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Exponential Lie Groups with Disconnected Near Cartan Subgroups

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Although Lie groups and their exponential function have been considered for 100 years, no necessary and sufficient conditions for the surjectivity of the exponential function are known. The survey [3] reports on recent progress. In particular, some conjectures concerning equivalent conditions to the surjectivity of the exponential function were given. One of them is formulated as Problem 5.5 in [3]. It says that an equivalent condition to the exponentiality of Lie groups, i.e. a Lie group with surjective exponential map, might be the connectedness of the near Cartan subgroups. For the definition of near Cartan subgroups see [4] or [3]. All Cartan subgroups are near Cartan subgroups. If all Cartan subgroups of a connected Lie group G are connected, its exponential function is surjective ([3], Paragraph preceding 5.5). Theorem 3.17 of [7] says that in solvable Lie groups with surjective exponential function all near Cartan subgroups are connected.

However, we point out here, that there are Lie groups with surjective exponential function and disconnected near Cartan subgroups. In fact, an example used by Hofmann and Djoković in [3] (cf. Example 5.3) to illustrate various unexpected properties of the exponential function also exemplifies this claim and thus disproves the conjecture formulated in [3], Problem 5.5. The author thanks K.H. Hofmann for useful hints concerning the presentation of this note.

We recall Example 5.3 of [3]. Denote by $\mathbb{S}^3$ the group of the unit quaternions. Let $M := \left\{ \begin{pmatrix} c & d \\ 0 & c \end{pmatrix} : c \in \mathbb{S}^3,\ d \in \mathbb{H} \right\}$. This group is isomorphic to the semidirect product $\mathbb{S}^3 \ltimes \mathbb{H}$. Now we have the following lemma:

Lemma. *Let G be an exponential Lie group and B a connected solvable subgroup of G containing a Cartan subgroup H of G. Then each near Cartan subgroup of B is a near Cartan subgroup of G.*

Proof. By Théorème VII.3.4.3 of [2] all Cartan subalgebras in a solvable Lie algebra are conjugate. In [5] is shown that in a solvable Lie group each Cartan subgroup is connected and thus is the exponential image of a Cartan subalgebra.

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Hence all Cartan subgroups of a solvable Lie group are conjugate. Since B contains one Cartan subgroup of G each Cartan subgroup of B is also a Cartan subgroup of G. Each near Cartan subgroup of B can be written as $\lim H_n$, where H_n is a conjugate of H. Thus $\lim H_n$ is also a near Cartan subgroup of G. ∎

Theorem. *M is an exponential Lie group with a disconnected near Cartan subgroup.*

Proof. In 5.3 of [3] is proved that M contains a maximal solvable connected subgroup B, which is not exponential, namely TN with $T := \{e^{it}\cdot \mathbf{1}_2 : t \in \mathbb{R}\}$ and $N := \left\{ \begin{pmatrix} 0 & h \\ 0 & 0 \end{pmatrix} : h \in \mathbb{H} \right\}$. According to [3], $H := \left\{ \begin{pmatrix} e^{ti} & z \\ 0 & e^{ti} \end{pmatrix} : t \in \mathbb{R},\ z \in \mathbb{C} \right\}$ is a Cartan subgroup of M, contained in B. By the lemma, M possesses a disconnected near Cartan subgroup. ∎

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