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$\mathbb{Z}$-gradations of Lie algebras and infinitesimal generators

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Abstract. In this paper we arrive at explicit formulae for the infinitesimal generators of the action of a complex simple Lie group G on the manifold $M = G/P$ where P is a maximal parabolic subgroup. These formulae are obtained by assuming that local coordinates on M are furnished by the nilpotent subalgebra $\mathfrak{n}$ complementary to the maximal parabolic subalgebra $\mathfrak{p}$ corresponding to P. For the classical isogeny classes A_r, B_r, C_r, and D_r, the components of the infinitesimal generators are never worse than quartic polynomials in the coordinate functions, but for the exceptional cases, G_2, F_4, and E_r, higher-degree polynomials frequently occur.

1. Introduction

This note explains a table (at the end of this paper) of expressions I obtained and outlines my solution to a problem associated with infinitesimal actions: Suppose G is a complex simple Lie group and P is a maximal parabolic subgroup. The group G acts transitively on the homogeneous space $M = G/P$ by translation of right cosets. We are interested in giving an infinitesimal version of this action explicitly. That is, we know that, given local coordinates on the manifold M, we can write down the infinitesimal generators of this action by giving a realization of the Lie algebra $\mathfrak{g}$ by vector fields where the isotropy subalgebra is the Lie subalgebra for P. We wish to find explicit formulae for the components of these generators. It turns out that this has a purely algebraic solution, (not surprisingly since this is a local problem), but the algebra gets somewhat intractable in some cases; in fact, new complications arise in the computations only when we want to consider the five exceptional simple Lie algebras. The algebraic solution involves the construction of various "tensor"-polynomials, denoted by F^i_j, which are constructed using the elementary Schur polynomials. It is interesting that the tensors F^i_j can be constructed in a universal way, independent of the choice of $\mathfrak{g}$. Thus it is the author's belief that it may be possible to extend these results to the more general case when $\mathfrak{g}$ is an indecomposable Kac-Moody-Lie algebra.

I first encountered this problem when studying primitive actions in connection with quasi-exactly solvable systems, [3]. (Recall that a transitive action

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of a Lie group G on a manifold M is *primitive* if the action admits no invariant foliations.) According to a result of Golubitsky [2], on the local level, if $\mathfrak{g}$ is simple and the isotropy subalgebra $\mathfrak{p}$ of the primitive action is not reductive, then $\mathfrak{p}$ is necessarily maximal parabolic. In the quasi-exact-solvability picture, it is useful to have infinitesimal realizations of these actions by spaces of vector fields, and thus the current problem presents itself. Using these actions, it is possible to construct, for each maximal parabolic subalgebra, a series of finite-dimensional representations of the Lie algebra $\mathfrak{g}$, confirming a conjecture by the authors of [3] concerning "quantization of cohomology". For these actions, however, this is a somewhat trivial result beyond the scope of the current problem, so we will not address this here.

2. Notation and Standard Results

First we need to make some well-known [1, 6, 9] comments about simple Lie algebras and $\mathbb{Z}$-gradations thereof. Suppose $\mathfrak{g}$ is a complex simple Lie algebra. As usual, $\mathfrak{g}$ has associated with it a Cartan subalgebra $\mathfrak{h}$, a set $\Delta \subset \mathfrak{h}^*$ of roots, and a set $\{X_\alpha : \alpha \in \Delta\}$ of root vectors. Choosing a hyperplane $\Omega \subset \mathfrak{h}^*$ not containing any roots partitions Δ into sets Δ^+ and Δ^- of positive roots and negative roots respectively. There is a maximal $\mathbb{Z}^+$-spanning set Σ^+ for Δ^+ known as the set of *simple* roots. The corresponding set for Δ^- is Σ^-, which happens to be the same as $-\Sigma^+$. The cardinality r of Σ^+ is known as the *rank* of $\mathfrak{g}$ and this is also equal to the dimension of the Cartan subalgebra $\mathfrak{h}$. Given a positive root $\alpha \in \Delta^+$, there is a unique way to express

$$\alpha = \sum n_i \alpha_i$$

as a sum of the simple roots α_i. The sum $l(\alpha) = \sum n_i$ is known as the *level* of α, and there is a unique *highest root* with maximal level. Similarly, the lowest root has the lowest level, and it is the negative of the highest root.

Having chosen a Cartan subalgebra $\mathfrak{h}$ and a hyperplane $\Omega \subset \mathfrak{h}^*$ not containing any roots, there are some important subalgebras known as parabolic subalgebras. To identify these we first form the Borel subalgebra $\mathfrak{b}$ as the sum of $\mathfrak{h}$ and all of the root vectors X_α with $\alpha \in \Delta^+$. A subalgebra $\mathfrak{p}$ is *parabolic* if it contains $\mathfrak{b}$. It is not difficult to see that the parabolic subalgebras correspond exactly with subsets of Σ^-. That is, suppose $\Sigma' \subset \Sigma^-$. Then there is a parabolic subalgebra which contains all of the root vectors X_α with $\alpha \in \Sigma'$. Thus, the parabolic subalgebras comprise a lattice with 2^r vertices, partially ordered by inclusion. The r parabolic subalgebras directly beneath $\mathfrak{g}$ by the partial ordering are known as *maximal* parabolic subalgebras. Evidently, we can construct a maximal parabolic subalgebra by first choosing a simple root, say $\alpha_0 \in \Sigma^-$, and forming the parabolic subalgebra which contains the root vectors for all simple roots except α_0. The maximal parabolic subalgebras are interesting because each one of them induces a $\mathbb{Z}$-gradation of $\mathfrak{g}$:

Theorem 2.1. *Let $\mathfrak{g}$ be a complex simple Lie algebra and $\mathfrak{p}$ be a maximal parabolic subalgebra. Then $\mathfrak{g}$ has a decomposition*

$$\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} V_j$$

where (a) $V_1 \neq 0$, (b) $\mathfrak{p}$ is the sum of all V_j with $j \geq 0$, (c) $[V_i, V_j] \subset V_{i+j}$, (in particular, V_0 acts on each V_j via the adjoint action), (d) $V_0 = \mathfrak{g}_0 \oplus \mathbb{C}E$ where E acts trivially on $\mathfrak{g}_0$, and (e) this decomposition has exactly $2n_0+1$ non-zero terms V_j, where n_0 is the coefficient of α_0 in the highest root.

Proof. Parts (a)-(d) are shown in [2], so we will prove only part (e). Notice that corresponding to this decomposition is a partition of the root system

$$\Delta = \bigcup_{j \in \mathbb{Z}} \Delta_j,$$

where $\alpha \in \Delta_j$ iff $X_\alpha \in V_j$. Thus, showing that there are $2n_0 + 1$ non-zero terms V_j is equivalent to showing that there are $2n_0 + 1$ non-trivial subsets Δ_j. Assume the simple roots are $\Sigma^+ = \{\alpha_0, ..., \alpha_r\}$ (so that $\mathfrak{g}$ has rank $r+1$) and that $\mathfrak{p}$ is the maximal parabolic subalgebra obtained by not including $-\alpha_0$ in Δ_0. Suppose $n_0\alpha_0 + \cdots n_r\alpha_r$ is the highest root. Then for every integer $j \in [-n_0, n_0]$, Δ contains a root of the form $j\alpha_0 + \alpha$ for some $\alpha \in \Delta_0$. This shows that each such Δ_j is non-trivial and if $|j| > n_0$, then Δ_j is trivial. Thus there are exactly $2n_0 + 1$ non-trivial subsets Δ_j. ∎

Later, we will also need the following notation: We use $\Delta_\pm$ to denote the union $\bigcup_{j>0} \Delta_{\pm j}$. Thus, $\Delta_0 \cup \Delta_+$ is the set of roots corresponding to the maximal parabolic subalgebra $\mathfrak{p}$ and Δ_- is the set of roots corresponding to the complementary nilpotent subalgebra $\mathfrak{n}$.

3. Infinitesimal Actions on Homogeneous Spaces

We are interested in giving a realization of $\mathfrak{g}$ in terms of vector fields on some manifold where $\mathfrak{p}$, a maximal parabolic subalgebra, is the isotropy subalgebra for the action. This means we must locally parameterize the manifold G/P with some neighborhood of the origin in $\mathbb{C}^d$ for some d. There is a natural choice for this space, namely the complementary nilpotent subalgebra $\mathfrak{n}$. Notice that we have a local diffeomorphism [5]

$$\mathfrak{n} \to N \to G/P,$$

where the first arrow is the exponential map and the second is the natural projection onto right cosets $\sigma \mapsto P\sigma$. Let $\mathbf{v} \in \mathfrak{g}$. In order to find the corresponding infinitesimal generator of the right-translation action on G/P, we differentiate the action of the one-parameter subgroup $g = \exp(t\mathbf{v})$ on $\mathfrak{n}$ and evaluate at $t = 0$. We have a basis for $\mathfrak{n}$, namely the set of root vectors X_α for $\alpha \in \Delta_-$. Thus, we assume local coordinates are furnished by prescribing that

$$Z = \sum_{\alpha \in \Delta_-} x_\alpha X_\alpha$$

be an arbitrary point in $\mathfrak{n}$. The corresponding point in N is $\sigma = \exp(Z)$. Assume that the one-parameter subgroup translates Z to

$$W = Z \cdot \exp(t\mathbf{v}) = Z + tF + o(t^2),$$

where

$$F = \sum_{\alpha\in\Delta_-} f_\alpha X_\alpha$$

is another element of $\mathfrak{n}$, the functions f_α being smooth functions in the coordinate functions x_α, $\alpha \in \Delta_-$. The corresponding infinitesimal generator is then

$$\hat{\mathbf{v}} = \sum_{\alpha\in\Delta_-} f_\alpha \frac{\partial}{\partial x_\alpha}.$$

We know that we have a local diffeomorphism, so we can assert that a unique F exists for each $\mathbf{v}$. We are now interested in determining F explicitly. Suppose the element in N corresponding to W is $\tau = \exp(W)$. The group G acts on G/P by right-translation of cosets, so we must impose the condition that σg and τ correspond to the same right coset. That is,

$$\begin{aligned} \sigma g \sim \tau \quad &\text{iff} \quad P\sigma g = P\tau \\ &\text{iff} \quad \sigma g \tau^{-1} \in P. \end{aligned}$$

This is precisely the constraint we need on τ to determine F. We can now state:

Proposition 3.1. *Let $z(\mathbf{v}) = ad_Z(\mathbf{v}) = [Z, \mathbf{v}]$. For sufficiently small $Z \in \mathfrak{n}^-$ and sufficiently small t, $\exp(Z)\exp(t\mathbf{v})\exp(-Z - tF + o(t^2))$ is an element of P iff*

$$\frac{e^z - 1}{z}(F) = e^z(\mathbf{v})\big|_{\mathfrak{n}} \tag{1}$$

and $F \in \mathfrak{n}$.

Remark 3.2. Looking at this constraint, we can see that for any $\mathbf{v} \in \mathfrak{g}$, the functions f_α are always polynomials, so it may be useful to introduce a little notation. The space $\mathfrak{n}$ has a vector-space dual $\mathfrak{n}^*$, which is spanned by the coordinate functions x_α for $\alpha \in \Delta_-$. Let us denote the ring $\mathrm{Sym}(\mathfrak{n}^*)$ of polynomials in the coordinate functions by $\mathbb{C}[x_\alpha]$. Given $\mathbf{v} \in \mathfrak{g}$, we construct the infinitesimal generator $\hat{\mathbf{v}}$ in two steps. First we determine $F(\mathbf{v})$ as some element of the space $\mathbb{C}[x_\alpha] \otimes \mathfrak{n}$. Then we identify the space $\mathfrak{n}$ with the dual of the space $\mathfrak{n}^*$ in order to write down a vector field in differential-operator form. The infinitesimal generator $\hat{\mathbf{v}}$ should therefore be considered (as usual) as lying in the space

$$\mathrm{Hom}(\mathfrak{n}^*, \mathbb{C}[x_\alpha])$$

of derivations on $\mathfrak{n}^*$. Notice that we have two isomorphic vector spaces with drastically differing Lie-algebra structures. The first, $\mathbb{C}[x_\alpha] \otimes \mathfrak{n}$ is nilpotent, and the second $\mathrm{Hom}(\mathfrak{n}^*, \mathbb{C}[x_\alpha])$ contains a faithful image of $\mathfrak{g}$.

Proof. As before, set $\sigma = \exp(Z)$, $g = \exp(t\mathbf{v}) = I + t\mathbf{v} + o(t^2)$, and $\tau = \exp(Z + tF + o(t^2))$. We must show that the constraint is equivalent to having $\sigma g \tau^{-1} \in P$. Notice that one may write

$$\begin{aligned} \tau^{-1} &= \exp(-Z - tF + o(t^2)) \\ &= I - (Z + tF) + \tfrac{1}{2}[Z^2 + t(FZ + ZF)] + \cdots + o(t^2) \\ &= I - Z + \tfrac{1}{2}Z^2 - \cdots - tG + o(t^2) \\ &= \exp(-Z) - tG + o(t^2) \end{aligned}$$

where, for brevity, $G = F - \frac{1}{2}(FZ + ZF) + \frac{1}{6}(FZ^2 + ZFZ + Z^2F) + \cdots$. Now write

$$\begin{aligned}
\sigma g \tau^{-1} &= \exp(Z)\exp(t\mathbf{v})\exp(-Z - tF + o(t^2)) \\
&= \exp(Z)(I + t\mathbf{v})\left[\exp(-Z) - tG\right] + o(t^2) \\
&= \left[\exp(Z) + t\exp(Z)\mathbf{v}\right]\left[\exp(-Z) - tG\right] + o(t^2) \\
&= I + t\left[-\exp(Z)G + \exp(Z)\mathbf{v}\exp(-Z)\right] + o(t^2).
\end{aligned}$$

Now it should be clear, since we want $\sigma g \tau^{-1} \in P$, that we must have

$$\exp(Z)G = \exp(Z)\mathbf{v}\exp(-Z)|_{\mathfrak{n}}.$$

One will quickly see that

$$\exp(Z)G = \frac{e^z - 1}{z}(F)$$

and

$$\exp(Z)\mathbf{v}\exp(-Z) = e^z(\mathbf{v}),$$

and these are the terms which appear in the proposition. ■

4. The Algebraic Solution

The constraint (1) developed in the preceeding section shows that there is a well-defined function $F : \mathfrak{g} \to \mathbb{C}[x_\alpha] \otimes \mathfrak{n}$ which, after we identify $\mathfrak{n}$ with differentiation operators, imbeds $\mathfrak{g}$ as a set of local vector fields. We now wish to learn more about the nature of this function F. It turns out that F is much easier to study if we restrict it to the subspaces V_j in our $\mathbb{Z}$-gradation of $\mathfrak{g}$. To see this, we will need a bit of algebra.

First, let A be the vector space of finite linear combinations of the symbols $\{z_i : i = 1, 2, 3, ...\}$ and let $\mathcal{T}(A) = \bigotimes A$ be the tensor algebra generated by A. We define a Lie bracket on $\mathcal{T}(A)$ by writing

$$[u, v] = u \otimes v - v \otimes u,$$

for $u, v \in \mathcal{T}(A)$. We can construct a representation of $\mathcal{T}(A)$ on $\mathbb{C}[x_\alpha] \otimes \mathfrak{g}$ as follows. Notice that we have projections $\pi_i : \mathbb{C}[x_\alpha] \otimes \mathfrak{g} \to \mathbb{C}[x_\alpha] \otimes V_i$. Suppose we set $Z_i = \pi_{-i}(Z)$ where Z is defined as before. Evidently we have

$$Z_i = \sum_{\alpha \in \Delta_{-i}} x_\alpha X_\alpha.$$

For a generator z_i in A and a vector $\mathbf{v} \in \mathbb{C}[x_\alpha] \otimes \mathfrak{g}$, we set

$$z_i(\mathbf{v}) = [Z_i, \mathbf{v}],$$

and for a decomposable tensor $v_1 \otimes \cdots \otimes v_n \in \mathcal{T}(A)$, set

$$v_1 \otimes \cdots \otimes v_n(\mathbf{v}) = v_1(\cdots(v_n(\mathbf{v}))\cdots).$$

Linearly extending this gives our action of $\mathcal{T}(A)$ on $\mathbb{C}[x_\alpha] \otimes \mathfrak{g}$. By construction, this action preserves Lie brackets. Obviously, since z_i maps $\mathbb{C}[x_\alpha] \otimes V_j \to \mathbb{C}[x_\alpha] \otimes$

V_{j-i}, this action restricts to the subspace $\mathbb{C}[x_\alpha]\otimes\mathfrak{n}$. Finally, the Lie algebra $\mathcal{T}(A)$ has a $\mathbb{Z}$-gradation

$$\mathcal{T}(A)=\sum_{i>0}\mathcal{T}^i(A),$$

where $\mathcal{T}^i(A)$ is spanned by all decomposable tensors whose subscripts sum to i. Notice that if u lies in $\mathcal{T}^i(A)$, then $u(\mathbb{C}[x_\alpha]\otimes V_j)\subset\mathbb{C}[x_\alpha]\otimes V_{j-i}$. Let us use ρ^i to denote the natural projections $\mathcal{T}(A)\to\mathcal{T}^i(A)$.

For later convenience, we will need elements R_i, S_i, and T_i of $\mathcal{T}(A)$ defined as follows. Write $z=\sum_{i>0}z_i$ and $e^z=1+z+\frac{1}{2}z\otimes z+\cdots$, and set

$$R_i=S_i=T_i=0 \text{ if } i<0,$$

and

$$S_i=\rho^i\left(e^z\right),$$

$$R_i=\rho^i\left(\frac{z}{e^z-1}\right),$$

and

$$T_i=\rho^i\left(\frac{e^z-1}{z}\right)$$

for $i\geq 0$. Notice that if we imbed the ring of polynomials $\mathrm{Sym}(A)$ in the tensor algebra $\mathcal{T}(A)$ in the natural way, then the S_i defined here are identified with the usual Schur polynomials defined by

$$\sum_{i\in\mathbb{Z}}S_i(z)x^i=\exp\sum_{i>0}z_ix^i.$$

Suppose $\mathbf{v}\in\mathfrak{g}$ and recall the algebraic criterion that $F(\mathbf{v})$ be an infinitesimal generator:

$$\frac{e^z-1}{z}(F(\mathbf{v}))=e^z(\mathbf{v})|_{\mathfrak{n}}.$$

Given an arbitrary $\mathbf{v}\in\mathfrak{g}$, it is not so easy to find the element $F\in\mathbb{C}[x_\alpha]\otimes\mathfrak{n}$ which satisfies this constraint. However, under this algebraic framework, we do have the following:

Theorem 4.1. *If $\mathbf{v}\in V_j$, then there is an element $F_j\in\mathcal{T}(A)$ such that*

$$\left(\frac{e^z-1}{z}\otimes F_j\right)(\mathbf{v})=e^z(\mathbf{v})|_{\mathfrak{n}},$$

and $F_j(\mathbf{v})$ satisfies the constraint (1). The element $F_j\in\mathcal{T}(A)$ is given by

$$F_j=\sum_{i>0}F_j^i,$$

where

$$F_j^i=\sum_{k=0}^{i-1}R_k\otimes S_{i+j-k}. \tag{2}$$

Proof. Suppose $\mathbf{v} \in V_j$. First of all, notice that since $\mathbf{v} \in V_j$, we have $F_j(\mathbf{v}) \in \mathbb{C}[x_\alpha] \otimes \mathfrak{n}$. We can also show that $F_j(\mathbf{v})$ is the solution to (1): Notice that $R_0 = T_0 = 1$, so we can write

$$\sum_{k=0}^{i-1} T_k \otimes F_j^{i-k} = S_{i+j},$$

simply inverting the linear system (2). Now, notice that for $i > 0$,

$$\begin{aligned} \pi_{-i}\left(\tfrac{e^z-1}{z}(F_j(\mathbf{v}))\right) &= \left(\rho^{i+j}\left(\tfrac{e^z-1}{z} \otimes F_j\right)\right)(\mathbf{v}) \\ &= \left(\textstyle\sum_{k=0}^{i-1} T_k \otimes F_j^{i-k}\right)(\mathbf{v}), \end{aligned}$$

and, because $\mathbf{v} \in V_j$,

$$\begin{aligned} \pi_{-i}(e^z(\mathbf{v})|_{\mathfrak{n}}) &= \pi_{-i}(e^z(\mathbf{v})) \\ &= (\rho^{i+j}(e^z))(\mathbf{v}) \\ &= S_{i+j}(\mathbf{v}). \end{aligned}$$

Since $\pi_{-i}\left(\frac{e^z-1}{z}(F_j(\mathbf{v}))\right)$ and $\pi_{-i}(e^z(\mathbf{v})|_{\mathfrak{n}})$ agree for all $i > 0$, we have

$$\frac{e^z-1}{z}(F_j(\mathbf{v})) = e^z(\mathbf{v})|_{\mathfrak{n}}. \qquad \blacksquare$$

This theorem gives us an algorithm for obtaining the infinitesimal generators of the right-multiplication action of G on G/P where P is a maximal parabolic subgroup. Start with a complex simple Lie algebra $\mathfrak{g}$ and a maximal parabolic subalgebra $\mathfrak{p}$. Then for each piece V_j in the induced $\mathbb{Z}$-gradation, apply the appropriate tensor F_j to each element $\mathbf{v}$ of a spanning set for V_j. The number of different tensors F_j we need to do this is identical to the number of non-zero terms V_j in our $\mathbb{Z}$-gradation, so this is a task reasonable enough to ask, say, Mathematica to perform. Also, if we know more about the structure of $\mathfrak{g}$, we can simplify our computations somewhat.

As a first specialization, consider the case when the choice of $\mathfrak{p}$ induces a gradation with at most five non-zero terms. For example, this always occurs when $\mathfrak{g}$ lies in one of the classical families $\{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}\}$ of complex simple Lie algebras. For these cases, (a) z_i always acts trivially when $i > 2$, and (b) the actions of z_1 and z_2 commute (since $[Z_1, Z_2] = 0$). Condition (a) says that when writing down our solutions, we can set $z_i = 0$ when $i > 2$ and condition (b) says that we don't have to pay attention to the order in which z_1 and z_2 are written; that is, $z_1 \otimes z_2$ acts in a manner identical to that of $z_2 \otimes z_1$, so we write $z_1 \otimes z_2 = z_2 \otimes z_1 = z_1 z_2$. Thus we have

Proposition 4.2. *If $\mathfrak{g}$ is a complex simple Lie algebra and $\mathfrak{p}$ is a maximal parabolic subalgebra inducing a $\mathbb{Z}$-gradation with at most five non-zero terms V_j, then the formulae for the infinitesimal generators $F_j(\mathbf{v})$ for $\mathbf{v} \in V_j$ are given in* *the following table:*

j	$F_j(\mathbf{v})$
-2	$\mathbf{v}$
-1	$(1+\frac{1}{2}z_1)(\mathbf{v})$
0	$(z_1+z_2)(\mathbf{v})$
1	$(z_2+\frac{1}{2}z_1^2+\frac{1}{2}z_1z_2-\frac{1}{12}z_1^3)(\mathbf{v})$
2	$(z_1z_2+\frac{1}{6}z_1^3+\frac{1}{2}z_2^2-\frac{1}{24}z_1^4)(\mathbf{v})$

(The solutions as elements F_j^i of $\mathcal{T}(A)$ may be found in the table at the end of this paper, if the reader is interested.)

Frequently the $\mathbb{Z}$-gradation has exactly three non-zero terms. In this case, z_2 acts trivially and $z = z_1 + z_2 + \cdots$ acts just like z_1. Thus, restricting the preceeding case, we have

Proposition 4.3. *Suppose $\mathfrak{g} = V_{-1} \oplus V_0 \oplus V_1$ is a $\mathbb{Z}$-gradation and $\mathfrak{p} = V_0 \oplus V_1$ is maximal, parabolic. Then (a) if $\mathbf{v} \in V_{-1}$ then $F(\mathbf{v}) = \mathbf{v}$, (b) if $\mathbf{v} \in V_0$, then $F(\mathbf{v}) = z(\mathbf{v})$, and (c) if $\mathbf{v} \in V_1$, then $F(\mathbf{v}) = \frac{1}{2}z^2(\mathbf{v})$.*

These formulae yield the familiar realization

$$\mathfrak{sl}_2\mathbb{C} \cong \left\{ \frac{\partial}{\partial x}, 2x\frac{\partial}{\partial x}, -x^2\frac{\partial}{\partial x} \right\},$$

the differentiated action of $SL_2\mathbb{C}$ on $\mathbb{C}P^1$. The cases with three non-zero terms are interesting also because these cases coincide with the classification of irreducible Hermitian symmetric spaces [5].

5. An Example

We now wish to illustrate the application of these formulae to the case when $\mathfrak{g}$ is the Lie algebra $\mathfrak{b}_2$ and the choice of a maximal parabolic subalgebra $\mathfrak{p}$ induces a $\mathbb{Z}$-gradation with exactly five non-zero terms V_j. Suppose $\mathfrak{g}$ has the basis $\{X_1, X_2, X_3, X_4, Y_1, Y_2, Y_3, Y_4, H_2, H_4\}$ where H_2 and H_4 span a Cartan subalgebra and the remaining eight vectors are root vectors for this Cartan subalgebra. In order to perform any computations, we need to know the commutation relations among the elements of the basis. To determine these structural constants, assume the commutation relations are in accordance with choosing $\mathfrak{g}$ as spanned by the following set of 4-by-4 matrices:

$$\begin{array}{rclrcl}
X_1 &=& E_{21} - E_{34}, & Y_1 &=& E_{12} - E_{43},\\
X_2 &=& E_{31}, & Y_2 &=& E_{13},\\
X_3 &=& E_{41} + E_{32}, & Y_3 &=& E_{14} + E_{23},\\
X_4 &=& E_{42}, & Y_4 &=& E_{24},\\
H_2 &=& -E_{11} + E_{33}, & H_4 &=& -E_{22} + E_{44}.
\end{array}$$

(This is quite arbitrary; all we need to know are the structural constants, so we will get by with any faithful representation of $\mathfrak{b}_2$.) Set

$$\begin{aligned} V_{-2} &= \operatorname{span}\{Y_2\}, \\ V_{-1} &= \operatorname{span}\{Y_1, Y_3\}, \\ V_0 &= \operatorname{span}\{H_2, H_4, X_4, Y_4\}, \\ V_1 &= \operatorname{span}\{X_1, X_3\}, \\ V_2 &= \operatorname{span}\{X_2\}. \end{aligned}$$

It is easy to check that $\mathfrak{p} = V_0 \oplus V_1 \oplus V_2$ is a maximal parabolic subalgebra and that this defines a $\mathbb{Z}$-gradation of $\mathfrak{g}$.

We now wish to apply our formulae to obtain a realization of $\mathfrak{g}$ by vector fields. First we write

$$Z = Z_1 + Z_2$$

where $Z_1 = x_1Y_1 + x_3Y_3 \in \mathbb{C}[x_1, x_2, x_3] \otimes V_1$ and $Z_2 = x_2Y_2 \in \mathbb{C}[x_1, x_2, x_3] \otimes V_2$. By definition, $z_i(\mathbf{v}) = [Z_i, \mathbf{v}]$ for $i = 1, 2$, and, since $[Z_1, Z_2] = 0$, we do not need to pay any attention to the order in which the z_i are applied to any vector in $\mathbb{C}[x_1, x_2, x_3] \otimes \mathfrak{g}$. Let us determine the infinitesimal generator corresponding to X_1. Since $X_1 \in V_1$, we must apply the tensors F_1^i to obtain the infinitesimal generator. According to the table,

$$\begin{aligned} F(X_1) &= F_1^1(X_1) + F_1^2(X_1) \\ &= (z_2 + \tfrac{1}{2}z_1^2 + \tfrac{1}{2}z_1z_2 - \tfrac{1}{12}z_1^3)(X_1) \\ &= [Z_2, X_1] + \tfrac{1}{2}[Z_1, [Z_1, X_1]] + \tfrac{1}{2}[Z_1, [Z_2, X_1]] - \tfrac{1}{12}[Z_1[Z_1[Z_1, X_1]]] \\ &= -x_1^2Y_1 - x_1x_2Y_2 - (x_2 + x_1x_3)Y_3. \end{aligned}$$

Thus the infinitesimal generator corresponding to X_1 is

$$\hat{X}_1 = -x_1^2\frac{\partial}{\partial x_1} - x_1x_2\frac{\partial}{\partial x_2} - (x_2 + x_1x_3)\frac{\partial}{\partial x_3}.$$

Collecting the results when we apply the appropriate tensor to each element of our basis, we have

$$\begin{array}{rclrcl} \hat{X}_1 &=& -x_1^2p_1 - x_1x_2p_2 - (x_2 + x_1x_3)p_3 & \hat{Y}_1 &=& p_1 - x_3p_2 \\ \hat{X}_2 &=& -x_1x_2p_1 - x_2^2p_2 - x_2x_3p_3 & \hat{Y}_2 &=& p_2 \\ \hat{X}_3 &=& (x_2 - x_1x_3)p_1 - x_2x_3p_3 - x_3^2p_3 & \hat{Y}_3 &=& x_1p_2 + p_3 \\ \hat{X}_4 &=& x_3p_1 & \hat{Y}_4 &=& x_1p_3 \\ \hat{H}_2 &=& x_1p_1 + 2x_2p_2 + x_3p_3 & \hat{H}_4 &=& -x_1p_1 + x_3p_3 \end{array}$$

where, for brevity, we have written p_i in place of $\frac{\partial}{\partial x_i}$. (Note: Lie gave this realization of $\mathfrak{b}_2$ in his classification of transitive actions on $\mathbb{C}^3$ [8].)

6. Conclusion

As a concluding remark, it may be interesting to note that the associated spaces $\mathbb{C}[x_\alpha]$ carry a natural generalization of "degree". Notice that in our example we obtained the operator

$$\hat{H}_2 = x_1p_1 + 2x_2p_2 + x_3p_3.$$

If we assign the monomial x_2 a degree of 2 and the monomials x_1, x_3 each degree 1, (as our $\mathbb{Z}$-gradation suggests), then $\hat{H}_2$ resembles Euler's degree operator on homogeneous polynomials:

$$\hat{H}_2(f) = \deg(f) \cdot f,$$

where f lives in $\mathbb{C}[x_1, x_2, x_3]$. Actually, this occurs in general. Recall that V_0 is the sum of a semisimple ideal $\mathfrak{g}_0$ with a central element E. By properly scaling E, it is not hard to show that the corresponding infinitesimal generator is

$$\hat{E} = \sum_{j>0} \sum_{\alpha \in \Delta_{-j}} j x_\alpha \frac{\partial}{\partial x_\alpha},$$

which, operating on $\mathbb{C}[x_\alpha]$, generalizes the usual degree operator.

TABLE 1.

j	F_j^1	F_j^2
-2	0	1
-1	1	$\frac{1}{2}z_1$
0	z_1	z_2
1	$z_2 + \frac{1}{2}z_1^2$	$\frac{1}{2}z_1 z_2 - \frac{1}{12}z_1^3$
2	$z_1 z_2 + \frac{1}{6}z_1^3$	$\frac{1}{2}z_2^2 - \frac{1}{24}z_1^4$

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