Plate 1 : Sophus Lie. Oil–painting by Erik Theodor Werenskiold from the year 1902 (owned by the University of Oslo, catalogue–num. 818).

Journal of Lie Theory
Volume **9** (1999) 1–38

Sophus Lie
A Sketch of his Life and Work

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Communicated by K. H. Hofmann

Introduction

Sophus Lie (1842-1899) was one of the most important mathematicians of the nineteenth century. His work on line-sphere transformation and the creation of the theory of continuous groups and his application of these to other areas of mathematics was ground-breaking and has had a lasting effect on the further development in the field. Indeed, a new discipline of mathematics known as Lie theory today has resulted. This unique mathematician Sophus Lie summarized his life and work in the draft of an intellectual testimonial as follows [1]:

> My life is actually quite incomprehensible to me. As a young man, I had no idea that I was blessed with originality. Then, as a 26-year-old, I suddenly realized that I could create. I read a little and began to produce. In these years, 1869-1874, I had a lot of ideas which, in the course of time, I have developed only very imperfectly.
>
> In particular, it was group theory and its great importance for the differential equations which interested me. But publication in this area went woefully slow. I could not structure it properly, and I was always afraid of making mistakes. Not the small inessential mistakes ... No, it was the deep-rooted errors I feared. I am glad that my group theory in its present state doesn't contain any fundamental errors.

This aphoristic summary shall now be expanded by a more detailed account.

1842 - 1886

Marius Sophus Lie was born on December 17, 1842, in Nordfjordeid in Norway, where his father was a protestant minister. There he spent the first eight years of his life. In 1851, his father moved to Moss where the young Lie went

ISSN 0949–5932 / $2.50

Figure 1: Sophus Lie's birth-place Nordfjordeid. View from parish garden of the Eidsfjord.

to school until 1857. From 1857 to 1859, he was enrolled in Nissen's grammar school in Christiania (Oslo since 1924). Sophus Lie did well in all subjects, and, in his choice of studies, he wavered between languages and the sciences. Finally, he chose to study science at the University of Christiania. Remarkably, he did not demonstrate any particular preference for mathematics while studying. In 1865 he completed his final exam in mathematics and the natural sciences after which he taught both at a school and gave private lessons. In addition, he held well-attended lectures on astronomy directed toward a wide audience.

The turning point in Lie's relationship to mathematics and for his later life occurred in 1868. He had become interested in the works of Poncelet and Plücker, who, with Möbius, were the founders of projective geometry. Their ideas inspired Lie to intensive independent geometrical research. The period of searching was over; Lie felt himself drawn to the field of mathematics. Early in the year of 1869, he wrote the first of his 245 publications. At his own expense, he published a very concise presentation (a booklet of eight pages) of a segment of his investigations on geometry under the title *Repräsentation der Imaginären der Plangeometrie* (Representation of Imaginaries in Plane Geometry). The same year, the work was accepted by the *Journal für die reine und angewandte Mathematik.* As a result, Lie obtained a scholarship to study abroad for a year.

Lie spent the winter semester of 1869/70 in Berlin, which was considered one of the important centres of mathematical research at the time because of Kronecker, Kummer, and Weierstrass and their work. Young scholars like Lie flocked to Berlin to complete their education. In the seminars given by Kummer and Weierstrass, students also had the opportunity to submit an original research paper. Lie participated in these and in Kummer's seminar, he gave a presentation which caught people's attention. However, for the most part, he found the methods of the Berlin school alien. Instead of the "arithmetization of mathematics," he preferred a "synthetic" way of thinking, which, for him, meant a geometrical approach to mathematical problems. Thus, it is not surprising that, while in Berlin, Lie became friends with a like-minded student of Plücker, seven years his junior, named Felix Klein (1849-1925). Max Noether [2], who knew Lie and Klein personally, characterised the relationship between them as follows:

> Klein had been entrusted with the administration of the work that Plücker had left behind, and he had already published the second volume on line geometry as well as a number of his own works on second–degree general line complexes. From Klein, Lie obtained not only insights into areas hitherto foreign to him, but, above all, his interests in line geometry were met, something which helped him to gain clarity on his own ideas, unresolved as they were at the time. The two mathematicians met eye–to–eye in the area of transformations, even though Klein, taking Clebsch's modern algebraic-geometrical approach, emphasized the aspect of projective and constructive goemetry, while Lie highlighted their universal nature without limiting himself to the projective situation in the more limited sense of the word. The bond between the two men was strengthened by the fact that Weierstrass did nothing to spur them on in this direction. Only in Kummer's seminar did they receive a certain amount of encouragement.

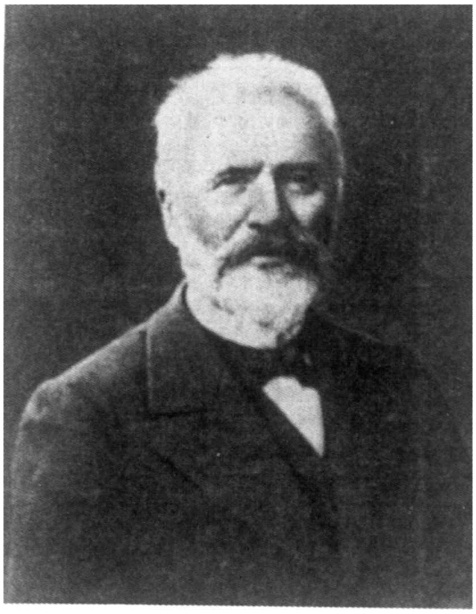


Figure 2: Camille Jordan (1838 – 1922)

In the spring of 1870, Lie left Berlin. He spent a few days in Göttingen to visit Clebsch before travelling on to Paris, where Klein also arrived at the end of April. Both men sought contact with young French mathematicians, especially with Darboux and Jordan. Jordan, whose work *Traité des substitutions et des équations algébriques* (Treatise on Substitutions and Algebraic Equations) had just appeared, drew Lie's and Klein's attention to Galois' theory. The two friends had already applied the concept of a group intuitively, but it was in Paris that they first became truly familiar with it. This is somewhat surprising since Lie had already had the opportunity to listen to Ludvig Sylows lectures on substitution theory in 1863 during his studies. Lie and Klein immediately recognized the importance of this concept and applied it to geometrical configurations and transformations, and in this fashion the idea of the transformation group and its invariants was born. In particular, Lie and Klein examined curves and surfaces which allow for an infinite number of interchangeable projective transformations and they co-published two articles in *Comptes Rendus*. This fruitful exchange was abruptly interrupted by political events. On July 19, 1870, the French declared war on Prussia which marked the beginning the Franco-German War of 1870/71 forcing Klein to leave Paris. As a citizen of neutral Norway, Lie stayed behind and decided to hike through France to Italy. Unfortunately, this innocent tour was cut short sixty kilometres outside of Paris in Fontainebleau, where Lie was arrested as a suspected spy for the Prussians. However, with the help of Darboux, the charges were withdrawn and Lie was released after four weeks of detention. A free man again, he travelled through Switzerland and Italy to Düsseldorf, arriving in November. There, he and Klein authored their third joint publication. Upon its completion, Lie returned to Christiania in December of 1870 after a year absence. (A fourth and final collaboration between Lie and Klein was printed in *Mathematische Annalen* in June, 1871. Its content complemented the two Paris articles, and Klein acted as an editor.)

Figure 3: The University of Christiana during Sophus Lie's time.

In 1871, Lie received a scholarship from the University of Christiania to work on his doctoral degree. Since the stipend was insufficient to live on, he supplemented his income by teaching mathematics at his former grammar school. However, he spent most of his time in writing his doctoral thesis, *Over en Classe geometriske Transformationer* [1] (On a Class of Geometrical Transformations). It was here that, for the first time, Lie gave a detailed presentation of his line-sphere

[1]In German translation in Lie, S., *Gesammelte Abhandlungen*, (Teubner-Verlag and Aschehoug, Leipzig / Oslo, 1922 - 1960), **1**, 105-152, annotations, 674-696.

transformation. Its discovery dated back to July 1870 when Lie found a contact transformation, which with a suitable choice of constants, had the property of transforming straight lines to spheres in space. Thus, Lie's research on projective geometry was viewed in relation to a metric geometry based on reciprocal radii which formed an equally-valid spherical geometry aside Plücker's line geometry.

In June of 1871, Lie defended his doctoral thesis and, some months later, applied for a professorship at the University of Lund in Sweden. This step surely was not easy for him as a patriotic Norwegian. Norway had been bound to Sweden since 1814 and the tensions between the two nations only ended in 1905 when Norway received its independence. Lie's intentions were made known in Norway, and as a result the Norwegian Parliament created a personal professorship for him (on March 26, 1872). On May 3, 1872, he was elected a member of the Science Society of Christiania and, on July 1, he was appointed professor to the university. Now, Lie could finally devote himself entirely to mathematics. With the improvement of his pecuniary situation he not longer needed to teach at the grammar school and the lectures he had to deliver did not require much work. A year later, Lie recalled the situation in a letter [3]:

> I, who, for the most part, had been very healthy, had become somewhat weakened because of the enormous strain of the last few years. This reached its peak last summer, but luckily, with my appointment to the professorship, it has been possible for me to live more sensibly. Since then, my health has gotten steadily better and I hope to have my old energy back soon. —You mustn't misunderstand me. In my whole life I have not yet been really sick; it's just that last year I began to be a bit nervous.

In September, 1872, Lie travelled with Klein to Göttingen and, at the end of the month both continued on to Erlangen, where Klein had obtained a position as professor of mathematics. On the way, Lie visited Lüroth and Noether and arrived in the Franconian university town on October 2. Through the personal contact a renewed, very fruitful influence resulted which is witnessed by Lie's publications and Klein's *Erlanger Programm* [2].

In Göttingen, Lie had become acquainted with Adolph Mayer (1839-1908), who also worked with first-order partial differential equations. At almost the same time, and completely independently of each other, they had come up with identical results. In order to continue this exchange of ideas, the amiable and financially independent Mayer invited the Norwegian to his hometown of Leipzig where Lie visited him after spending three weeks in Erlangen. Lie felt comfortable at Mayer's hospitable, cultivated home, and they parted as close friends. A lively exchange of letters between the two men continued until Lie was recruited as a professor of mathematics of the University of Leipzig in 1886. Engel described the relationship between Mayer and Lie as follows [4]:

> The relationship with A. Mayer had a considerable influence on Lie. Lie had, until then, developed his theory on partial differential equations

[2]The original manuscript was entitled *Vergleichende Betrachtungen über neuere geometrische Forschungen* (Comparitive Considerations Regarding Recent Geometric Research).

Figure 4: Anna Lie, née Birch (1854 – 1920) Sophus Lie (c. 1872)

> with the help of conceptual considerations, while Mayer had taken a purely analytical approach. Lie himself expressed this once by saying: "But, unfortunately, we speak two different mathematical languages. Or we argue, in any case, in a completely different manner." (Letter, December, 1872).
>
> He endeavoured from then on, therefore, to translate his conceptual arguments into analytical language, not only so that he could be better understood by Mayer and by mathematicians in general, but also because he had difficulties in giving his conceptual considerations a satisfactory form. Lie had tried this already in the articles on first-order partial differential equations which he published in Christiania, but he intensified his eforts in those works chosen for *Mathematische Annalen*. It was to the latter that he applied the greatest effort, leaving Mayer a rather free hand in making some of the lesser changes. In spite of this, he failed to satisfy the analytic school because it was still obvious that his representations were originally synthetical arguments dressed up in analytical clothing. It took a long time before he was able to give the analytical apparatus a form which was useful for his purposes. This has impeded the dissemination of his ideas.

Lie's economically secure position enabled him to think about founding a family. During the Christmas holidays of 1872, he became engaged to Anna Birch (1854-1920). They married in 1874 and had two daughters and one son.

In the spring of 1873, Lie and his former teacher, Ludvig Sylow, received an assignment from the Science Society of Christiania to reissue the works of Niels Henrik Abel which was to include his letters. This research project, which was principally carried out by Sylow, was completed in December 1881. Surprisingly, Lie seems not to have been particularly familiar with Abel's theories, as his disciple Gerhard Kowalewski explained [5]:

> Algebraic problems were alien to him even though he was a compatriot of Abel and, together with Sylow, had published Abel's collected works.

At the end of 1874, Lie's first work on group theory was published in the *Göttinger Nachrichten* under the title *Ueber Gruppen von Transformationen* (On

Transformation Groups). Lie had originally used the theory of transformation groups as a mathematical tool for the further development of his integration theory. However, he became interested in the theory itself so much so that in the following years he worked intensely on substantiating and developing it. In order to publish work on his new group theory more quickly and in more detail than it was possible in the Science Society's transactions, Lie established together with the scientists J. Worm Müller and G. O. Sars his own periodical in 1876 - *Archiv for Mathematik og Naturvidenskab* (The Archive for Mathematics and Natural Science). Nontheless, neither his integration theory nor his group theory received much attention. Because of this, Lie turned with new vigour towards geometry and published detailed work on minimal surfaces, a classification of surfaces according to the transformation groups of their geodesic lines, surfaces of constant curvature, and translational surfaces.

The French mathematicians Halphén and Laguerre, had, since 1875 and 1879 respectively, conducted investigations on differential invariants of the finite projective group of the plane and the corresponding methods of integration. This led Lie to resume work on his integration theory of a complete system with known infinitesimal transformations. By the summer of 1882, he was convinced that the differential invariants could be used as solutions of such a system. With this knowledge, Lie met with Klein and Mayer in Leipzig in September, 1882, and continued on to Paris. Here, on November 3, he gave an important lecture at the Société Mathématique on his integration theory for a complete system with known infinitesimal transformations. In a letter which was probably written around the end of 1883, Lie explained to Klein [6]:

> I have not written anything to you about the invariant theory for infinite groups. I have known since 1874 that finite groups determine differential invariants, all of which can be specified. Indeed, I have taken this to be self-evident. But that each infinite group also determines differential invariants, differential covariants which can all be specified, is something that has only come to me during the last year. This is extremely curious and gives rise to a great number of theories.

Lie's first comprehensive presentation of the theory of differential invariants of finite and infinite groups was published towards the end of 1884 in Volume XXIV of *Mathematische Annalen* under the title *Ueber Differentialinvarianten* (On Differential Invariants), Unfortunately, no part of his work—with the exception of that which concerned geometry—attracted the attention it deserved. Lie lamented this fact in a letter to Mayer [7]:

> If I only knew how I could get mathematicians interested in transformation groups and the treatment of differential equations which arises from them. I am certain, absolutely certain, that, at some point in the future, these theories will be recognized as fundamental. If I wish to create such an understanding *sooner*, it's because, among other things, I could then do ten times more.

And, in the previously mentioned draft of his intellectual testament, he states [1]:

> My actual goal has always been the integration of differential equations. To this end, I have outlined a large number of theories, especially in Math. Ann., volumes 24 and 25. In volume 24, I treated the important question of how one decides whether given expressions or a given system of differential equations can be brought to given forms through the transformation of a certain group. My theory of differential invariants gave me the necessary criteria. But I was not able, at that time, to reduce these *sufficient* criteria to their simplest form. Later, I was able to do this for *finite* groups. For finite groups there exists, most certainly, a full system of differential invariants, *all* allowing themselves to be derived by differentiation. ... I have now attempted to expand this to infinite groups. For a series of infinite groups, I have *verified* that they have a completely analogous invariant theory. Unfortunately, I have *not* been successful in proving the existence of a complete system of invariants. But, I am convinced that this conjecture is correct. Everything comes down to the question of whether all invariant characteristics of one form can be expressed by equations of the form
>
> $$J_1 = \varphi_1(J_1, J_2, \ldots), \ \ldots, J_r = \varphi_r(J_1, J_2, \ldots)$$
>
> I ask you to keep this question in mind. If you do not want to deal with it yourself, then make sure that *Vessiot* does. *Vessiot should certainly be the heir to my integration theories. ...* It is especially my comprehensive (if not quite perfect) work in volume 25 of Math. Ann. that I want to recommend to Vessiot. ... The theories I have set forward in the aforementioned volume 25 have, in my opinion, a very broad scope. I consider them my most important contributions.

From 1868 to 1884, Lie worked tirelessly to draft all of the theories which constitute his life's work. However, since he was not able to describe his mathematical ideas either understandably or convincingly enough, his ground-breaking results remained unnoticed amongst his peers. Lie formulated his problem in a letter to Mayer in 1884 [8]:

> My wife recently gave birth to a son. ... It is in connection to this that certain aspects of my ideas have not been sufficiently worked through. As you know, I initially reach my theories by synthesis. When it comes to the analytical execution, I always find it difficult to present everything in a sufficiently general way. Unfortunately, the analytical description requires a particular form in which, quite often, much is lost. This is natural, given that I haven't sufficiently mastered the analytical form.

And, in a letter to Klein from the same year Lie wrote [9]:

> I'm ashamed that in my latest works I quote you only with a degree of reservation. You see, I've learned that I have to be careful because if I quote someone without qualification, one will believe that the other person has done everything. I don't understand why. Probably because one assumes that my ideas are proportional to my editorial capabilities.

With the phrase "editorial capabilities," Lie means the ability to make oneself understandable in written form. In addition to the realisation of his insufficient "editorial capabilities," there was the problem that, in Christiania, he did not have any students or colleagues who were interested in his work. For this reason his second stay in Leipzig in September 1882, where he met Mayer and Klein (who, since 1880, had been professor of geometry at Alma mater Lipsiensis) must have had a strong impact on him. In September of 1883 he wrote to Klein reflecting on the past [10]:

> I hope that you are in good health. It's exactly one year since I visited Leipzig. The next time I come to Germany I will stay longer. Let's hope that it soon will be possible. It is lonely, frightfully lonely, here in Christiania where nobody understands my work and interests.

Thus, it was of great importance that Klein and Mayer, understanding Lie's difficult situation, decided to send their student, Friedrich Engel, on a mission to assist Lie with the preparation of a large work on the theory of transformation groups. Towards the end of 1883 Klein worked out this plan to help his Norwegian friend, and on June 30, 1884, Engel received a letter from Christiania [11,12]:

> Dear Mr. Engel!
> At the end of 1883, when Klein first told me about the plan for you to come to Christiania, the idea seemed so fantastic that I didn't even respond to it. As he has brought it up again in the meantime, I've jumped on the idea. If this plan can be realized, it will be a great gain for myself and for my investigations. I am well aware of your competence, not only from Mayer and Klein's glowing reports, but also from your own interesting, independent work (for which I hereby thank you most sincerely) as well as from your valuable comments on my latest notes, which I'll again be sending to you soon in Leipzig. Whether you will be reasonably satisfied here is questionable. I can only promise to do my utmost. I particularly want to place a lot of time at your disposal so long as I maintain my normal capacity for work which is not particularly great.
>
> The general ideas you touch upon in your letter are actually very good. With my investigations of differential equations which permit a finite continuous group, I've always had a vague idea of the analogy between substitution theory and transformation theory. I've always operated with such concepts as sub-groups, invariant (or preferential) sub-groups, commutative transformations, transitivity, *transitivity in the infinitesimal*, primitivity, etc., etc. When I refer to my method of discovery as *synthetic*, I mean to say that, on the one hand, I operate with the concept of manifold and, on the other hand, *that I operate as a whole with concepts.* One can prove that certain problems in integration can be reduced to certain ancillary equations of *particular* order and with particular characteristics, *while further reduction is impossible in general.* How far the analogy with the algebraic equations can be carried through, I can't say for the good reason that I have almost

> no knowledge of equation theory. In this area I will be expecting to gain a lot from you. As to the analogies with modern function theory, I have really no idea.
>
> From 1871–1876, I lived and breathed only transformation groups and integration problems. But when nobody took any interest in these things, I grew a bit weary and turned to geometry for a time. Now, just in the last few years, I have again taken up these old pursuits of mine. If you will support me with the further development and editing of these things, you would be doing me a great service, especially in that, for once, a mathematician finally has a serious interest in these theories. Here in Christiania, a specialist like myself is terribly lonely. No interest, no understanding.
>
> Lately, politics that has taken all the attention here. In these days, remarkable things are taking place which also completely occupy me, even though, of course, I am spending some hours on editing work. My first paper for *[Mathematische] Annalen* was already finished in May. The second is proceeding rapidly. They are going to be rather large.
>
> Please say hello to Mayer and Klein for me. I owe both of them letters. If you actually come to Christiania, you will be most welcome.
> Yours sincerely, Sophus Lie.

This young and gifted mathematician named Friedrich Engel was born in Lugau, near Chemnitz, on December 26, 1861, the son of a protestant minister. In 1865, the family moved to Greiz where his father taught religious studies at the local high school, and it was here that the boy went to school from 1868 to 1879. Beginning at Easter 1879, he studied mathematics, primarily in Leipzig, but also in Berlin where he became acquainted with Weierstrass' school of mathematics without overestimating it. In the beginning of 1883, Engel passed the state examination in Leipzig and, in the summer, received his doctorate for the work *Zur Theorie der Berührungstransformationen* (Towards a Theory of Contact Transformations), with Adolph Mayer as his supervisor. From April 1, 1883 to April 1, 1884, Engel completed his military service in Dresden. In 1884, he returned to Leipzig for the summer semester in order to take part in the meetings of the Mathematical Seminar. By the beginning of his stay in Norway, Engel had already published three major papers in *Leipziger Berichte* and *Mathematische Annalen*. In September, 1884, he arrived in Christiania on a scholarship from the University of Leipzig and the Royal Saxon Society of the Sciences in Leipzig. Gerhard Kowalewski, a close student of Lie and Engel, characterised this assignment, which had been initiated by Klein and Mayer, in the following way [13]:

> For Engel, it was of major importance that, in September of 1884, at the request of his teachers in Leipzig, he travelled to Christiania with an assignment to help Sophus Lie with a comprehensive description of his sensational theories. Lie would never have been able to produce such an account by himself. He would have drowned in the sea of ideas which filled his mind at that time. Engel succeeded in bringing a systematic order to this chaotic mass of thought ...

Lie and Engel worked together intensely over a nine-month period in Christiania. In the person of this German, the Norwegian finally found his first disciple. The painful feeling of isolation, which Lie had so often complained about, was ended. Engel later recalled this collaboration, which was so important for both of them [14]:

> Lie had, for some time, thought of writing a larger work on transformation groups, but, without the impetus from the outside which he now was getting, it would have quite surely gone the way of the work on first-order partial differential equations which he had made plans to do in the [eighteen-]seventies. But, now, Lie decided to tackle a major piece on transformation groups, which was certainly intended to be much more than a simple introduction to the elements of the theory. It was not to be a popular book, if one may use this expression about a mathematical work. Quite the contrary, we should, as Lie put it, begin "with a full orchestra": it should be a systematic and strict-as-possible account that would retain its worth for a long time. We met two times a day – in the mornings in my apartment and in the afternoons at Lie's. We immediately got down to preliminary editorial work of a series of chapters which, according to Lie's plan, would be included in the work. Orally, Lie developed the content of each chapter and gave me a short sketch as a basis for the compilation, a kind of skeleton to which I would supply the flesh and blood. In this way, I also received the best introduction to his group theory which, when arriving in Christiania, I had only scant knowledge of. Every day, I was newly astonished by the magnificence of the structure which Lie had built entirely on his own, and about which his publications, up to then, gave only a vague idea. The preliminary editorial work was completed by Christmas 1884, after which Lie devoted some weeks to working through all of the material in order to lay down the final draft. Starting at the end of January 1885, the editorial work began anew; the finished chapters were reworked and new ones were added. When I left Christiania in June of 1885, there was a pile of manuscripts which Lie figured would eventually fill approximately thirty printer's sheets. That it would be eight years before the work was completed and that the thirty sheets would become one hundred and twenty-five was something neither of us could have imagined at that time.

In Leipzig, Engel obtained his qualification as a lecturer with his postdoctoral thesis *Ueber die Definitionsgleichungen der continuirlichen Transformationsgruppen* (On the Defining Equations of the Continuous Transformation Groups). On July 15, 1885, Engel successfully defended his thesis and, on October 14, gave a lecture entitled *Anwendungen der Gruppentheorie auf Differentialgleichungen* (Applications of Group Theory to Differential Equations). Now that all of the proficiency requirements had been fulfilled, Engel could function as a *Privatdocent* (i.e. unsalaried lecturer at a university) at the University of Leipzig. In the winter semester of 1885/86, he taught his first course. It was fitting that it should deal with the theory of first-order partial differential equations.

In August, 1885, Klein received an offer for a professorship at the University of Göttingen. Fascinated by the thought of working where Gauss, Riemann and Clebsch were active, he accepted after a bit of hesitation and moved to the *Georgia Augusta* for the beginning of the 1886 summer semester. At the same time, the Royal Saxon Ministry for Cultural Affairs and Education in Dresden recommended Sophus Lie to the Philosophical Faculty as Klein's successor. In the recommendation, for which Klein had written the rough draft, they comment on the Norwegian [15]:

> Regarding proposals on the appointment to the professorship in geometry which will soon be vacant, we must above all name a man who, with his independent ideas, the consistency of his work and his lofty goals, without question stands out among all the others ... There are other geometricians who, through their many-faceted developments, have realized important results in one area or another, and who know how to present their knowledge of geometry in lectures. But Lie is the only one who, by force of his personality and in the originality of his thinking, is capable of establishing an independent school of geometry. We received proof of this when Kregel von Sternbach's scholarship was to be awarded. We sent a young mathematician – our present *Privatdocent*, Dr. Engel - to Lie in Christiania, from where he returned with a plethora of new ideas ... In closing, we wish to state that Lie, from what he has said lately, seems to be positively inclined towards an invitation to Germany. With such a move, he sees the means to escape the scientific isolation which he is forced to live with in Christiania.

Karl Weierstrass had wanted Hermann Amandus Schwarz to fill this professorship, and, when he became aware of the plan to bring Lie to Leipzig, he stated [16]:

> I cannot deny that Lie has produced his share of good work. But neither as a scholar nor as a teacher is he so important that there is a justification in preferring him, a foreigner, to all of those, our countrymen, who are available. It now seems that he is being seen as a second Abel who must be secured at any cost.

Weierstrass' rejection of "the foreigner's" importance in the development of mathematics based on different theoretical standpoints was shared by the entire Berlin school and was directed not only towards Lie, but also towards his students and colleagues. This continued to hold true after Lie's death and prevented, among other things, Friedrich Engel and Friedrich Schur even from being considered for a professorship in Berlin [17].

Under the express condition that Adolph Mayer be contented with a Honorarprofessorship (professorship without voice in faculty matters), and with reference to the aforementioned report, the Ministry for Cultural Affairs and Education in Dresden decided, on December 18, 1885, to negotiate with Lie. (In Saxony, unlike in Prussia, Weierstrass had no influence on university appointments.) Just one month later, it was possible to report to the Philosophical Faculty that Lie had accepted the position of a professor of geometry.

1886 - 1898

In February, 1886, Lie and his family arrived in Leipzig and moved into an apartment in *Seeburgstrasse.* On May 29, 1886 in the presence of King Albert I of Saxony, he held his inaugural lecture *Ueber den Einfluss der Geometrie auf die Entwicklung der Mathematik* (On the Influence of Geometry in the Development of Mathematics) [18] and was sworn in as a professor.

Figure 5: Main building of the University of Leipzig before the modification from 1894 to 1896. The *Aula*, i.e. the ceremonial hall of the University, where Lie held his inaugural lecture, was in the middle part of the building.

In 1886, Leipzig had a population of about 170,000. In this city, there were a large number of scientific societies, an academy, a university, several colleges, around one hundred printing shops and over two hundred publishing companies. The major scientific publisher, B. G. Teubner, would come to publish all of Lie's books, and, after his death, his collected papers.

The founding of a Royal Saxon Society of the Sciences was originally a plan of Gottfried Wilhelm Leibniz. He negotiated with August the Great in 1700 to establish a Saxon-Polish Academy for the Sciences and Arts. Despite the sovereigns interested position, the plan was scuttled when the Swedes under Karl XII marched into Saxony in 1706 as a consequence of the Nordic War. On July 1, 1846 the bicentennial of the birth of Leibniz, the Royal Saxon Society of the Sciences was founded in his native town Leipzig. In accordance with the original plan, it was divided into a mathematical-natural scientific section and a philological-historical section, and it published transactions named *Leipziger Abhandlungen* and *Leipziger Berichte.* In 1886, Lie became a member of this prestigious academy.

The second half of the nineteenth century saw the beginning of a period of growth for the University of Leipzig (founded 1409) which lasted right up until that fateful year of 1933. Its flourishing was closely bound to the reign of King Johann I of Saxony, who lived from 1801 to 1873 and governed from 1854 to his death. He was himself an important historian and philologist who had specialised in research on Dante. Under the pseudonym Philalethes (friend of the truth), he published his major work, a metrical translation of *The Divine Comedy*, which included a critical and historical commentary. With all his power, he promoted Saxony's educational system and, especially, the country's university. His efforts were supported and carried on by three highly gifted ministers for cultural affairs and education, Johann von Falkenstein (term: 1853-1871), Carl von Gerber (term: 1871-1891) and Paul von Seydewitz (term: 1892-1906). Carl von Gerber managed to obtain the services of Oskar Schlömilch for the Royal Saxon Ministry for Cultural Affairs and Education from 1874 to 1885. Schlömilch, who was known as the editor of *Zeitschrift für Mathematik und Physik* which he was for many years; in addition he worked as a mathematics consultant and advisor to the B. G. Teubner publishing house. He was particularly concerned with looking after the state of mathematics at the educational establishments in Saxony. Felix Klein was appointed to the newly established professorship in geometry at the University of Leipzig in 1880, and it was on his initiative that the Mathematical Seminar (later called the Mathematical Institute) was founded in 1881.

In 1886, the Mathematical Seminar consisted of the directors, Sophus Lie, Adolph Mayer and Carl von der Mühll, and of the assistant Friedrich Schur. It had two sections. Section I, at 24 *Ritterstrasse*, was furnished with a work room and a reading room. It was here that Friedrich Engel worked as a librarian. Section II, in the *Czermak's Spectatorium* at 32 *Brüderstrasse*, had a larger lecture-hall (where Lie usually lectured) and a model collection which was run by Friedrich Schur.

In 1889 Carl von der Mühll left as co-director for a position as professor in Basel. In 1888 Friedrich Schur took a position in Dorpat, and Friedrich Engel filled the vacant position of an assistant. Through his work as the librarian and assistant in the Mathematical Seminar, the poor *Privatdocent* (since 1889, associate professor) was able to earn a little money. Lie's other colleagues included the professors Heinrich Bruns, Carl Neumann, Wilhelm Scheibner, and until 1888, *Privatdocent* Eduard Study. Further members of the department were *Privatdocent* Georg Scheffers, who started 1891, became an associate professor in 1896, and went to Darmstadt in 1897, and, from 1895 on, *Privatdocent* Felix Hausdorff.

In Leipzig, Lie and Engel resumed their intensive work on the theory of transformation groups. In 1888, B. G. Teubner published the 632-page first part of *Theorie der Transformationsgruppen* (The Theory of Transformation Groups).

At the beginning of the winter semester of 1889/90, Lie was in poor health. A doctor's report dated November 16, 1889 [19] spoke of a "group of threatening symptoms of the nervous sort," which were probably the mental consequences of a yet undiagnosed pernicious anaemia. Lie sought treatment at a psychiatric clinic in Ilten near Hannover and then returned to Leipzig in July of 1890. During the winter semester of 1889/90 and the following summer semester, his lectures were held by Engel. It was not until the winter semester of 1890/91 that Lie could resume his teaching duties.

In 1890, the 555–page second part of *Theorie der Transformationsgruppen* was published, with the 831–page third part following in 1893. With this last part, this fundamental project was completed after nearly nine years of tireless work by Lie and Engel. Lie provided it with a lengthy dedication, where we find the following paragraph written about Engel [20]:

> In the elaboration of my ideas, both in terms of their details and of their systematic representation, I have, since 1884, benefitted greatly from the tireless assistance of Professor *Friedrich Engel*, my distinguished colleague from the University of *Leipzig*.

And, in the foreword to the third part he stated [21]:

> For me, Professor Engel occupies a special position. On the initiative of F. Klein and A. Mayer, he travelled to Christiania in 1884 to assist me in the preparation of a coherent description of my theories. He tackled this assignment, the size of which was not known at that time, with the perseverance and skill which typifies a man of his calibre. He has also, during this time, developed a series of important ideas of his own, but has in a most unselfish manner declined to describe them here in any great detail or continuity, satisfying himself with submitting short pieces to *Mathematische Annalen* and, particularly, *Leipziger Berichte*. He has, instead, unceasingly dedicated his talent and free time which his teaching allowed him to spend, to work on the presentation of my theories as fully, as completely and systematically, and, above all, as *precisely* as is in any manner possible. For this selfless work which has stretched over a period of nine years, I, and, in my opinion, the entire scientific world, owe him the highest gratitude.

Gerhard Kowalewski, a student close to Lie and Engel, has also emphasized the importance of the latter in *Theorie der Transformationsgruppen*. Six decades after the completion of the fundamental work—time enough to allow the importance of the research to be seen in the proper light—he wrote [22]:

> One demanded an analytical dressing, even for things which could be achieved through purely geometric or conceptual considerations, which was Lie's usual modus operandi. This often came out in his lectures, given in broken German, with the words: "Let us reason with concepts!" How often were we presented with a geometric figure instead of an analytical proof! Engel, who had been Lie's interpreter (not only linguistically but also in a mathematical sense) during the preparation of Lie's three-volume work, had studied in Berlin with Weierstrass as well as in Leipzig. He knew very well that if one wanted to win understanding for these new ideas, one could not do without a function-theoretical foundation in structuring the new theory. One had to limit oneself to analytical groups so that, right off the bat, one worked with power-series, and then one was later pleased if there was an analytical continuation that went beyond the narrow limits of the domains of convergence.

Lie knew that his ideas had a much broader scope, and he sometimes expressed unhappiness, when Engel restricted them with what were necessary conditions.

In this way Engel fought for mathematical exactness and against any extremely cumbersome and therefore unrealisable constructions.

After the first part of *Theorie der Transformationsgruppen* had appeared, Eduard Study, who had read the second proofs [23], wrote a thorough, 21–page scholarly report in *Zeitschrift für Mathematik und Physik*. It began as follows [24]:

> The work in question gives a comprehensive description of an extensive theory which Mr. Lie has developed over a number of years in a large number of individual articles in journals such as *Mathematische Annalen* and *Archiv for Mathematik og Naturvidenskab*, the latter being published in Christiania. Because most of these articles are not well known, and, because of their concise format, the content [of Lie's theory], in spite of its enormous value, has remained virtually unknown to the scientific community. But by the same token we can also be thankful that the author has had the rare opportunity of being able to let his thoughts mature in peace, to form them in harmony and think them through independently, away from the breathless competitiveness of our time. We do not have here a *textbook* written by a host of authors who have worked together to introduce their theories to a wider audience, but rather the creation of one man, an *original work* which, from beginning to end, deals with completely new things.

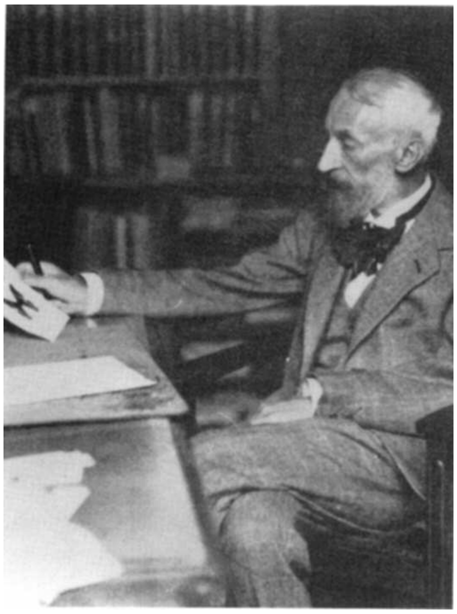


Figure 6: Eduard Study (1862 – 1930)

In the final part of his review, Study states, prophetically:

> It is hoped that what I have just said will awaken a desire to become familiar with the important content of this work in the wider scientific community. We do not believe that we are saying too much when we claim that there are few areas of mathematical science which will not be enriched by the fundamental ideas of this new discipline. But, above

> all, it is in those areas which are closely related to differential equations and geometry that a wealth of new approaches will be provided, and the ideas for further research will be planted.

And, in concluding Study wrote:

> Mr. Lie deserves to find here, with us, that which he looked for when he left his native country. Hopefully he will succeed in gathering a circle of students around him who will follow him down these newly trodden and promising paths, and, in joyful collaboration with him, will develop knowledge to ever more beautiful and higher form.

Here, Study expressed Lie's hopes and aims which were initially made difficult due to a decrease in the number of mathematics students. At the University of Leipzig, the number fell from 80 in the summer semester of 1886 to a low of 20 in the winter semester of 1893/94. After that, the number of students began to increase, and by the summer of 1898 it was up to 90. In spite of these variations, which were a consequence of an over–production of mathematicians in the 1870's and 1880's, Lie managed, with Engel's help, to establish a group theoretical school in Leipzig. In his basic lectures on geometry, Lie took up the analytic geometry of the plane and space, the theory of curves and surfaces in space, as well as metrical geometry, while Engel lectured on algebraic and function-theory problems - topics outside their own areas of specialisation. On the other hand, they offered special courses and training in their own areas of research and endeavoured to sell their ideas and win new followers. That they were successful in their efforts can be seen from the number of doctoral students under Lie's supervision. From 1890, 26 of the 56 doctoral students receiving Ph.Ds under five professors (Bruns, Lie, Mayer, Neumann and Scheibner), were supervised by Lie. Not surprisingly, given these figures, Lie publicly acknowledged Engel's teaching services. In the foreword to the third part of *Theorie der Transformationsgruppen*, Lie writes [25]:

> Personally, I must also thank him [Engel] for the support he has given me with respect to my teaching duties at the University of Leipzig by his lectures.

An increasing number of foreign students especially from the United States and France, came to Leipzig to study, not only under Lie, but also under Engel. Therefore, Lie had dedicated *Theorie der Transformationsgruppen* to the École Normale Supérieure in Paris, where he thanked professors Darboux, Picard and Tannery for the fact that [20]

...France's most capable young mathematicians, in competition with a range of young German mathematicians, study my investigations of *continuous groups, geometry* and *differential equations*, and employ them with excellent results.

The relationship with the École Normale Supérieure was deepened on the occasion of the school's centenary in April, 1895. To honor this event Lie completed a contribution on the influence of Galois in the development of mathematics in November, 1894 which was translated into French and published under the title *Influence de Galois sur le développement des Mathématiques* [26] in the commemorative volume. This article included his final conceptual definition of a group [27,28]. From 1880 onwards, Lie was also intensively engaged in the foundations of geometry. He had been spurred on by Helmholtz's treatise, *Ueber die Thatsachen, die der Geometrie zu Grunde liegen* (On the Facts Forming the Basis of Geometry) from 1868. From the standpoint of group theory, Lie undertook a massive critique of this treatise. He was able to prove that Helmholtz's axiom of monodromy is redundant in a three-dimensional space. In the third part of *Theorie der Transformationsgruppen*, he gave a solution to the problem which, on the one hand, operated only with the concepts of points, line elements, and surface elements, and, on the other hand, made only assumptions about transitivity of infinitesimal movements. In this way, Helmholtz's arguments and derivations were improved, and Lie demonstrated his group theory on a problem which was of interest to all mathematicians. For this work, he was awarded the first Lobachevsky prize by the International Lobachevsky Foundation in Kazan in 1897. [3]

Lie's most important students were Friedrich Engel (1861-1941), Gerhard Kowalewski (1876-1950), Georg Scheffers (1866-1945), Wladimir de Tannenberg (b. 1860), Arthur Tresse (b. 1868), and Kasimir Żorawski (1866-1953). In addition, he left lasting impressions on the mathematicians Élie Cartan [4] (1869-1951), Wilhelm Killing (1847-1923), Ludwig Maurer (1859-1927), Emile Picard (1856-1941), Henri Poincaré (1854-1912), Friedrich Schur (1856-1932), Eduard Study (1862-1930) and Ernest Vessiot (1865-1952), all of whom, along with his students, developed his theories in a variety of directions. (In Chapter 29 of the third part of *Theorie der Transformationsgruppen*, Lie discussed their group-theoretical works, which were available right up until the summer of 1893.) Felix Hausdorff (1868-1942), who later went on to work in other areas of mathematics [5], numbered among Lie's students.

Thus, Lie's labour bore rich fruit. However, this impressive progress in his discipline was overshadowed by his illness. The previously mentioned doctor's report from November 16, 1889 which consciously avoided a diagnosis, the time of the onset of the illness, the symptoms of sleeplessness and hypersensitivity to light,

---

[3] In his doctoral thesis *Über eine Kategorie von Transformationsgruppen einer vierdimensionalen Mannigfaltigkeit* (On a Category of Transformation Groups of a Four-Dimensional Set) from 1898 which was inspired by Sophus Lie, Gerhard Kowalewski was able to extend his teacher's important results to higher-dimensional spaces. In his treatise *Gruppentheorie und Grundlagen der Geometrie* (Group Theory and Foundations of Geometry), Friedrich Engel enriched and honed Lie's and Kowalewski's results. This work, finished in 1924, was designed to be used as a commentary in the collected works of N. I. Lobachevsky, which was to be published in Moscow. However, this edition was never realized in its projected form. And so it was not until just after the end of the war in 1945, that Karl Faber and Egon Ullrich published this important treatise in stencil form in *Gedenkband für Friedrich Engel* (Memorial Volume for Friedrich Engel). It remains practically unknown and unavailable.

[4] É. Cartan did not study with S. Lie in Leipzig as G. Kowalewski claimed on page 73 of *Bestand und Wandel* (Immutability and Change), but was made aware of Lie's ideas by his (Cartan's) friend A. Tresse (see [54]). (Personal communication from H. Cartan and J. Dixmier, August 25, 1992)

[5] With regard to F. Hausdorff's relationship to his teachers S. Lie and F. Engel, see the letters printed in *Vorlesungen zum Gedenken an Felix Hausdorff* (Lectures in Memorial of Felix Hausdorff) published by E. Eichhorn and E.-J. Thiele (see [60]).

as well as observations made by his close associates, all pointed to the fact that, from 1889, Lie suffered from pernicious anaemia [29]. Lie began the aforementioned draft of an intellectual testament which seems to stem from 1892, in the form of a letter on which he states [1]:

> My esteemed Mr. Picard! I must once again express my gratitude for all you have done for the theory of transformation groups. Perhaps you know that I have been ill for the past one and a half years. I am plagued by almost complete sleeplessness. This and the medicine used to combat it have taken such a toll on me that my endeavours as a scholar have come to an end, even though I am strong enough in body to live still longer.
>
> Under these unfathomably sad circumstances, it has been a great joy to me that you have supported group theory so strongly. In my life, scholarlship has been my greatest joy. I have been very lucky to have been granted the opportunity to contribute to the development of science.

By Christmas, 1898, the diagnosis of pernicious anaemia was confirmed [30]. Even if the diagnosis had been made earlier, his doctors (his doctor in Leipzig was the neurologist and psychiatrist, Paul Flechsig) still could not have helped him. It was not until 1926 that the North Americans Minot and Murphy developed a successful treatment for this disease.

Lie never really felt at home in Leipzig. With his roots being predominantly rural and with his particular values and world-view, he felt himself a stranger to the attitudes of upper middle-class society in the German urban environment. As a Scandinavian liberal hostile towards Bismarck, he also felt estranged from his colleagues who were predominantly conservative and, for the most part, supporters of the Imperial Chancellor. Lie also missed the Norwegian landscape with its fjords and mountains [31,32].

In 1893, probably for reasons connected to his illness, Lie severed most of his personal contacts in Germany, including his close relationship with Friedrich Engel. The following passage, from the foreword of the third part of *Theorie der Transformationsgruppen*, documents his break with Felix Klein [33]:

> F. Klein, whom I kept abreast of all of my ideas during these years, was occasioned to develop similar viewpoints for discontinuous groups. In his Erlangen Program, where he reports on his and on my ideas, he, in addition, talks about groups which, according to my terminology, are neither continuous nor discontinuous. For example, he speaks of the group of all Cremona transformations and of the group of distortions. The fact that there is an essential difference between these types of groups and the groups which I have called continuous (given the fact that my continuous groups can be defined with the help of differential equations) is something that has apparently escaped him. Also, there is almost no mention of the important concept of a differential invariant in Klein's program. Klein shares no credit for this concept, upon which a general invariant theory can be built, and it was from me that

Figure 7: Family photograph from the Leipzig period.

> he learned that each and every group defined by differential equations determines differential invariants which can be found through integration of complete systems.
>
> I feel these remarks are called for since Klein's students and friends have repeatedly represented the relationship between his work and my work wrongly. Moreover, some remarks which have accompanied the new editions of Klein's interesting program (so far, in four different journals) could be taken the wrong way. I am no student of Klein and neither is the opposite the case, though the latter might be nearer to the truth.
>
> By saying all this, of course, I do not mean to criticize Klein's original work in the theory of algebraic equations and function theory. I regard Klein's talent highly and will never forget the sympathetic interest he has taken in my research endeavours. Nonetheless, I don't believe he distinguishes sufficiently between induction and proof, between a concept and its use.

After this, the contact between the two friends ceased. But, some months after he received the Lobachevsky prize, Lie thanked Klein for his kind words to the International Lobachevsky Foundation [6]. In the same letter, Lie expressed his concerns with regard to the naming of a suitable successor. He wrote [34]:

> For twelve years, I have held the banner of geometry as high as I possibly could. It would be painful for me to think that my leaving should bring with it the loss of a professorship of geometry. Yet, I know

[6] The International Lobachevsky Foundation which gave awards for outstanding mathematical results in the field of geometry, was founded in 1894 in Kazan by A. Wassiljew (see [29], p. 198 and [56], p. 707). From the available sources, it must be assumed that this foundation no longer existed after the Russian Revolution in 1917.

> of no one that I can recommend. I've thought about *Study*, whom I consider to be a competent geometrician. I've thought a little about *Stäckel*, too. If *Study* has prepared himself as a lecturer, one could consider him. That is Mayer's opinion, in any case.
>
> If an expert in analysis is appointed, then there is a danger that he will have no more interest in geometry than both Neumann and Mayer, and that Germany will lose its sole professorship of geometry.

On May 12, 1898, Klein answered with a detailed opinion regarding the question of the appointment. The "king-maker in mathematics" (Gerhard Kowalewski) came out against Study and Engel, in particular (see also Felix Hausdorff's view in [60], pp. 70-87).

Because of the break between Lie and Engel, the projected book on differential invariants and infinitely continuous groups, mentioned in the foreword of the third part of *Theorie der Transformationsgruppen* (and which was also to include the application of group theory to the integration of differential equations), went unwritten. Nor did a planned work on Lie's theory for first-order partial differential equations see the light of day. Felix Hausdorff, who intended to work on this topic, felt treated unkindly and chose another direction in mathematics [36]. Only the collaboration with Georg Scheffers came to fruition. Scheffers had already come out with two substantial accounts of Lie's basic ideas. These were of a more pedagogical nature and set aside absolute mathematical rigour. They were entitled *Vorlesungen über Differentialgleichungen mit bekannten infinitesimalen Transformationen* (Lectures on Differential Equations with Known Infinitesimal Transformations) and *Vorlesungen über continuierliche Gruppen mit geometrischen und anderen Anwendungen* (Lectures on Continuous Groups with Geometric and Other Applications), and were published in 1891 and 1893, respectively, by B. G. Teubner. They were followed, in 1896, by the first volume of *Geometrie der Berührungstransformationen* (Geometry of Contact Transformations) by the same publisher. Because of Scheffers transfer to Darmstadt, the planned second volume never got past the preparation stage.

1898 - 1899

When it became known that Lie longed to come home, the Norwegian writer and poet, Bjørnstjerne Bjørnson, took the initiative to enable his return to Norway, and, in 1894, the Norwegian Parliament established a highly paid personal professorship for him to teach the theory of transformation groups at the University of Christiania. However, Lie hesitated. It was not until May 22, 1898 that he submitted his resignation to the University of Leipzig, to take effect at the end of the summer semester. The same day, Lie explained his decision to the Royal Saxon Minister for Cultural Affairs and Education, Paul von Seydewitz in a personal letter [37]:

> Your Excellency! In a letter to the Royal Ministry for Cultural Affairs and Education, I have just asked to be released from my position as professor of geometry at the University of Leipzig. I must admit that it is not easy for me to give up my teaching duties at this wonderful

Figure 8: Sophus Lie (towards the end of the Leipzig period)

> university and to leave its excellent Mathematical Institute. Nevertheless, I have to leave Leipzig. The primary reason for this is that the climatic and physical conditions here are not at all favourable to me. I still need many years to realize my literary plans, and I consider it most probable that Christiania will be more favourable than Leipzig for my health and for my capacity to do work. The kindness that Your Excellency has accorded to me recently has moved me deeply. I will always remember with thankfulness the good will that Your Excellency has shown to me on many occasions. [7]
>
> In deepest respect and sincere gratitude,
> Professor Sophus Lie

Because of Lie's imminent departure, the Faculty of Philosophy was requested to suggest a suitable successor. A majority preferred either Heinrich Weber from Strassburg or David Hilbert from Göttingen, both of whom had been around at the time of Scheibner's departure. The majority agreed that, if neither Weber nor Hilbert accepted, they would suggest Otto Hölder. In a special resolution written by Wilhelm Scheibner (and which Carl Neumann also signed), Friedrich Engel and the function-theoretician, Martin Krause, were suggested. Neumann, with whom Lie had already a tense relationship, explained [38]:

> If those at our university wish to hold fast to the course that Lie has laid out, then it is only *Prof. Engel* who could be considered for Lie's professorship.
>
> If, on the other hand, they wish to let the direction that Lie has taken fall, then, in my opinion, it is *Prof. Krause* who must be given primary consideration.

[7] For example, on April 12, 1894, Lie was awarded, the Knight's Cross of the First Class of the Order of Merit for his accomplishments - one of the highest Saxon honours.

In a personal statement, Lie came out against both Engel and Krause [39]. As a result neither were offered the position and when both Weber and Hilbert turned down the offer, Otto Hölder was declared Lie's successor at Alma mater Lipsiensis.

Lie returned to Christiania in September, 1898, a mortally ill man. For a time, he could still teach those students who had followed him from Leipzig at his home, but he eventually had to give up this, his final teaching activity. He died of pernicious anaemia on February 18, 1899.

He was buried at *Vår Frelsers Gravelund* (Our Saviour's Cemetery). On the 24th of February, Ludvig Sylow gave a memorial address to the Science Society of Christiania honoring his former student.

Figure 9: The family grave of Sophus Lie, his wife and his son at the cemetery *Vår Frelsers Gravelund* in Oslo.

The Royal Saxon Society of the Sciences remembered its outstanding member at a plenary meeting on November 14, 1899, on the 183$^{\text{rd}}$ anniversary of Leibniz's death. Friedrich Engel gave the memorial speech. He began with the words [40]:

> If the capacity for discovery is the true measure of a mathematician's greatness, then Sophus Lie must be ranked among the foremost mathematicians of all time. Only extremely few have opened up so many vast areas for mathematical research and created such rich and wide-ranging methods as he. He paved the way to finding solutions to problems which will provide an abundant resource to mathematicians for generations to come. It will be a long time before everything he knew and mastered, but introduced in only concise suggestions, can be treated. The wealth of ideas which he has disseminated will have a still more lasting effect, and no one can say at this time when their force will cease to be felt or whether this will ever happen at all.

> In addition to a capacity for discovery, we expect a mathematician to possess a penetrating mind, and Lie was really an exceptionally gifted mathematician. But he did not waste his mathematical acumen on problems just because they were difficult or because many had applied their skills to them in vain. Nor was he out to track down problems which must be hidden in a seemingly complete theory. His efforts were based on tackling problems which are important, but solvable, and it often happened that he was able to solve problems which had withstood the efforts of other eminent mathematicians. He could certainly be proud of such results, but only because they proved that his way of posing the questions and treating the problems were important and fruitful. He did not need such proof for himself. He was absolutely certain, and this certainty that he would prevail, that his methods and theories would persevere, never left him, even in the difficult times when appreciation and acknowledgement was not forthcoming.
>
> But the most characteristic thing about Lie is, and will remain, his capacity for discovery, his original mathematical thinking. He rejected the beaten path, but rather went his own way. I would like to compare him to a Scout in the primeval forest who, when others had given up trying to break through the undergrowth in desperation, always knows how to find a way, the way that affords the best views of unfamiliar, yet romantic mountains and valleys.

A quarter of a century later, a mature Eduard Study, Lie's like-minded colleague, gave his final appraisal of Lie and his work [41]:

> Sophus Lie had the shortcomings of an autodidact, but he was also one of the most brilliant mathematicians who ever lived. He possessed something which is not found very often and which is now becoming even rarer, and he possessed it in abundance: creative imagination. Coming generations will learn to appreciate this visionary's mind better than the present generation, who can only appreciate the mathematicians' sharp intellect. The all-encompassing scope of this man's vision, which, above all, demands recognition, is nearly completely lost. But, the coming generation ... will understand the importance of the theory of transformation groups and ensure the scientific status that this magnificent work deserves.

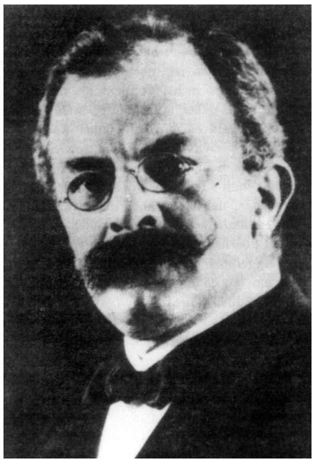
Ludvig Sylow (1832 – 1918)

Felix Klein (1849 – 1925)

Gaston Darboux (1842 – 1917)

Adolph Mayer (1839 – 1908)

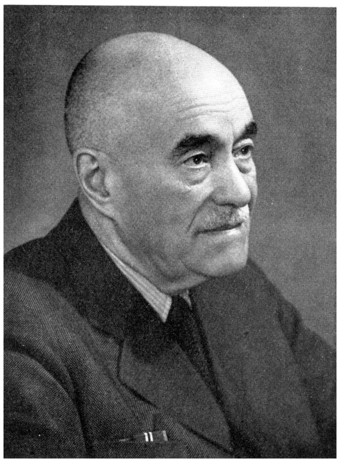
Plate 2



Friedrich Engel (1861 – 1941)

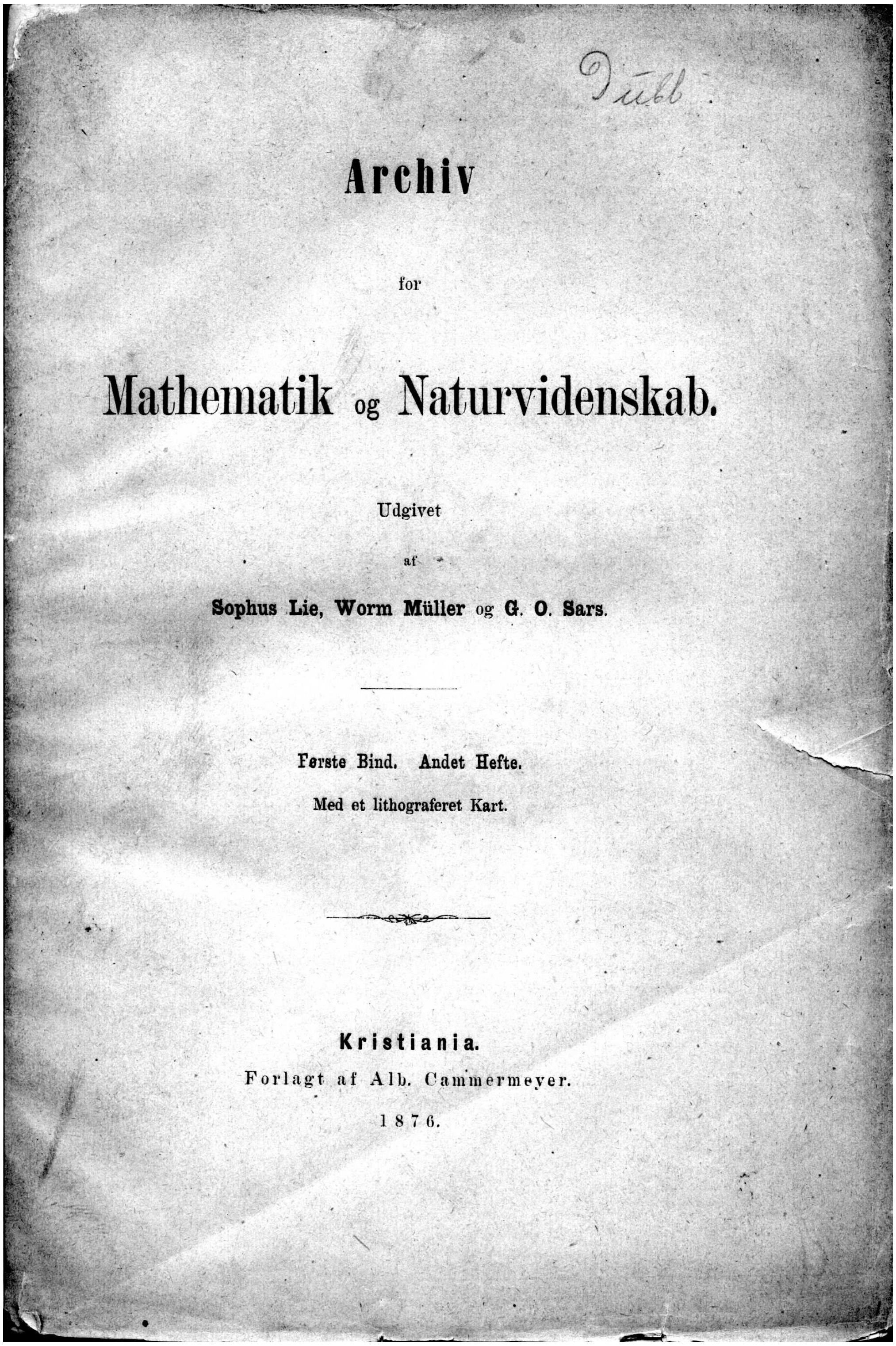
Georg Scheffers (1866 – 1945)

Felix Hausdorff (1868 – 1942)



Gerhard Kowalewski (1876 – 1950)

Plate 3



Archiv

for

Mathematik og Naturvidenskab.

Udgivet

af

Sophus Lie, Worm Müller og G. O. Sars.

Første Bind. Andet Hefte.

Med et lithograferet Kart.

Kristiania.

Forlagt af Alb. Cammermeyer.

1876.

Plate 4 : Title-page of *Archiv for Mathematik og Naturvidenskab*, which was co-established and co-edited by Sophus Lie.

Archiv for Mathematik og Naturvidenskab udkommer med 1 Bind (4 Hefter) aarlig. Subscriptionsprisen er fastsat til 2 Spd. pr. Bind.

Andet Hefte indeholder følgende Afhandlinger:

Amund Helland (Forts.) Om de isfyldte Fjorde og de glaciale Dannelser i Nordgrønland, med 1 Kart.

Worm Müller. Om Tællingen af de røde Blodlegemer efter Malassez's Methode (3 Træsnit).

Worm Müller. Om Forholdet mellem Blodlegemernes Antal og Blodets Farvekraft.

Sophus Lie. Theorie der Transformations-Gruppen. II, mit einer Note über Berührungs-Transformationen.

S. A. Sexe. Hvorfor blæser det ikke bestandigt fra Øst i det saakaldte stille Belte?

K. Pettersen. Saltens Geologie.

Tredie Hefte vil indeholde følgende Afhandlinger:

Amund Helland. Om Klormængden i Nordsøen, Atlanterhavet og Davisstrædet.

Münster. Om Hytteproduktet, Sten.

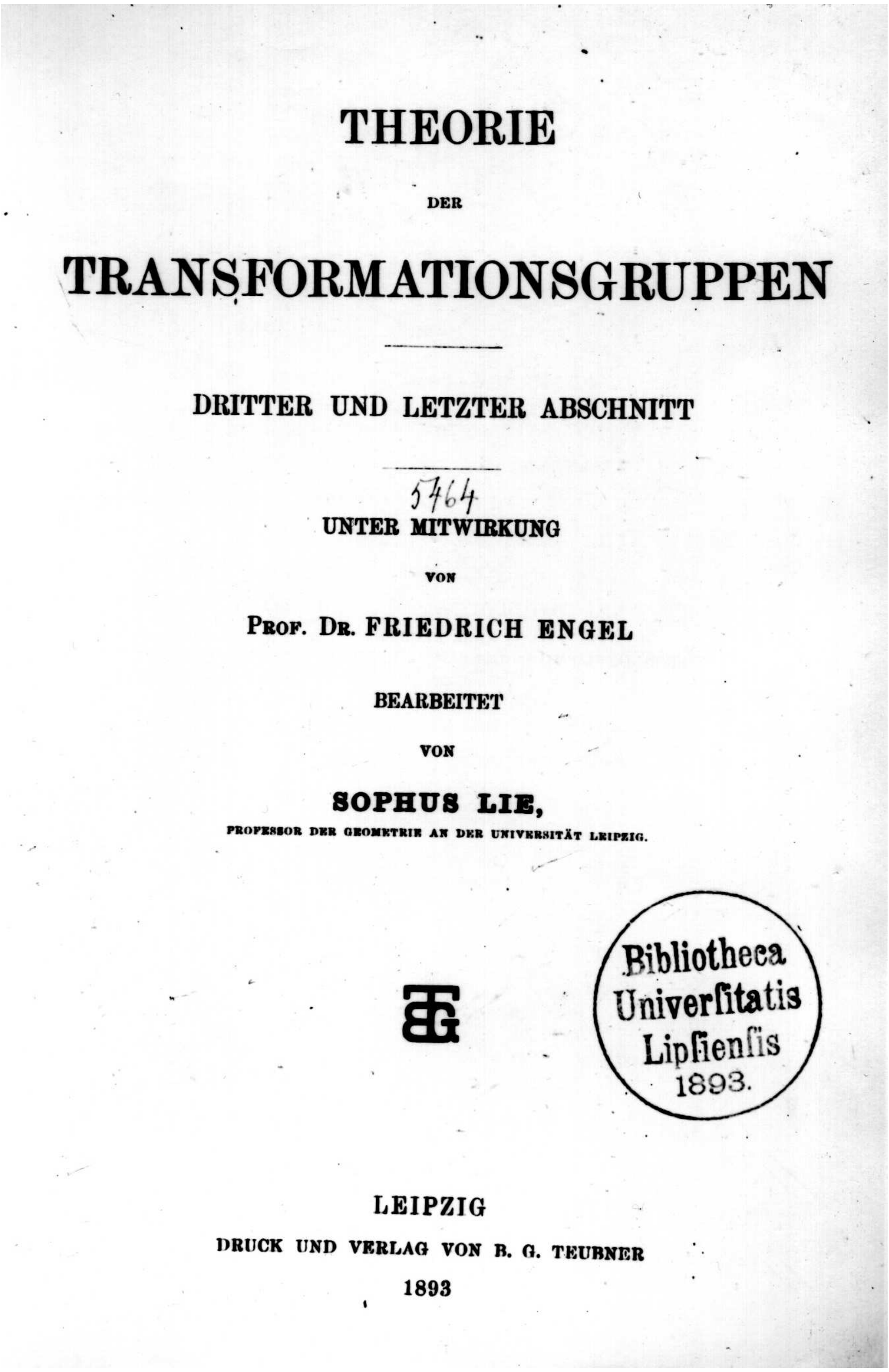
G. O. Sars. Nye Bidrag til Middelhavets Invertebratfauna. I. Middelhavets Mysider.

Sophus Lie. Resumé einer neuen Integrations-Theorie.

Kristiania i Mai 1876.

Alb. Cammermeyer.

Plate 5 : Index of the second part of the first volume of *Archiv for Mathematik og Naturvidenskab*.



THEORIE

DER

TRANSFORMATIONSGRUPPEN

DRITTER UND LETZTER ABSCHNITT

5464

UNTER MITWIRKUNG

VON

PROF. DR. FRIEDRICH ENGEL

BEARBEITET

VON

SOPHUS LIE,

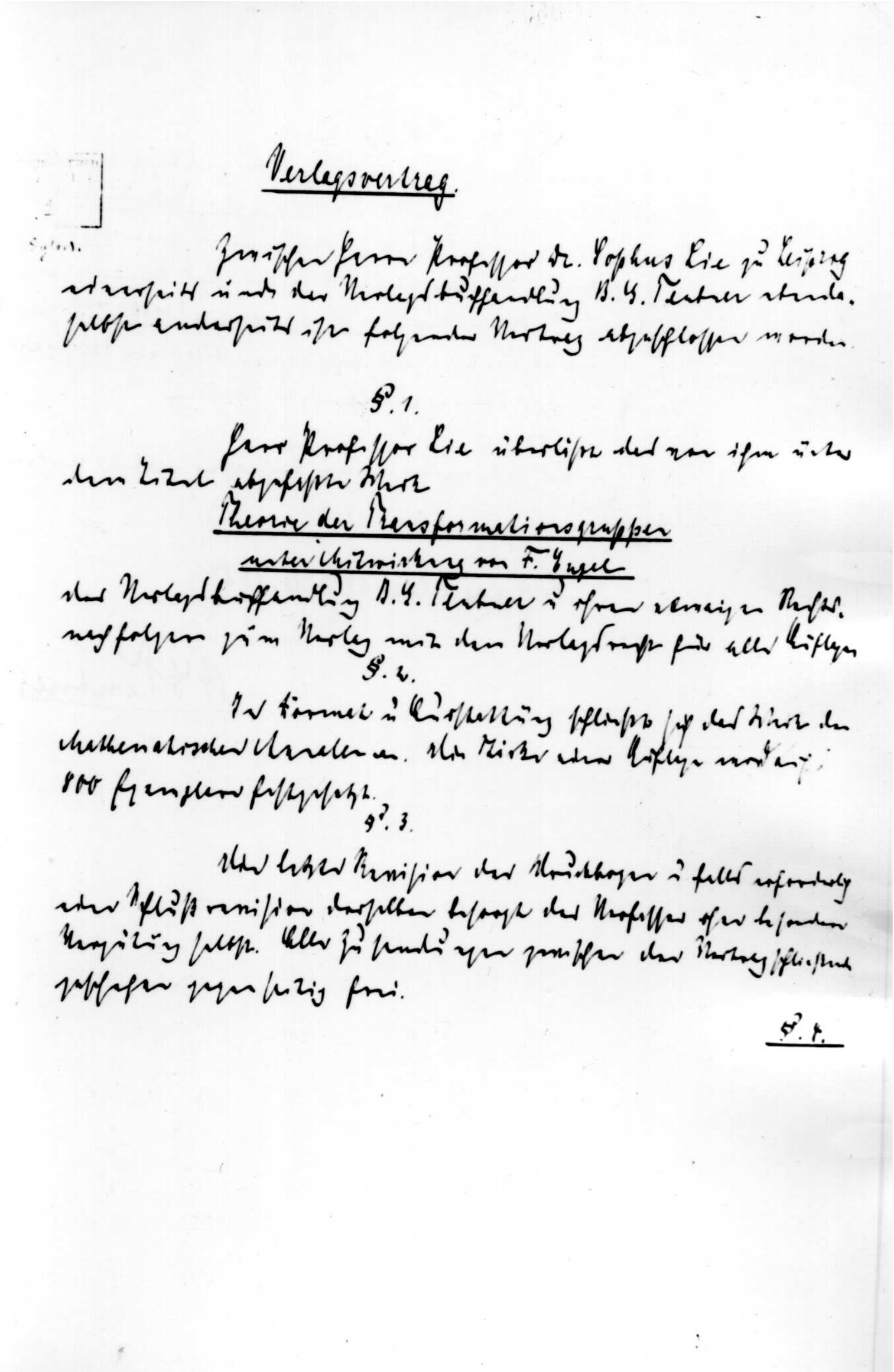
PROFESSOR DER GEOMETRIE AN DER UNIVERSITÄT LEIPZIG.

LEIPZIG

DRUCK UND VERLAG VON B. G. TEUBNER

1893

Plate 6 : Title-page of the third part of *Theorie der Transformationsgruppen.*



Verlagsvertrag.

Zwischen Herrn Professor Dr. Sophus Lie zu Leipzig einerseits und der Verlagsbuchhandlung B. G. Teubner ebendaselbst andererseits ist folgender Vertrag abgeschlossen worden.

§. 1.

Herr Professor Lie überläßt das von ihm unter dem Titel abgefaßte Werk

Theorie der Transformationsgruppen

unter Mitwirkung von F. Engel

der Verlagsbuchhandlung B. G. Teubner und ihren etwaigen Rechtsnachfolgern zum Verlag mit dem Verlagsrechte für alle Auflagen.

§. 2.

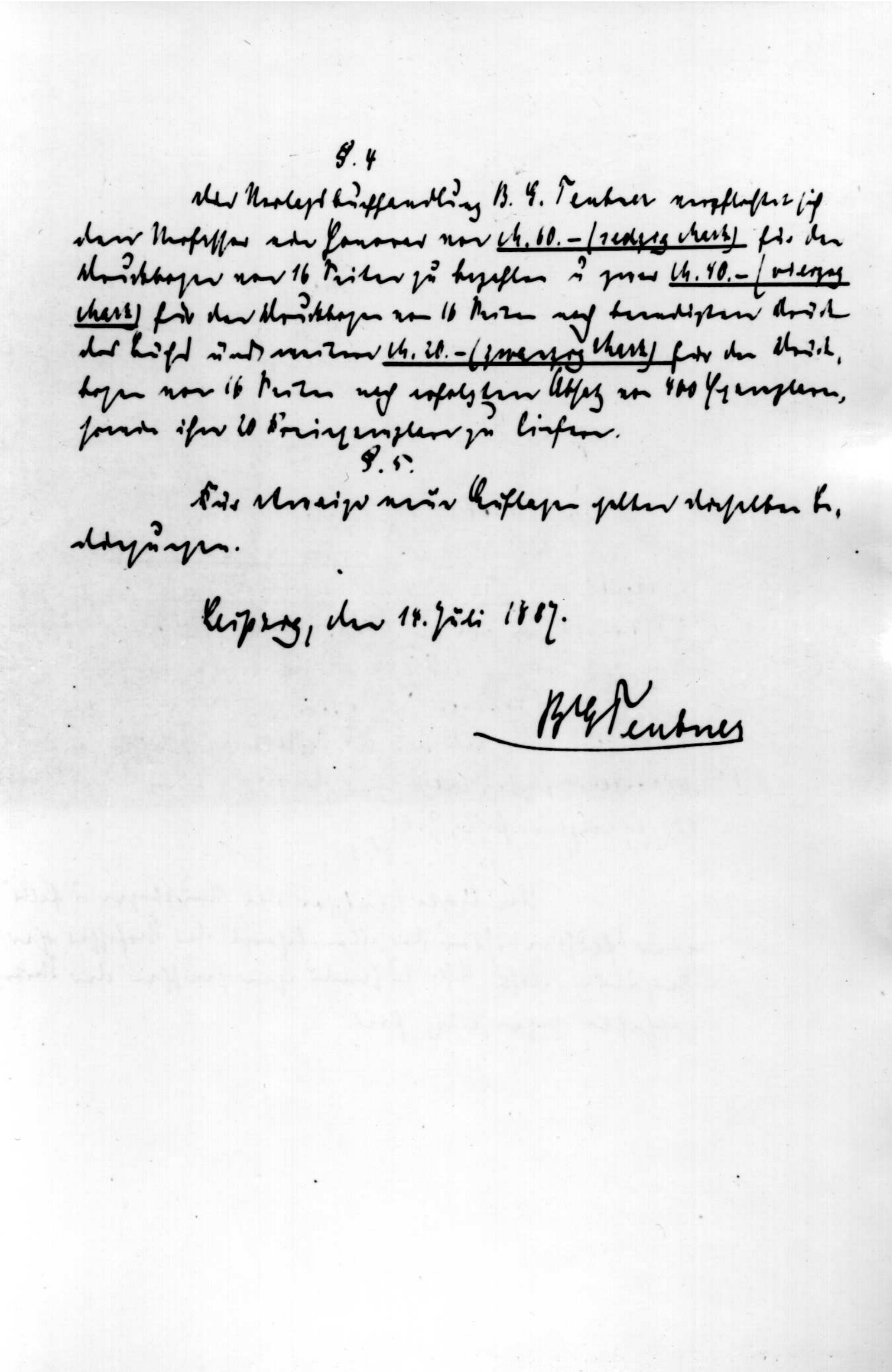
Das Format und Ausstattung schließt sich dem Werke der mathematischen Annalen an. Die Stärke einer Auflage wird auf 800 Exemplare festgesetzt.

§. 3.

Die letzte Revision der Druckbogen und falls erforderlich eine Schlußrevision derselben besorgt der Verfasser ohne besondere Vergütung selbst. Alle Zusendungen zwischen den Vertragschließenden geschehen gegenseitig frei.

§. 4.

Plate 7



§. 4

Die Verlagsbuchhandlung B. G. Teubner verpflichtet sich dem Verfasser ein Honorar von M. 60.– (sechzig Mark) für den Druckbogen von 16 Seiten zu bezahlen und zwar M. 40.– (vierzig Mark) für den Druckbogen von 16 Seiten nach beendigtem Druck des Heftes und weitere M. 20.– (zwanzig Mark) für den Druckbogen von 16 Seiten nach erfolgtem Absatz von 400 Exemplaren, sowie ihm 10 Freiexemplare zu liefern.

§. 5.

Für Abdrucke neuer Auflagen gelten dieselben Bedingungen.

Leipzig, den 14. Juli 1887.

B. G. Teubner

Plates 7 and 8 : Text of the contract between Sophus Lie and the publishing house B. G. Teubner on *Theorie der Transformationsgruppen* from July 14, 1887.

Widmung.

Die neuen Theorien, die in diesem Werke dargestellt sind, habe ich in den Jahren von 1869 bis 1884 entwickelt, wo mir die Liberalität meines

Geburtslandes Norwegen

gestattete, ungestört meine volle Arbeitskraft der Wissenschaft zu widmen, die durch Abels Werke in *Norwegens* wissenschaftlicher Literatur den ersten Platz erhalten hat.

Bei der Durchführung meiner Ideen im Einzelnen und bei ihrer systematischen Darstellung genoss ich seit 1884 in der grössten Ausdehnung die unermüdliche Unterstützung des Professors

Friedrich Engel

meines ausgezeichneten Kollegen an der Universität *Leipzig*.

Das in dieser Weise entstandene Werk widme ich *Frankreichs*

École Normale Supérieure,

deren unsterblicher Schüler Galois zuerst die Bedeutung des Begriffs *discontinuirliche Gruppe* erkannte. Den hervorragenden Lehrern dieses Instituts, besonders den Herren G. Darboux, E. Picard und J. Tannery verdanke ich es, dass die tüchtigsten jungen Mathematiker Frankreichs wetteifernd mit einer Reihe junger *deutscher* Mathematiker meine Untersuchungen über *continuirliche Gruppen*, über *Geometrie* und über *Differentialgleichungen* studiren und mit glänzendem Erfolge verwerthen.

Sophus Lie.

Plate 9 : Dedication at the beginning of Sophus Lie's third part of *Theorie der Transformationsgruppen.*

SOPHUS LIE

GESAMMELTE ABHANDLUNGEN

AUF GRUND EINER BEWILLIGUNG AUS DEM
NORWEGISCHEN FORSCHUNGSFONDS VON 1919
MIT UNTERSTÜTZUNG DER
VIDENSKAPSSELSKAP ZU KRISTIANIA
UND DER
AKADEMIE DER WISSENSCHAFTEN ZU LEIPZIG
HERAUSGEGEBEN VON DEM
NORWEGISCHEN MATHEMATISCHEN VEREIN

DURCH

FRIEDRICH ENGEL
PROFESSOR AN DER UNIVERSITÄT
GIESSEN

POUL HEEGAARD
PROFESSOR AN DER UNIVERSITÄT
KRISTIANIA

DRITTER BAND

LEIPZIG
B. G. TEUBNER
1922

KRISTIANIA
H. ASCHEHOUG & CO.
1922

Plate 10 : Title–page of the third volume of *Gesammelte Abhandlungen* by Sophus Lie. This volume from 1922 marked the beginning of this series. Volumes one and two were printed in the thirties. The seventh and final volume was not published until 1960.

This work is an expanded version of a talk on Sophus Lie given at The University of Leipzig to commemorate the 150th anniversary of Lie's birth. It was also delivered at Seminar Sophus Lie in August, 1996 in Vienna. It was translated into English by Terrence Baine from the Norwegian translation of the original German text, as done by Torger Holtsmark, The Department of Physics, University of Oslo, 1995. The final reading of the English text was completed by Jacqueline Vansant.

The photographs of F. Engel, F. Hausdorff, S. Lie in Leipzig and E. Study are published with the permission of the Teubner-Verlag. The remaining photographs were provided by Bernd Fritzsche and by Universities in Leipzig and Oslo.

The author wishes to thank the SFVTN for permission to reprint this revised and expanded version of the article *Sophus Lie – En skisse av hans liv og verk* published in *SYMMETRI – Nordic Journal for Physics and Mathematics* **2**, (1995), 7 – 20.

The author wishes to thank Karl Heinrich Hofmann for his generous and consistent support of the author's research on Sophus Lie's life and work.

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Received September 28, 1997
and in final form November 9, 1998