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Relation between invertibility of Casimir operators and semisimplicity of quadratic Lie superalgebras.

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Abstract. We choose a definition of semisimplicity for Lie superalgebras. As a consequence of this definition: semisimple Lie superalgebras are rigid. We study relation between semisimplicity and invertibility of Casimir operator in the case of a quadratic Lie superalgebra $(\mathfrak{g}=\mathfrak{g}_{\bar{0}}\oplus\mathfrak{g}_{\bar{1}}, B)$. In particular, we show that if the representation of the Lie algebra $\mathfrak{g}_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$ is completely reducible then semisimplicity is equivalent to the invertibility of Casimir operator.

1. Introduction and main results

Lie superalgebras considered in this paper are finite dimensional over a commutative field $\mathbb{K}$ with characteristic zero.

It is well known that, if $\mathfrak{g}$ is a Lie algebra, the following properties are equivalent and characterize the semisimple Lie algebras:

$\mathfrak{g}$ does not contain any non-zero solvable ideal, (1.1)
$\mathfrak{g}$ is the direct sum of finitely many simple Lie algebras, (1.2)
the Killing form of $\mathfrak{g}$ is non-degenerate, (1.3)
All the finite dimensional representations of $\mathfrak{g}$ are completely reducible. (1.4)

It is not the case for Lie superalgebras. When we replace ideal by graded ideal and representation by graded representation, the next properties are not equivalent: each statement is strictly stronger than the foregoing [10]. The structure of Lie superalgebras satisfying (1.1) and which are not direct sums of simple ones can be recovered using a theorem due to Kac [8]. In [7] one can find examples of such structure.

First statement is the definition given by Kac [8] for semisimplicity of Lie superalgebras. But semisimple Lie superalgebras are not necessary rigid: it is the case for the simple Lie superalgebra $D(2,1)$ [2] and [8].

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Fourth condition is very restrictive; it rules out all simple Lie superalgebras except the family $osp(1,2n)$ [10].

Then it seems to be reasonable if we choose third condition as a definition for semisimplicity: *a Lie superalgebra* $\mathfrak{g}$ *is semisimple if its Killing form is non-degenerate.* A consequence of this choice that semisimple Lie superalgebras are rigid.

In section 2, we study the structure of semisimple Lie superalgebras; some known theorems are direct consequences of results given in this study.

We close this section by given a radical of semisimple structure, in general this radical is not solvable, but it coincides with the solvable radical in the case of Lie algebras.

In section 3, in the case of a quadratic Lie superalgebra $(\mathfrak{g}, B)$ (B is a consistent, $\mathfrak{g}$-invariant, non-degenerate bilinear form), we discuss relation between semisimplicity of a such superalgebra and the invertibility of its Casimir operator Ω_B associated to B. Recall that a quadratic Lie algebra is semisimple if and only if its Casimir operator is invertible [3] and that by a theorem due to Kac and Wakimoto [9], if $(\mathfrak{g}, B)$ is a quadratic simple Lie superalgebra, then $\mathfrak{g}$ is semisimple if and only if its Casimir operator is invertible. Without simplicity the question steel open. In some interesting cases we obtain the following theorems which give equivalence between semisimplicity and invertibility of Casimir operator.

Theorem 1. *If $(\mathfrak{g}, B)$ is a quadratic Lie superalgebra such that $\mathcal{R}(\mathfrak{g}) = \{0\}$, then the following assertions are equivalent:*
1. $\mathfrak{g}$ is semisimple,
2. the Casimir operator associated to B is invertible.

Theorem 2. *Let $(\mathfrak{g}, B)$ be a quadratic Lie superalgebra such that the representation of the Lie algebra $\mathfrak{g}_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$ is completely reducible. Then $\mathfrak{g}$ is semisimple if and only if Ω_B is invertible.*

Theorem 3. *Let $(\mathfrak{g}, B)$ be a quadratic B-irreducible Lie superalgebra such that $\mathfrak{g}_{\bar{0}}$ is semisimple. Then $\mathfrak{g}$ is simple.*

As a consequence of this theorem, if $(\mathfrak{g}, B)$ is a quadratic Lie superalgebra such that $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}]$ is semisimple, then $\mathfrak{g}$ is semisimple if and only if the Casimir operator Ω_B is invertible. It is well known that the invertibility of the Casimir operator is very useful in cohomological theory of Lie superalgebras. The above theorems show that, in the case of quadratic Lie superalgebras which are not semisimple and satisfy at least one of the following conditions,

- $\mathcal{R}(\mathfrak{g}) = \{0\}$
- The representation of the Lie algebra $\mathfrak{g}_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$ is completely reducible
- $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}]$ is semisimple
- $\dim \mathfrak{g}_{\bar{1}} = 2$;

other methods would be necessary for the cohomological theory.

On the other hand, we give some conditions which imply that the Casimir operator is not invertible.

Theorem 4. *Let $(\mathfrak{g}, B)$ be a quadratic Lie superalgebra such that $\mathfrak{g}_{\bar{1}} \neq \{0\}$ and $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}]$ is a degenerate ideal of $\mathfrak{g}_{\bar{0}}$. Then the Casimir operator Ω_B is not invertible.*

Theorem 5. *Let $(\mathfrak{g}, B)$ be a quadratic Lie superalgebra such that $\mathfrak{g}_{\bar{1}} \neq \{0\}$ and $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}]$ is solvable. Then the Casimir operator Ω_B is not invertible.*

2. Structure of semisimple Lie superalgebras

Semisimple Lie superalgebra $\mathfrak{g}$, in the sense of Kac [8] (i.e. $\mathfrak{g}$ does not contain any non-zero solvable graded ideal), is not necessary rigid. In fact, simple Lie superalgebras $D(2,1;\alpha), \alpha \in \mathbb{K}\backslash\{0,-1\}$, are deformations of the simple Lie superalgebra $D(2,1)$ [8], [2]. The principal motivation of this part is to choose a good" definition for semisimplicity of Lie superalgebras which implies the rigidity by deformation. We choose the condition (1.3) as definition of semisimplicity of Lie superalgebras: *A Lie superalgebra $\mathfrak{g}$ is semisimple if its Killing form is non-degenerate.* We note that a semisimple Lie superalgebra is semisimlpe in the sense of Kac [8]: in fact

Proposition 2.1. [10] *A Lie superalgebra with non-degenerate Killing form does not contain any non zero solvable graded ideals.*

Let us remark that this definition of semisimplicity of Lie superalgebra seems to be the good definition: first it is not very restrictive, there exist many classes of Lie superalgebras with non-degenerate Killing form (the liste of simple Lie superalgebras with non-degenerate Killing form is given in [8], [10]). Second, and it is the important fact, a semisimple Lie superalgebra $\mathfrak{g}$ is rigid: in fact, as we shall see in this section $\mathfrak{g} = \overset{r}{\underset{i=1}{\oplus}} \mathfrak{g}_i$, where $\mathfrak{g}_i$ are simple, semisimple graded ideals of $\mathfrak{g}$. Let Ω_i be a Casimir operator of $\mathfrak{g}_i$. For all $X \in \mathfrak{g}$, $\Omega_i \circ \mathrm{ad}(X) = \mathrm{ad}(X) \circ \Omega_i$. By [9], Ω_i is invertible. So by [8], $H^k(\mathfrak{g}, \mathfrak{g}_i) = \{0\}$. We conclude that $H^k(\mathfrak{g}, \mathfrak{g}) = \bigoplus_{i=1}^r H^k(\mathfrak{g}, \mathfrak{g}_i) = \{0\}$ and by [2], $\mathfrak{g}$ is rigid.

Definition 2.2 Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a Lie superalgebra, B is a bilinear form on $\mathfrak{g}$; then

a. B is said to be *supersymmetric* if $B(X,Y) = (-1)^{xy}B(Y,X)$ for all $X \in \mathfrak{g}_x$, $Y \in \mathfrak{g}_y$; $x, y \in \mathbb{Z}_2$.
b. B is said to be *invariant* if $B(X,[Y,Z]) = B([X,Y],Z)$ for all $X, Y, Z \in \mathfrak{g}$.
c. B is said to be *consistent* if, for all $X \in \mathfrak{g}_{\bar{0}}$ and all $Y \in \mathfrak{g}_{\bar{1}}$, $B(X,Y) = 0$.

Proposition 2.3. [8] *The Killing form $\mathcal{K}$ of a Lie superalgebra is invariant, supersymmetric and consistent bilinear form.*

Definition 2.4 Let $\mathfrak{g}$ be a Lie superalgebra with bilinear form B. $(\mathfrak{g}, B)$ is called *quadratic* if B is supersymmetric, consistent, non-degenerate and $\mathfrak{g}$-invariant. In this case, B is called an invariant scalar product on $\mathfrak{g}$.

The class of semisimple Lie superalgebras is strictly contained in the class of quadratic Lie superalgebras. For example Lie suparalgebras $D(2,1;\alpha)$ are quadratic not semisimple. Note that there exist solvable quadratic Lie superalgebras [1].

Lemma 2.5. *Let $(\mathfrak{g}, B)$ be a quadratic Lie superalgebra.*
1. *If I is a graded ideal of $\mathfrak{g}$, then $I^{\perp}$ is a graded ideal of $\mathfrak{g}$.*
2. *$[\mathfrak{g},\mathfrak{g}]^{\perp} = \mathfrak{Z}(\mathfrak{g})$ where $\mathfrak{Z}(\mathfrak{g})$ is the cneter of $\mathfrak{g}$.*

Proof. 1. Let $X = X_{\bar{0}} + X_{\bar{1}}$ be an element of $I^{\perp}$. To prove that $I^{\perp}$ is graded, by [10], we need to show that $X_{\bar{0}} - X_{\bar{1}}$ is in $I^{\perp}$. Let Y be an homogeneous element of the graded ideal I, since B is even one has $B(Y, X_{\bar{0}} - X_{\bar{1}}) = B(Y, X_{\bar{0}} + X_{\bar{1}}) = 0$. This proves 1.

2. Let X be an element of $\mathfrak{Z}(\mathfrak{g})$, then for all Y, Z in $\mathfrak{g}$,
$B(X,[Y,Z]) = B([X,Y],Z) = 0$; thus X is in $[\mathfrak{g},\mathfrak{g}]^{\perp}$.

Conversely: let X in $[\mathfrak{g},\mathfrak{g}]^{\perp}$, then for all Y, Z in $\mathfrak{g}$, $0 = B([Y,Z],X) = B(Y,[Z,X])$. Since B is non-degenerate, then $[Z,X] = 0$, for all Z in $\mathfrak{g}$, and X is in $\mathfrak{Z}(\mathfrak{g})$. ◇

Proposition 2.6. *If $\mathfrak{g}$ is a semisimple Lie superalgebra, then $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$.*

Proof. Since $\mathfrak{Z}(\mathfrak{g}) = \{0\}$, the result is a consequence of Lemma 2.5.

Definition 2.7 A Lie superalgebra $\mathfrak{g}$ is perfect if $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$.

Definition 2.8 A graded ideal I of a Lie superalgebra $\mathfrak{g}$ is said to be direct factor if there exits a graded ideal J of $\mathfrak{g}$ such that $\mathfrak{g} = I \oplus J$.

Definition 2.9 A Lie superalgebra $\mathfrak{g}$ is irreducible if $\mathfrak{g}$ has no nontrivial direct factor.

Lemma 2.10. *Let $\mathfrak{g}$ be a Lie superalgebra, then $\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_r$, where $\mathfrak{g}_i$ are irreducible graded ideals of $\mathfrak{g}$.*

Proof. By induction on the dimension of $\mathfrak{g}$.

Lemma 2.11. *Let $\mathfrak{g}$ be a perfect Lie superalgebra, I a graded ideal of $\mathfrak{g}$. If I is a direct factor of $\mathfrak{g}$ then I is perfect.*

Theorem 2.12. *Let $\mathfrak{g}$ be a perfect Lie superalgebra. Then $\mathfrak{g}$ splits into direct sum of all his perfect irreducible direct factor graded ideals.*

Proof. By using previous lemmas one can write $\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_r$, where $\mathfrak{g}_i$ are graded perfect irreducible ideals. Let I be a perfect irreducible graded ideal. Suppose that I is a direct factor, we have to show that there exists $1 * i * r$ such that $I = \mathfrak{g}_i$. I is perfect, then $I = [I, \mathfrak{g}]$ and then

$$I = \bigoplus_{i=1}^{r} [I, \mathfrak{g}_i],$$

since I is irreducible, there exists a unique i_0 in $\{1,\ldots,r\}$ such that $I=[I,\mathfrak{g}_{i_0}]$; which implies that $I\subset\mathfrak{g}_{i_0}$. On the other hand I is a direct factor, then there exists a graded ideal J of $\mathfrak{g}$ such that $\mathfrak{g}=I\oplus J$. Hence, since $\mathfrak{g}_{i_0}=[\mathfrak{g},\mathfrak{g}_{i_0}]$, $\mathfrak{g}_{i_0}=[I,\mathfrak{g}_{i_0}]\oplus[J,\mathfrak{g}_{i_0}]$. But $\mathfrak{g}_{i_0}$ is irreducible, then $\mathfrak{g}_{i_0}=[I,\mathfrak{g}_{i_0}]$ or $\mathfrak{g}_{i_0}=[J,\mathfrak{g}_{i_0}]$. Now $I\cap J=\{0\}$, then $\mathfrak{g}_{i_0}=[I,\mathfrak{g}_{i_0}]$ and $\mathfrak{g}_{i_0}\subset I$. ■

Lemma 2.13. *Let $(\mathfrak{g},B)$ be a quadratic perfect Lie superalgebra. Let I be a graded ideal of $\mathfrak{g}$. The following statements are equivalent:*
1. The restriction $B_{|_I}$ of B to $I\times I$ is non-degenerate,
2. $\mathfrak{Z}(I)=\{0\}$.

Proof. 1. implies 2. is obvious. Let X be in $I^{\perp}$, then for all Y in I, $B(X,Y)=0$. For all Z in $\mathfrak{g}$, one has $0=B(X,[Y,Z])=B([X,Y],Z)$. Since B is non-degenerate, then for all Y in I, $[X,Y]=0$, which says that X is in $\mathfrak{Z}(I)$; so B_I is non-degenerate. ◇

Proposition 2.14. *Let $\mathfrak{g}$ be a semisimple Lie superalgebra, I a graded ideal of $\mathfrak{g}$. Then I is a semisimple direct factor.*

Proof. $\mathfrak{Z}(I)$ is an abelian graded ideal of the semisimple Lie superalgebra $\mathfrak{g}$, then $\mathfrak{Z}(I)=\{0\}$. By Lemma 2.13, the restriction K_I of the Killing form of $\mathfrak{g}$ is non-degenerate, but this restriction is exactly the Killing form of I [10], and I is semisimple. Now from the semisimplicity of I we deduce that every derivation of I is inner [10]. Let $J=\{X\in\mathfrak{g}:\ [X,I]=\{0\}\ \}$. Then J is a graded ideal of $\mathfrak{g}$. On the other hand, since $\mathfrak{Z}(I)=\{0\}$, then $I\cap J=\{0\}$. For all Y in $\mathfrak{g}$, $\mathrm{ad}(Y)_{|_I}$ is a derivation on I because $[I,I]=I$, there exits X in I such that $\mathrm{ad}_{\mathfrak{g}}(Y)_{|_I}=\mathrm{ad}_I(X)$. Then $[Y,T]=[X,T]$ for all T in I. We deduce that for all Y in $\mathfrak{g}$ there exits X in I such that $Y-X\in J$. i.e. $\mathfrak{g}=I+J$. ◇

Corollary 2.15. [8] *Let $\mathfrak{g}$ be a semisimple Lie superalgebra. Then $\mathfrak{g}$ splits into direct sum of its simple, semisimple ideals.*

Proof. Since $\mathfrak{g}$ is perfect, by Theorem 2.12: $\mathfrak{g}=\mathfrak{g}_1\oplus\ldots\oplus\mathfrak{g}_r$, where $\mathfrak{g}_i$ are graded perfect irreducible ideals. Every $\mathfrak{g}_i$ is simple: In fact if I is a grade ideal of $\mathfrak{g}_i$ then I is a graded ideal of $\mathfrak{g}$. By Proposition 2.14, I is a direct factor of $\mathfrak{g}$, then of $\mathfrak{g}_i$. But $\mathfrak{g}_i$ is irreducible, then $I=\{0\}$ or $I=\mathfrak{g}_i$. On the other hand $\mathfrak{g}_i$ is perfect then $\mathfrak{g}_i=[\mathfrak{g}_i,\mathfrak{g}_i]\neq\{0\}$. Since, for $i\neq j$, $\mathfrak{g}_i$ and $\mathfrak{g}_j$ are orthogonal, then semisimplicity of $\mathfrak{g}_i$ is obvious. ■

Remark 1.: Theorem 2.12. shows that the semisimplicity is not a necessary condition for a such decomposition.

In the following we give the radical of semisimple structure of Lie superalgebra. This radical coincides with the solvable radical in the case of Lie algebra.

Lemma 2.16. *Let $\mathfrak{g}$ be a Lie superalgebra, I a graded ideal such that $\mathfrak{Z}(I)=\{0\}$ and $Der(I)=\mathrm{ad}(I)$. Then I is a direct factor of $\mathfrak{g}$.*

Remark 2.: Every graded semisimple ideal of a Lie superalgebra is a direct factor.

Lemma 2.17. *If $\mathfrak{g}$ is a Lie superalgebra direct sum of two graded semisimple ideals I and J, then $\mathfrak{g}$ is semisimple.*

Definition 2.18 Let $\mathfrak{g}$ be a Lie superalgebra. A graded ideal I of $\mathfrak{g}$ is characteristic if I is invariant by $Der(\mathfrak{g})$.

Lemma 2.19. *Let $\mathfrak{g}$ be a Lie superalgebra and I a graded perfect ideal of $\mathfrak{g}$. Then I is characteristic.*

Proposition 2.20. *Let $\mathfrak{g}$ be a Lie superalgebra, I and J two graded semisimple ideals of $\mathfrak{g}$. Then $I \cap J$ and $I + J$ are two graded semisimple ideals of $\mathfrak{g}$.*

Proof. Firstly, $I \cap J$ is semisimple by Proposition 2.14. Secondly, by using Corollary 2.15, one has $I = \bigoplus_{i=1}^{r} I_i$; $J = \bigoplus_{i=1}^{s} J_i$, where I_i and J_i are graded, simple, semisimple ideals. Since $[I, I] = I$ and $[J, J] = J$, then for all $i = 1, \ldots, r$; $j = 1, \ldots, s$, one has $[I_i, I_i] = I_i$ and $[J_j, J_j] = J_j$. Hence I_i and J_j are characteristic ideals of $\mathfrak{g}$. Let X be in $\mathfrak{g}$, then $[X, I_i] \subset I_i$, $[X, J_i] \subset J_i$. On the other hand for all i, j one has $I_i \cap J_j = \{0\}$ or $I_i = J_j$. There exists $p * \min(r, s)$ such that

$$\forall i \in \{1, \ldots, p-1\},\, I_i = J_i; \quad \forall j * p,\, I_i \cap J_j = \{0\}.$$

Then $I + J = (I_1 \oplus I_2 \oplus \ldots \oplus I_r) + (J_p \oplus \ldots \oplus J_s)$. But it is easy to see that $[I, J_p \oplus \ldots \oplus J_s] = \{0\}$. Now if $X \in I \cap (J_p \oplus \ldots \oplus J_s)$, then $[X, I] = \{0\}$ and $X \in \mathfrak{Z}(I)$. We deduce that $I + J = (I_1 \oplus I_2 \oplus \ldots \oplus I_r) \oplus (J_p \oplus \ldots \oplus J_s)$, thus $I + J$ is semisimple by Lemma 2.17. ∎

Corollary 2.21. *Every Lie superalgebra $\mathfrak{g}$ contains a greatest graded semisimple ideal.*

Proof. Let $D = \{\dim I : I \text{ is a graded semisimple ideal of } \mathfrak{g}\}$. Let d be the greatest element of D. There exists a semisimple ideal S of $\mathfrak{g}$ such that $d = \dim\, S$. If I is a graded semisimple ideal of $\mathfrak{g}$, then $I + S$ is a graded semisimple ideal of $\mathfrak{g}$. Then $I \subset S$. ∎

Proposition 2.22. *Let $\mathfrak{g}$ be a Lie superalgebra.*
1. If I and J are two graded solvable ideals. Then $I \cap J$ and $I + J$ are solvable graded ideals.
2. $\mathfrak{g}$ has a greatest graded solvable ideal $\mathfrak{r}(\mathfrak{g})$.

Let us denote by $\mathcal{J}$ the set of all graded perfect ideals with zero Killing form. Let $\mathfrak{h}(\mathfrak{g}) = \sum_{I \in \mathcal{J}} I$.

Theorem 2.23. *Let $\mathfrak{g}$ be a Lie superalgebra, $\mathfrak{r}(\mathfrak{g})$ the greatest solvable graded ideal of $\mathfrak{g}$. Let $\mathfrak{b}(\mathfrak{g}) = \mathfrak{r}(\mathfrak{g}) + \mathfrak{h}(\mathfrak{g})$. Then $\mathfrak{g}$ is semisimple if and only if $\mathfrak{b}(\mathfrak{g}) = \{0\}$.*

Proof. If $\mathfrak{g}$ is semisimple, it is clear that $\mathfrak{b}(\mathfrak{g}) = \{0\}$. Suppose that $\mathfrak{b}(\mathfrak{g}) = \{0\}$. We denote by $\mathcal{K}$ the Killing form of $\mathfrak{g}$. By Lemma 2.16 $\mathfrak{g} = S \oplus M$, where S (resp. M) is the greatest graded semisimple ideal (resp. graded ideal) of $\mathfrak{g}$. We have to show that $M = \{0\}$. Suppose that $M \neq \{0\}$. M is not simple, because every simple ideal which is not semisimple is contained in $\mathfrak{h}(\mathfrak{g})$. Let m be a minimal non zero graded ideal of M. Then $[m,m]=\{0\}$ or $[m,m]=m$. Since $\mathfrak{r}(\mathfrak{g})=\{0\}$, first case can not occur. If $m^{\perp}$ is the orthogonal of m with respect to $\mathcal{K}_{|m}$, $m^{\perp}$ is a graded ideal of m. ¿From $\mathcal{K}_{|m}\neq 0$ and the fact that m is not semisimple, we deduce that $m^{\perp}=m$. Contradiction. So $M=\{0\}$. ■

We call $\mathfrak{b}(\mathfrak{g})$ the radical of $\mathfrak{g}$. Let us remark that for a Lie algebra $\mathfrak{g}$, if I is a perfect ideal of $\mathfrak{g}$ with Killing form equal to zero, then by the first Cartan's criterion one has I is solvable; and then $I=\{0\}$. So $\mathfrak{h}(\mathfrak{g})=\{0\}$. One can give a better definition of radical if the greatest graded solvable ideal of a Lie superalgebra is characteristic.

3. Semisimplicity and Casimir operators

If $(\mathfrak{g},B)$ is a quadratic Lie algebra, then $\mathfrak{g}$ is semisimple if and only if the Casimir operator, $\mathfrak{C}$, associated to B is invertible [3].

Theorem 3.1. [3] *If $(\mathfrak{g},B)$ is a quadratic Lie algebra, then the following assertions are equivalent:*
1. $\mathfrak{g}$ is semisimple,
2. the Casimir operator Ω_B is invertible.

On the other hand for any simple quadratic Lie superalgebra $(\mathfrak{g},B)$ one has equivalence between semisimplicity of $\mathfrak{g}$ and invertibility of Casimir operator, Ω_B, associated to B [9].

Theorem 3.2. [9] *If $(\mathfrak{g},B)$ is a simple quadratic Lie superalgebra, then the following assertions are equivalent:*
1. $\mathfrak{g}$ is semisimple,
2. the Casimir operator Ω_B is invertible.

In this section we study relation between semisimplicity and invertibility of Casimir operator for a quadratic Lie superalgebra.

Definition 3.3 Let $(\mathfrak{g},B)$ be a quadratic Lie superalgebra. A graded ideal I of $\mathfrak{g}$ is called non-degenerate (resp. degenerate) if the restriction of B to $I\times I$ is a non-degenerate (resp. degenerate) bilinear form.

Proposition 3.4. *Let $(\mathfrak{g},B)$ be a quadratic Lie superalgebra. Then $\mathfrak{g}=\bigoplus_{i=1}^{r}\mathfrak{g}_i$ such that, for all $1*i*r$,*
(i) *$\mathfrak{g}_i$ is a non-degenerate graded ideal of $\mathfrak{g}$,*
(ii) *$\mathfrak{g}_i$ contains no nontrivial non-degenerate graded ideal,*
(iii) *for all $i\neq j$, $\mathfrak{g}_i$ and $\mathfrak{g}_j$ are orthogonal.*

Lemma 3.5. *Let $(\mathfrak{g},B)$ be a quadratic Lie superalgebra. Let I be graded ideal of $\mathfrak{g}$. If I is a direct factor and $[I,I]=I$, then I is non-degenerate.*

Proof. I is a direct factor, by definition there exists a graded ideal J of $\mathfrak{g}$ such that $\mathfrak{g} = I \oplus J$. Let $i \in I$ is such that $B(i, I) = \{0\}$. Since $I = [I, I]$, by invariance we deduce that $B(i, J) = \{0\}$ and $B(i, \mathfrak{g}) = \{0\}$. This implies that $i = 0$. ∎

Definition 3.6 We say that a quadratic Lie superalgebra $(\mathfrak{g}, B)$ is B-irreducible if $\mathfrak{g}$ contains no nontrivial non-degenerate graded ideal.

Let $(\mathfrak{g}, B)$ be a quadratic Lie superalgebra. We choose $(e_i)_{1*i*n}$, $(e'_i)_{1*i*n}$ two basis of $\mathfrak{g}_{\bar{0}}$ such that $B(e_i, e'_j) = \delta_{ij}$, where δ_{ij} is the Kronecker's symbol. We choose $(f_i)_{1*i*m}$, $(f'_i)_{1*i*m}$ two basis of $\mathfrak{g}_{\bar{1}}$ such that $B(f_i, f'_j) = \delta_{ij}$. Let ω_B be the element of the enveloping algebra $\mathfrak{U}(\mathfrak{g})$ of $\mathfrak{g}$ defined by

$$\omega_B = \sum_{i=1}^{n} e_i e'_i - \sum_{i=1}^{m} f_i f'_i.$$

ω_B is called the Casimir element of $(\mathfrak{g}, B)$. By [10], ω_B is a central element in $\mathfrak{U}(\mathfrak{g})$ and ω_B does not depend on the choice of the basis.

Let T denote the extension of the adjoint representation of $\mathfrak{g}$ to $\mathfrak{U}(\mathfrak{g})$. $T(\omega_B) \in End(\mathfrak{g})$ is called the Casimir operator of $(\mathfrak{g}, B)$. We denote $\Omega_B = T(\omega_B)$.

Lemma 3.7. *If $(\mathfrak{g}, B)$ is a quadratic Lie algebra and $\mathfrak{C}$ its Casimir operator, then*

1. *there exists $n \in \mathbb{N}^*$: $\mathfrak{g} = \operatorname{Ker} \mathfrak{C}^n \oplus \operatorname{Im} \mathfrak{C}^n$, where $\operatorname{Im} \mathfrak{C}^n$ is the greatest semisimple ideal of $\mathfrak{g}$ and $\operatorname{Ker} \mathfrak{C}^n$ is an ideal of $\mathfrak{g}$.*
2. *If I is an ideal of $\mathfrak{g}$, then $I = I \cap \operatorname{Ker} \mathfrak{C}^n \oplus I \cap \operatorname{Im} \mathfrak{C}^n$. $I \cap \operatorname{Im} \mathfrak{C}^n$ is the greatest semisimple ideal of I.*

Proof. 1. See [[5], Proposition 3].
2. If $I = S \oplus M$, with S the greatest semisimple ideal of I and M an ideal of I. By [[6], Lemma (4.1)], M is a characteristic ideal of I. Then I is an ideal of $\mathfrak{g}$. We deduce that $M \cap \operatorname{Im} \mathfrak{C}^n = \{0\}$ and $[\operatorname{Im} \mathfrak{C}^n, M + \operatorname{Ker} \mathfrak{C}^n] = \{0\}$, so $\operatorname{Im} \mathfrak{C}^n \cap (M + \operatorname{Ker} \mathfrak{C}^n) = \{0\}$. Let $x \in I$, then there exist $s \in S$ and $m \in M$ such that: $x = s + m$; but $x \in \mathfrak{g}$, so there exist $a \in \operatorname{Im} \mathfrak{C}^n$ and $b \in \operatorname{Ker} \mathfrak{C}^n$ such that $x = a + b$. Then $s - a = b - m \in \operatorname{Im} \mathfrak{C}^n \cap (M + \operatorname{Ker} \mathfrak{C}^n) = \{0\}$. We conclude that

$$I = I \cap \operatorname{Ker} \mathfrak{C}^n \oplus I \cap \operatorname{Im} \mathfrak{C}^n.$$

∎

Now in the following we give some conditions which implies that the Casimir operator is not invertible.

Theorem 3.8. *Let $(\mathfrak{g}, B)$ be a quadratic Lie superalgebra such that $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}]$ is a degenerate ideal of $\mathfrak{g}_{\bar{0}}$. Then the Casimir operator Ω_B is not invertible.*

Proof. Let $\mathfrak{C}$ be the Casimir operator of the quadratic Lie algebra $(\mathfrak{g}_{\bar{0}}, B_0 = B_{|\mathfrak{g}_{\bar{0}} \times \mathfrak{g}_{\bar{0}}})$. By Lemma 3.7, there exists a non-negative integer n such that

$$\mathfrak{g}_{\bar{0}} = \operatorname{Ker} \mathfrak{C}^n \oplus \operatorname{Im} \mathfrak{C}^n,$$

where $\operatorname{Im}\mathfrak{C}^n$ is the greatest semisimple ideal of $\mathfrak{g}_{\bar 0}$ and $\operatorname{Ker}\mathfrak{C}^n$ is an ideal of $\mathfrak{g}_{\bar 0}$. $[\mathfrak{g}_{\bar 1},\mathfrak{g}_{\bar 1}]$ is an ideal of $\mathfrak{g}_{\bar 0}$, by Lemma 3.7

$$[\mathfrak{g}_{\bar 1},\mathfrak{g}_{\bar 1}] = [\mathfrak{g}_{\bar 1},\mathfrak{g}_{\bar 1}]\cap \operatorname{Ker}\mathfrak{C}^n \oplus [\mathfrak{g}_{\bar 1},\mathfrak{g}_{\bar 1}]\cap \operatorname{Im}\mathfrak{C}^n,$$

where $[\mathfrak{g}_{\bar 1},\mathfrak{g}_{\bar 1}]\cap \operatorname{Im}\mathfrak{C}^n$ is the greatest semisimple ideal of $[\mathfrak{g}_{\bar 1},\mathfrak{g}_{\bar 1}]$. Now, since $[\mathfrak{g}_{\bar 1},\mathfrak{g}_{\bar 1}]$ is degenerate, by Lemma 3.5, it is not semisimple. Then $[\mathfrak{g}_{\bar 1},\mathfrak{g}_{\bar 1}]\cap\operatorname{Ker}\mathfrak{C}^n \neq \{0\}$. $[\mathfrak{g}_{\bar 1},\mathfrak{g}_{\bar 1}]$ is degenerate, then there exists $0\neq \ell \in [\mathfrak{g}_{\bar 1},\mathfrak{g}_{\bar 1}]$ such that $B(\ell,[\mathfrak{g}_{\bar 1},\mathfrak{g}_{\bar 1}]) = \{0\}$. ℓ can be written as $\ell = s+i$, where $s\in[\mathfrak{g}_{\bar 1},\mathfrak{g}_{\bar 1}]\cap\operatorname{Im}\mathfrak{C}^n$ and $i\in[\mathfrak{g}_{\bar 1},\mathfrak{g}_{\bar 1}]\cap\operatorname{Ker}\mathfrak{C}^n$. We show that $s=0$. By using $[[\mathfrak{g}_{\bar 1},\mathfrak{g}_{\bar 1}]\cap\operatorname{Im}\mathfrak{C}^n,[\mathfrak{g}_{\bar 1},\mathfrak{g}_{\bar 1}]\cap\operatorname{Ker}\mathfrak{C}^n] = \{0\}$, $[[\mathfrak{g}_{\bar 1},\mathfrak{g}_{\bar 1}]\cap\operatorname{Im}\mathfrak{C}^n,[\mathfrak{g}_{\bar 1},\mathfrak{g}_{\bar 1}]\cap\operatorname{Im}\mathfrak{C}^n] = [\mathfrak{g}_{\bar 1},\mathfrak{g}_{\bar 1}]\cap\operatorname{Im}\mathfrak{C}^n$ and by the invariance of B, we deduce that $B(i,[\mathfrak{g}_{\bar 1},\mathfrak{g}_{\bar 1}]\cap\operatorname{Im}\mathfrak{C}^n) = \{0\}$. Consequently $B(s,[\mathfrak{g}_{\bar 1},\mathfrak{g}_{\bar 1}]\cap\operatorname{Im}\mathfrak{C}^n) = B(\ell,[\mathfrak{g}_{\bar 1},\mathfrak{g}_{\bar 1}]\cap\operatorname{Im}\mathfrak{C}^n) = \{0\}$. By Lemma 3.5, $s=0$. Thus $\ell\in\operatorname{Ker}\mathfrak{C}^n$, then $\mathfrak{C}^n(\ell)=0$. Consequently, there exists a non-negative integer m such that $\mathfrak{C}^m(\ell)\neq 0$ and $\mathfrak{C}^{m+1}(\ell)=0$. Denote $b=\mathfrak{C}^m(\ell)$, $b\neq 0$ and $\mathfrak{C}(b)=0$. Let X and Y be in $\mathfrak{g}_{\bar 1}$, then by [[5], Lemma 2]

$$B(b,[X,Y]) = B(\mathfrak{C}^m(\ell),[X,Y]) = B(\ell,\mathfrak{C}^m([X,Y]).$$

Since $\mathfrak{C}^m([\mathfrak{g}_{\bar 1},\mathfrak{g}_{\bar 1}])\subset[\mathfrak{g}_{\bar 1},\mathfrak{g}_{\bar 1}]$ and $B(\ell,[\mathfrak{g}_{\bar 1},\mathfrak{g}_{\bar 1}])=\{0\}$, then $B(b,[X,Y]) = 0$. We have proved that $B(b,[\mathfrak{g}_{\bar 1},\mathfrak{g}_{\bar 1}])=\{0\}$. Consequently, by invariance, we obtain $B([b,\mathfrak{g}_{\bar 1}],\mathfrak{g}_{\bar 1})=\{0\}$. On the other hand, since B is consistent and b is in $\mathfrak{g}_{\bar 0}$, one has $B([b,\mathfrak{g}_{\bar 1}],\mathfrak{g}_{\bar 0})=\{0\}$. It follows that $[b,\mathfrak{g}_{\bar 1}]=\{0\}$.

Let $(e_i)_{1*i*n}$ and $(e'_i)_{1*i*n}$ be two basis of $\mathfrak{g}_{\bar 0}$ such that $B(e_i,e'_j)=\delta_{ij}$.

Let $(f_i)_{1*i*m}$ and $(f'_i)_{1*i*m}$ be two basis of $\mathfrak{g}_{\bar 1}$ such that $B(f_i,f'_j)=\delta_{ij}$. Then

$$\Omega_B = \sum_{i=1}^{n}\operatorname{ad}_{\mathfrak{g}} e_i\circ\operatorname{ad}_{\mathfrak{g}} e'_i - \sum_{i=1}^{m}\operatorname{ad}_{\mathfrak{g}} f_i\circ\operatorname{ad}_{\mathfrak{g}} f'_i.$$

So $\Omega_B(b) = \sum_{i=1}^n[e_i,[e'_i,b]] - \sum_{i=1}^m[f_i,[f'_i,b]] = \Big(\sum_{i=1}^n\operatorname{ad}_{\mathfrak{g}_{\bar 0}} e_i\circ\operatorname{ad}_{\mathfrak{g}_{\bar 0}} e'_i\Big)(b) = \mathfrak{C}(b)=0$ and Ω_B is not invertible. ∎

Theorem 3.9. *Let $(\mathfrak{g},B)$ be a quadratic Lie superalgebra such that $\mathfrak{g}_{\bar 1}\neq\{0\}$ and $[\mathfrak{g}_{\bar 1},\mathfrak{g}_{\bar 1}]$ is solvable. Then the Casimir operator Ω_B is not invertible.*

Proof. If $[\mathfrak{g}_{\bar 1},\mathfrak{g}_{\bar 1}]$ is degenerate, the result follows from Theorem 3.8.

If $[\mathfrak{g}_{\bar 1},\mathfrak{g}_{\bar 1}]$ is a non-degenerate ideal of $\mathfrak{g}_{\bar 0}$, then $I=[\mathfrak{g}_{\bar 1},\mathfrak{g}_{\bar 1}]\oplus\mathfrak{g}_{\bar 1}$ is a graded non-degenerate ideal of $\mathfrak{g}$. So $\mathfrak{g}=I\oplus I^{\perp}$, where $I^{\perp}\subset\mathfrak{g}_{\bar 0}$ is a graded non-degenerate ideal of $\mathfrak{g}$. Since $[\mathfrak{g}_{\bar 1},\mathfrak{g}_{\bar 1}]$ is a solvable, then by [10], I is solvable. So $[I,I]\neq I$ and by Lemma 2.5, $\mathfrak{Z}(I)\neq\{0\}$. Now $[I,I^{\perp}]=\{0\}$ implies that $\mathfrak{Z}(I)\subset\mathfrak{Z}(\mathfrak{g})$; consequently $\mathfrak{Z}(I)\subset\operatorname{Ker}\Omega_B$, so Ω_B is not invertible. ∎

As a consequence of the previous theorem, one has the folowing corollary.

Corollary 3.10. *Let $(\mathfrak{g},B)$ be a quadratic Lie superalgebra such that $[\mathfrak{g}_{\bar 1},\mathfrak{g}_{\bar 1}]$ is solvable. Then the Casimir operator Ω_B is invertible if and only if $\mathfrak{g}$ is a semisimple Lie algebra.*

Let us consider the class $\mathcal{A}$ of quadratic Lie superalgebras $(\mathfrak{g},B)$ such that $\mathcal{R}(\mathfrak{g})=\{0\}$. We show that, if $\mathfrak{g}\in\mathcal{A}$, $\mathfrak{g}$ is semisimple if and only if the Casimir operator Ω_B is invertible.

Lemma 3.11. *If* $(\mathfrak{g}, B)$ *is a quadratic Lie superalgebra such that* $\mathcal{R}(\mathfrak{g}) = \{0\}$*, then* $\mathfrak{g} = \bigoplus_{i=1}^{r} \mathfrak{g}_i$ *such that, for all* $1 * i * r$*,*

(i) $\mathfrak{g}_i$ *is a non-degenerate graded ideal of* $\mathfrak{g}$*,*

(ii) $\mathfrak{g}_i$ *contains no nontrivial non-degenerate graded ideal.*

(iii) *for all* $i \neq j$*,* $\mathfrak{g}_i$ *and* $\mathfrak{g}_j$ *are orthogonal,*

(iv) *for all* i*,* $\mathfrak{g}_i$ *is simple.*

Proof. Three first points follow from Proposition 3.4.

(iv) Suppose that there exists $i_0 \in \{1, \cdots, r\}$ such that $\mathfrak{g}_{i_0}$ is not simple. Then there exists a nontrivial graded ideal I of $\mathfrak{g}_{i_0}$. Then $I^{\perp}$ is a graded ideal of $\mathfrak{g}$. We suppose that I is minimal. By (i), I is degenerate and then $I \cap I^{\perp} \neq \{0\}$. Now I is minimal, then $I = I \cap I^{\perp}$. So $B_{|_I} = 0$. Since B is invariant, then I is abelian and $I \subset \mathcal{R}(\mathfrak{g}_{i_0}) = \{0\}$. Contradiction. ∎

Remark 3.: This lemma proves that for a quadratic Lie superalgebra the conditions (1.1) and (1.2) are equivalent.

Theorem 3.12. *If* $(\mathfrak{g}, B)$ *is a quadratic Lie superalgebra such that* $\mathcal{R}(\mathfrak{g}) = \{0\}$*, then the following assertions are equivalent:*

1. $\mathfrak{g}$ *is semisimple,*
2. *the Casimir operator* Ω_B *is invertible.*

Proof. By Lemma 3.11 one has

$$\mathfrak{g} = \bigoplus_{i=1}^{r} \mathfrak{g}_i,$$

where each $\mathfrak{g}_i$ is a non-degenerate simple graded ideal and for all $i \neq j$, $B(\mathfrak{g}_i, \mathfrak{g}_j) = \{0\}$. We denote by $B_i = B_{|_{\mathfrak{g}_i \times \mathfrak{g}_i}}$. Let $\omega^i_{B_i}$ be the Casimir element of $(\mathfrak{g}_i, B_i)$. Let T_i be the representation of $\mathfrak{U}(\mathfrak{g}_i)$, the unique extension of the adjointe representation of $\mathfrak{g}_i$. Denote by ${\Omega^i}_{B_i} = T_i(\omega^i_{B_i})$ the Casimir operator of $(\mathfrak{g}_i, B_i)$. Now $\omega_B = {\omega^1}_{B_1} + \cdots + {\omega^r}_{B_r}$. Then it is easy to see that

- $T({\omega^i}_{B_i})(\mathfrak{g}_j) = 0$ if $i \neq j$,
- $T({\omega^i}_{B_i})_{|_{\mathfrak{g}_i}} = T_i({\omega^i}_{B_i})$,
- $T({\omega^i}_{B_i}) \circ T({\omega^j}_{B_j}) = T({\omega^j}_{B_j}) \circ T({\omega^i}_{B_i})$,
- $T(\omega_B) = \sum_{i=1}^{r} T({\omega^i}_{B_i})$.

1. $\Longrightarrow$ 2.: Let $\mathcal{K}$ be the Killing form of $\mathfrak{g}$. Since $\mathfrak{g}$ is semisimple, then for all $i \in \{1, \cdots, r\}$, $\mathfrak{g}_i$ is a simple semisimple Lie superalgebras [10]. Thus, by Theorem 3.2, for all $i \in \{1, \cdots, r\}$, the Casimir operator ${\Omega^i}_{B_i}$ of $(\mathfrak{g}_i, B_i)$ is invertible.

Let $X = \sum_{i=1}^{r} X_i$ where $X_i \in \mathfrak{g}_i$ such that $T(\omega_B)(X) = 0$. Then

$$\sum_{i=1}^{r} T({\omega^i}_{B_i})(X) = \sum_{i=1}^{r} T_i({\omega^i}_{B_i})(X_i) = \sum_{i=1}^{r} {\Omega^i}_{B_i}(X_i) = 0,$$

then, for all $i \in \{1, \cdots, r\}$ we have $\Omega^i_{B_i}(X_i) = 0$,which implies that $X_i = 0$, for $1 * i * r$, since $\Omega^i_{B_i}$ is invertible. So $X = 0$ and Ω_B is invertible.

2. $\Longrightarrow$ 1.: Suppose that the Casimir operator Ω_B of $(\mathfrak{g}, B)$ is invertible. Then, for all $i \in \{1, \cdots r\}$, $\Omega^i_{B_i}$ is invertible. Since $\mathfrak{g}_i$ is simple, then by Theorem 3.2, $\mathfrak{g}_i$ is semisimple and Proposition 2.20 implies that $\mathfrak{g}$ is semisimple. ■

Lemma 3.13. *Let $(\mathfrak{g}, B)$ be a quadratic Lie superalgebra and Ω_B its Casimir operator. Then*

1. $\Omega_B \circ \mathrm{ad}_{\mathfrak{g}}(X) = \mathrm{ad}_{\mathfrak{g}}(X) \circ \Omega_B$ *for all* $X \in \mathfrak{g}$.
2. $B(\Omega_B(X), Y) = B(X, \Omega_B(Y))$ *for all* $X, Y \in \mathfrak{g}$.
3. *For all integer* $n * 1$, $B(\mathrm{Ker}(\Omega_B)^n, \mathrm{Im}(\Omega_B)^n) = \{0\}$.
4. *For all integer* $n * 1$, $\mathrm{Ker}(\Omega_B)^n$ *is an ideal of* $\mathfrak{g}$.

Proof. This is an easy computation.

Proposition 3.14. *If $(\mathfrak{g}, B)$ is a quadratic, B-irreducible Lie superalgebra and Ω_B its Casimir operator, then Ω_B is nilpotent or invertible.*

Proof. There exists an integer $n * 1$ such that

$$\mathfrak{g} = \mathrm{Ker}(\Omega_B)^n \oplus \mathrm{Im}(\Omega_B)^n.$$

By Lemma 3.13, we deduce that $\mathrm{Ker}(\Omega_B)^n$ is a non-degenerate ideal of $\mathfrak{g}$. Since $\mathfrak{g}$ is B-irreducible, then $\mathrm{Ker}(\Omega_B)^n = \mathfrak{g}$ or $\mathrm{Ker}(\Omega_B)^n = \{0\}$. So Ω_B is nilpotent or invertible. ■

Corollary 3.15. *Let $(\mathfrak{g}, B)$ be a simple quadratic Lie superalgebra. Let Ω_B its Casimir operator. Then the following assertions are equivalent:*

1. Ω_B is nilpotent

2. $\mathfrak{g}$ is not semisimple.

Corollary 3.16. *Let $(\mathfrak{g}, B)$ be a quadratic Lie superalgebra such that $\mathcal{R}(\mathfrak{g}) = \{0\}$. Let Ω_B its Casimir operator . Then the following assertions are equivalent:*

1. Ω_B *is nilpotent*
2. $\mathfrak{g}$ *does not contain any nontrivial semisimple ideal.*

Remark 4. Let $(\mathfrak{g}, B)$ be a quadratic Lie superalgebra. If $\mathfrak{Z}(\mathfrak{g}) \neq \{0\}$ then Ω_B is not invertible. Up to now we shall only consider Lie superalgebras $\mathfrak{g}$ with $\mathfrak{Z}(\mathfrak{g}) = \{0\}$.

Let $\mathcal{B}$ be the class of quadratic Lie superalgebras $(\mathfrak{g}, B)$ such that the representation of the Lie algebra $\mathfrak{g}_{\bar 0}$ on $\mathfrak{g}_{\bar 1}$ is completely reducible. In the following we show that if $(\mathfrak{g}, B)$ is in $\mathcal{B}$ then $\mathfrak{g}$ is semisimple if and only if the Casimir operator associated to B is invertible.

Lemma 3.17. *Let $(\mathfrak{g}, B)$ be a quadratic B-irreducible Lie superalgebra. Suppose that the $\mathfrak{g}_{\bar 0}$-module $\mathfrak{g}_{\bar 1}$ is not irreducible. If Ω_B is invertible then any nontrivial $\mathfrak{g}_{\bar 0}$-submodule of $\mathfrak{g}_{\bar 1}$ is degenerate.*

Proof. If V is a nontrivial non-degenerate $\mathfrak{g}_{\bar 0}$-submodule of $\mathfrak{g}_{\bar 1}$, then $\mathfrak{g}_{\bar 1} = V \oplus V^{\perp}$.

Firstly, we show that $V^\perp$ is a $\mathfrak{g}_{\bar 0}$-module. If $X \in \mathfrak{g}_{\bar 0}$, $Y \in V^\perp$ and $Z \in V$, then $B([X,Y],Z) = -B(Y,[X,Z]) = 0$. So $[X,Y] \in V^\perp$ and $[\mathfrak{g}_{\bar 0}, V^\perp] \subset V^\perp$. Secondly, we show that $[V, V^\perp] = \{0\}$. Indeed

$$B([V,V^\perp],\mathfrak{g}_{\bar 0}) = B(V,[V^\perp,\mathfrak{g}_{\bar 0}]) = \{0\},$$

so $[V,V^\perp] = \{0\}$, since $[V,V^\perp] \subset \mathfrak{g}_{\bar 0}$ and $B_{|\mathfrak{g}_{\bar 0}\times\mathfrak{g}_{\bar 0}}$ is non-degenerate. Thirdly, we show that $[[V,V],V^\perp] = \{0\}$: $B([[V,V],V^\perp],V^\perp) = B([V,[V,V^\perp]],V^\perp) + B([V,[V^\perp,V]],V^\perp) = \{0\}$, so $[[V,V],V^\perp] = \{0\}$. Let $I = [V,V] \oplus V$. Since V is a $\mathfrak{g}_{\bar 0}$-module and $[\mathfrak{g}_{\bar 1}, V] = [V,V]$, then I is an ideal of $\mathfrak{g}$. The fact that $\mathfrak{g}$ is B-irreducible implies that I is a degenerate ideal of $\mathfrak{g}$ and $[V,V]$ is a degenerate ideal of $\mathfrak{g}_{\bar 0}$. Let $\mathfrak{C}$ be the Casimir operator of the quadratic Lie algebra $(\mathfrak{g}_{\bar 0}, B_0 = B_{|\mathfrak{g}_{\bar 0}\times\mathfrak{g}_{\bar 0}})$, then by Lemma 3.7, firstly, there exists a non-negative integer n such that $\mathfrak{g}_{\bar 0} = \operatorname{Ker}\mathfrak{C}^n \oplus \operatorname{Im}\mathfrak{C}^n$, where $\operatorname{Im}\mathfrak{C}^n$ is the greatest semisimple ideal of $\mathfrak{g}_{\bar 0}$ and $\operatorname{Ker}\mathfrak{C}^n$ is an ideal of $\mathfrak{g}_{\bar 0}$; secondly, $[V,V] = [V,V]\cap\operatorname{Ker}\mathfrak{C}^n \oplus [V,V]\cap\operatorname{Im}\mathfrak{C}^n$.

By Lemma 3.5, $[V,V]$ is not semisimple, then $[V,V]\cap\operatorname{Ker}\mathfrak{C}^n \neq \{0\}$. Since $[V,V]$ is degenerate, then there exists $0 \neq \ell = x+y$, $x \in [V,V]\cap\operatorname{Ker}\mathfrak{C}^n$, $y \in [V,V]\cap\operatorname{Im}\mathfrak{C}^n$ such that $B(\ell,[V,V]) = \{0\}$.

By the same argument using in the proof of Theorem 3.8, $y = 0$ and $\ell \in [V,V]\cap\operatorname{Ker}\mathfrak{C}^n$. Then there exists a non-negative integer m such that $\mathfrak{C}^m(\ell) \neq 0$ and $\mathfrak{C}^{m+1}(\ell) = 0$. Let $b = \mathfrak{C}^m(\ell)$, then by Lemma 3.13 applied to $(\mathfrak{g}_{\bar 0}, B_0)$, $[b,V] = \{0\}$. Now since b is in $[V,V]$, we deduce that $[b,V^\perp] = \{0\}$. So $[b,\mathfrak{g}_{\bar 1}] = \{0\}$. Thus $\Omega_B(b) = 0$ and Ω_B is not invertible. ∎

Corollary 3.18. *Let $(\mathfrak{g},B)$ be a quadratic B-irreducible Lie superalgebra. Suppose that the $\mathfrak{g}_{\bar 0}$-module $\mathfrak{g}_{\bar 1}$ is not irreducible. Suppose that $\mathfrak{g}_{\bar 1}$ does not contain any completely isotropic $\mathfrak{g}_{\bar 0}$-module. Then Ω_B is not invertible.*

Proof. Suppose that Ω_B is invertible. If V is a nontrivial $\mathfrak{g}_{\bar 0}$-submodule of $\mathfrak{g}_{\bar 1}$. By Lemma 3.17, V is a degenerate submodule. So $V \cap V^\perp$ is a nontrivial $\mathfrak{g}_{\bar 0}$-submodule comletely isotropic. Contradiction. ∎

Theorem 3.19. *Let $(\mathfrak{g},B)$ be a quadratic Lie superalgebra such that the representation of the Lie algebra $\mathfrak{g}_{\bar 0}$ on $\mathfrak{g}_{\bar 1}$ is completely reducible. Then $\mathfrak{g}$ is semisimple if and only if Ω_B is invertible.*

Proof. If $\mathfrak{g}_{\bar 1} = \{0\}$, then $\mathfrak{g}$ is a quadratic Lie algebra and the result follows from Theorem 3.1. In the following we suppose that $\mathfrak{g}_{\bar 1} \neq \{0\}$. By Proposition 3.4, it suffices to prove the theorem for B-irreducible Lie superalgebra $\mathfrak{g}$. We show that if Ω_B is invertible then $\mathfrak{g}$ is semisimple.

Suppose that Ω_B is invertible, by Theorem 3.8 $[\mathfrak{g}_{\bar 1}, \mathfrak{g}_{\bar 1}]$ is a non degenerate graded ideal of $\mathfrak{g}_{\bar 0}$. Then $I = [\mathfrak{g}_{\bar 1}, \mathfrak{g}_{\bar 1}] \oplus \mathfrak{g}_{\bar 1}$ is a non-degenerate graded ideal of $\mathfrak{g}$. Since $\mathfrak{g}$ is B-irreducible, then $I = \mathfrak{g}$ and $\mathfrak{g}_{\bar 0} = [\mathfrak{g}_{\bar 1}, \mathfrak{g}_{\bar 1}]$.

First case: Suppose that $\mathfrak{g}_{\bar 1}$ is an irreducible $\mathfrak{g}_{\bar 0}$-module. We show that the action of $\mathfrak{g}_{\bar 0}$ on $\mathfrak{g}_{\bar 1}$ is faithful.

Let $x \in \mathfrak{g}_{\bar 0}$ such that $[x,\mathfrak{g}_{\bar 1}] = \{0\}$. Then $[x,\mathfrak{g}_{\bar 0}] = [x,[\mathfrak{g}_{\bar 1},\mathfrak{g}_{\bar 1}]] = \{0\}$ and $x \in \mathfrak{Z}(\mathfrak{g})$. But Ω_B is invertible, this implies that $\mathfrak{Z}(\mathfrak{g}) = \{0\}$, consequently

$x = 0$. Now by [[10], Ch 2 2 Lemma 3] the Lie superlgebra $\mathfrak{g}$ is simple and by Theorem 3.2 $\mathfrak{g}$ is semisimple.

Second case: Suppose that $\mathfrak{g}_{\bar{1}}$ is not irreducible. Then by Lemma 3.17 any nontrivial $\mathfrak{g}_{\bar{0}}$-submodule V of $\mathfrak{g}_{\bar{1}}$ is degenerate.

Claim 1: Let V be a nontrivial $\mathfrak{g}_{\bar{0}}$-submodule of $\mathfrak{g}_{\bar{1}}$, then V is completely isotropic.

Proof of the claim 1: Since V is degenerate, then $V \cap V^{\perp} \neq \{0\}$. The representation of $\mathfrak{g}_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$ is completely reducible, then there exists a $\mathfrak{g}_{\bar{0}}$-submodule W of $\mathfrak{g}_{\bar{1}}$ such that $\mathfrak{g}_{\bar{1}} = V \cap V^{\perp} \oplus W$. So $V = V \cap V^{\perp} \oplus V \cap W$, which implies that $V \cap W$ is a non-degenerate $\mathfrak{g}_{\bar{0}}$-submodule of $\mathfrak{g}_{\bar{1}}$. Then $V \cap W = \{0\}$ and $V = V \cap V^{\perp}$, i.e. V is completely isotropic, which complete the proof of the claim 1.

We show that $\mathfrak{g}$ is simple. Let $I = I_{\bar{0}} \oplus I_{\bar{1}}$ be a graded ideal of $\mathfrak{g}$.

Claim 2: $I_{\bar{1}} = \{0\}$.

Proof of the claim 2: Suppose that $I_{\bar{1}} \neq \{0\}$. The representation of $\mathfrak{g}_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$ is completely reducible, then there exists a $\mathfrak{g}_{\bar{0}}$-module W of $\mathfrak{g}_{\bar{1}}$ such that $\mathfrak{g}_{\bar{1}} = I_{\bar{1}} \oplus W$. By using invariance of B and claim 1, one has $[W, W] = [I_{\bar{1}}, I_{\bar{1}}] = \{0\}$. Thus

$$[W, I_{\bar{1}}] = [\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] = \mathfrak{g}_{\bar{0}} \subset I_{\bar{0}}$$

which implies that $\mathfrak{g}_{\bar{0}} = I_{\bar{0}}$.

On the other hand, Ω_B is invertible, so $\mathfrak{Z}(\mathfrak{g}) = \{0\}$ and by Lemma 2.5, $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. So $\mathfrak{g}_{\bar{1}} = I_{\bar{1}}$, then $W = \{0\}$ and $\mathfrak{g}_{\bar{0}} = \{0\}$. Consequently $\mathfrak{g} = \mathfrak{g}_{\bar{1}}$ is an abelian Lie superalgebra. Since Ω_B is invertible, then $\mathfrak{g} = \{0\}$. Contradiction with $I_{\bar{1}} \neq \{0\}$. This complete the proof of the claim 2. Now $I_{\bar{1}} = \{0\}$, then $[\mathfrak{g}_{\bar{1}}, I_{\bar{0}}] = \{0\}$. Since Ω_B is invertible, then $(\mathrm{Ker}\,\mathfrak{C}) \cap I_{\bar{0}} = \{0\}$, where $\mathfrak{C}$ is the Casimir operator of $(\mathfrak{g}_{\bar{0}}, B_0)$. We show that $I_{\bar{0}}$ is semisimple. By Lemma 3.7, there exists a non-negative integer n such that

1) $\mathfrak{g}_{\bar{0}} = \mathrm{Ker}\,\mathfrak{C}^n \oplus \mathrm{Im}\,\mathfrak{C}^n$, where $\mathrm{Im}\,\mathfrak{C}^n$ is the greatest semisimple ideal of $\mathfrak{g}_{\bar{0}}$ and $\mathrm{Ker}\,\mathfrak{C}^n$ is an ideal of $\mathfrak{g}_{\bar{0}}$,
2) $I_{\bar{0}} = I_{\bar{0}} \cap \mathrm{Ker}\,\mathfrak{C}^n \oplus I_{\bar{0}} \cap \mathrm{Im}\,\mathfrak{C}^n$, where $I_{\bar{0}} \cap \mathrm{Im}\,\mathfrak{C}^n$ is the greatest semisimple ideal of $I_{\bar{0}}$.

Since $\mathrm{Ker}\,\mathfrak{C} \cap I_{\bar{0}} = \{0\}$ and $I_{\bar{0}}$ is an ideal of $\mathfrak{g}_{\bar{0}}$, then $I_{\bar{0}} \cap \mathrm{Ker}\,\mathfrak{C}^n = \{0\}$, which implies that $I_{\bar{0}}$ is semisimple. Now $I = I_{\bar{0}}$ is a graded semisimple ideal of $\mathfrak{g}$ and by Lemma 3.5 I is non-degenerate, so $I = \{0\}$ or $I = \mathfrak{g}$. We have prove that $\mathfrak{g}$ is simple, by Theorem 3.2 $\mathfrak{g}$ is semisimple. ∎

Let $(\mathfrak{g}, B)$ be in the class $\mathcal{C}$ of quadratic Lie superalgebras such that $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}]$ is semisimple, then we have: $\mathfrak{g}$ is semisimple if and only if Casimir operator is invertible.

Theorem 3.20. *Let $(\mathfrak{g}, B)$ be a B-irreducible quadratic Lie superalgebra such that $\mathfrak{g}_{\bar{0}}$ is semisimple. Then $\mathfrak{g}$ is simple.*

Proof. We start by showing that $\mathfrak{Z}(\mathfrak{g}) = \{0\}$. $\mathfrak{g}_{\bar{0}}$ is semisimple implies that

$\mathfrak{Z}(\mathfrak{g}) \cap \mathfrak{g}_{\bar{0}} = \{0\}$; so $\mathfrak{Z}(\mathfrak{g}) = \mathfrak{Z}(\mathfrak{g}) \cap \mathfrak{g}_{\bar{1}}$.

Now, $\mathfrak{g}_{\bar{1}}$ is a completely reducible $\mathfrak{g}_{\bar{0}}$-module, there exists a sub-$\mathfrak{g}_{\bar{0}}$-module V of $\mathfrak{g}_{\bar{1}}$ such that

$$\mathfrak{g}_{\bar{1}} = \mathfrak{Z}(\mathfrak{g}) \oplus V.$$

Denote by $V^{\perp}$ the orthogonal of V in $\mathfrak{g}_{\bar{1}}$ with respect to the bilinear form B. Then

$$B([V \cap V^{\perp}, V], \mathfrak{g}_{\bar{0}}) = B(V \cap V^{\perp}, [V, \mathfrak{g}_{\bar{0}}]) = \{0\},$$

which implies that $[V \cap V^{\perp}, V] = \{0\}$ and $[V \cap V^{\perp}, \mathfrak{g}_{\bar{1}}] = \{0\}$. On the other hand

$$B([V \cap V^{\perp}, \mathfrak{g}_{\bar{0}}], \mathfrak{g}_{\bar{1}}) = B(\mathfrak{g}_{\bar{0}}, [V \cap V^{\perp}, \mathfrak{g}_{\bar{1}}]) = \{0\},$$

so $[V \cap V^{\perp}, \mathfrak{g}_{\bar{0}}] = \{0\}$ and $V \cap V^{\perp} \subset \mathfrak{Z}(\mathfrak{g})$. Consequently $V \cap V^{\perp} = \{0\}$, then V is a non-degenerate $\mathfrak{g}_{\bar{0}}$-module. Let $I = \mathfrak{g}_{\bar{0}} \oplus V$. Since $[\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{1}}] = [\mathfrak{g}_{\bar{0}}, V] \subset V$, then I is a non-degenerate graded ideal of $\mathfrak{g}$. This implies that $I = \mathfrak{g}$ and then $\mathfrak{Z}(\mathfrak{g}) = \{0\}$. So by lemma 2.5, $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$; thus $[\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{1}}] = \mathfrak{g}_{\bar{1}}$.

$\mathfrak{g}_{\bar{0}}$ is semisimple, then $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}]$ is a non-degenerate ideal of $\mathfrak{g}_{\bar{0}}$. This implies that $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] \oplus \mathfrak{g}_{\bar{1}}$ is a non-degenerate graded ideal of $\mathfrak{g}$. So $\mathfrak{g} = [\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] \oplus \mathfrak{g}_{\bar{1}}$ and $\mathfrak{g}_{\bar{0}} = [\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}]$. We have to show that $\mathfrak{g}$ is simple. Let H be an ideal of $\mathfrak{g}$, then $H_{\bar{0}}$ is an ideal of $\mathfrak{g}_{\bar{0}}$. So $H_{\bar{0}}$ is semisimple, since $\mathfrak{g}_{\bar{0}}$ is semisimple. $\mathfrak{g}_{\bar{0}} = H_{\bar{0}} \oplus S$, where S is a semisimple ideal of $\mathfrak{g}_{\bar{0}}$. Now $[H_{\bar{0}}, H_{\bar{0}}] = H_{\bar{0}}$, $[S, S] = S$ and B is invariant, then $B(H_{\bar{0}}, S) = \{0\}$. So $H_{\bar{0}}$ is a non-degenerate ideal of $\mathfrak{g}_{\bar{0}}$. On the other hand, since $\mathfrak{g}_{\bar{0}}$ is semisimple and $[\mathfrak{g}_{\bar{0}}, H_{\bar{1}}] \subset H_{\bar{1}}$, there exists a subspace L of $\mathfrak{g}_{\bar{1}}$ such that $[\mathfrak{g}_{\bar{0}}, L] \subset L$ and $\mathfrak{g}_{\bar{1}} = H_{\bar{1}} \oplus L$.

Since $[\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{1}}] = \mathfrak{g}_{\bar{1}}$, then $[\mathfrak{g}_{\bar{0}}, H_{\bar{1}}] = H_{\bar{1}}$ and $[\mathfrak{g}_{\bar{0}}, L] = L$. But $[H_{\bar{0}}, L] \subset L$ and $[H_{\bar{0}}, L] \subset H_{\bar{1}}$, so $[H_{\bar{0}}, L] \subset L \cap H_{\bar{1}} = \{0\}$ and $[H_{\bar{0}}, L] = \{0\}$. We deduce that $[S, L] = L$. This implies that L and $H_{\bar{1}}$ are orthogonal with respect to B. Indeed:

$$B(L, H_{\bar{1}}) = B([S, L], H_{\bar{1}}) = B(S, [L, H_{\bar{1}}]) = \{0\},$$

because $[L, H_{\bar{1}}] = H_{\bar{0}}$ and $B(H_{\bar{0}}, S) = \{0\}$. We deduce that $H_{\bar{1}}$ is a non-degenerate subspace of $\mathfrak{g}_{\bar{1}}$. Thus H is a non-degenerate ideal of $\mathfrak{g}$. But $\mathfrak{g}$ is B-irreducible, then $H = \{0\}$ or $H = \mathfrak{g}$. This proves that $\mathfrak{g}$ is simple. ■

Corollary 3.21. *Let $(\mathfrak{g}, B)$ be a quadratic Lie superalgebra such that $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] \neq \{0\}$ is semisimple. Then the following assertions are equivalent:*

1. The Casimir operator Ω_B is invertible,

2. $\mathfrak{g}$ is semisimple.

Proof. By Proposition 3.4, $\mathfrak{g} = \bigoplus_{i=1}^{r} \mathfrak{g}_i$ where, for all $1 * i * r$, $\mathfrak{g}_i \neq \{0\}$ satisfies the following conditions

(i) $\mathfrak{g}_i$ is a non-degenerate graded ideal of $\mathfrak{g}$,

(ii) $\mathfrak{g}_i$ is B_i-irreducible,

(iii) for all $i \neq j$, $\mathfrak{g}_i$ and $\mathfrak{g}_j$ are orthogonal.

Suppose that Ω_B is invertible, then ${\Omega^i}_{B_i}$ is invertible for all $1 * i * r$.

• If $[\mathfrak{g}_{i\bar{1}}, \mathfrak{g}_{i\bar{1}}] = \{0\}$, then by invariance $\mathfrak{g}_{i\bar{1}} \subset \mathfrak{Z}(\mathfrak{g}_i)$. Now $\mathfrak{g}_{i\bar{0}}$ and $\mathfrak{g}_{i\bar{1}}$ are non-degenerate ideals of $\mathfrak{g}_i$. But $\mathfrak{g}_i$ is B_i-irreducible, so $\mathfrak{g}_{i\bar{0}} = \{0\}$ or $\mathfrak{g}_{i\bar{1}} = \{0\}$. If $\mathfrak{g}_{i\bar{0}} = \{0\}$, then $\mathfrak{g}_i = \mathfrak{g}_{i\bar{1}}$ is an abelian Lie superalgebra. The fact that ${\Omega^i}_{B_i}$ is invertible implies that $\mathfrak{g}_i = \{0\}$. So $\mathfrak{g}_i = \mathfrak{g}_{i\bar{0}}$ is a quadratic Lie algebra and by Theorem 3.1, $\mathfrak{g}_i$ is semisimple.

• If $[\mathfrak{g}_{i\bar{1}}, \mathfrak{g}_{i\bar{1}}] \neq \{0\}$, then par Theorem 3.8 $[\mathfrak{g}_{i\bar{1}}, \mathfrak{g}_{i\bar{1}}]$ is a non-degenerate ideal of $\mathfrak{g}_{i\bar{0}}$. This implies that $[\mathfrak{g}_{i\bar{1}}, \mathfrak{g}_{i\bar{1}}] \oplus \mathfrak{g}_{i\bar{1}}$ is a non-degenerate ideal of $\mathfrak{g}_i$ which is B_i-irreducible. So

$$\mathfrak{g}_i = [\mathfrak{g}_{i\bar{1}}, \mathfrak{g}_{i\bar{1}}] \oplus \mathfrak{g}_{i\bar{1}},$$

thus $\mathfrak{g}_{i\bar{0}} = [\mathfrak{g}_{i\bar{1}}, \mathfrak{g}_{i\bar{1}}]$. Since $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}]$ is a semisimple ideal of $\mathfrak{g}$, then $\mathfrak{g}_{i\bar{0}}$ is semisimple and by Theorem 3.20 $\mathfrak{g}_i$ is simple. Since ${\Omega^i}_{B_i}$ is invertible, by the Theorem 3.2, $\mathfrak{g}_i$ is semisimple. We deduce by Proposition 2.20 that $\mathfrak{g}$ is semisimple Lie superalgebra. ■

Let us remark that if $(\mathfrak{g}, B)$ is quadratic Lie superalgebra, then the dimension of $\mathfrak{g}_{\bar{1}}$ is even. We obtain the following result for the first nontrivial case ($\mathfrak{g}_{\bar{1}} \neq \{0\}$).

Corollary 3.22. *Let $(\mathfrak{g}, B)$ is quadratic Lie superalgebra such that $\delta \operatorname{Im} \mathfrak{g}_{\bar{1}} = 2$. Then the following assertions are equivalent:*

1. *The Casimir operator Ω_B is invertible,*
2. *$\mathfrak{g}$ is semisimple.*

Proof. If $\delta \operatorname{Im} \mathfrak{g}_{\bar{1}} = 2$ then $\delta \operatorname{Im} [\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] * 3$. This implies that $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}]$ is solvable or semisimple. If Ω_B is invertible, then by Theorem 3.9, $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}]$ is semisimple and the corollary follows from Corollary 3.21. ■

We close this paper by the following open question: Let $\mathcal{S}$ the class of quadratic B-irreducible Lie superalgebras $(\mathfrak{g}, B)$ such that:

- $\mathfrak{Z}(\mathfrak{g}) = \{0\}$,
- $\mathfrak{g}_{\bar{1}}$ is not completely reducible as $\mathfrak{g}_{\bar{0}}$-module,
- $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}]$ is a non-degenerate ideal of $\mathfrak{g}_{\bar{0}}$,
- Any $\mathfrak{g}_{\bar{0}}$-submodule of $\mathfrak{g}_{\bar{1}}$ is degenerate.

If this class is not empty, what about invertibility of Casimir operators of an elelment of $\mathcal{S}$? An answer to this question would complete the study of the relation between invertibility of Casimir operator and semisimplicity of a quadratic Lie superalgebra.

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