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Dipolarizations in semisimple Lie algebras and homogeneous parakähler manifolds

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Abstract. A dipolarization in a Lie algebra $\mathfrak{g}$ is two polarizations $\mathfrak{g}^{\pm}$ in $\mathfrak{g}$ at a common linear form on $\mathfrak{g}$ satisfying $\mathfrak{g}=\mathfrak{g}^{+}+\mathfrak{g}^{-}$. We study dipolarizations in semisimple Lie algebras, especially, the relation between dipolarizations and gradations. As an application, we give a relation between semisimple homogeneous parakähler manifolds and hyperbolic semisimple orbits. For $\mathfrak{g}$ real semisimple, we determine the characteristic elements, from which dipolarizations can be constructed.

Key words: Semisimple Lie algebra, graded Lie algebra, parabolic subalgebra, dipolarization, parakähler manifold.

MS classification: 17B20, 22E46, 53C12, 15, 30.

Introduction

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{F}$ ($=\mathbb{R}$ or $\mathbb{C}$), and $\mathfrak{g}^{\pm}$ two subalgebras of $\mathfrak{g}$ and let f be a linear form on $\mathfrak{g}$. Then the triple $\{\mathfrak{g}^{\pm}, f\}$ is called a *dipolarization* in $\mathfrak{g}$, if $\mathfrak{g}^{\pm}$ are totally isotropic subspaces with respect to df and $\mathfrak{g}=\mathfrak{g}^{+}+\mathfrak{g}^{-}$ and further $\mathfrak{g}^{+}\cap\mathfrak{g}^{-}$ coincides with the centralizer $\mathfrak{g}^{f}$ of f in $\mathfrak{g}$ (cf. Definition 1.2). A dipolarization, or more generally, a weak dipolarization was introduced by Kaneyuki [8]. If $\mathfrak{g}$ is semisimple, weak dipolarizations reduce to dipolarizations. A significant property is that there is a one-to-one correspondence between weak dipolarizations and infinitesimal classes of homogeneous parakähler manifolds ([8]). A homogeneous parakähler manifold is, by definition, a homogeneous symplectic manifold (of a Lie group G) which admits a pair of invariant Lagrangean foliations. Note that a parahermitian symmetric space is a symmetric coset space with homogeneous parakähler structure.

A fundamental problem on homogeneous parakähler manifolds is the classification and construction of such manifolds. In this paper, we settle this problem

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in some sense, for the case where $\mathfrak{g}$ is semisimple (cf. Theorems 3.7 and 3.8). Let us go back to dipolarizations. Lemma 1.3 says that $\{\mathfrak{g}^{\pm}, f\}$ is a dipolarization if and only if $\mathfrak{g}^{\pm}$ are two polarizations at f and $\mathfrak{g} = \mathfrak{g}^{+} + \mathfrak{g}^{-}$ is valid. This reduces the study of dipolarizations to that of polarizations. Dipolarizations in solvable Lie algebras were studied in [2], [11] and [3], and weak dipolarizations in compact Lie algebras were studied in [7].

In this paper, we are concerned with dipolarizations in real or complex semisimple Lie algebras. Let $\{\mathfrak{g}^{\pm}, f\}$ be a dipolarization in a semisimple Lie algebra $\mathfrak{g}$. Then it follows from Ozeki-Wakimoto [12] that $\mathfrak{g}^{\pm}$ are parabolic subalgebras. The Killing dual Z of f, called the *characteristic element* of $\{\mathfrak{g}^{\pm}, f\}$, is a semisimple element of $\mathfrak{g}$, and further the centralizer $\mathfrak{c}(Z)$ in $\mathfrak{g}$ is a Levi subalgebra of $\mathfrak{g}^{\pm}$ (Proposition 2.3). Our main concerns are:

Problem A. Are the parabolic subalgebras $\mathfrak{g}^{\pm}$ opposite to each other?

Problem B. Which semisimple element can be the characteristic element of a dipolarization?

Problem A is related to the classification of dipolarizations in semisimple Lie algebras. In Section 2, we consider the above problems for the case where $\mathfrak{g}$ is complex semisimple. Problem A is settled affirmatively (Theorem 2.8), and any semisimple element is the characteristic element of a dipolarization (Theorem 2.5). In the case where $\mathfrak{g}$ is real semisimple, the situation is more complicated. Let $\mathfrak{h}$ be a Cartan subalgebra containing Z. Let $\widetilde{\Delta}$ be the root system of the complexification of $\mathfrak{g}$ with respect to that of $\mathfrak{h}$, and $\widetilde{\Delta}_0(Z)$ be the totality of roots which vanish on Z. Then one can prove that every imaginary root lies in $\widetilde{\Delta}_0(Z)$ (Proposition 3.1). By using this, we have that Z can be imbedded in an Iwasawa (= maximally split) Cartan subalgebra (Lemma 3.3). These two things simplify the subsequent considerations.

The first main result is that any dipolarization $\{\mathfrak{g}^{\pm}, f\}$ comes from a gradation of $\mathfrak{g}$, that is, there exists a gradation $\mathfrak{g} = \sum_{k=-\nu}^{\nu} \mathfrak{g}_k$ such that $\mathfrak{g}^{\pm} = \sum_{k=0}^{\nu} \mathfrak{g}_{\pm k}$ (Theorem 3.6). This settles Problem A affirmatively for $\mathfrak{g}$ real semisimple. As an application, we have a characterization of semisimple homogeneous parakähler manifolds: Let G be a connected semisimple Lie group. Then a coset space G/H is a homogeneous parakähler manifold if and only if it is a G-equivariant covering manifold of a hyperbolic semisimple $\mathrm{Ad}\, G$-orbit (Theorem 3.7). This can be also viewed as a geometric characterization of hyperbolic semisimple orbits.

Problem B is settled in Section 4. For $\mathfrak{g}$ real semisimple, we give two kinds of characterizations for a semisimple element Z to be the characteristic element of a dipolarization, in terms of $\widetilde{\Delta}_0(Z)$ (Proposition 4.1, Theorem 4.3). For $\mathfrak{g}$ real simple, we give more explicit determination of the characteristic elements of dipolarizations, and indicate in which case non-hyperbolic characteristic elements occur (Propositions 4.5 – 4.14). On the other hand, there are real simple Lie algebras admitting only dipolarizations whose characteristic elements are hyperbolic semisimple elements (Theorem 4.4).

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Notation:

$\overline{\mathfrak{g}}$ denotes the complexification of a Lie algebra $\mathfrak{g}$ or a vector space $\mathfrak{g}$. $\mathfrak{c}(E)$ denotes the centralizer of an element E in a Lie algebra. $\widetilde{\Delta}_0(E)$ denotes the totality of roots which vanish on an element E. $\Delta_0(E)$ denotes the totality of restricted roots which vanish on E, unless otherwise stated.

1. Polarizations and dipolarizations

Definition 1.1. [e.g. Dixmier [4]] Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{F}$ $(= \mathbb{R}$ or $\mathbb{C})$. Then a pair $\{\mathfrak{m}, f\}$ is called a *polarization* in $\mathfrak{g}$, if the following conditions are satisfied:

P0) $\mathfrak{m}$ is a subalgebra (over $\mathbb{F}$) of $\mathfrak{g}$ and f is a $\mathbb{F}$-linear form on $\mathfrak{g}$.
P1) $\mathfrak{m}$ is totally isotropic with respect to f, that is, $f([\mathfrak{m}, \mathfrak{m}]) = 0$.
P2) $\mathfrak{m}$ is maximal among subspaces of $\mathfrak{g}$ which satisfy P1).

Definition 1.2. [cf. [8]] Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{F}$. Then a triple $\{\mathfrak{g}^{\pm}, f\}$ is called a *dipolarization* (over $\mathbb{F}$) in $\mathfrak{g}$, if the following conditions are satisfied:

DP0) $\mathfrak{g}^+$ and $\mathfrak{g}^-$ are two subalgebras of $\mathfrak{g}$ and f is an $\mathbb{F}$-linear form on $\mathfrak{g}$.
DP1) $f([\mathfrak{g}^+, \mathfrak{g}^+]) = f([\mathfrak{g}^-, \mathfrak{g}^-]) = 0$.
DP2) $f([X, \mathfrak{g}]) = 0$ if and only if $X \in \mathfrak{g}^+ \cap \mathfrak{g}^-$.
DP3) $\mathfrak{g} = \mathfrak{g}^+ + \mathfrak{g}^-$

The following lemma shows how dipolarizations and polarizations are related.

Lemma 1.3. *Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{F}$. Then $\{\mathfrak{g}^{\pm}, f\}$ is a dipolarization in $\mathfrak{g}$ if and only if*

ND1) $\{\mathfrak{g}^+, f\}$ *and* $\{\mathfrak{g}^-, f\}$ *are two polarizations in* $\mathfrak{g}$, *and*
ND2) $\mathfrak{g} = \mathfrak{g}^+ + \mathfrak{g}^-$.

Proof. Suppose first that (ND1) and (ND2) are valid. Consider the centralizer $\mathfrak{g}^f$ of f in $\mathfrak{g}$, that is,

$$\mathfrak{g}^f = \{X \in \mathfrak{g} : f([X, \mathfrak{g}]) = 0\}. \tag{1}$$

To prove the "if"part, we only have to show (DP2), which is equivalent to the equality

$$\mathfrak{g}^f = \mathfrak{g}^+ \cap \mathfrak{g}^-. \tag{2}$$

The inclusion $\supset$ in (2) follows easily from (ND2) and (ND1). To prove the converse inclusion, consider the two subspaces $\mathfrak{m}^{\pm} = \mathfrak{g}^{\pm} + \mathfrak{g}^f$ of $\mathfrak{g}$. Then, by (ND1) and (1), we have

$$f([\mathfrak{m}^+, \mathfrak{m}^+]) = f([\mathfrak{g}^+ + \mathfrak{g}^f, \mathfrak{g}^+ + \mathfrak{g}^f]) = f([\mathfrak{g}^+, \mathfrak{g}^+]) + f([\mathfrak{g}^+, \mathfrak{g}^f]) + f([\mathfrak{g}^f, \mathfrak{g}^f]) = 0,$$

which implies that $\mathfrak{m}^+$ is a totally isotropic subspace of $\mathfrak{g}$ containing $\mathfrak{g}^+$. By the maximality of $\mathfrak{g}^+$ ((ND1) and (P2)), we have $\mathfrak{m}^+ = \mathfrak{g}^+$ and hence $\mathfrak{g}^f \subset \mathfrak{g}^+$. Similarly we have $\mathfrak{g}^f \subset \mathfrak{g}^-$. We have thus proved (2). The "only if" part was already proved in [7]. ∎

Definition 1.4. Let $\{\mathfrak{g}_i^{\pm}, f_i\}$ $(i = 1, 2)$ be two dipolarizations in a Lie algebra $\mathfrak{g}$. We say that $\{\mathfrak{g}_1^{\pm}, f_1\}$ and $\{\mathfrak{g}_2^{\pm}, f_2\}$ are *weakly isomorphic*, if there exists an automorphism φ of the Lie algebra $\mathfrak{g}$ such that $\varphi(\mathfrak{g}_1^{+}) = \mathfrak{g}_2^{+}$ and $\varphi(\mathfrak{g}_1^{-}) = \mathfrak{g}_2^{-}$. We say that the two dipolarizations are *isomorphic*, if they are weakly isomorphic and further $f_1 = f_2 \circ \varphi$ is valid.

Definition 1.5. Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{F}$ $(= \mathbb{R}$ or $\mathbb{C})$, and let $\{\mathfrak{g}^{\pm}, f\}$ be a dipolarization in $\mathfrak{g}$. Let $Z \in \mathfrak{g}$ be a unique element defined by

$$(Z, X) = f(X), \qquad X \in \mathfrak{g}, \tag{3}$$

where $(\ ,\ )$ is the Killing form of $\mathfrak{g}$. Z is called the *characteristic element* of the dipolarization $\{\mathfrak{g}^{\pm}, f\}$.

In view of the above definition, we often say a dipolarization $\{\mathfrak{g}^{\pm}, Z\}$ instead of $\{\mathfrak{g}^{\pm}, f\}$. In the same way one can define the characteristic element of a polarization in $\mathfrak{g}$. We have easily

Lemma 1.6. *Let $\mathfrak{g}$ be a semisimple Lie algebra, and let Z be the characteristic element of a dipolarization $\{\mathfrak{g}^{\pm}, f\}$. Then $\mathfrak{g}^f$ coincides with the centralizer $\mathfrak{c}(Z)$ of Z in $\mathfrak{g}$.*

2. Dipolarizations in complex semisimple Lie algebras

First we need

Proposition 2.1. Ozeki-Wakimoto [12, Th. 2.2] *Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$. Let $\{\mathfrak{m}, Z\}$ be a polarization in $\mathfrak{g}$ with characteristic element Z. Then $\mathfrak{m}$ is a parabolic subalgebra of $\mathfrak{g}$, and $[Z, \mathfrak{m}]$ is the nilradical of $\mathfrak{m}$.*

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{R}$ and $\{\mathfrak{g}^{\pm}, f\}$ be a dipolarization in $\mathfrak{g}$. Let $\overline{\mathfrak{g}}^{+}$ and $\overline{\mathfrak{g}}^{-}$ be the complexifications of $\mathfrak{g}^{+}$ and $\mathfrak{g}^{-}$, respectively, and let $\overline{f}$ be the $\mathbb{C}$-linear extension of f to the complexification $\overline{\mathfrak{g}}$ of $\mathfrak{g}$.

Lemma 2.2. *$\{\overline{\mathfrak{g}}^{\pm}, \overline{f}\}$ is a dipolarization in $\overline{\mathfrak{g}}$.*

Proof. We only have to verify (DP2) for $\{\overline{\mathfrak{g}}^{\pm}, \overline{f}\}$. Let $X = X_1 + iX_2 \in \overline{\mathfrak{g}}$, $X_1, X_2 \in \mathfrak{g}$. Suppose that $\overline{f}([X, Y]) = 0$ for an arbitrary element $Y = Y_1 + iY_2 \in \overline{\mathfrak{g}}$, $Y_1, Y_2 \in \mathfrak{g}$. Then we have that $f([X_1, Y_1]) - f([X_2, Y_2]) = 0$ and $f([X_1, Y_2]) + f([X_2, Y_1]) = 0$. Putting $Y_2 = 0$, we get $f([X_1, \mathfrak{g}]) = f([X_2, \mathfrak{g}]) = 0$, which implies that $X_1, X_2 \in \mathfrak{g}^f = \mathfrak{g}^{+} \cap \mathfrak{g}^{-}$. Therefore $X = X_1 + iX_2 \in \overline{\mathfrak{g}^{+} \cap \mathfrak{g}^{-}} = \overline{\mathfrak{g}}^{+} \cap \overline{\mathfrak{g}}^{-}$, or equivalently, $\overline{\mathfrak{g}}^{\overline{f}} \subset \overline{\mathfrak{g}}^{+} \cap \overline{\mathfrak{g}}^{-}$. Similarly we have the converse inclusion. ■

By Proposition 2.1, Lemmas 2.2 and 1.3, $\mathfrak{g}^{\pm}$ are parabolic subalgebras of $\mathfrak{g}$. Note that the characteristic element of $\{\overline{\mathfrak{g}}^{\pm}, \overline{f}\}$ is the same as that of $\{\mathfrak{g}^{\pm}, f\}$.

Proposition 2.3. *Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{F}$ $(= \mathbb{R}$ or $\mathbb{C})$. Let $\{\mathfrak{g}^{\pm}, f\}$ be a dipolarization in $\mathfrak{g}$. Then the characteristic element Z of $\{\mathfrak{g}^{\pm}, f\}$ is semisimple, and further, the centralizer $\mathfrak{c}(Z)$ in $\mathfrak{g}$ is a Levi subalgebra of the parabolic subalgebras $\mathfrak{g}^{+}$ and $\mathfrak{g}^{-}$.*

Proof. Without loss of generality, one can assume $\mathbb{F} = \mathbb{C}$. By Lemma 1.3 and Proposition 2.1, $\mathfrak{g}^{\pm}$ are parabolic subalgebras of $\mathfrak{g}$. Choose two Borel subalgebras $\mathfrak{b}^{\pm} \subset \mathfrak{g}$ such that $\mathfrak{b}^{+} \subset \mathfrak{g}^{+}$ and $\mathfrak{b}^{-} \subset \mathfrak{g}^{-}$. The intersection $\mathfrak{b}^{+} \cap \mathfrak{b}^{-}$ contains a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ (cf. Dixmier [4, Prop. 1.10.18]). Therefore it follows from the equality $\mathfrak{g}^{+} \cap \mathfrak{g}^{-} = \mathfrak{c}(Z)$ (cf. (2)) that $\mathfrak{h}$ is contained in $\mathfrak{c}(Z)$. The maximality of $\mathfrak{h}$ implies that the abelian subalgebra $\mathfrak{h}' = \mathfrak{h} + \mathbb{C}Z$ coincides with $\mathfrak{h}$. Consequently Z is contained in $\mathfrak{h}$ and hence semisimple. As a centralizer of a semisimple element, $\mathfrak{c}(Z)$ is reductive.

By semisimplicity of Z, the sum $\mathfrak{c}(Z) + [Z, \mathfrak{g}^{+}]$ is a direct sum. Since $\dim \mathfrak{g}^{+} = \dim \mathfrak{c}(Z) + \dim [Z, \mathfrak{g}^{+}]$, we have $\mathfrak{g}^{+} = \mathfrak{c}(Z) \oplus [Z, \mathfrak{g}^{+}]$, which implies that $\mathfrak{c}(Z)$ is a Levi subalgebra of $\mathfrak{g}^{+}$. As for $\mathfrak{g}^{-}$, we can proceed in the same way. ■

Now let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$. We denote the Killing form of $\mathfrak{g}$ by $(\ ,\ )$. Let $Z \in \mathfrak{g}$ be a non-zero semisimple element. Choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ containing Z and let $\widetilde{\Delta} = \Delta(\mathfrak{g}, \mathfrak{h})$, the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Put

$$\widetilde{\Delta}_0(Z) = \{\alpha \in \widetilde{\Delta} : \alpha(Z) = 0\} \tag{4}$$

Then we have

Lemma 2.4. *There exists a fundamental system $\widetilde{\Pi}$ of $\widetilde{\Delta}$ such that $\widetilde{\Pi}_0 := \widetilde{\Pi} \cap \widetilde{\Delta}_0(Z)$ is a fundamental system of $\widetilde{\Delta}_0(Z)$.*

Proof. We write $\mathfrak{h}$ as

$$\mathfrak{h} = \mathfrak{h}^{+} + \mathfrak{h}^{-}, \tag{5}$$

where $\mathfrak{h}^{+}$ and $\mathfrak{h}^{-}$ are the toroidal part and the vector part of $\mathfrak{h}$, respectively. Then Z can be written as

$$Z = Z^{+} + Z^{-}, \tag{6}$$

where $Z^{+} \in \mathfrak{h}^{+}$ and $Z^{-} \in \mathfrak{h}^{-}$ are the elliptic component and the hyperbolic component of Z, respectively. Furthermore, in our case we have that $\mathfrak{h}^{-} = i\mathfrak{h}^{+}$ and $\mathfrak{h}^{-}$ is the real part of $\mathfrak{h}$. The root system $\widetilde{\Delta}$ is naturally identified with a subset of $\mathfrak{h}^{-}$. Also we have

$$\alpha \in \widetilde{\Delta}_0(Z) \iff \alpha(Z^{-}) = \alpha(iZ^{+}) = 0. \tag{7}$$

Let us first consider the case (a) where Z^{-} and iZ^{+} are linearly independent over $\mathbb{R}$. Choose the linear order in $\widetilde{\Delta}$ defined by a basis of $\mathfrak{h}^{-}$ of the form $\{Z^{-}, iZ^{+}, \cdots\}$. Let $\widetilde{\Pi} = \{\alpha_1, \cdots, \alpha_l\}$ be the fundamental system of $\widetilde{\Delta}$ with respect to this order. Then $\widetilde{\Pi}$ is viewed as a basis of $\mathfrak{h}^{-}$ and we have the partition

$$\widetilde{\Pi} = \widetilde{\Pi}_0 \coprod \widetilde{\Pi}_1' \coprod \widetilde{\Pi}_1'', \tag{8}$$

where $\widetilde{\Pi}_0 = \widetilde{\Pi} \cap \widetilde{\Delta}_0(Z)$ and

$$\begin{aligned} \widetilde{\Pi}_1' &= \{\alpha_k \in \widetilde{\Pi} - \widetilde{\Pi}_0 : (\alpha_k, Z^{-}) > 0\}, \\ \widetilde{\Pi}_1'' &= \{\alpha_k \in \widetilde{\Pi} - \widetilde{\Pi}_0 : (\alpha_k, Z^{-}) = 0, (\alpha_k, iZ^{+}) > 0\}. \end{aligned} \tag{9}$$

By using (7) – (9) , we have that a root $\alpha \in \widetilde{\Delta}_0(Z)$ is expressed as a linear combination of simple roots in $\widetilde{\Pi}_0$. Next let us consider the case (b) where Z^{-}

and iZ^+ are linearly dependent. This case breaks up further into the two cases; (1) $Z^- \neq 0$, (2) $Z^- = 0$ and $Z^+ \neq 0$. For the case (1), choose a basis $\{Z^-, \cdots\}$ of $\mathfrak{h}^-$. Then (8) reduces to the partition $\widetilde{\Pi} = \widetilde{\Pi}_0 \coprod \widetilde{\Pi}_1'$. For the case (2), choose a basis $\{iZ^+, \cdots\}$ of $\mathfrak{h}^-$. Then (8) reduces to the partition $\widetilde{\Pi} = \widetilde{\Pi}_0 \coprod \widetilde{\Pi}_1''$. Therefore we obtain the lemma also for the case (b). ■

Theorem 2.5. *Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$, and let Z be a nonzero element of $\mathfrak{g}$. Then Z is the characteristic element of a dipolarization in $\mathfrak{g}$ if and only if Z is semisimple.*

Proof. The "only if" part is just Proposition 2.3. Suppose that Z is semisimple. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ containing Z. Choose a fundamental system $\widetilde{\Pi}$ of $\widetilde{\Delta}$ given in Lemma 2.4. Then we have a partition $\widetilde{\Pi} = \widetilde{\Pi}_0 \coprod \widetilde{\Pi}_1$, where $\widetilde{\Pi}_1 = \widetilde{\Pi} - \widetilde{\Pi}_0$. To this partition there corresponds a gradation $\mathfrak{g} = \sum_{k=-\nu}^{\nu} \mathfrak{g}_k$ (cf. [9]). Let $Z^* \in \mathfrak{h}^-$ be the characteristic element of this gradation. Then it follows from the choice of Z^* (cf. [9], see also (17)) that

$$\widetilde{\Delta}_0(Z^*) = \widetilde{\Delta}_0(Z), \tag{10}$$

which implies that the centralizers $\mathfrak{c}(Z)$ and $\mathfrak{c}(Z^*)$ in $\mathfrak{g}$ coincides: $\mathfrak{c}(Z) = \mathfrak{c}(Z^*) = \mathfrak{g}_0$. Let $\mathfrak{g}^+ = \sum_{k \geq 0} \mathfrak{g}_k$ and $\mathfrak{g}^- = \sum_{k \leq 0} \mathfrak{g}_k$, and let f be the linear form on $\mathfrak{g}$ which is the dual of Z with respect to the Killing form of $\mathfrak{g}$ (cf. (3)). Taking account of the equality $\mathfrak{c}(Z) = \mathfrak{c}(Z^*)$, we have, by the same way as in the proof of Theorem 4.2 [8], that $\{\mathfrak{g}^\pm, f\}$ is a dipolarization in $\mathfrak{g}$ whose characteristic element is Z. ■

Remark 2.6. Let f^* be the Killing dual of the above Z^*. Then $\{\mathfrak{g}^\pm, f^*\}$ is also a dipolarization in $\mathfrak{g}$. Let G be a connected Lie group with $Lie(G) = \mathfrak{g}$, and let G_0 be the connected Lie subgroup of G with $Lie(G_0) = \mathfrak{c}(Z) = \mathfrak{c}(Z^*)$. Then the coset space G/G_0 has two homogeneous symplectic structures induced respectively by f and f^*. These two symplectic structures have the common Lagrangean foliations corresponding to $\mathfrak{g}^\pm$.

Definition 2.7. Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{R}$ or $\mathbb{C}$, and let $\{\mathfrak{g}^\pm, f\}$ be a dipolarization in $\mathfrak{g}$. We say that $\{\mathfrak{g}^\pm, f\}$ *comes from a gradation of* $\mathfrak{g}$, if there exists a gradation $\mathfrak{g} = \sum_k \mathfrak{g}_k$ such that $\mathfrak{g}^+ = \sum_{k\geq 0} \mathfrak{g}_k$ and $\mathfrak{g}^- = \sum_{k \leq 0} \mathfrak{g}_k$.

Theorem 2.8. *Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$. Then any dipolarization $\{\mathfrak{g}^\pm, f\}$ in $\mathfrak{g}$ comes from a gradation of $\mathfrak{g}$.*

Proof. Let Z be the characteristic element of $\{\mathfrak{g}^\pm, f\}$. Then we have $\mathfrak{g}^+ \cap \mathfrak{g}^- = \mathfrak{c}(Z)$. By Theorem 2.5, Z is semisimple. Choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ containing Z, and consider the root system $\widetilde{\Delta} = \Delta(\mathfrak{g}, \mathfrak{h})$ as before. Since $\mathfrak{g}^+$ is a parabolic subalgebra (Proposition 2.1), There is a linear order in $\widetilde{\Delta}$ such that $\mathfrak{g}^+$ is written as

$$\mathfrak{g}^+ = \mathfrak{c}(Z) + \sum_{\alpha \in \widetilde{\Delta}^+ - \widetilde{\Delta}_0(Z)} \mathfrak{g}^\alpha, \tag{11}$$

where $\widetilde{\Delta}^+$ denotes the totality of positive roots in $\widetilde{\Delta}$. Since $\mathfrak{g}^-$ contains $\mathfrak{h}$ and since $\mathfrak{g} = \mathfrak{g}^+ + \mathfrak{g}^-$ by (DP3), for any root $\alpha \in \widetilde{\Delta}$ the root space $\mathfrak{g}^\alpha$ lies either in $\mathfrak{g}^+$ or in $\mathfrak{g}^-$. On the other hand $\mathfrak{g}^-$ is a parabolic subalgebra with Levi subalgebra $\mathfrak{c}(Z)$. Therefore we have

$$\mathfrak{g}^- \supset \mathfrak{c}(Z) + \sum_{\alpha \in \widetilde{\Delta}^- - \widetilde{\Delta}_0(Z)} \mathfrak{g}^\alpha, \tag{12}$$

where $\widetilde{\Delta}^-$ denotes the totality of negative roots in $\widetilde{\Delta}$. Let $\widetilde{\Pi}$ be the fundamental system of $\widetilde{\Delta}$ corresponding to $\widetilde{\Delta}^+$. Then $\widetilde{\Pi}_0 := \widetilde{\Pi} \cap \widetilde{\Delta}_0(Z)$ is a fundamental system of $\widetilde{\Delta}_0(Z)$. We have a partition $\widetilde{\Pi} = \widetilde{\Pi}_0 \coprod \widetilde{\Pi}_1$, where $\widetilde{\Pi}_1 := \widetilde{\Pi} - \widetilde{\Pi}_0$. To this partition there corresponds a gradation $\mathfrak{g} = \sum_k \mathfrak{g}_k$. Let Z^* be the characteristic element of this gradation. Then, as in the proof of Theorem 2.5, we have $\mathfrak{c}(Z) = \mathfrak{c}(Z^*) = \mathfrak{g}_0$. Any root $\alpha \in \widetilde{\Delta}$ takes an integer value on Z^*. We have that $\alpha \in \widetilde{\Delta}^+ - \widetilde{\Delta}_0(Z)$ (resp. $\widetilde{\Delta}^- - \widetilde{\Delta}_0(Z)$) if and only if $(\alpha, Z^*) > 0$ (resp. < 0). this means that $\mathfrak{g}^+ = \sum_{k \geq 0} \mathfrak{g}_k$ and $\mathfrak{g}^- = \sum_{k \leq 0} \mathfrak{g}_k$. ∎

Remark 2.9. Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$. Theorem 2.5 means that there is an epimorphism of the set of isomorphism classes of dipolarization in $\mathfrak{g}$ onto the set of semisimple orbits in $\mathfrak{g}$. In fact, this epimorphism is given by assigning the characteristic element to a dipolarization.

3. Dipolarizations in real semisimple Lie algebras

Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{R}$ and $Z \in \mathfrak{g}$ be a semisimple element. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ containing Z. Let $\overline{\mathfrak{g}}$ and $\overline{\mathfrak{h}}$ be the complexifications of $\mathfrak{g}$ and $\mathfrak{h}$, respectively. Let $\widetilde{\Delta} = \Delta(\overline{\mathfrak{g}}, \overline{\mathfrak{h}})$, the root system of $\overline{\mathfrak{g}}$ with respect to $\overline{\mathfrak{h}}$, and let $\widetilde{\Delta}_I$ be the set of imaginary roots in $\widetilde{\Delta}$ with respect to $\mathfrak{h}$ (cf. Hirai [6]). Contrary to the complex case, not every semisimple element of $\mathfrak{g}$ is the characteristic element of a dipolarization in $\mathfrak{g}$.

Proposition 3.1. *Let $\mathfrak{g}, Z$ and $\widetilde{\Delta}$ be as above. Suppose that Z is the characteristic element of a dipolarization in $\mathfrak{g}$. Then $\widetilde{\Delta}_I \subset \widetilde{\Delta}_0(Z)$ (cf. (4)).*

Proof. Let $\{\mathfrak{g}^\pm, f\}$ be a dipolarization in $\mathfrak{g}$ with Z as its characteristic element, and let us consider the complexified one $\{\overline{\mathfrak{g}}^\pm, \overline{f}\}$ given in Lemma 2.2. Since $\mathfrak{h}$ is contained in $\mathfrak{c}(Z) = \mathfrak{g}^+ \cap \mathfrak{g}^-$, the complexification $\overline{\mathfrak{h}}$ is contained in $\overline{\mathfrak{g}}^+ \cap \overline{\mathfrak{g}}^-$. Now let $\alpha \in \widetilde{\Delta}_I$. By (DP3) for $\overline{\mathfrak{g}}^\pm$, the root space $\mathfrak{g}^\alpha$ in $\overline{\mathfrak{g}}$ is contained in $\overline{\mathfrak{g}}^+$ or in $\overline{\mathfrak{g}}^-$. Suppose that $\mathfrak{g}^\alpha \subset \overline{\mathfrak{g}}^+$. Let σ be the conjugation of $\overline{\mathfrak{g}}$ with respect to $\mathfrak{g}$. Then we have $\sigma(\alpha) = -\alpha$. Noting that $\overline{\mathfrak{g}}^+$ is stable under σ, we obtain

$$\mathfrak{g}^{-\alpha} = \mathfrak{g}^{\sigma(\alpha)} = \sigma(\mathfrak{g}^\alpha) \subset \overline{\mathfrak{g}}^+. \tag{13}$$

On the other hand

$$\overline{f}([\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}]) = (Z, [\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}]) = ([Z, \mathfrak{g}^\alpha], \mathfrak{g}^{-\alpha}]) = \alpha(Z)(\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}). \tag{14}$$

The first member of (14) is equal to zero, by (13) and (DP1), which implies that $\alpha(Z) = 0$, and hence $\alpha \in \widetilde{\Delta}_0(Z)$. ∎

As a corollary, we have Theorem 1 in Hou-Deng-Kaneyuki [7].

Corollary 3.2. *A compact semisimple Lie algebra $\mathfrak{g}$ never admits nontrivial dipolarizations.*

Proof. Let $\{\mathfrak{g}^{\pm}, f\}$ be a dipolarization in $\mathfrak{g}$ with characteristic element Z. Since $\mathfrak{g}$ is compact, $\widetilde{\Delta}_I = \widetilde{\Delta}$ is valid. Therefore $\widetilde{\Delta}_0(Z) = \widetilde{\Delta}$ by Proposition 3.1. This implies that $\mathfrak{c}(Z) = \mathfrak{g}^{\pm} = \mathfrak{g}$ and hence $Z = 0$, i.e., $f = 0$. ∎

Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{R}$. A Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is said to be an *Iwasawa Cartan subalgebra*, if the vector part of $\mathfrak{h}$ has maximal possible dimension.

Lemma 3.3. *Let $\mathfrak{g}$ be as above and Z be the characteristic element of a dipolarization in $\mathfrak{g}$. Then Z is imbedded in an Iwasawa Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$.*

Proof. Choose a Cartan subalgebra $\mathfrak{h}'$ of $\mathfrak{g}$ containing Z (cf. Proposition 2.3). Let τ be a Cartan involution of $\mathfrak{g}$ leaving $\mathfrak{h}'$ stable, and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition by τ. By Sugiura [13], there exist complete representatives $\mathfrak{h}_1, \cdots, \mathfrak{h}_s$ of conjugate classes of Cartan subalgebras of $\mathfrak{g}$ such that they are τ-stable and that (1) $\mathfrak{h}_i^+ \subset \mathfrak{h}_1^+$ $(1 \leq i \leq s)$, (2) $\mathfrak{h}_i^- \subset \mathfrak{h}_s^-$ $(1 \leq i \leq s)$ and (3) $\dim \mathfrak{h}_i^- \leq \dim \mathfrak{h}_{i+1}^-$ $(1 \leq i \leq s-1)$; here $\mathfrak{h}_j^+$ and $\mathfrak{h}_j^-$ are the toroidal part and the vector part of $\mathfrak{h}_j$, respectively. Since $\mathfrak{h}'$ is conjugate to one of the above representatives under $\mathrm{Ad}\,\mathfrak{g}$, one can assume that $\mathfrak{h}' = \mathfrak{h}_k$ $(1 \leq k \leq s)$. Let i_0 be the greatest possible integer among $1 \leq i \leq s$ such that $\mathfrak{h}_i \ni Z$. Put $\mathfrak{h} := \mathfrak{h}_{i_0}$ and let $\widetilde{\Delta} = \Delta(\overline{\mathfrak{g}}, \overline{\mathfrak{h}})$. Then the set $\widetilde{\Delta}_I$ of imaginary roots in $\widetilde{\Delta}$ breaks up into two parts : $\widetilde{\Delta}_I = \widetilde{\Delta}_{I,\mathfrak{k}} \coprod \widetilde{\Delta}_{I,\mathfrak{p}}$, where $\widetilde{\Delta}_{I,\mathfrak{k}}$ (resp. $\widetilde{\Delta}_{I,\mathfrak{p}}$) is the totality of roots $\alpha \in \widetilde{\Delta}_I$ whose root spaces $\mathfrak{g}^{\alpha}$ in $\overline{\mathfrak{g}}$ are contained in the complexification $\overline{\mathfrak{k}}$ (resp. $\overline{\mathfrak{p}}$). Suppose now that $\widetilde{\Delta}_{I,\mathfrak{p}}$ is not empty. Then, by a result of [12], any root $\alpha \in \widetilde{\Delta}_{I,\mathfrak{p}}$ takes a nonzero value on Z. But this contradicts Proposition 3.1. Therefore $\widetilde{\Delta}_{I,\mathfrak{p}}$ must be empty, and hence $\mathfrak{h}$ is an Iwasawa Cartan subalgebra (cf. Araki [1]). ∎

Let $\{\mathfrak{g}^{\pm}, Z\}$ be a dipolarization in $\mathfrak{g}$ with characteristic element Z, and let $\mathfrak{h}$ be an Iwasawa Cartan subalgebra of $\mathfrak{g}$ containing Z. In the following, in view of Lemma 3.3, we will use the conventional notation $\widetilde{\Delta}_{\bullet}$ instead of $\widetilde{\Delta}_I$. The complexification $\overline{\mathfrak{c}(Z)}$ of the centralizer $\mathfrak{c}(Z)$ in $\mathfrak{g}$ can be written as

$$\overline{\mathfrak{c}(Z)} = \overline{\mathfrak{h}} + \sum_{\alpha \in \widetilde{\Delta}_0(Z)} \mathfrak{g}^{\alpha}. \tag{15}$$

Let σ be the conjugation of $\overline{\mathfrak{g}}$ with respect to $\mathfrak{g}$. Then $\widetilde{\Delta}_0(Z)$ is σ-stable and contains $\widetilde{\Delta}_{\bullet}$. Let ϖ be the orthogonal projection of $\mathfrak{h}_{\mathbb{R}} := i\mathfrak{h}^+ + \mathfrak{h}^-$ onto $\mathfrak{h}^-$ with respect to the Killing form of $\mathfrak{g}$, and let $\Delta = \varpi(\widetilde{\Delta} - \widetilde{\Delta}_{\bullet})$. Then Δ is the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}^-$ (restricted root system). Now let $\Delta_0(Z) := \varpi(\widetilde{\Delta}_0(Z) - \widetilde{\Delta}_{\bullet})$. Note that $\Delta_0(Z)$ is not equal to the set $\{\gamma \in \Delta : \gamma(Z) = 0\}$, unless Z is hyperbolic.

Lemma 3.4.

$$\mathfrak{c}(Z) = \mathfrak{c}(\mathfrak{h}^-) + \sum_{\gamma \in \Delta_0(Z)} \mathfrak{g}^{\gamma},$$

where $\mathfrak{c}(\mathfrak{h}^-)$ denotes the centralizer of $\mathfrak{h}^-$ in $\mathfrak{g}$.

Proof. The complexification of the right-hand side is equal to (cf. Proposition 3.1 and (15))

$$\overline{\mathfrak{c}(\mathfrak{h}^-)} + \sum_{\gamma\in\Delta_0(Z)} \overline{\mathfrak{g}^\gamma} = \left(\bar{\mathfrak{h}} + \sum_{\alpha\in\widetilde{\Delta}_\bullet} \mathfrak{g}^\alpha\right) + \sum_{\alpha\in\widetilde{\Delta}_0(Z)-\widetilde{\Delta}_\bullet} \mathfrak{g}^\alpha = \bar{\mathfrak{h}} + \sum_{\alpha\in\widetilde{\Delta}_0(Z)} \mathfrak{g}^\alpha = \overline{\mathfrak{c}(Z)},$$

which implies the lemma. ∎

Since $\mathfrak{g}^+$ is a parabolic subalgebra of $\mathfrak{g}$ with $\mathfrak{c}(Z)$ as a Levi subalgebra, there exists a linear order in $\mathfrak{h}^-$ such that $\mathfrak{g}^+$ can be written as

$$\mathfrak{g}^+ = \mathfrak{c}(Z) + \sum_{\gamma\in\Delta^+-\Delta_0(Z)} \mathfrak{g}^\gamma = \mathfrak{c}(\mathfrak{h}^-) + \sum_{\gamma\in\Delta^+\cup\Delta_0(Z)} \mathfrak{g}^\gamma, \tag{16}$$

where Δ^+ is the set of positive roots in Δ with respect to that order. Let Π be the corresponding fundamental system for Δ. Choose a σ-order in $\widetilde{\Delta}$ which induces the original order on $\mathfrak{h}^-$. Let $\widetilde{\Pi}$ be the corresponding σ-fundamental system of $\widetilde{\Delta}$, and let $\widetilde{\Pi}_\bullet = \widetilde{\Pi}\cap\widetilde{\Delta}_\bullet$. Then we have $\Pi = \varpi(\widetilde{\Pi}-\widetilde{\Pi}_\bullet)$. Furthermore, if we put $\widetilde{\Pi}_0(Z) = \widetilde{\Pi}\cap\widetilde{\Delta}_0(Z)$, then we have $\widetilde{\Pi}_\bullet \subset \widetilde{\Pi}_0(Z)$ by Proposition 3.1. Now let us consider the subset of $\Delta_0(Z)$: $\Pi_0(Z) := \varpi(\widetilde{\Pi}_0(Z)-\widetilde{\Pi}_\bullet)$.

Lemma 3.5. *$\Pi_0(Z)$ is a fundamental system for $\Delta_0(Z)$. In particular, $\Pi_0(Z) = \Pi\cap\Delta_0(Z)$ holds.*

Proof. $\widetilde{\Pi}_0(Z)$ is clearly a fundamental system of $\widetilde{\Delta}_0(Z)$, from which the lemma follows. ∎

Theorem 3.6. *Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{R}$. Then any dipolarization $\{\mathfrak{g}^\pm, f\}$ in $\mathfrak{g}$ comes from a gradation of $\mathfrak{g}$.*

Proof. Let Z be the characteristic element of the dipolarization $\{\mathfrak{g}^\pm, f\}$. Then all the previous arguments are valid. We retain the notation before. Put $\Pi_0 = \Pi_0(Z)$ and $\Pi_1 = \Pi - \Pi_0$. Let $\Pi = \{\gamma_1, \cdots, \gamma_r\}$. We define an element $Z^* \in \mathfrak{h}^-$ by

$$\begin{cases} (\gamma_i, Z^*) = 0, & \gamma_i \in \Pi_0, \\ (\gamma_j, Z^*) = 1, & \gamma_j \in \Pi_1. \end{cases} \tag{17}$$

Let $\Delta_0(Z^*)$ denote the totality of the roots in Δ which vanish on Z^*. Then we have

$$\Delta_0(Z^*) = \Delta_0(Z). \tag{18}$$

This implies that $\mathfrak{c}(Z^*) = \mathfrak{c}(Z)$. Since Π is compatible with the positive system Δ^+, we can conclude that $\gamma \in \Delta^+ - \Delta_0(Z)$ if and only if $(\gamma, Z^*) > 0$. Now let us consider the gradation $\mathfrak{g} = \sum_k \mathfrak{g}_k$ whose characteristic element is Z^*. Then (16), (18) and the above argument show that $\mathfrak{g}^+ = \sum_{k\geq 0} \mathfrak{g}_k$. On the other hand, we know that $\mathfrak{g}^\pm \supset \mathfrak{c}(Z) \supset \mathfrak{h} \supset \mathfrak{h}^- \ni Z^*$. Hence $\mathfrak{g}^-$ is spanned by $\mathfrak{c}(\mathfrak{h}^-)$ and the root spaces belonging to certain roots in Δ. By (DP3), the root space $\mathfrak{g}^\gamma$ for any root $\gamma \in \Delta$ is contained either in $\mathfrak{g}^+$ or in $\mathfrak{g}^-$. Therefore, if we denote by Δ^- the set of negative roots in Δ, then a root in $\Delta^- - \Delta_0(Z)$ must be a root belonging to $\mathfrak{g}^-$. Consequently we have the inclusion $\mathfrak{g}^- \supset \mathfrak{c}(Z) + \sum_{\gamma\in\Delta^- - \Delta_0(Z)} \mathfrak{g}^\gamma$. Since $\mathfrak{g}^+$ and $\mathfrak{g}^-$ are equidimensional by a property of dipolarizations ([8]), (16) implies that the above inclusion must be an equality. Now the equality $\mathfrak{g}^- = \sum_{k\le 0} \mathfrak{g}_k$ is an immediate consequence from this and $\mathfrak{g}^+ = \sum_{k\ge 0} \mathfrak{g}_k$. ■

The above theorem and its proof tell us that any dipolarization in $\mathfrak{g}$ can be reconstructed by its characteristic element.

The gradation of $\mathfrak{g}$ constructed in Theorem 3.6 is called the *gradation associated to the dipolarization* (or simply *the associated gradation*). We give a necessary and sufficient condition for a coset space G/H of a semisimple Lie group G to be a homogeneous parakähler manifold.

Theorem 3.7. *Let G be a connected semisimple Lie group and H be a closed subgroup of G. Then the coset space $M = G/H$ is a homogeneous parakähler manifold if and only if H is an open subgroup of a Levi subgroup of a parabolic subgroup of G.*

Proof. Let $\mathfrak{g} = Lie(G)$. Suppose first that M is a homogeneous parakähler manifold. Then, by [8], there exists a dipolarization $\{\mathfrak{g}^\pm, Z\}$ in $\mathfrak{g}$ such that $Lie(H) = \mathfrak{c}(Z) = \mathfrak{g}^+ \cap \mathfrak{g}^-$. Let $\mathfrak{g} = \sum_{k=-\nu}^{\nu} \mathfrak{g}_k$ be the associated gradation of $\{\mathfrak{g}^\pm, Z\}$ and let $Z^* \in \mathfrak{g}$ be its characteristic element. Then we have that $Lie(H) = \mathfrak{g}_0, \mathfrak{g}^+ = \sum_{k=0}^{\nu} \mathfrak{g}_k$ and $\mathfrak{g}^- = \sum_{k=0}^{\nu} \mathfrak{g}_{-k}$. Let $N(\mathfrak{g}^\pm)$ be the respective normalizers of $\mathfrak{g}^\pm$ in G, which are the parabolic subgroups of G corresponding to $\mathfrak{g}^\pm$. It follows from a result of [10, Prop. 7.83] that

$$\begin{aligned} N(\mathfrak{g}^+) &= C(Z^*) \exp\left(\sum_{k=1}^{\nu} \mathfrak{g}_k\right), \\ N(\mathfrak{g}^-) &= C(Z^*) \exp\left(\sum_{k=1}^{\nu} \mathfrak{g}_{-k}\right), \end{aligned} \tag{19}$$

where $C(Z^*)$ denotes the centralizer of Z^* in G. The G-invariant Lagrangean distributions on M are induced by the subalgebras $\mathfrak{g}^\pm$. The G-invariance is equivalent to the condition

$$(\mathrm{Ad}_{\mathfrak{g}}\, H)\, \mathfrak{g}^\pm \subset \mathfrak{g}^\pm. \tag{20}$$

Consequently we have $H \subset N(\mathfrak{g}^+) \cap N(\mathfrak{g}^-) = C(Z^*)$. Therefore H is an open subgroup of the Levi subgroup $C(Z^*)$ of $N(\mathfrak{g}^\pm)$.

Conversely suppose that H is an open subgroup of a Levi subgroup L of a parabolic subgroup G^+ of G. Let $\mathfrak{g}^+ = Lie(G^+)$. Then, as a property of parabolic subgroups, G^+ coincides with the normalizer $N(\mathfrak{g}^+)$ in G. Let $\mathfrak{l} = Lie(L)$. One can choose a gradation $\mathfrak{g} = \sum_{k=-\nu}^{\nu} \mathfrak{g}_k$, whose characteristic element is denoted by Z^*, such that $\mathfrak{l} = \mathfrak{g}_0 = \mathfrak{c}(Z^*)$ and $\mathfrak{g}^+ = \sum_{k=0}^{\nu} \mathfrak{g}_k$. It follows that $N(\mathfrak{g}^+) = C(Z^*) \exp(\sum_{k=1}^{\nu} \mathfrak{g}_k)$ and $L = C(Z^*)$. Let $\mathfrak{g}^- := \sum_{k=0}^{\nu} \mathfrak{g}_{-k}$ and let f^* be the Killing dual of Z^*. Then $\{\mathfrak{g}^\pm, f^*\}$ is a dipolarization in $\mathfrak{g}$ satisfying $Lie(H) = \mathfrak{g}^+ \cap \mathfrak{g}^-$ (cf. [8]). The inclusion $N(\mathfrak{g}^\pm) \supset C(Z^*) = L \supset H$ implies (20). The equality $(\mathrm{Ad}_{\mathfrak{g}}\, H)\, Z^* = Z^*$ implies that f^* is $\mathrm{Ad}_{\mathfrak{g}}\, H$-invariant. Therefore, by [8], M is a homogeneous parakähler manifold. ■

Theorem 3.8. *Let G be a connected semisimple Lie group and H be a closed subgroup of G. Then the coset space $M = G/H$ is a homogeneous parakähler manifold if and only if M is a G-equivariant covering space of a hyperbolic semisimple* $\mathrm{Ad}\, G$*-orbit in* $\mathfrak{g} = Lie(G)$.

Proof. Suppose that M is homogeneous parakähler. Then, from Theorem 3.7 and its proof it follows that M is a G-equivariant covering space of $G/C(Z^*) = (\mathrm{Ad}\, G)\, Z^*$, where Z^* is a hyperbolic semisimple element.

To prove the converse, let $\mathcal{O}$ be a hyperbolic semisimple $\mathrm{Ad}\, G$-orbit in $\mathfrak{g}$ covered G-equivariantly by M. Let π be the natural projection of M onto $\mathcal{O}$, and let Z' be the image of the origin of M under π. Z' is a hyperbolic semisimple element in $\mathfrak{g}$, and $\mathcal{O} = G \cdot Z' = G/C(Z')$. Let $\mathfrak{g}^+$ be the sum of the eigenspaces of $\mathrm{ad}\, Z'$ in $\mathfrak{g}$ corresponding to nonnegative eigenvalues. $\mathfrak{g}^+$ is a parabolic subalgebra of $\mathfrak{g}$ and $\mathfrak{c}(Z')$ is a Levi subalgebra of $\mathfrak{g}^+$. Hence $C(Z')$ is a Levi subgroup of the parabolic subgroup $N(\mathfrak{g}^+)$. By the G-equivariancy of π, we have that H is an open subgroup of $C(Z')$. Thus Theorem 3.7 implies that G/H is a homogeneous parakähler manifold. ∎

Remark 3.9. Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{R}$, $\{\mathfrak{g}^{\pm}, Z\}$ a dipolarization in $\mathfrak{g}$ with characteristic element Z, and let $\mathfrak{g} = \sum_{k=-\nu}^{\nu} \mathfrak{g}_k$ be the associated gradation. Then Z lies in the center $\mathfrak{z}(\mathfrak{g}_0)$ of $\mathfrak{g}_0$ (cf. (18)). Let $\mathfrak{z}(\mathfrak{g}_0)^+$ and $\mathfrak{z}(\mathfrak{g}_0)^-$ denote the elliptic component and the hyperbolic component of $\mathfrak{z}(\mathfrak{g}_0)$ respectively. Then we have $Z^{\pm} \in \mathfrak{z}(\mathfrak{g}_0)^{\pm}$ (cf. (6)). Suppose $\mathfrak{z}(\mathfrak{g}_0)^+ = (0)$, which is the case for most real simple graded Lie algebras. In this case, Z itself is a hyperbolic semisimple element (see Theorem 4.4 below).

4. Characterization and determination of characteristic elements

Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{R}$. By Lemma 3.3, the characteristic element of a dipolarization belongs to an Iwasawa Cartan subalgebra $\mathfrak{h} = \mathfrak{h}^+ \oplus \mathfrak{h}^-$. In this section, we study a necessary and sufficient condition for an element $Z \in \mathfrak{h}$ to be the characteristic element of a dipolarization.

4.1. General criterion

By the results in §§1. – 3., we know the following.

Proposition 4.1. *Let Z be an element of Iwasawa Cartan subalgebra $\mathfrak{h}$. Then the following conditions are mutually equivalent.*

(1) *$Z \in \mathfrak{h}$ is the characteristic element of a dipolarization.*
(2) *There exists $Z^* \in \mathfrak{h}^-$ such that $\mathfrak{c}(Z^*) = \mathfrak{c}(Z)$.*
(3) *There exists $Z^* \in \mathfrak{h}^-$ such that $\Delta_0(Z^*) = \Delta_0(Z)$ (cf. Lemma 3.4).*
(4) *There exists $Z^* \in \mathfrak{h}^-$ such that $\widetilde{\Delta}_0(Z^*) = \widetilde{\Delta}_0(Z)$.*

Proof. For the equivalence of (1), (2) and (3), see the proof of Theorem 3.6 (cf. (18)). On the other hand, $\mathfrak{c}(Z)$ determines $\widetilde{\Delta}_0(Z)$, and vice versa (cf. (15)). Therefore the equivalence of (2) and (4) follows. ∎

We denote the set of all subsets of $\widetilde{\Delta} = \Delta(\overline{\mathfrak{g}}, \overline{\mathfrak{h}})$ by $\mathsf{Sub}(\widetilde{\Delta})$. Similarly, $\mathsf{Sub}(\mathfrak{h}^-)$ denotes the set of all linear subspaces of $\mathfrak{h}^-$ over $\mathbb{R}$. Let us consider the following operations I and V on them. For $\mathfrak{a} \in \mathsf{Sub}(\mathfrak{h}^-)$, put

$$I(\mathfrak{a}) = \{\alpha \in \widetilde{\Delta} : \alpha(X) = 0 \ (\forall X \in \mathfrak{a})\} = \{\alpha \in \widetilde{\Delta} : \alpha|_{\mathfrak{a}} = 0\}.$$

Similarly, for $\Sigma \in \mathsf{Sub}(\widetilde{\Delta})$, we put

$$V(\Sigma) = \{X \in \mathfrak{h}^- : \alpha(X) = 0 \ (\forall \alpha \in \Sigma)\} = \bigcap_{\alpha \in \Sigma} \left(H_\alpha \cap \mathfrak{h}^-\right),$$

where H_α is the hyperplane corresponding to a root α. Clearly, $V \circ I \circ V = V$ and $I \circ V \circ I = I$ hold. Note that $\widetilde{\Delta}_0(Z) = I(\mathbb{R}Z)$.

Definition 4.2. A subset $\Sigma \subset \widetilde{\Delta}$ is called *primitive* if $I(V(\Sigma)) = \Sigma$ holds.

It is easy to see that, if $\Sigma \subset \widetilde{\Delta}$ is primitive, then Σ contains the set of imaginary roots $\widetilde{\Delta}_I$.

Theorem 4.3. *Let $\mathfrak{h}$ be an Iwasawa Cartan subalgebra of $\mathfrak{g}$. Then an element $Z \in \mathfrak{h}$ is the characteristic element of a dipolarization if and only if $\widetilde{\Delta}_0(Z)$ is primitive.*

Proof. Let us prove that the condition is necessary. Assume that Z is the characteristic element of a dipolarization. Then, there exists $Z^* \in \mathfrak{h}^-$ which satisfies $\widetilde{\Delta}_0(Z^*) = \widetilde{\Delta}_0(Z)$. If we put $\Sigma = \widetilde{\Delta}_0(Z) = \widetilde{\Delta}_0(Z^*)$, Z^* belongs to $V(\Sigma)$ by definition. Therefore we have

$$\widetilde{\Delta}_0(Z^*) = I(\mathbb{R}Z^*) \supset I(V(\Sigma)) \supset \Sigma = \widetilde{\Delta}_0(Z).$$

Hence we conclude that the inclusions above are all equalities, and that $\Sigma = \widetilde{\Delta}_0(Z)$ is primitive.

Conversely, suppose that $\Sigma = \widetilde{\Delta}_0(Z)$ is primitive. If $\alpha \in \widetilde{\Delta} - \Sigma$, α is not identically zero on $V(\Sigma)$ by the primitivity of Σ. Therefore $H_\alpha \cap V(\Sigma)$ is a hyperplane in $V(\Sigma)$. Now it is clear that the set

$$V(\Sigma) - \bigcap_{\alpha \in \widetilde{\Delta} - \Sigma} H_\alpha$$

is not empty, and we can choose $Z^* \in \mathfrak{h}^-$ from it. We conclude that $\widetilde{\Delta}_0(Z^*) = \widetilde{\Delta}_0(Z)$ by the choice of Z^*, or equivalently, Z is the characteristic element of a dipolarization by Proposition 4.1. ∎

For almost all cases, characteristic elements are hyperbolic. In fact, for real simple Lie algebras, we have the following theorem.

Theorem 4.4. *Let $\mathfrak{g}$ be a real simple Lie algebra. If the Satake diagram of $\mathfrak{g}$ does not have any arrow, then the characteristic element of a dipolarization of $\mathfrak{g}$ is hyperbolic.*

Proof. If the Satake diagram does not contain any arrow, the set of imaginary simple roots becomes a basis of $(\mathfrak{h}^+)^*$. So the toroidal part of a characteristic element Z must vanish, because $\widetilde{\Delta}_0(Z)$ contains all the imaginary roots. This means that Z is hyperbolic. ■

Contrary to the theorem above, there are real simple Lie algebras which have non-hyperbolic characteristic elements. We will consider these cases explicitly in the following subsections.

4.2. Classical cases

In this subsection, we assume that $\mathfrak{g}$ is a real simple Lie algebra of classical type, and denote an Iwasawa Cartan subalgebra by $\mathfrak{h}$. By Theorem 4.4, we have only to treat $\mathfrak{g}$ whose Satake diagram has arrows. Namely, $\mathfrak{g}$ is one of type AIII, AIV, DI (with only white vertices, and its restricted root system is of type B), or DIII (with odd rank) in Cartan's notation (see, e.g., [5, Ch. X, Table VI]).

Case DI. In this case, $\mathfrak{g} \simeq \mathfrak{so}(l, l+2)$ $(l \geq 3)$. In the notation of [5, Ch. X, Table VI], the root system $\widetilde{\Delta} = \Delta(\overline{\mathfrak{g}}, \overline{\mathfrak{h}})$ has a fundamental system $\widetilde{\Pi} = \{\alpha_1, \alpha_2, \cdots, \alpha_{l+1}\}$ $(l \geq 3)$. The roots $\alpha_1, \cdots, \alpha_{l-1}$ are real. The roots α_l, α_{l+1} are complex and are jointed by an arrow.

Proposition 4.5. *Take $Z \in \mathfrak{h}$. Up to conjugation by the little Weyl group $W(\Delta)$, we can assume:*

$$\alpha_i(Z) \geq 0 \quad (1 \leq i \leq l-1), \qquad (\alpha_l + \alpha_{l+1})(Z) \geq 0.$$

Then Z is the characteristic element of a dipolarization if and only if the following condition (1) *or* (2) *holds.*

(1) $(\alpha_l + \alpha_{l+1})(Z) \neq 0$ *or* $\alpha_{l-1}(Z) \neq 0$. *In this case, Z is possibly non-hyperbolic.*
(2) $\alpha_i(Z) = 0$ $(l-1 \leq i \leq l+1)$. *In this case, Z is necessarily hyperbolic.*

Remark 4.6. Z is non-hyperbolic if and only if $(\alpha_l - \alpha_{l+1})(Z) \neq 0$ holds. Therefore, the above proposition tells us that a non-hyperbolic element Z is *not* a characteristic element if and only if $(\alpha_l + \alpha_{l+1})(Z) = \alpha_{l-1}(Z) = 0$.

Proof. Since $\{\alpha_1, \cdots, \alpha_{l-1}, (\alpha_l + \alpha_{l+1})/2\}$ is a simple system of the restricted roots, we can assume the values of those roots at Z are all positive after some conjugation by the little Weyl group $W(\Delta)$.

First, let us suppose that Z is the characteristic element of a dipolarization. Then there exists a hyperbolic element Z^* such that $\widetilde{\Delta}_0(Z) = \widetilde{\Delta}_0(Z^*)$. Assume that $(\alpha_l + \alpha_{l+1})(Z) = \alpha_{l-1}(Z) = 0$. Since $\alpha_l + \alpha_{l+1} + \alpha_{l-1}$ is a positive root and $(\alpha_l + \alpha_{l+1} + \alpha_{l-1})(Z) = 0$, we have that $\alpha_{l-1}, \alpha_{l-1} + \alpha_l + \alpha_{l+1} \in \widetilde{\Delta}_0(Z^*)$. This means that

$$(\alpha_l + \alpha_{l+1})(Z^*) = (\alpha_{l-1} + \alpha_l + \alpha_{l+1})(Z^*) - \alpha_{l-1}(Z^*) = 0.$$

On the other hand, we have $(\alpha_l - \alpha_{l+1})(Z^*) = 0$ because Z^* is hyperbolic. Now we conclude that $\alpha_i \in \widetilde{\Delta}_0(Z^*) = \widetilde{\Delta}_0(Z)$ for $l-1 \leq i \leq l+1$.

Conversely, let us suppose that the condition (1) or (2) holds. If (2) holds, then Z is hyperbolic. Therefore it is a characteristic element by Proposition 4.1, and we are done.

Next, we suppose (1). We divide the case into two: (a) $(\alpha_l + \alpha_{l+1})(Z) > 0$; or (b) $(\alpha_l + \alpha_{l+1})(Z) = 0$ and $\alpha_{l-1}(Z) > 0$. We discuss these two cases separately.

Case (a). Since $(\alpha_l + \alpha_{l+1})(Z) > 0$, $\widetilde{\Delta}_0(Z)$ does not contain any complex root. Therefore, if we put $Z = Z^+ + Z^- \in \mathfrak{h}^+ + \mathfrak{h}^-$ (cf. (6)), we have $\widetilde{\Delta}_0(Z) = \widetilde{\Delta}_0(Z^-)$. This means that Z is the characteristic element of a dipolarization.

Case (b). We can choose $Z^* \in \mathfrak{h}^-$ such that

$$\alpha_i(Z) = \alpha_i(Z^*) \quad (1 \le i \le l-1), \qquad (\alpha_l + \alpha_{l+1})(Z^*) > 0.$$

A positive root β can be expressed as a sum of simple roots: $\beta = \sum_i m_i \alpha_i$. If $m_l = m_{l+1} = 0$, then it is easy to see that $\beta \in \widetilde{\Delta}_0(Z)$ if and only if $\beta \in \widetilde{\Delta}_0(Z^*)$. On the other hand, we claim that if $m_l \neq 0$ or $m_{l+1} \neq 0$, then both $\widetilde{\Delta}_0(Z)$ and $\widetilde{\Delta}_0(Z^*)$ do not contain β. In fact, since $\alpha_l(Z^*) = \alpha_{l+1}(Z^*) > 0$, $\beta(Z^*) > 0$. This means that $\beta \notin \widetilde{\Delta}_0(Z^*)$. We can assume that $\alpha_l(Z) = -\alpha_{l+1}(Z)$ is nonzero. Otherwise Z is hyperbolic and hence it is a characteristic element. Since $\alpha_{l+1} = \sigma(\alpha_l)$ (complex conjugation) and $(\alpha_l + \alpha_{l+1})(Z) = 0$, the value $\alpha_l(Z) = -\alpha_{l+1}(Z) \neq 0$ is purely imaginary. Suppose first that $\beta = \alpha_l$ or α_{l+1}. Then we have $\beta(Z) \neq 0$ by the assumption, and consequently $\beta \notin \widetilde{\Delta}_0(Z)$. So let us assume that $\beta \neq \alpha_l, \alpha_{l+1}$. Then, since $m_l \neq 0$ or $m_{l+1} \neq 0$, it is easy to see that $m_{l-1} \neq 0$ by the property of the root system D_{l+1}. This implies that the real part of $\beta(Z)$ is positive:

$$\Re\beta(Z) = \sum_{i=1}^{l-1} m_i \alpha_i(Z) \ge m_{l-1}\alpha_{l-1}(Z) > 0.$$

Hence $\beta \notin \widetilde{\Delta}_0(Z)$. For negative roots, we can proceed in the same way. We have shown that $\widetilde{\Delta}_0(Z)$ and $\widetilde{\Delta}_0(Z^*)$ consists only of real roots. Now we conclude that $\widetilde{\Delta}_0(Z) = \widetilde{\Delta}_0(Z^*)$ for the hyperbolic element Z^*, which means that Z is a characteristic element. ∎

Case DIII. In this case $\mathfrak{g} \simeq \mathfrak{u}^*_{2l+1}(\mathbb{H})$ $(= \mathfrak{so}^*(4l+2))$, with rank $r = 2l + 1 \ge 5$. In the notation of [5, Ch. X, Table VI], the root system $\widetilde{\Delta} = \Delta(\overline{\mathfrak{g}}, \overline{\mathfrak{h}})$ has a fundamental system $\widetilde{\Pi} = \{\alpha_1, \alpha_2, \cdots, \alpha_r\}$. The roots α_{2k-1} $(1 \le k \le l)$ are imaginary (black vertices) and α_{2k} $(1 \le k \le l-1)$ are real (white vertices). The roots α_{r-1}, α_r are complex and are jointed by an arrow. (Remark. In [5, Ch. X, Table VI], there is a misprint in the diagram of DIII. The black vertex labeled α_{r-1} should be read as α_{r-2}.)

Proposition 4.7. *Take $Z \in \mathfrak{h}$ and let us suppose that Z is killed by imaginary roots (cf. Proposition 3.1). Up to conjugation by the little Weyl group $W(\Delta)$, we can assume:*

$$\alpha_{2k-1}(Z) = 0 \quad (1 \le k \le l), \qquad \alpha_{2k}(Z) \ge 0 \quad (1 \le k \le l-1), \qquad (\alpha_r + \alpha_{r-1})(Z) \ge 0.$$

Then Z is the characteristic element of a dipolarization if and only if the following condition (1) or (2) holds.

(1) $(\alpha_{r-1} + \alpha_r)(Z) \neq 0$ or $\alpha_{r-3}(Z) \neq 0$. In this case, Z is possibly non-hyperbolic.

(2) $\alpha_{r-1}(Z) = \alpha_r(Z) = 0$. In this case, Z is necessarily hyperbolic.

Remark 4.8. Z is non-hyperbolic if and only if $(\alpha_{r-1} - \alpha_r)(Z) \neq 0$ holds. Therefore, the above proposition tells us that a non-hyperbolic element Z is *not* a characteristic element if and only if $(\alpha_{r-1} + \alpha_r)(Z) = \alpha_{r-3}(Z) = 0$.

Proof. We omit the proof, since it is similar to that of Proposition 4.5. ■

Cases AIII and AIV. In this case, $\mathfrak{g} = \mathfrak{su}(p,q)$. We divide this case into two parts: (a) $\mathfrak{g} = \mathfrak{su}(p,p)$ $(p \geq 2)$; and (b) $\mathfrak{g} = \mathfrak{su}(p,q)$ $(1 \leq p < q)$.

In the case (a), we put $p = q = l \geq 2$. Then $\mathfrak{g} = \mathfrak{su}(l,l)$, and its restricted root system is of type C. The Satake diagram has only white vertices. Take a fundamental system of roots $\widetilde{\Pi} = \{\alpha_i, \overline{\alpha_i}, \alpha_l : 1 \leq i \leq l-1\}$ as in [5, Ch. X, Table VI], where $\alpha_i, \overline{\alpha_i}$ $(1 \leq i \leq l-1)$ are complex roots and α_l is real.

Proposition 4.9. *Let $\mathfrak{g} = \mathfrak{su}(p,p)$ $(p = l \geq 2)$ and let $Z \in \mathfrak{h}$. Up to conjugation by the little Weyl group $W(\Delta)$, we can assume:*

$$(\alpha_i + \overline{\alpha_i})(Z) \geq 0 \quad (1 \leq i \leq l-1), \qquad \alpha_l(Z) \geq 0.$$

Let $0 \leq j_0 \leq l-1$ be the smallest possible integer satisfying the following property:

$$(\alpha_i + \overline{\alpha_i})(Z) = 0 \quad (j_0 < i \leq l-1).$$

Then Z is the characteristic element of a dipolarization if and only if the following condition (1) *or* (2) *holds.*
(1) $\alpha_l(Z) \neq 0$.
(2) $\alpha_i(Z) = \overline{\alpha_i}(Z) = 0$ *for all i satisfying $j_0 < i \leq l-1$.*

Remark 4.10. Z is hyperbolic if and only if $(\alpha_i - \overline{\alpha_i})(Z) = 0$ for $1 \leq i \leq l-1$.

Proof. Let us suppose that Z is the characteristic element of a dipolarization. Then there exists a hyperbolic Z^* such that $\widetilde{\Delta}_0(Z) = \widetilde{\Delta}_0(Z^*)$. Assume that the condition (1) does not hold, i.e., $\alpha_l(Z) = 0$. Clearly we have $\alpha_l(Z^*) = 0$ also. If, in addition, $(\alpha_{l-1} + \overline{\alpha_{l-1}})(Z) = 0$, then we have $\alpha_{l-1} + \alpha_l + \overline{\alpha_{l-1}} \in \widetilde{\Delta}_0(Z) = \widetilde{\Delta}_0(Z^*)$. This means

$$(\alpha_{l-1} + \overline{\alpha_{l-1}})(Z^*) = (\alpha_{l-1} + \alpha_l + \overline{\alpha_{l-1}})(Z^*) - \alpha_l(Z^*) = 0.$$

Since Z^* is hyperbolic, $(\alpha_{l-1} - \overline{\alpha_{l-1}})(Z^*) = 0$. Therefore we have $\alpha_{l-1}, \overline{\alpha_{l-1}} \in \widetilde{\Delta}_0(Z^*) = \widetilde{\Delta}_0(Z)$, which means $\alpha_{l-1}(Z) = \overline{\alpha_{l-1}}(Z) = 0$. We can argue exactly in the same way to go down inductively from $l-1$ to the first nonzero position j_0. So we conclude that the condition (2) holds.

Let us prove the converse implication. First, we suppose that $\alpha_l(Z) > 0$. In this case, $\widetilde{\Delta}_0(Z)$ breaks up into a disjoint union: $\widetilde{\Delta}_0(Z) = \widetilde{\Delta}_0(Z)^\flat \coprod \widetilde{\Delta}_0(Z)^\sharp$, where $\widetilde{\Delta}_0(Z)^\flat = \widetilde{\Delta}_0(Z) \cap \sum_{i=1}^{l-1} \mathbb{Z}\alpha_i$ and $\widetilde{\Delta}_0(Z)^\sharp$ is complex conjugate to $\widetilde{\Delta}_0(Z)^\flat$.

Using the arguments for complex Lie algebra $\mathfrak{sl}(l,\mathbb{C})$, we conclude that there exists a hyperbolic element $Z' \in \mathfrak{h}^-$ such that $\widetilde{\Delta}_0(Z)^\flat = \widetilde{\Delta}_0(Z') \cap \sum_{i=1}^{l-1} \mathbb{Z}\alpha_i$. Note that $\widetilde{\Delta}_0(Z')$ contains the complex conjugation $\widetilde{\Delta}_0(Z)^\sharp$ also. Choose $E \in \mathfrak{h}^-$ such that $\alpha_l(E) = 1, \alpha_i(E) = 0$ $(1 \leq i \leq l-1)$. In the real affine line $Z' + \mathbb{R}E$, we can choose Z^* such that $\beta(Z^*) \neq 0$ for any root β which has nonzero α_l-component. This is obvious because there are only finitely many roots. By the above construction, it is easy to see that

$$\widetilde{\Delta}_0(Z^*) = \widetilde{\Delta}_0(Z)^\flat \coprod \widetilde{\Delta}_0(Z)^\sharp = \widetilde{\Delta}_0(Z).$$

This means that Z is a characteristic element.

Second, let us assume $\alpha_l(Z) = 0$. Then by condition (2), we see that $\alpha_i, \overline{\alpha_i} \in \widetilde{\Delta}_0(Z)$ for $j_0 < i \leq l$. Since the real part of $\alpha_{j_0}(Z)$ is positive by the assumption, we can use α_{j_0} instead of α_l in the first part. Let $\widetilde{\Delta}^{j_0}$ be a root subsystem spanned by $\{\alpha_i, \overline{\alpha_i}\ (j_0 < i \leq l-1), \alpha_l\}$, and put $\widetilde{\Delta}_0(Z)^\flat = \widetilde{\Delta}_0(Z) \cap \sum_{i=1}^{j_0-1} \mathbb{Z}\alpha_i$. We denote the complex conjugate of $\widetilde{\Delta}_0(Z)^\flat$ by $\widetilde{\Delta}_0(Z)^\sharp$ as before. Then we have the following decomposition of $\widetilde{\Delta}_0(Z)$:

$$\widetilde{\Delta}_0(Z) = \widetilde{\Delta}_0(Z)^\flat \coprod \widetilde{\Delta}^{j_0} \coprod \widetilde{\Delta}_0(Z)^\sharp.$$

By the similar arguments as those in the first part, we can choose Z^* such that $\widetilde{\Delta}_0(Z) = \widetilde{\Delta}_0(Z^*)$ holds. We omit the details. ■

In the case (b), we put $p = l \geq 1, q - p - 1 = l' \geq 0$. In this case, $\mathfrak{g} \simeq \mathfrak{su}(l, l + l' + 1)$ is of type AIII or AIV in Helgason's notation. Take a fundamental system of roots $\widetilde{\Pi} = \{\alpha_i, \overline{\alpha_i}, \beta_j : 1 \leq i \leq l,\ 1 \leq j \leq l'\}$ as in [5, Ch. X, Table VI], where $\alpha_i, \overline{\alpha_i}$ $(1 \leq i \leq l)$ are complex roots and β_j $(1 \leq j \leq l')$ are imaginary ones.

Proposition 4.11. *Let $\mathfrak{g} = \mathfrak{su}(p,q)$ $(1 \leq p < q)$ and let us use the above notation. Let Z be an element of $\mathfrak{h}$ killed by imaginary roots (cf. Proposition 3.1). Up to conjugation by the little Weyl group $W(\Delta)$, we can assume:*

$$(\alpha_i + \overline{\alpha_i})(Z) \geq 0 \quad (1 \leq i \leq l), \qquad \beta_j(Z) = 0 \quad (1 \leq j \leq l').$$

Let $0 \leq j_0 \leq l$ be the smallest possible integer satisfying the following property:

$$(\alpha_i + \overline{\alpha_i})(Z) = 0 \quad (j_0 < i \leq l).$$

Then Z is the characteristic element of a dipolarization if and only if

$$\alpha_i(Z) = \overline{\alpha_i}(Z) = 0 \quad (j_0 < i \leq l).$$

Proof. We omit the proof because it is almost the same as the second part of Proposition 4.9. ■

4.3. Exceptional cases

In exceptional cases, we have only to consider the types EII and EIII in Cartan's notation. The other types have no arrows in their Satake diagrams (cf. Theorem 4.4). The both types have the same complexification E_6. We denote a real simple Lie algebra by $\mathfrak{g}$ and one of its Iwasawa Cartan subalgebras by $\mathfrak{h}$.

Case EII. In this case, $\mathfrak{g} = \mathfrak{e}_{6(2)}$. Choose a fundamental system $\widetilde{\Pi} = \{\alpha_1, \alpha_2, \cdots, \alpha_6\}$ of roots as in [5, Ch. X, Table VI], where α_2, α_4 are real roots and $\overline{\alpha_1} = \alpha_6$, $\overline{\alpha_3} = \alpha_5$. The restricted root system is of type F_4.

Proposition 4.12. *Let $Z \in \mathfrak{h}$. Up to conjugation by the little Weyl group $W(\Delta)$, we can assume:*

$$\alpha_2(Z) \geq 0, \qquad \alpha_4(Z) \geq 0, \qquad (\alpha_1+\alpha_6)(Z) \geq 0, \qquad (\alpha_3+\alpha_5)(Z) \geq 0.$$

Then Z is the characteristic element of a dipolarization if and only if one of the following conditions (1) – (3) *holds.*
(1) $(\alpha_3+\alpha_5)(Z) \neq 0$ *or* $\alpha_4(Z) \neq 0$. *In this case, Z is possibly non-hyperbolic.*
(2) $(\alpha_1+\alpha_6)(Z) \neq 0$ *and* $\alpha_i(Z) = 0$ $(i = 3,4,5)$. *In this case, Z is possibly non-hyperbolic.*
(3) $\alpha_i(Z) = 0$ $(i = 1,3,4,5,6)$. *In this case, Z is necessarily hyperbolic.*

Remark 4.13. Z is hyperbolic if and only if $(\alpha_1-\alpha_6)(Z) = (\alpha_3-\alpha_5)(Z) = 0$ holds.

Proof. We suppose that Z is the characteristic element of a dipolarization. Suppose that the condition (1) does not hold. Then we have $(\alpha_3+\alpha_5)(Z) = \alpha_4(Z) = 0$, hence $(\alpha_3+\alpha_5)+\alpha_4 \in \widetilde{\Delta}_0(Z)$. We can prove $\alpha_i(Z) = 0$ $(i = 3,4,5)$ in the same way as in the proof of Proposition 4.5. If $(\alpha_1+\alpha_6)(Z) = 0$ in addition, we have $(\alpha_1+\alpha_6)+(\alpha_3+\alpha_5)+\alpha_4 \in \widetilde{\Delta}_0(Z)$. Using this, we can prove that (3) holds.

The converse implication is proved by explicit and tiresome calculations. Here we indicate a rough sketch. Note that if $\alpha_4(Z) > 0$, then $\widetilde{\Delta}_0(Z)$ is contained in the root subsystem generated by $\{\alpha_i : i \neq 4\}$ which is of type $2A_2 + A_1$. The subsystem $2A_2$ corresponds to a complex Lie algebra isomorphic to $\mathfrak{sl}(3,\mathbb{C})$, and we can reduce the argument to the classical case $\mathfrak{sl}(3,\mathbb{C}) + \mathfrak{sl}(2,\mathbb{R})$. Next, we consider a degenerate case where $\alpha_4(Z) = 0$ and $(\alpha_3+\alpha_5)(Z)$ is positive. In this case, $\widetilde{\Delta}_0(Z)$ is contained in the subsystem of $\widetilde{\Delta}$ generated by $\{\alpha_1,\alpha_2,\alpha_4,\alpha_6\}$. Therefore the argument reduces to the case $2A_1 + A_2$, which is isomorphic to $\mathfrak{sl}(2,\mathbb{C}) + \mathfrak{sl}(3,\mathbb{R})$. The last case is: $\alpha_i(Z) = 0$ $(i = 3,4,5)$ and $(\alpha_1+\alpha_6)(Z)$ is positive. This reduces to the case DI of rank 4 considered above. We omit the details. ■

Case EIII. In this case, $\mathfrak{g} = \mathfrak{e}_{6(-14)}$. Again choose a fundamental system $\widetilde{\Pi} = \{\alpha_1, \alpha_2, \cdots, \alpha_6\}$ of roots as in [5, Ch. X, Table VI], where α_2 is a real root; $\alpha_3, \alpha_4, \alpha_5$ are imaginary; and $\overline{\alpha_1} = \alpha_6$. The restricted root system is of type BC_2.

Proposition 4.14. *Let Z be an element of $\mathfrak{h}$ killed by imaginary roots* (*cf. Proposition* 3.1). *Up to conjugation by the little Weyl group $W(\Delta)$, we can assume:*

$$\alpha_i(Z) = 0 \ (i = 3,4,5), \qquad \alpha_2(Z) \geq 0, \qquad (\alpha_1+\alpha_6)(Z) \geq 0.$$

Then Z is the characteristic element of a dipolarization if and only if the following condition (1) *or* (2) *holds.*
(1) $(\alpha_1+\alpha_6)(Z) \neq 0$. *In this case, Z is possibly non-hyperbolic.*
(2) $\alpha_1(Z) = \alpha_6(Z) = 0$. *In this case, Z is necessarily hyperbolic.*

Remark 4.15. Z is hyperbolic if and only if $(\alpha_1-\alpha_6)(Z) = 0$.

Proof. We omit the proof. ■

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