Journal of Lie Theory
Volume **9** (1999) 249–269

Multi-Parameter Symmetries of First Order Ordinary Differential Equations

Sebastian Walcher

Communicated by P. J. Olver

Abstract. This is a collection of results on the use of infinitesimal orbital symmetries of first-order ordinary differential equations. Some of these results are classical, dating back to Lie and Bianchi, and some new results are added.

1. Introduction

Sophus Lie's work on symmetries of differential equations (see [11], for instance) has been rediscovered and revived in recent years, and it forms the foundation for a wealth of new results. It seems that most of the recent work in the tradition of Lie has been focussed on partial differential equations or on ordinary differential equations of higher order. In contrast, this article deals exclusively with first-order ordinary differential equations and their (infinitesimal orbital) symmetries. We present several results of Lie and his school, sometimes rephrased to fit the given situation, and add some new results.

We will consider autonomous, analytic ordinary differential equations $\dot{x} = f(x)$, which are defined on an open and connected subset U of $\mathbf{K}^n$ (with $\mathbf{K}$ standing for the real or complex numbers). To every vector field f we assign the *Lie derivative* L_f which sends any scalar-valued function ψ to the function $L_f(\psi)$ defined by $L_f(\psi)(x) := D\psi(x)f(x)$. Specifically, we are interested in (orbital) symmetries of first order differential equations. As is well-known (see [17], for instance), an analytic vector field g (defined on some open subset $\tilde{U}$ of U) generates a local one-parameter group of orbital symmetries of such a differential equation (thus, the transformations map solution orbits of $\dot{x} = f(x)$ to solution orbits, not necessarily preserving the parameterization) if and only if there is an analytic function λ such that $[g, f] = \lambda f$ on $\tilde{U}$. (The Lie bracket of vector fields is defined as usual: $[g, f](x) = Df(x)g(x) - Dg(x)f(x)$. It follows that $L_{[g,f]} = L_g L_f - L_f L_g$.) In the case $\lambda = 0$ the parameterizations will also be preserved. Symmetries of a non-autonomous equation $\dot{x} = h(t, x)$, as introduced by S. Lie, correspond to orbital symmetries of the "autonomized"

ISSN 0949–5932 / \$2.50

system $\dot{t} = 1$, $\dot{x} = h(t, x)$. Moreover, f and μf, with μ analytic and $\mu(x) \neq 0$ for all x in an open subset $\tilde{U}$ of U, have the same orbits, hence the same orbital symmetries on any open subset of $\tilde{U}$. In this sense, it is harmless to pass from f to μf. It should also be noted that orbital symmetries of $\dot{x} = f(x)$ stand in correspondence with the symmetries of the linear, homogeneous partial differential equation $L_f(\phi) = 0$. Several of Lie's results (cf. Lie [11], Bianchi [2], Engel/Faber [3]) were originally stated for the latter type.

The vector fields g satisfying $[g, f] = \lambda f$ on some open $\tilde{U} \subseteq U$ form a Lie algebra but not necessarily a finite dimensional Lie algebra, and moreover, even a finite set of such vector fields will not, in general, generate a finite dimensional Lie algebra of vector fields. To illustrate this, consider the simple case $f = (1, 0, \ldots, 0)^t$. Then

$$[g, f] = \lambda f \quad \text{iff} \quad g(x) = \begin{pmatrix} \gamma(x) \\ \hat{g}(x_2, \ldots, x_n) \end{pmatrix},$$

with $\partial\gamma/\partial x_1 = \lambda$. In general, due to the straightening theorem, a similar observation holds locally for any differential equation near a nonstationary point. Therefore, it would be unnatural to restrict attention to finite dimensional Lie algebras of infinitesimal orbital symmetries. (Things are different in the case of higher-order ordinary differential equations, where the infinitesimal symmetries automatically form a finite dimensional Lie algebra. This is due to a geometrically motivated restriction imposed on the admissible symmetries of the equivalent first-order system; a discussion can be found in Gaeta [4].)

The "multiparameter systems of infinitesimal orbital symmetries" to be discussed here are finite involution systems of vector fields generating local one-parameter groups of orbital symmetries. Recall that a set $g_1, \ldots, g_r$ of analytic vector fields defined on an open, connected subset U of $\mathbf{K}^n$ is called an involution system if there are analytic, scalar-valued functions μ_{ijk} on U such that $[g_i, g_j] = \sum_k \mu_{ijk} g_k$ for all i and j. Any finite set $h_1, \ldots, h_s$ of analytic vector fields on U can be extended to a (finite) set that is in involution on an open-dense subset U^* of U. To see this, consider the h_i as elements of an n-dimensional vector space over the field $\mathbf{L}$ of all meromorphic functions on U. If some $[h_i, h_j]$ is not a linear combination of $h_1, \ldots, h_s$ (equivalently, there is a point y in $\mathbf{K}^n$ such that $[h_i, h_j](y)$ is not a linear combination of $h_1(y), \ldots, h_s(y)$), augment the given set by this vector field. Due to finite dimension over $\mathbf{L}$, this process will terminate. (If convenient, it may also be assumed that the vector fields forming the involution system are linearly independent over $\mathbf{L}$.) If, additionally, $[h_i, f] = \lambda_i f$ for analytic λ_i, $1 \leq i \leq s$, a similar property will also hold for the Lie brackets $[h_i, h_j]$, and therefore for the involution system generated by the h_i.

Let us introduce a little more notation. For an open-dense $U^* \subseteq U$, let $\mathcal{N}_{U^*}(f)$ be the set of all vector fields g on U^* such that $[g, f] = \lambda f$ for an analytic function λ on U^*, and let $\mathcal{C}_{U^*}(f)$ be the set of all vector fields g on U^* such that $[g, f] = 0$. These will be called, respectively, the normalizer and centralizer of f on U^*. Both the normalizer and the centralizer of f are Lie algebras over $\mathbf{K}$, and also modules over the ring of first integrals of f on U^*. (It is said that ψ is a first integral of f if $L_f(\psi) = 0$. The module property follows from the general rule $[\psi g, f] = \psi\,[g, f] - L_f(\psi)g$.) Using this language, it may be said that this article is about subalgebras of $\mathcal{N}_{U^*}(f)$, resp. $\mathcal{C}_{U^*}(f)$ which are also finitely generated submodules. The exposition will include several classical results, due to Lie and Bianchi. (Sources for these are Lie [11], Engel/Faber [3], Bianchi [1], and Hermann [8], and more details can be found in these references. For a general overview of symmetries of differential equations we refer to Olver [12], [13]. A short account of Lie's method for first-order ordinary differential equations in dimension two can also be found in Ince [9].) We will discuss the use of such systems of infinitesimal (orbital) symmetries to obtain nontrivial information about (and possibly simplify) $\dot{x} = f(x)$.

Specifically, we will discuss the use of infinitesimal symmetries to find solution-preserving maps to equations of smaller dimension, and, ideally, to obtain solutions of $\dot{x} = f(x)$ by "algebraic" operations and quadratures alone. (Since "integration" is an ambiguous term in our context, we will use the term "quadrature" to describe the determination of antiderivatives.) It has to be emphasized, though, that this is not the only useful application of symmetries. For instance, there is a large body of work (from various perspectives) on the influence that a compact linear symmetry group has on the qualitative behavior of a differential equation; see Golubitsky et al. [5], [6], and Scheurle [15], among others. Hadeler [7] discusses qualitative features of equations admitting the (non-compact) symmetry group $\mathbf{R}^+$. Bluman and Kumei [2] briefly mention that local one-parameter groups of symmetries stabilize certain invariant sets, like separatrices of real equations in dimension two. As will be shown here, orbital symmetries enforce the existence of certain invariant sets for a differential equation under very general circumstances. (This result may be seen as a generalization of the familiar "stratification of phase space" in the case of a compact linear symmetry group.) Therefore, even if Lie's general local reduction procedure (which relies on non-constructive tools like the inverse function theorem) cannot be carried out for a given equation, one still gets nontrivial information in a relatively easy way. (Occasionally there will be a comment on the computational aspect of a general result, but we will not discuss this systematically.) A few examples and applications will be presented in the final section.

We will not discuss the (serious) problem of how to determine infinitesimal symmetries of first order ordinary differential equations. It is a regrettable fact that there is no general computational approach to these, and this distinguishes the class of first-order ordinary differential equations. We will usually assume that some symmetries are known (and note that it is frequently possible to start with a set of symmetries-to-be and then construct the symmetric vector fields). Most of the examples we discuss will therefore have a prescribed set of symmetries, although, as will be indicated, it is sometimes possible to find these symmetries in a constructive manner.

2. Reduction of dimension via invariants

Throughout this section let $\dot{x} = f(x)$ be given on U, and vector fields $g_1, \ldots, g_r$ on U such that $[g_i, f] = \lambda_i f$, and $[g_i, g_j] = \sum_k \mu_{ijk} g_k$ for all i and j, with the λ_i and μ_{ijk} analytic on an open(-dense) $U^* \subseteq U$. Denote by s the rank of $g_1, \ldots, g_r$ over the field $\mathbf{L}$. In this section it will be assumed that $s < n$.

A well-known theorem of Frobenius states that there is an open and dense subset U^{**} of U^* and analytic functions $\phi_1, \ldots, \phi_{n-s}$ on U^{**} that are independent (i.e., their functional matrix has rank $n-s$ everywhere) common first integrals of $g_1, \ldots, g_r$. Moreover, any point of U^{**} has a neighborhood such that every common first integral of $g_1, \ldots, g_r$ in this neighborhood is a function of the ϕ_j. ¿From $[g_i, f] = \lambda_i f$ it follows that $L_{g_i} L_f - L_f L_{g_i} = \lambda_i L_f$, and in particular $L_{g_i} L_f(\phi_j) = \lambda_i L_f(\phi_j)$ for all i and j. This property is the basis of the following "reduction of dimension theorem", which goes back to Lie [11]; see also Engel/Faber [3] and Hermann [8].

Proposition 2.1. *Assume that $[g_i, f] = 0$ for $i = 1, \ldots, r$.*
(a) *Then every point of U^{**} has a neighborhood such that the restriction of $\Phi := (\phi_1, \ldots, \phi_{n-s})^t$ is solution-preserving from $\dot{x} = f(x)$ to an equation $\dot{x} = h(x)$ on an open subset of $\mathbf{K}^{n-s}$.*
(b) *Moreover, for every point of U^{**} there is a neighborhood and analytic functions $\phi_{n-s+1}, \ldots, \phi_n$ such that $\Psi := (\phi_1, \ldots, \phi_n)^t$ is invertible and solution-preserving from $\dot{x} = f(x)$ to an equation $\dot{x} = f^*(x)$, with*

$$f^*(x) = \begin{pmatrix} f_1^*(x_1, \ldots, x_{n-s}) \\ \vdots \\ f_{n-s}^*(x_1, \ldots, x_{n-s}) \\ f_{n-s+1}^*(x_1, \ldots, x_n) \\ \vdots \\ f_n^*(x_1, \ldots, x_n) \end{pmatrix}.$$

Here, the g_i are transformed to $g_i^* = (0, \ldots, 0, *, \ldots, *)^t$, with zeros in the first $n-s$ entries. Furthermore, one has $[g_i^*, f^*] = 0$, and the g_i^* are locally in involution.

Proof. (a) We have $L_{g_i} L_f = L_f L_{g_i}$ for all i. Therefore, whenever ψ is a common first integral of the g_i then so is $L_f(\psi)$, and hence it can be expressed as a function of the ϕ_j in a neighborhood of every point of U^{**}. It follows that the entries of $D\Phi(x)f(x)$ are common first integrals of the g_i, and this shows the existence of h such that $D\Phi(x)f(x) = h(\Phi(x))$ for all x in this neighborhood. The proof of part (b), with arbitrary $\phi_{n-s+1}, \ldots, \phi_n$ such that $\phi_1, \ldots, \phi_n$ are functionally independent at the point in question, is similar. Since solution-preserving maps preserve Lie brackets, the final assertions follow. ∎

Remark 2.2. The results of (2.1), with the exception of the last statement, also hold for vector fields of the form $f + \sum \sigma_i g_i$, with analytic functions σ_i on U, and thus for a larger class of vector fields than those admitting the g_i as infinitesimal symmetries. Therefore, the proposition does not use all the information available from the symmetries. ∎

To extend the procedure to the orbitally symmetric case (with the λ_i not necessarily zero), we use the following auxiliary result.

Lemma 2.3. *Assume that* $[g_i, f] = \lambda_i f$ *for* $i = 1, \dots, r$, *and let* ψ *be a common first integral of the* g_i *such that* $L_f(\psi) \neq 0$. *Then*

$$\left[g_i, \frac{1}{L_f(\psi)} f\right] = 0 \quad \text{for} \quad 1 \leq i \leq r.$$

Proof. The bracket conditions imply $L_{g_i} L_f(\psi) = \lambda_i L_f(\psi)$, thus

$$L_{g_i}(L_f(\psi))^{-1} = -L_f(\psi)^{-2} L_{g_i} L_f(\psi) = -\lambda_i (L_f(\psi))^{-1},$$

and the assertion follows. ∎

This transfer from normalizer to centralizer (on a smaller, but still open and dense subset of U where $L_f(\psi)$ has no zeros) is not quite satisfactory in some respects. For instance, the zero set of $L_f(\psi)$ contains all the stationary points of f, where interesting local behavior may occur.

The exceptional case that every common first integral of the g_i is also a first integral of f has been excluded in the lemma. This exceptional case occurs if and only if f is a linear combination of the g_i over $\mathbf{L}$. (An equivalent characterization is that for all y in an open-dense subset of U, $f(y)$ is a linear combination of $g_1(y), \dots, g_r(y)$ over $\mathbf{K}$.)

Proposition 2.4. *Let the hypotheses and notation be as at the beginning of this section. Furthermore, assume that there is a common first integral* ψ *of* $g_1, \dots, g_r$ *such that* $L_f(\psi) \neq 0$, *and let* $\hat{U} := \{z \in U^{**} : L_f(\psi)(z) \neq 0\}$.

(a) Then every point of $\hat{U}$ *has a neighborhood such that the restriction of* $\Phi := (\phi_1, \dots, \phi_{n-s})^t$ *is locally orbit-preserving from* $\dot{x} = f(x)$ *to an equation* $\dot{x} = h(x)$ *on an open subset of* $\mathbf{K}^{n-s}$.

(b) Moreover, for every point of $\hat{U}$ *there is a neighborhood and analytic functions* $\phi_{n-s+1}, \dots, \phi_n$ *such that* $\Psi := (\phi_1, \dots, \phi_n)^t$ *is invertible and locally orbit-preserving from* $\dot{x} = f(x)$ *to an equation* $\dot{x} = f^*(x)$, *with*

$$f^*(x) = \begin{pmatrix} f_1^*(x_1, \dots, x_{n-s}) \\ \vdots \\ f_{n-s}^*(x_1, \dots, x_{n-s}) \\ f_{n-s+1}^*(x_1, \dots, x_n) \\ \vdots \\ f_n^*(x_1, \dots, x_n) \end{pmatrix}.$$

The g_i^* *still satisfy* $[g_i^*, f^*] = \lambda_i^* f^*$, *with* $\lambda_i^* = \lambda_i \circ \Psi$.

Proof. This follows from (2.1) for $(L_f(\psi))^{-1} f$. (The cautious phrase "locally orbit-preserving" is only necessary for the complex case. If $\mathbf{K}$ is the real number field then Φ resp. Ψ map solution orbits of $\dot{x} = f(x)$ on $\hat{U}$ to solution orbits of the respective image equations.) ∎

Note that the assertion remains true with $h = 0$, resp.

$$f^* = (0, \dots, 0, *, \dots, *)^t$$

if every common first integral of the g_i is also a first integral of f, but then it does not provide any new information.

Remark 2.5. Finding common first integrals of the g_i may not be a simple matter, although the proof of Frobenius' theorem (see Hermann [8]) is, up to some point, constructive provided that the vector fields in question can be integrated in an elementary manner. The following observation may be useful in the case that all $[g_i, f] = 0$: If ψ is a common first integral of the g_i, then the same is true for $L_f(\psi)$, and repeated application of L_f to some nonempty set of common first integrals of $g_1, \ldots, g_r$ will produce an independent set $\psi_1, \ldots, \psi_q$ (with $q \leq s$) of common first integrals such that every $L_f(\psi_j)$ can locally be expressed as a function of $\psi_1, \ldots, \psi_q$. It is obvious that a variant of (2.1) works in this situation. Thus, it is not necessary to know a maximal independent system of common first integrals from the start. ∎

Under certain circumstances it is possible to carry out the reduction of dimension step-by-step. We only discuss the case of infinitesimal symmetries; the orbital symmetry case follows with (2.3).

Proposition 2.6. *Assume that, in addition to the general hypotheses,* $[g_i, f] = 0$ *for all* i*, and furthermore that there is a* $p < r$ *such that* $[g_i, g_j] = \mu_{ij1}g_1 + \cdots + \mu_{ijp}g_p$ *for all* $i \leq p$ *and all* j*.*
(a) *Then* $g_1, \ldots, g_p$ *are in involution on* U^**. Let* $\gamma_1, \ldots, \gamma_{n-q}$ *(with* q *denoting the rank of* $g_1, \ldots, g_p$ *over* $\mathbf{L}$*) be a functionally independent system of common first integrals of* $g_1, \ldots, g_p$ *on the open-dense subset* U^{**} *of* U^**. Then every point of* U^{**} *has a neighborhood such that every* $L_{g_j}(\gamma_k)$ *is a function of* $\gamma_1, \ldots, \gamma_{n-q}$ *in this neighborhood.*
(b) *Let* $\Gamma : U^{**} \to \mathbf{K}^{n-q}$ *be defined by* $\Gamma(x) := (\gamma_1(x), \ldots, \gamma_{n-q}(x))^t$*. Then there are analytic vector fields* $\tilde{f}$*,* $\tilde{g}_{p+1}, \ldots, \tilde{g}_r$ *on an open and dense subset* $\tilde{U}$ *of* $\Gamma(U^{**}) \subseteq \mathbf{K}^{n-q}$ *such that* $D\Gamma(x)f(x) = \tilde{f}(\Gamma(x))$*, and* $D\Gamma(x)g_i(x) = \tilde{g}_i(\Gamma(x))$ *for* $p+1 \leq i \leq r$*. Moreover,* $[\tilde{g}_i, \tilde{f}] = 0$*, and the* $\tilde{g}_i$ *are in involution on an open and dense subset of* $\tilde{U}$*.*

Proof. (a) The hypothesis implies in particular that $g_1, \ldots, g_p$ are in involution. For all $i \leq p$, all j, and for $1 \leq k \leq n-q$ one has

$$L_{g_i}L_{g_j}(\gamma_k) = L_{g_j}L_{g_i}(\gamma_k) + \mu_{ij1}L_{g_1}(\gamma_k) + \ldots + \mu_{ijp}L_{g_p}(\gamma_k) = 0,$$

thus $L_{g_i}(\gamma_k)$ is again a first integral of $g_1, \ldots, g_p$.
(b) The existence of $\tilde{f}$ and the $\tilde{g}_j$ follows as in (2.1). Since solution-preserving maps respect Lie brackets, it is also clear that $\tilde{f}$ and the $\tilde{g}_j$ commute. For $j, l \geq p+1$ one gets

$$\begin{aligned} [\tilde{g}_j, \tilde{g}_l](\Gamma(x)) = D\Gamma(x)[g_j, g_l](x) &= \sum_{m=1}^{r} \mu_{jlm}(x) D\Gamma(x) g_m(x) \\ &= \sum_{m=p+1}^{r} \mu_{jlm}(x)\tilde{g}_m(\Gamma(x)), \end{aligned}$$

as $D\Gamma(x)g_m(x) = 0$ for $m \leq p$. Therefore, Γ being locally onto, every $[\tilde{g}_j, \tilde{g}_l](z)$ is a linear combination of the $\tilde{g}_m(z)$ (with $m > p$) for all z in an open and dense subset of $\tilde{U}$, and the $\tilde{g}_j$ are in involution. ∎

This opens up the possibility to employ common first integrals of the $\tilde{g}_j$ in the next step, and thus obtain a stronger version of (2.1). One interesting special case for which (2.6) is applicable is when the g_i span a finite dimensional Lie algebra over $\mathbf{K}$, and $g_1, \dots, g_p$ span an ideal of this algebra. In this case, Bianchi [1] refined the result, using composition series of the Lie algebra. (One may, in general, have to modify the given involution system in order to apply (2.6). Note that "linear combinations" with first integrals of f as coefficients are allowed.)

Let us recall the reduction procedure in presence of a single infinitesimal orbital symmetry. If $[g, f] = \lambda f$, and g is such that one knows (locally) a solution-preserving map Ψ from $\dot{x} = (1, 0, \dots, 0)^t$ to $\dot{x} = g(x)$, then this same map is solution-preserving from $\dot{x} = f^*(x) := \mu(x)\tilde{f}(x_2, \dots, x_n)$ to $\dot{x} = f(x)$, with $\partial\mu/\partial x_1 = \lambda$. Since $\tilde{f}$ depends only on $x_2, \dots, x_n$, the solution of $\dot{x} = \tilde{f}(x)$ amounts to solving an $n-1$-dimensional differential equation and quadratures. From this every solution of $\dot{x} = f^*(x)$ is obtained by another quadrature. (Note that, due to the straightening theorem, a map Ψ with the desired properties exists locally near every nonstationary point of g. To carry out the procedure, however, one needs to know Ψ explicitly.) The above "reduction of dimension plus quadratures" has no counterpart in the multi-parameter situation, in general. There is, however, one special case worth mentioning:

Proposition 2.7. *Let the hypotheses and notation be as at the beginning of this section, and assume furthermore that $g_1 \neq 0$ and $[g_1, g_j] = \mu_{1j1} g_1$ for all j. Let Ψ be an invertible analytic map from an open $V \subseteq \mathbf{K}^n$ to a neighborhood of some point of U^* such that Ψ is solution-preserving from $\dot{x} = (1, 0, \dots, 0)$ to $\dot{x} = g_1(x)$.*

Then there are analytic vector fields $g_2^*, \dots, g_r^*$ and f^* on V such that Ψ is solution-preserving from $\dot{x} = f^*(x)$ to $\dot{x} = f(x)$, and from $\dot{x} = g_j^*(x)$ to $\dot{x} = g_j(x)$ for all j. Moreover,

$$g_j^*(x) = \begin{pmatrix} \gamma_j(x) \\ \hat{g}_j(x_2, \dots, x_n) \end{pmatrix} \quad \text{and} \quad f^*(x) = \mu(x) \begin{pmatrix} \phi(x_2, \dots, x_n) \\ \hat{f}(x_2, \dots, x_n) \end{pmatrix}.$$

The $\hat{g}_j$ are in involution on a suitable open subset $\hat{V}$ of $\mathbf{K}^{n-1}$, and there are analytic $\hat{\lambda}_j$ on $\hat{V}$ such that $[\hat{g}_j, \hat{f}] = \hat{\lambda}_j \hat{f}$ for $2 \le j \le r$.

Proof. The existence of the vector fields on V follows from invertibility of Ψ, and it was seen above that f^* has the asserted form. Now $[g_1, g_j] = \mu_{1j1} g_1$ implies $[g_1^*, g_j^*] = \mu_{1j1}^* g_1^*$, or $\partial g_j^*/\partial x_1 = \mu_{1j1}^* \cdot (1, 0, \dots, 0)^t$, whence entries number $2, \dots, n$ of g_j^* are independent of x_1.

A simple computation shows that

$$[g_j^*, g_l^*] = \begin{pmatrix} \star \\ [\hat{g}_j, \hat{g}_l] \end{pmatrix},$$

with the Lie bracket in the last entry being taken in $\mathbf{K}^{n-1}$, and it follows that the $\hat{g}_j$ are in involution. Similarly, it follows that $[\hat{g}_j, \hat{f}] = \hat{\lambda}_j \hat{f}$ if μ has no zeros in a neighborhood of the point under consideration. ∎

Thus, g_1 reduces the solution of $\dot{x} = f(x)$ to solving a differential equation in dimension $n-1$, plus quadratures, and the remaining symmetries from $g_2, \ldots, g_r$ survive for the reduced differential equation. This latter property is the important one. The following consequence (when applied to finite-dimensional solvable Lie algebras) is sometimes called "Bianchi's theorem".

Corollary 2.8. *In addition to the hypotheses of* (2.7), *assume that*

$$\begin{array}{rcl} [g_2, g_j] & = & \mu_{2j1}g_1 + \mu_{2j2}g_2 \\ \vdots & & \vdots \\ [g_{r-1}, g_j] & = & \mu_{r-1,j,1}g_1 + \ldots + \mu_{r-1,j,r-1}g_{r-1} \end{array}$$

for all j, *and that* $g_1, \ldots, g_r$ *has rank* r *over* $\mathbf{L}$.

Then solving $\dot{x} = f(x)$ *can (locally) be reduced to solving an* $(n-r)$*-dimensional differential equation and quadratures.* ∎

It seems that Bianchi formulated this only for finite dimensional solvable Lie algebras, although in other parts of [1] he also considered general involution systems. The proper formulation of Bianchi's theorem for higher-order differential equations was first given by Olver; cf. [12], Thm. 2.64. (We remark that whenever the condition on the orbit dimensions of the prolongated actions in this theorem is not satisfied then one necessarily gets a nontrivial first integral by (4.2), which also allows a reduction of the order.)

Even if the vector fields at hand can be explicitly integrated, it may be troublesome or even impossible to implement any of the reduction procedures discussed in this section. The implicit function theorem and the inverse function theorem are the basis for many of the results presented here, and these are not constructive. One may be able to actually carry out a variant of the reduction from (2.1), for instance, if all the functions and vector fields involved are polynomial, and one uses a more subtle approach. For (compact) linear symmetry groups, this procedure is quite familiar, see Rumberger/Scheurle [14] and Scheurle [15], for instance, where the symmetric vector fields are quite general, and qualitative features are of primary interest. (In the case of compact linear symmetry groups qualitative investigations are greatly facilitated by the fact that the reducing map is proper.) Hadeler [7] discusses qualitative features of homogeneous vector fields, with the (non-compact) symmetry group $\mathbf{R}^+$. The case of orbital linear symmetries is dicussed in [10].

3. Invariant sets

This section is motivated by the well-known strategy to find group-invariant solutions (cf. Olver [12]), and also by the "stratification of the phase space" of differential equations admitting a (compact) linear symmetry group, see Gaeta [4], Golubitsky et al. [5], [6], and Scheurle [15].

In this section it will be assumed that f and $g_1, \ldots, g_r$ are analytic vector fields on U, and that $[g_i, f] = \lambda_i f$ with analytic functions λ_i on U. (We do not assume that the g_i are in involution.) The basic question we are interested in is the following: What invariant sets does $\dot{x} = f(x)$ necessarily admit as a consequence of the existence of the g_i?

One answer is the following.

Theorem 3.1. (a) *The set*

$$Y := \{y \in U : f(y), g_1(y), \ldots, g_r(y) \text{ are linearly dependent in } \mathbf{K}^n\}$$

is invariant for $\dot{x} = f(x)$.
(b) *If, in addition, all* $[g_i, f] = 0$ *then*

$$Z := \{z \in U : g_1(y), \ldots, g_r(y) \text{ are linearly dependent in } \mathbf{K}^n\}$$

is invariant for $\dot{x} = f(x)$.

Proof. (i) We will use the following invariance criterion (see [17], for instance): Let ρ_i and σ_{ij} $(1 \leq i, j \leq q)$ be analytic on U, and $L_f(\rho_i) = \sum_j \sigma_{ij}\rho_j$ for all i. Then the set of common zeros of $\rho_1, \ldots, \rho_q$ is invariant for $\dot{x} = f(x)$.

(To see this, let $v(t)$ be a solution of $\dot{x} = f(x)$. Then $w(t) := (\rho_1(v(t)), \ldots, \rho_q(v(t)))^t$ solves a homogeneous linear differential equation. Thus, $w(0) = 0$ implies $w(t) = 0$ for all t.)
(ii) We also need an auxiliary result from (multi-)linear algebra. Denote by $\nu_1, \ldots, \nu_n$ the dual basis of $\mathbf{K}^n$, thus $\nu_i((x_1, \ldots, x_n)^t) = x_i$, and fix some $q \leq n$.

Let $1 \leq i_0 < \ldots < i_q \leq n$ be a strictly increasing sequence of integers, and consider the map

$$\Delta_{i_0,\ldots,i_q} : (\mathbf{K}^n)^{q+1} \to \mathbf{K}, \quad \Delta_{i_0,\ldots,i_q}(v_0, \ldots, v_q) = \det\left(\nu_{i_k}(v_l)\right)_{k,l}.$$

This map is multilinear and alternating, and (by the universal property of the Grassmann algebra) every multilinear and alternating map from $(\mathbf{K}^n)^{q+1}$ to $\mathbf{K}$ is a linear combination of such $\Delta_{i_0,\ldots,i_q}$.

Now let $B : \mathbf{K}^n \to \mathbf{K}^n$ be a linear map, and $j_0 < \ldots < j_q$ fixed. Then it is easy to verify that

$$\delta_B : (v_0, \ldots, v_q) \mapsto \sum_{i=0}^{q} \Delta_{j_0,\ldots,j_q}(v_0, \ldots, v_{i-1}, Bv_i, v_{i+1}, \ldots, v_q)$$

is multilinear and alternating, and therefore a linear combination of the $\Delta_{i_0,\ldots,i_q}$. Moreover, the coefficients are linear functions of the entries of B.

(iii) We now prove part (a). Note that Y is the common zero set of all $\rho_{i_0,\ldots,i_r}(x) := \Delta_{i_0,\ldots,i_r}(f(x), g_1(x), \ldots, g_r(x))$. Fix $j_0 < \ldots < j_r$, and abbreviate $\Delta := \Delta_{j_0,\ldots,j_r}$, and $\rho := \rho_{j_0,\ldots,j_r}$. By multilinearity of the determinant, the Lie derivative of ρ is as follows:

$$\begin{aligned}
&L_f(\rho)(x) = D\rho(x)f(x)\\
&= \Delta\left(Df(x)f(x), g_1(x), \ldots, g_r(x)\right)\\
&\quad + \sum_i \Delta\left(f(x), g_1(x), \ldots, g_{i-1}(x), Dg_i(x)f(x), g_{i+1}(x), \ldots, g_r(x)\right)\\
&= \Delta\left(Df(x)f(x), g_1(x), \ldots, g_r(x)\right)\\
&\quad + \sum_i \Delta\left(f(x), g_1(x), \ldots, g_{i-1}(x), Df(x)g_i(x), g_{i+1}(x), \ldots, g_r(x)\right)\\
&\quad - \sum_i \Delta\left(f(x), g_1(x), \ldots, g_{i-1}(x), \lambda_i(x)f(x), g_{i+1}(x), \ldots, g_r(x)\right),
\end{aligned}$$

since $[g_i, f] = \lambda_i f$. The last term vanishes, and the remaining sum can, according to (ii), be expressed as a linear combination of the $\rho_{i_0,\ldots,i_r}$, with the coefficients being linear in the entries of $Df(x)$, and hence analytic on U. Now the criterion from (i) applies.

(iv) The proof of (b) is an easier variant of (iii). ∎

Since any intersection, or set-theoretic difference, of invariant sets is invariant, we get

Corollary 3.2. *Let $q \le r$. In the situation of (3.1 a), the set*

$$Y_q := \{y \in U : \operatorname{rank}\left(f(y), g_1(y), \ldots, g_r(y)\right) = q+1\}$$

is invariant for $\dot{x} = f(x)$. In the situation of (3.1 b), the set

$$Z_q := \{z \in U : \operatorname{rank}\left(g_1(y), \ldots, g_r(y)\right) = q\}$$

is invariant for $\dot{x} = f(x)$. ∎

If $\lambda_i \neq 0$ then Z is not, in general, invariant for $\dot{x} = f(x)$. A simple example is given by $g(x) = x$ and f a nonzero constant. Here we have $[g, f] = -f$, while $Z = \{0\}$ is not invariant for $\dot{x} = f(x)$. To illustrate that (3.1) can be useful, consider

$$f(x) := \begin{pmatrix} x_1^2 - x_2x_3 \\ 2x_1x_2 \\ 2x_1x_3 \end{pmatrix}, \quad g(x) := \begin{pmatrix} x_1 \\ 3x_2 \\ -x_3 \end{pmatrix}, \quad \text{and } h(x) := \begin{pmatrix} x_1x_2 \\ x_2^2 \\ -x_1^2 \end{pmatrix}.$$

It is easy to verify that g and h normalize f, and therefore the set of zeros of

$$\rho(x) := \det\left(f(x), g(x), h(x)\right) = -x_2\left(x_1^2 + x_2x_3\right)^2$$

is invariant for $\dot{x} = f(x)$, due to (3.1). Thus we have found the invariant plane $x_2 = 0$ and the (less obvious) invariant cone $x_1^2 + x_2x_3 = 0$. Application of (3.1) to f and g alone turns out to produce the invariant straight lines $x_1 = x_2 = 0$, $x_1 = x_3 = 0$, and $x_2 = x_3 = 0$, and so on.

In the recent article [16] by Ünal, the result of (3.1) is proven (from a somewhat different perspective) in the special case $r = n-1$ (see [16], Theorem 1), and some examples are given.

The following may also be worth recording.

Remark 3.3. If the g_i are in involution on $U^* \subseteq U$, then the sets $Y_q \cap U^*$ and $Z_q \cap U^*$ are invariant for all g_i. The proof of this is similar to the one of (3.1) and (3.2). ∎

Let us briefly return to the situation of §2, assuming that the g_i are analytic on U, and the μ_{ijk} analytic on U^*. Then $U \setminus U^*$ is contained in Z, as an application of Cramer's rule over $\mathbf{L}$ shows. Thus, if all $\lambda_i = 0$ then (2.1) takes care of the points in U^* while (3.1) and (3.2) take care of Z (and there are local coordinates on $Z_q \setminus Z_{q-1}$ which allow, in principle, to transfer the restriction of $\dot{x} = f(x)$ to an open subset of $\mathbf{K}^q$). Thus, the points in $U \setminus U^*$ are actually quite interesting. Moreover, the equations of the invariant sets determined so far can actually be computed.

The following result works for arbitrary solution-preserving maps, whether they are determined from symmetries or not.

Proposition 3.4. *Let the analytic differential equation $\dot{x} = h(x)$ be defined on $V \subseteq \mathbf{K}^m$, and $\Phi : U \to V$ be solution-preserving from $\dot{x} = f(x)$ to $\dot{x} = h(x)$. Then every set $W_q := \{x \in U : \operatorname{rank} D\Phi(x) = q\}$ is invariant for $\dot{x} = f(x)$.*

Proof. Denote by $F(t, y)$ (resp. $H(t, y)$) the solution of $\dot{x} = f(x)$, $x(0) = y$ (resp. $\dot{x} = h(x)$, $h(0) = y$). Then F and H are analytic, and, by definition, $\Phi(F(t, y)) = H(t, \Phi(y))$. Differentiate this with respect to y to obtain

$$D\Phi(F(t, y))\, D_2 F(t, y) = D_2 H(t, \Phi(y))\, D\Phi(y),$$

with D_2 symbolizing the partial derivative with respect to the second variable. Now, $D_2 F(t, y)$ solves a linear differential equation with the identity matrix as initial value, and hence is always invertible, and a similar observation holds for $D_2 H$. Therefore, the rank of $D\Phi(F(t, y))$ is equal to the rank of $D\Phi(y)$ for all t. ∎

A look at (2.1)ff. may give the impression that this is not a useful result, since the rank of Φ is, by construction, constant on the set under consideration. But in fact one often obtains nontrivial information. For instance, assume that $U \subseteq \mathbf{K}^4$ and that $\Phi(x) := \binom{x_1^2 + x_2^2}{x_3^2 + x_4^2}$ is solution-preserving from $\dot{x} = f(x)$ to some equation $\dot{x} = h(x)$. Then the intersections of U with the planes defined by $x_1 = x_2 = 0$, resp. $x_3 = x_4 = 0$ are invariant for $\dot{x} = f(x)$, as the criterion from (3.4) shows. (Such a situation occurs, for instance, if f is $so(2) \times so(2)$-symmetric, but there are more general equations admitting this solution-preserving map; see (2.2).) It should also be noted that in the case of a finite linear symmetry group, with Φ constructed from a set of generators of the invariant algebra, (3.4) yields all the strata.

To finish this section, let us investigate how first integrals of f behave in presence of the g_i. As will be seen, this may yield an approach to finding new invariant sets from given ones.

Proposition 3.5. (a) *If ψ is a first integral of f on U then so is every $L_{g_i}(\psi)$.*

(b) *Any functionally independent set of first integrals of* f *(on an open-dense subset of* U *) can be extended to a functionally independent set* $\psi_1, \ldots, \psi_p$ *of first integrals of* f *such that every* $L_{g_i}(\psi_j)$ *is locally a function of the* ψ_l *.*

Proof. Part (a) is a consequence of $L_{g_i} L_f - L_f L_{g_i} = \lambda_i L_f$, while part (b) follows from repeatedly applying the L_{g_i} to the given set. ∎

Corollary 3.6. *Let the hypotheses and* $\psi_1, \ldots, \psi_p$ *be as in (3.5) on the open and dense subset* U^* *of* U *, with* $p \geq 1$ *.*

(a) *Then for every point of* U^* *there is a neighborhood and functions* $\psi_{p+1}, \ldots, \psi_n$ *such that* $\Psi := (\psi_1, \ldots, \psi_n)^t$ *is invertible on this neighborhood. There are vector fields* f^* *,* $g_1^*, \ldots, g_r^*$ *such that* $D\Psi(x)f(x) = f^*(\Psi(x))$ *, and* $D\Psi(x)g_i(x) = g_i^*(\Psi(x))$ *. Moreover,*

$$f^*(x) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \hat{f}(x) \end{pmatrix}, \quad \text{and} \quad g_i^*(x) = \begin{pmatrix} \gamma_{i,1}(x_1, \ldots, x_p) \\ \vdots \\ \gamma_{i,p}(x_1, \ldots, x_p) \\ \tilde{g}_i(x) \end{pmatrix}.$$

(b) *There is a* $q \leq p$ *(actually,* q *is the rank of the matrix* $(\gamma_{i,k}(x_1, \ldots, x_p))$ *in an open and dense subset* V *of the given neighborhood) and linear combinations* $g_{q+1}^{**}, \ldots, g_r^{**}$ *of the* g_i^* *(with analytic first integrals of* f *as coefficients) such that*

$$g_j^{**}(x) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \hat{g}_j(x) \end{pmatrix} \quad \text{for} \quad j > q.$$

(c) *Fix* $z \in V$ *. Then*

$$\begin{aligned} &\left[\hat{f}(z_1, \ldots, z_p, x_{p+1}, \ldots, x_n), \hat{g}_j(z_1, \ldots, z_p, x_{p+1}, \ldots, x_n)\right] \\ &= \hat{\lambda}_j(x_{p+1}, \ldots, x_n)\hat{f}(z_1, \ldots, z_p, x_{p+1}, \ldots, x_n) \end{aligned}$$

(the bracket is taken in $\mathbf{K}^{n-q}$ *) for all* $j > q$ *. Moreover, if the* g_i *are in involution then the* $\hat{g}_j(z_1, \ldots, z_p, x_{p+1}, \ldots, x_n)$ *are in involution.*

Proof. In part (a), the existence of the ψ_l (with $l > p$) is clear, as is the existence of the g_i^* and of f^*. The special form of f^* follows from $\psi_1, \ldots, \psi_p$ being first integrals, while the special form of the g_i^* is a consequence of (3.5). Part (b) follows with Gauss' algorithm (over $\mathbf{L}$) applied to the g_i^*. (In case $\gamma_{1,1} \neq 0$ the first step is to change g_j^* to $g_j^* - (\gamma_{j,1}/\gamma_{1,1})g_1^*$ $(j > 1)$ to create zeros in the first entry. Note that the coefficients are indeed first integrals of f.) Part (c) follows from simple computations. ∎

There is the usual problem with this proposition: It may be impossible to carry out the necessary coordinate transformations explicitly. A weaker version, which avoids this problem, can be stated as follows: Whenever σ is a first integral of f and, for instance, $L_{g_1}(\sigma) \neq 0$, then $\tilde{g}_j := L_{g_1}(\sigma)g_j - L_{g_j}(\sigma)g_1$

(for $j \geq 2$) is also contained in the normalizer of f and admits the first integral σ. Moreover the $\tilde{g}_j$ are in involution on an open and dense subset of U if the g_i are in involution on an open and dense subset of U. In this sense, there are infinitesimal symmetries of f that allow restriction to the level sets of σ. Obviously, this process can be continued.

4. First integrals from involution systems

At the end of §3 it has been shown how given first integrals of f can be put to use in presence of infinitesimal symmetries. In this section we will see that first integrals of f may actually be found from an involution system of infinitesimal symmetries.

Throughout this section we will assume that $f, g_1, \ldots, g_r$ are analytic vector fields on U, that $[g_i, f] = \lambda_i f$ for all i, with analytic λ_i, that $[g_i, g_j] = \sum_k \mu_{ijk} g_k$, with analytic μ_{ijk}, on an open and dense subset U^* of U, and that $g_1, \ldots, g_r$ are linearly independent over $\mathbf{L}$.

We will distinguish two cases. The first (which will be referred to as "nondegenerate") is characterized by the condition that $f, g_1, \ldots, g_r$ are linearly independent over $\mathbf{L}$, and its discussion goes back to Lie [11]; see the exposition in Engel/Faber [3]. The second ("degenerate") case is that $f, g_1, \ldots, g_r$ are linearly dependent over $\mathbf{L}$, and it seems that it was not discussed so far. (From the point of view of linear partial differential equations this is understandable.) In any case, it will turn out that there is a systematic reduction to the case of finite dimensional Lie algebras of infinitesimal (orbital) symmetries, by way of first integrals.

Proposition 4.1. *In the nondegenerate case, the following holds:*
(a) *The μ_{ijk} are first integrals of f.*
(b) *If not all the μ_{ijk} are constant then there exist functionally independent first integrals $\phi_1, \ldots, \phi_q$ (with $q > 0$) on an open-dense $U^{**} \subseteq U^*$ with the following property: For each point of U^{**} there is a neighborhood and analytic functions $\phi_{q+1}, \ldots, \phi_n$ such that $\Phi := (\phi_1, \ldots, \phi_n)^t$ is invertible on this neighborhood, and solution-preserving from $\dot{x} = f(x)$ to $\dot{x} = f^*(x)$, and from $\dot{x} = g_i(x)$ to $\dot{x} = g_i^*(x)$, $1 \leq i \leq r$. Moreover,*

$$f^*(x) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \hat{f}(x) \end{pmatrix} \quad \text{and} \quad g_i^*(x) = \begin{pmatrix} \gamma_{i,1}(x_1, \ldots, x_q) \\ \vdots \\ \gamma_{i,q}(x_1, \ldots, x_q) \\ \tilde{g}_i(x) \end{pmatrix},$$

and $[g_i^*, g_j^*] = \sum_k \mu_{ijk}^* g_k^*$, with the μ_{ijk}^* functions of $x_1, \ldots, x_q$. Furthermore, there is an integer $p \leq q$ and linear combinations g_j^{**} of the g_i^* ($j > q$), with the coefficients being functions of $x_1, \ldots, x_q$, and

$$g_j^{**}(x) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \hat{g}_j(x) \end{pmatrix}.$$

(c) *Fix z in this neighborhood. Then the $\hat{g}_j(z_1, \dots, z_q, x_{q+1}, \dots x_n)$ form a finite dimensional Lie algebra of infinitesimal orbital symmetries for*

$$\hat{f}(z_1, \dots, z_q, x_{q+1}, \dots x_n)$$

on a suitable open subset of $\mathbf{K}^{n-q}$.

Proof. As to part (a), we have $[[g_i, g_j], f] = \alpha_{ij} f$, with suitable α_{ij}, from the Jacobi identity, and $[[g_i, g_j], f] = \sum_k [\mu_{ijk} g_k, f] = \sum_k \mu_{ijk}\lambda_k f - \sum_k L_f(\mu_{ijk}) g_k$. The linear independence of $f, g_1, \dots, g_r$ over $\mathbf{L}$ now shows $L_f(\mu_{ijk}) = 0$. Parts (b) and (c) follow from an application of (3.6) to a maximal functionally independent subset of the μ_{ijk}. The assertion about finite dimension in (c) follows from the fact that the coefficients occurring in the involution systems are functions of $x_1, \dots, x_q$. ∎

Thus, a "proper" involution system may provide more readily accessible information about $\dot{x} = f(x)$ than a finite dimensional Lie algebra of infinitesimal symmetries. Actually, the argument of (4.1) is also useful for certain finite dimensional Lie algebras.

Remark 4.2. The dimension of a finite dimensional Lie algebra of vector fields on U is greater than or equal to the dimension over $\mathbf{L}$ of the corresponding involution system. The latter is equal to the "generic" dimension of the local group orbits, and it is equal to the dimension of the Lie algebra if and only if the local group action is semi-regular on an open and dense subset of U. In all other cases, (4.1) yields nontrivial first integrals.

For a specific example, consider the vector fields

$$g_1(x) = \begin{pmatrix} 0 \\ -x_3 \\ x_2 \\ 0 \end{pmatrix} \quad \text{and} \quad g_2(x) = \begin{pmatrix} 0 \\ 0 \\ -x_4 \\ x_3 \end{pmatrix}$$

on $\mathbf{K}^4$. These generate a three-dimensional Lie algebra (isomorphic to $so(3)$), and the corresponding involution system has dimension 2 over $\mathbf{L}$, as follows from $[g_1, g_2] = \frac{x_4}{x_3} g_1 + \frac{x_2}{x_3} g_2$. Therefore, every vector field f that is defined on an open subset U of $\mathbf{K}^4$ and admits the infinitesimal orbital symmetries g_1 and g_2 also admits the first integrals $\rho_1(x) = \frac{x_4}{x_3}$ and $\rho_2(x) = \frac{x_2}{x_3}$, provided that the nondegeneracy condition holds. It follows that

$$f(x) = \begin{pmatrix} \phi(x) \\ \mu(x)x_2 \\ \mu(x)x_3 \\ \mu(x)x_4 \end{pmatrix} = \mu(x) \begin{pmatrix} \psi(x) \\ x_2 \\ x_3 \\ x_4 \end{pmatrix},$$

with analytic functions μ, ϕ and ψ on an open-dense subset of U. (A simple computation shows that the g_i are infinitesimal orbital symmetries if and only if ψ is a common first integral of the g_i.) There are other methods to obtain this result, in particular since the g_i are linear vector fields, but the method from (4.1) seems straightforward and useful in quite general circumstances. ∎

Now let us turn to the degenerate case. It follows from the general hypothesis of this section that there are analytic functions $\beta_1, \ldots, \beta_r$ on an open and dense subset U^* of U such that $f = \sum_i \beta_i g_i$. (We will assume that $f \neq 0$.) The following can be said in this situation.

Proposition 4.3. (a) *Assume that $\beta_1 \neq 0$. Then all β_i/β_1 $(i > 1)$ are first integrals of f, and passage to the orbit-equivalent vector field $\tilde{f} := \beta_1^{-1} f$ yields $\tilde{f} = \sum_i \tilde{\beta}_i g_i$, with all $\tilde{\beta}_i$ first integrals of $\tilde{f}$. If all $\lambda_i = 0$ then all β_i are first integrals of f.*
(b) *If all β_i are first integrals of f then all $\beta_l \mu_{ijk} - \beta_k \mu_{ijl}$ are first integrals of f.*

Proof. Part (a) follows from $0 = [f, f] = \sum [f, \beta_i g_i] = -\alpha f + \sum L_f(\beta_i) g_i$, with $\alpha := \sum \beta_i \lambda_i$. The linear independence of the g_i over $\mathbf{L}$ forces $L_f(\beta_i) = \alpha \beta_i$ for all i, and from this the assertion follows. Part (b) follows similarly: There are analytic functions λ_{ij} such that $[[g_i, g_j], f] = \lambda_{ij} f$; on the other hand $[[g_i, g_j], f] = \sum [\mu_{ijk} g_k, f] = (\sum \mu_{ijk} \lambda_k) f - \sum L_f(\mu_{ijk}) g_k$, whence $L_f(\mu_{ijk}) = \nu_{ij} \beta_k$, and $L_f(\beta_l \mu_{ijk}) = \nu_{ij} \beta_k \beta_l$. ∎

This gives rise to a (rather obvious) application of (3.6), which will not be written down explicitly, but a brief look at the case when no nonconstant first integrals can be gained from this procedure may be in order. First, we may assume that all β_i are constant, and then there is no loss of generality in assuming that $f = g_1$. Part (b) then shows that all μ_{ijk} with $k > 1$ are constants.

In a sense, it is possible to reduce the degenerate to the nondegenerate case, as the following shows. For a given set of vector fields g_i, however, this may not be the most advisable strategy.

Proposition 4.4. *Let the hypotheses and notation be as in (4.3). Let ϕ be analytic on U such that $L_f(\phi) \neq 0$, and define $\tilde{g}_i := g_i - (L_{g_i}(\phi)/L_f(\phi)) f$. Then the system $\tilde{g}_1, \ldots, \tilde{g}_r$ has rank $r - 1$ over $\mathbf{L}$, and $[\tilde{g}_i, f] = \tilde{\lambda}_i f$ for suitable analytic $\tilde{\lambda}_i$, $1 \leq i \leq r$.*

If $\tilde{g}_2, \ldots, \tilde{g}_r$ are linearly independent over $\mathbf{L}$ then they are in involution. Moreover, there is a nonzero analytic function σ on an open-dense subset of U such that $[\tilde{g}_i, \sigma f] = 0$ for $i > 1$.

Proof. By construction, the $\tilde{g}_i$ have the common first integral ϕ, and $[\tilde{g}_i, f] = \tilde{\lambda}_i f$. If $\tilde{g}_2, \ldots, \tilde{g}_r$ are linearly independent over $\mathbf{L}$ then $f, \tilde{g}_2, \ldots, \tilde{g}_r$ are linearly independent over $\mathbf{L}$, since a relation $f = \sum_{i>1} \tilde{\beta}_i \tilde{g}_i$ contradicts $L_f(\phi) \neq 0$. Hence $f, \tilde{g}_2, \ldots, \tilde{g}_r$ are in involution, and therefore, for all $i, j \geq 2$ there are $\tilde{\nu}_{ij}$ and $\tilde{\mu}_{ijk}$ such that $[\tilde{g}_i, \tilde{g}_j] = \tilde{\nu}_{ij} f + \sum_{k>1} \tilde{\mu}_{ijk} \tilde{g}_k$, and $L_{[\tilde{g}_i, \tilde{g}_j]}(\phi) = 0$ shows that $\tilde{\nu}_{ij} = 0$. The last assertion, with $\sigma = 1/L_f(\phi)$, follows from (2.3). ∎

In case $f = g_1$, elementary linear algebra shows that $\tilde{\mu}_{ijk} = \mu_{ijk}$ whenever $i, j, k \geq 2$.

5. Some examples and applications

5.1. Equations in dimension three with two infinitesimal symmetries. It is well-known that knowledge of a nontrivial infinitesimal (orbital) symmetry g of an equation $\dot{x} = f(x)$ on $U \subseteq \mathbf{K}^2$ reduces the determination of a first integral to a quadrature, since an integrating factor can be constructed from g; cf. Olver [12], Theorem 2.48. (Furthermore, up to determining the inverse of a certain transformation, the solution of $\dot{x} = f(x)$ is thus reduced to quadratures.) Here we will discuss the case of an equation $\dot{x} = f(x)$ on $U \subseteq \mathbf{K}^3$, with two given normalizer elements g, h. We will always assume that f, g and h are linearly independent over $\mathbf{L}$. (This excludes pathological cases like $h = g + \rho f$, which should not qualify as a second nontrivial infinitesimal symmetry.) We will take the traditional point of view that $\dot{x} = f(x)$ is "solved" as soon as there are two independent first integrals, and that necessary coordinate transformations do not cause problems. It seems unnatural to assume a priori that g and h are in involution, and therefore we will not do so. The discussion of the degenerate case in §4 turns out to be quite useful here.

By Cramer's rule over $\mathbf{L}$, there are analytic α, σ and τ on an open and dense subset of U such that $[g, h] = \alpha f + \sigma g + \tau h$. Then (4.3 b) shows that $L_f(\sigma) = L_f(\tau) = 0$, and it was shown in (3.5) that applying L_g or L_h to a first integral of f will again produce a first integral of f. Now there are three cases to be distinguished.

In the first case, there are two functionally independent first integrals of f among σ, τ, $L_g(\sigma)$, $L_g(\tau)$, $L_h(\sigma)$, $L_h(\tau)$. (If there are less than two in this list, then further applications of L_g or L_h are to no avail.) In this case, we consider $\dot{x} = f(x)$ as essentially solved.

In the second case the above list contains one, and only one, independent first integral of f, say σ, and $\tilde{g} := L_g(\sigma)h - L_h(\sigma)g$ is a nonzero vector field that normalizes f, yet is not a multiple of f over $\mathbf{L}$. (This would only be possible if $L_g(\sigma) = L_h(\sigma) = 0$, a contradiction to the linear independence of f, g and h over $\mathbf{L}$.) In appropriate local coordinates one has

$$f = \begin{pmatrix} 0 \\ \hat{f} \end{pmatrix} \quad \text{and} \quad g = \begin{pmatrix} 0 \\ \hat{g} \end{pmatrix},$$

and the problem has been reduced to dimension 2.

The third case is characterized by $\sigma =$ const. and $\tau =$ const. Consequently, if one defines $\mathcal{J} := \{\mu f : \mu \text{ analytic}\}$, then $(\mathbf{K} \cdot g + \mathbf{K} \cdot h + \mathcal{J})/\mathcal{J}$ is a two-dimensional Lie algebra over $\mathbf{K}$, and we can assume that $\sigma = 0$, $\tau = \epsilon$, with $\epsilon \in \{0, 1\}$.

Now assume, with no loss of generality, that x_1 is not a first integral of f, and define $\tilde{g} := g - (L_g(x_1)/L_f(x_1))\, f$, and $\tilde{h} := h - (L_h(x_1)/L_f(x_1))\, f$, according to (4.4). Then $\tilde{g}$ and $\tilde{h}$ normalize f, and $\left[\tilde{g}, \tilde{h}\right] = \epsilon\tilde{h}$, as follows from (4.4) and the observation thereafter. Moreover, we may assume (by going to $(1/L_f(x_1))f$, if necessary) that $[\tilde{g}, f] = \left[\tilde{h}, f\right] = 0$. Note that $L_f(x_1)$ is, by (2.5), a common first integral of g and h, and thus a function of x_1 alone. (If the normalization according to (2.3) has been carried out then one even has

$L_f(x_1) = 1$.) Thus far we have (dropping the tilde)

$$g = \begin{pmatrix} 0 \\ \hat{g} \end{pmatrix}, \quad h = \begin{pmatrix} 0 \\ \hat{h} \end{pmatrix}, \quad f(x) = \begin{pmatrix} \phi_1(x_1) \\ \phi_2(x) \\ \phi_3(x) \end{pmatrix}.$$

As far as $\hat{g}$ and $\hat{h}$ are concerned, this is a two-dimensional system, with x_1 as a parameter, and for every fixed value of x_1 one has $\left[\hat{g}, \hat{h}\right] = \epsilon h$. Here $1/\det(\hat{g}, \hat{h})$ is an integrating factor for $\hat{h}$, and thus determining a first integral for $\hat{h}$ is a quadrature problem. This is also a first integral for h, and it is necessarily functionally independent from x_1. Thus a suitable coordinate transformation leads to $h = (0, 0, *)^t$, and then to $h = (0, 0, 1)^t$. (These transformations can be chosen so that they fix the first coordinate. One should use a new name for the transformed h, but there is little danger of confusion.) Using $[h, f] = 0$ and $[g, h] = \epsilon h$, one obtains

$$f = \begin{pmatrix} \phi_1(x_1) \\ \phi_2(x_1, x_2) \\ \phi_3(x_1, x_2) \end{pmatrix}, \quad g = \begin{pmatrix} 0 \\ \gamma_2(x_1, x_2) \\ \gamma_3(x) \end{pmatrix}, \quad h = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Note that solving $\dot{x}_3 = \phi_3(x_1, x_2)$ requires only quadratures once x_1 and x_2 are determined. Now $[g, f] = 0$ forces

$$\left[\begin{pmatrix} \phi_1(x_1) \\ \phi_2(x_1, x_2) \end{pmatrix}, \begin{pmatrix} 0 \\ \gamma_2(x_1, x_2) \end{pmatrix}\right] = 0,$$

and this is, once again, a two-dimensional problem. Thus, up to quadratures and coordinate transformations, the equation $\dot{x} = f(x)$ has been solved.

5.2. Involution systems of rank 2. The simplest nontrivial case of multi-parameter symmetries occurs when $\dot{x} = f(x)$ admits two infinitesimal orbital symmetries g and h on $U \subseteq \mathbf{K}^n$ such that f, g and h are linearly independent over $\mathbf{L}$, and g and h are in involution. (In particular, $n \geq 3$.) Therefore we will take a closer look at involution systems g, h such that g and h are linearly independent over $\mathbf{L}$, and $[g, h] = \sigma g + \tau h$. Locally, near any point y such that $g(y)$ and $h(y)$ are linearly independent, there are $n - 2$ common first integrals (due to Frobenius' theorem), which, after a coordinate transformation, may be taken as $x_3, \ldots, x_n$. Moreover, h can be straightened, hence there is no loss of generality in assuming that

$$h = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad g = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \text{and } [g, h] = -\begin{pmatrix} \partial\gamma_1/\partial x_1 \\ \partial\gamma_2/\partial x_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

with $\gamma_2 \neq 0$, and

$$\sigma = -\frac{\partial\gamma_2/\partial x_1}{\gamma_2}, \quad \tau = \frac{(\partial\gamma_2/\partial x_1)\,\gamma_1 - (\partial\gamma_1/\partial x_1)\,\gamma_2}{\gamma_2}.$$

It is sometimes useful to further normalize g, while leaving h and the common first integrals intact. The admissible transformations are of the form

$$\Psi(x) = \begin{pmatrix} x_1 + \psi_1(x_2, \ldots, x_n) \\ \psi_2(x_2, \ldots, x_n) \\ \psi_3(x_3, \ldots, x_n) \\ \vdots \\ \psi_n(x_3, \ldots, x_n) \end{pmatrix},$$

as is easily verified, and one has $D\Psi(x)g(x) = \hat{g}(\Psi(x))$, with the entries of $\hat{g}$ satisfying $\gamma_1(x) + (\partial\psi_1/\partial x_2)(x)\gamma_2(x) = \hat{\gamma}_1(\Psi(x))$, $(\partial\psi_2/\partial x_2)(x)\gamma_2(x) = \hat{\gamma}_2(\Psi(x))$, and $\hat{\gamma}_i = 0$ whenever $i > 2$.

We will not attempt a detailed study of rank 2 involution systems, but we will look at a few special cases. There is no need to discuss the finite dimensional Lie algebra case, with σ and τ constant (and $\sigma = 0$ w.l.o.g.). Here (2.7) and (2.8) are applicable. In the following it will always be assumed that σ or τ is not constant. As was seen in sections 3 and 4, both σ and τ are first integrals of f, and application of L_g and L_h produces more first integrals. In fact, for this reason there are rank 2 involution systems which do not occur as systems of infinitesimal orbital symmetries of a nonzero vector field f. For example, let $\gamma_2 = \exp\left(-\left(x_1x_2 + x_1^2x_3 + \ldots + x_1^{n-1}x_n + x_1^{n+1}\right)\right)$, and hence $\sigma = x_2 + 2x_1x_3 + \ldots + (n-1)x_1^{n-2}x_n + (n+1)x_1^n$. Now σ and all $\partial^k\sigma/\partial x_1^k$ are first integrals of any vector field f admitting the infinitesimal orbital symmetries g and h, and this set contains n independent functions, forcing $f = 0$.

In any case, whenever the procedure outlined above produces at least two independent first integrals, the intuitive expectation that there should be a reduction of dimension by at least two is satisfied. Let us therefore consider the "borderline" case when this procedure yields exactly one first integral, up to functional dependence. We may assume that σ is not constant. If $L_g(\sigma) = L_h(\sigma) = 0$ then restricting to a level set of σ will lead to the finite dimensional Lie algebra case, with the dimension reduced by one. If $L_g(\sigma) \neq 0$ then $\tilde{h} := L_g(\sigma)h - L_h(\sigma)g$ still normalizes f, and admits the first integral σ. Furthermore $\left[g, \tilde{h}\right] = \tilde{\sigma}g + \tilde{\tau}\tilde{h}$ and $\tilde{\sigma} = 0$ follows from $L_{[g,\tilde{h}]}(\sigma) = L_{\tilde{h}}L_g(\sigma) - L_gL_{\tilde{h}}(\sigma) = 0$. (Recall that $L_g(\sigma)$ is locally a function of σ on an open-dense subset of U, and is therefore annihilated by $L_{\tilde{h}}$.) Therefore we may assume that $[g, h] = \tau h$, with τ not constant. (The case $L_g(\tau) \neq 0$ works similarly.)

Locally, we have $h = (1, 0, \ldots, 0)^t$, and $g = (\gamma_1, \gamma_2, 0, \ldots, 0)^t$ after a suitable choice of coordinates, and $\partial\gamma_2/\partial x_1 = 0$, $\tau = -\partial\gamma_1/\partial x_1$. Since γ_2 is a function of $x_2, \ldots, x_n$ only, there is a function ψ_2 of $x_2, \ldots, x_n$ such that $(\partial\psi_2/\partial x_2)\gamma_2 = 1$. Therefore, after applying the invertible transformation $\Psi(x) = (x_1, \psi_2(x_2, \ldots, x_n), x_3, \ldots, x_n)^t$, there remains $g = (\gamma_1, 1, 0, \ldots, 0)^t$.

Now $L_h(\tau) = \partial\tau/\partial x_1$ is locally a function of τ if and only if there are functions ρ (of $n-1$ variables) and ν (of one variable) such that $\tau(x) = \nu(x_1 + \rho(x_2, \ldots, x_n))$. (To see "if", note that ν' is locally a function of ν near any point z such that $\nu'(z) \neq 0$. The other implication follows from solving an elementary differential equation $\partial\tau/\partial x_1 = \beta(\tau)$, with $x_2, \ldots, x_n$ as parameters.) Therefore, locally $\tau(x) = \mu'(x_1 + \rho(x_2, \ldots, x_n))$, and $\gamma_1(x) = -\mu(x_1 + \rho(x_2, \ldots, x_n)) + \sigma(x_2, \ldots, x_n)$, with suitable functions μ and σ. With $L_g(\tau) = \mu''(x_1 + \rho)(-\mu(x_1 + \rho) + \sigma + \partial\rho/\partial x_2)$, and the observation that μ'' and μ are locally functions of μ', it follows that $L_g(\tau)$ is locally a function of τ (equivalently, of $x_1 + \rho$) if and only if $\sigma + \partial\rho/\partial x_2$ is. But this is possible only if $\sigma + \partial\rho/\partial x_2 = c = \text{const.}$, as differentiation with respect to x_1 shows. We have found

$$h(x) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \; g(x) = \begin{pmatrix} -\mu(x_1 + \rho(x_2, \ldots, x_n)) - (\partial\rho/\partial x_2)(x_2, \ldots, x_n) + c \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

and one may assume $c = 0$.

Let us take a brief look at a vector field f centralized by both g and h: Such a vector field is independent of x_1 (due to h), and admits the first integral $\psi := x_1 + \rho$. Evaluating $[g, f] = 0$ shows that, in addition, the last $n-1$ entries of f are independent of x_2. Thus,

$$f(x) = \begin{pmatrix} \phi(x_2, \ldots, x_n) \\ \hat{f}(x_3, \ldots, x_n) \end{pmatrix},$$

and this, together with the first integral condition, is necessary and sufficient. The solution of such an equation requires solving a differential equation in $\mathbf{K}^{n-2}$ for $x_3, \ldots, x_n$, then a quadrature (for x_2). Since ψ is a first integral, the first entry of a solution can be found from $x_1 + \rho(x_2, \ldots, x_n) = \text{const.}$ So it may be said that the first integral saves one quadrature, compared to the usual procedure.

Perhaps a "concrete" example is in order. On $\mathbf{K}^3$, consider

$$\tilde{f}(x) := \begin{pmatrix} x_1 + x_1^2 x_3 \\ 2x_1 + x_2 + x_1 x_2 x_3 - x_1^3 x_3^2 \\ -2x_3 - 2x_1 x_3^2 \end{pmatrix},$$

$$\tilde{g}(x) := \begin{pmatrix} x_1^3 x_3 \\ x_1^2 x_2 x_3 + x_1 \\ -x_1^2 x_3^2 \end{pmatrix}, \quad \tilde{h}(x) := \begin{pmatrix} x_1 \\ x_2 \\ -x_3 \end{pmatrix}.$$

Then a simple verification shows that $\tilde{g}$ and $\tilde{h}$ centralize $\tilde{f}$, while $\left[\tilde{g}, \tilde{h}\right] = -x_1^2 x_3 \tilde{h}$. (Admittedly, this example has been built in an artificial manner, but it should be noted that there is a manageable access to $\tilde{g}$ and $\tilde{h}$: One can restrict the search to centralizer elements that are polynomials of a prescribed degree, and determining these amounts to solving a finite system of linear equations. The unusual feature here is that one can find *nontrivial* polynomial vector fields centralizing $\tilde{f}$.)

The map $\Psi(x) := (\exp(x_1), x_2 \exp(x_1), x_3 \exp(-x_1))^t$ straightens $\tilde{h}$; more precisely, $D\Psi(x)h(x) = \tilde{h}(\Psi(x))$, with $h(x) = (1, 0, 0)^t$. (Although there are many straightening maps, this one may be considered natural.) Furthermore, Ψ transforms $\tilde{g}$ to g, and $\tilde{f}$ to f, with

$$g(x) = \begin{pmatrix} x_3 \exp(x_1) \\ 1 \\ 0 \end{pmatrix}, \quad \text{and } f(x) = \begin{pmatrix} 1 + x_3 \\ 2 - x_3^2 \\ -x_3 - x_3^2 \end{pmatrix},$$

as a routine computation shows. The equation $\dot{x} = f(x)$ can easily be solved by elementary functions, and applying Ψ produces the general solution of $\dot{x} = \tilde{f}(x)$. (There also is the first integral $x_1^2 x_3$ for $\tilde{f}$.)

5.3. Can there be too much reduction? From the point of view of (2.2), this can happen: If the g_i admit no common first integrals then this proposition is not applicable, and it is also of little use when there is only one common first integral, up to functional dependence, in the orbitally symmetric case. (It can be easily achieved that, say, the first entry of f is independent of $x_2, \ldots, x_n$: Just divide f by a suitable scalar function.) Let us see that things cannot be too bad even in this situation. (Hypotheses and notation are as in §2, except that there is no restriction on s.) Assume that there are no first integrals to be gained from the strategies discussed in §4, and that we deal with the nondegenerate case of §4, with the g_i centralizing f; compare (4.4) to see that there is no loss of generality in these assumptions. Then the g_i span a finite dimensional Lie algebra $\mathcal{L}$ of vector fields, of dimension $r < n$, and the local group orbits generically are r-dimensional. If $r = n - 1$, and thus there are no useful common first integrals of the g_i, structure theory of finite dimensional Lie algebras is helpful: The radical is a solvable ideal, and thus yields a preliminary reduction by (2.8), which requires only quadratures and finding inverses of transformations. By (2.6) and (2.7), the quotient modulo the radical still survives as algebra of infinitesimal symmetries for the reduced vector field. In the semisimple case, one can always resort to (maximal) toral subalgebras, or to solvable subalgebras, and obtain at least a partial reduction. There seem to be no serious problems in such situations.

Acknowledgement. I am grateful to Peter J. Olver for valuable comments on an earlier version of this paper. Also, I thank the referees for their careful reading of the manuscript and their suggestions.

References

[1] Bianchi, L., "Lezioni sulla teoria dei gruppi continui finiti di trasformazioni," Spoerri, Pisa, 1903.

[2] Bluman, G.W., and S. Kumei, "Symmetries and differential equations," Springer, New York Berlin, 1989.

[3] Engel, F., und K. Faber, "Die Liesche Theorie der partiellen Differentialgleichung erster Ordnung," Teubner, Leipzig, 1932.

[4] Gaeta, G., "Nonlinear symmetries and nonlinear equations," Kluwer, Dordrecht, 1994.

[5] Golubitsky, M., and D. G. Schaeffer, "Singularities and groups in bifurcation theory," vol. **1**, Springer-Verlag, New York etc., 1985.

[6] Golubitsky, M., I. Stewart, and D. G. Schaeffer, "Singularities and groups in bifurcation theory," vol. **2**, Springer-Verlag, New York etc., 1988.

[7] Hadeler, K.-P., *Periodic solutions of homogeneous equations,* J. Diff. Eqs. **95** (1992), 183–202.

[8] Hermann, R., "Differential geometry and the calculus of variations," Academic Press, New York, 1968.

[9] Ince, E.L., "Ordinary differential equations," Dover, New York, 1956.

[10] Kinyon, M., and S. Walcher, *On ordinary differential equations admitting a finite linear group of symmetries.*, J. Math. Analysis Appl. **216** (1997), 180–196.

[11] Lie, S., "Differentialgleichungen," Chelsea, New York, 1967.

[12] Olver, P. J., "Applications of Lie groups to differential equations," Springer-Verlag, New York etc., 1986.

[13] —, "Equivalence, invariants, and symmetry," Cambridge University Press, 1995.

[14] Rumberger, M., and J. Scheurle, *Invariant C^j-functions and center manifold reduction*, Progresss in Nonlin. Diff. Equations and Appl. **19** (1996), 145–153.

[15] Scheurle, J., *Some aspects of successive bifurcations in the Couette-Taylor problem*, Fields Institute Comm. **5** (1996), 335–345.

[16] Ünal, G., *Probability density functions, the rate of entropy change and symmetries of dynamical systems,* Phys. Lett. A **233** (1997), 193–202.

[17] Walcher, S., *Symmetries of ordinary differential equations,* Nova J. Algebra Geom. **2**(3) (1993), 245–275.

Zentrum Mathematik
TU München
D- 80290 München, Germany
walcher@mathematik.tu-muenchen.de

Received March 10, 1998
and in final form August 4, 1998