18] and is well known.

Proposition 3. *Let G/Γ be a homogeneous space determined by a discrete co-compact subgroup Γ. If G/Γ admits a G-ivariant symplectic form, G must be abelian.*

Proof. It is known [12, p. 183] that the commutator subgroup $[G, G]$ acts on G/Γ in a hamiltonia way: $i_{X_\xi}\omega = df_\xi$ for a function f_ξ on G/Γ and a fundamental vector field X_ξ on G/Γ determined by a vector $\xi \in [\mathfrak{g}, \mathfrak{g}]$. Since G/Γ is compact, the critical set $C_\xi = \{x \in G/\Gamma | df_\xi = 0\}$ is not empty. But then for any $x \in C_\xi$ one obtains $X_\xi(x) = 0$ and since Γ is discrete, $X_\xi = 0$ everywhere. Hence, $\xi = 0$ and $[\mathfrak{g}, \mathfrak{g}] = 0$ as required. ∎

Thus, we stress once more that our symplectic forms are not invariant in the sense of Proposition 3. On the other hand, it seems that this simple observation combined with the homotopic considerations of [20] may shed some new light on the problems discussed in this article.

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