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Homogeneous spaces of compact connected Lie groups which admit nontrivial invariant algebras

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Abstract. In 1965 Wolf [10] and Gangolli [1] proved that compact semisimple groups are distinguished in the class of all compact connected Lie groups by the following property: every uniformly closed function algebra which is invariant with respect to left and right translations is also invariant with respect to the complex conjugation. In this article we extend this result to the class of homogeneous spaces of compact connected Lie groups with connected stable subgroups: a homogeneous space admits only self-conjugated invariant function algebras if and only if the isotropy representation has no nonzero fixed vectors.

1. Introduction

Let M be a homogeneous space of a compact connected Lie group G acting on M by left. Denote by H the stable subgroup of some point $x_0 \in M$, by $N_G(H)$ the normalizer of H in G, and by N the group $N_G(H)/H$. Let $C(M)$ be the commutative Banach algebra of all complex-valued continuous functions on M. Then G acts on $C(M)$ by the formula

$$(L_g f)(x) = f(g^{-1}x), \qquad g \in G,\ x \in M.$$

We shall say that A is an *invariant algebra on M* if A is an invariant under this action of G uniformly closed subalgebra of $C(M)$. A function algebra A will be called *self-conjugated* if it contains the complex conjugated function $\bar{f}$ for any $f \in A$. The Stone-Weierstrass theorem implies that any self-conjugated uniformly closed function algebra on a compact Q may be identified with $C(\tilde{Q})$, where $\tilde{Q}$ is the factor of Q by the equivalence $x \sim y \Longleftrightarrow f(x) = f(y)$ for all $f \in A$. If A is an invariant algebra then the equivalence is invariant. Hence the factor space is a homogeneous space M' of G and the factorization is a continuous equivariant mapping $\pi : M \to M'$ such that $A = C(M') \circ \pi$. Therefore, self-conjugated algebras on M are in one-to-one correspondence with closed subgroups of G which include H.

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The main result of this paper is

Theorem. *Let G be a compact connected Lie group, H is its closed connected subgroup. Then the following conditions are equivalent:*

(1) *every uniformly closed invariant algebra on $M = G/H$ is self-conjugated;*
(2) *the group $N_G(H)/H$ is finite;*
(3) *the isotropy group of any point $x_0 \in M$ has no fixed nonzero vectors in the tangent space $T_{x_0}M$.*

Bi-invariant uniform algebras on a compact group G are $G \times G$-invariant algebras on G, where $G \times G$ acts on G by left anf right translations. The condition that the group $N_{G\times G}(G)/G$ is finite is equivalent to the condition that the group G is semisimple. Hence the theorem generalizes results of Wolf and Gangolli.

The case of invariant algebras on spheres is most investigated one among homogeneous spaces which are not of the type $G \times G/G$. de Leeuw and Mirkil [6] showed that every invariant algebra on the real sphere $S^n = \mathrm{SO}(n+1)/\mathrm{SO}(n)$ is self-conjugated. Nagel and Rudin [7] showed that there are invariant algebras on the complex speres $S^{2n-1} = \mathrm{U}(n)/\mathrm{U}(n-1)$ which are not self-conjugated. The other transitive actions on S^n which admit only self-conjugated invariant algebras were described in [5]. In [11] Wolf proved that almost all compact homogeneous symmetric spaces admit only self-conjugated invariant algebras.

The way of the proof of the theorem was suggested by Gichev [2].

2. Finitely generated invariant algebras

Definition. An invariant algebra A will be called *finitely generated* if A is generated as a Banach algebra by a finite-dimensional invariant vector space $V \subset C(M)$.

Let A be a finitely generated invariant algebra on M, V be a generating finite-dimensional invariant vector space.

The group G acts on the complex vector space V^* of all linear functionals on V. This action extends to the action of the complexification $G^{\mathbf{C}}$ of G in V^*.

Let δ be the evaluating functional for the point x_0 on V: $\delta(f) = f(x_0)$ for all $f \in V$. Let $O_{\delta,V}$ be the G-orbit of the vector δ in V^* and $O^{\mathbf{C}}_{\delta,V}$ be the $G^{\mathbf{C}}$-orbit of δ in V^*. Denote by $\mathrm{Stab}\,\delta$ and $\mathrm{Stab}_{\mathbf{C}}\,\delta$ the stabilizers of δ in G and $G^{\mathbf{C}}$ respectively; note that $H \subseteq \mathrm{Stab}\,\delta$. We obtain a continuous equivariant mapping $M \to O_{\delta,V}$. Since linear functions on V^* are in one-to-one correspondence with functions in V, we may identify A with the uniform closure of the algebra of all holomorphic polynomials on $O_{\delta,V}$.

Lemma 1. *Let A be an invariant algebra on M generated by a finite dimen-*

sional invariant subspace V. If $O^{\mathbf{c}}_{\delta,V}$ is closed and

$$(\mathrm{Stab}\ \delta)^{\mathbf{c}} = \mathrm{Stab}_{\mathbf{c}}\ \delta$$

then A is self-conjugated.

Proof. Let B be the algebra of all holomorphic polynomials on $O^{\mathbf{c}}_{\delta,V}$. Then A is the uniform closure of $B|_M$. Denote by $\#$ the involutive automorphism of $G^{\mathbf{c}}$ whose set of fixed points is the maximal compact subgroup G of $G^{\mathbf{c}}$. The condition of the lemma implies that $\#$ induces an involution in $O^{\mathbf{c}}_{\delta,V}$ with $O_{\delta,V}$ as the set of fixed points. For any holomorphic on $O^{\mathbf{c}}_{\delta,V}$ function f the function $f^{\#}(z) = \overline{f(z^{\#})}$ is also holomorphic. Hence the restriction to $O_{\delta,V}$ of the algebra of all holomorphic on $O^{\mathbf{c}}_{\delta,V}$ functions is self-conjugated. By the Stone-Weierstrass theorem, the uniform closure of the algebra of all holomorphic on $O^{\mathbf{c}}_{\delta,V}$ functions restricted to $O_{\delta,V}$ is $C(O_{\delta,V})$.

Since $O^{\mathbf{c}}_{\delta,V}$ is closed, any holomorphic on $O^{\mathbf{c}}_{\delta,V}$ function can be extended to an entire function by H.Cartan extension theorem (see, for example, [Chapter VIII A, Theorem 18, 4]. Hence any holomorphic on $O^{\mathbf{c}}_{\delta,V}$ function can be approximated by polynomials uniformly on compact sets (Taylor series), in particular on $O_{\delta,V}$. Therefore, polynomials are dense in $C(O_{\delta,V})$. This easily implies that the restriction of the algebra of polynomials on $O_{\delta,V}$ is self-conjugated. ∎

Lemma 2. *If every finitely generated invariant subalgebra of an invariant algebra A on M is self-conjugated then A is self-conjugated.*

Proof. Let A_0 be the subset of A consisting of functions which generate finite dimensional invariant subspaces. If $f \in A_0$ then $\bar{f} \in A$ by the assumption of the lemma. Clearly, $\bar{f} \in A_0$. Hence A_0 is self-conjugated. It remains to note that A_0 is dense in A because G is compact. ∎

3. Proof of the theorem

Lemma 3. *The conditions* (2) *and* (3) *of the theorem are equivalent.*

Proof. The Lie algebra of $N_G(H)$ coincides with the normalizer $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h})$ of $\mathfrak{h}$ in $\mathfrak{g}$ because H is connected. There is an $\mathrm{Ad}\,H$-invariant decomposition of $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h})$: $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h} \oplus \mathfrak{n}$. Since $\mathfrak{n}$ is an ideal in $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h})$ the Lie algebra $\mathfrak{n}$ can be identified with the set of all $\mathrm{Ad}\,H$-invariant vectors in the tangent space $T_{x_0}M$. Therefore the group N is finite if and only if $\mathfrak{n} = 0$. ∎

Lemma 4. *If the group N is infinite then there exists an invariant algebra on M which is not self-conjugated.*

Proof. Suppose the group N is infinite. Since $gHn = gnH$ for all $g \in G$ and $n \in N$, N acts on G/H by right translations. Hence there is an action of the group $\mathrm{U}(1) = \{e^{i\varphi}\}$ commuting with the action of G.

For every integer m set

$$S_m = \{f \in C(M) | \rho(e^{i\varphi})f = e^{im\varphi}f\}.$$

The spaces S_m are G-invariant since the actions of the groups G and $\mathrm{U}(1)$ commute. Then the algebra

$$A = \sum_{m \in \mathbb{Z}_+} S_m$$

of all functions extending on φ holomorphically to the unit disc is an invariant algebra on M which is not self-conjugated. ■

Let $N_{\mathbb{C}}$ be the factor of the normalizer of $H^{\mathbb{C}}$ in $G^{\mathbb{C}}$ by $H^{\mathbb{C}}$. Denote by τ the antilinear involutive automorphism which distinguishes the compact real form $\mathfrak{g}$ of the Lie algebra $\mathfrak{g}^{\mathbb{C}}$.

Lemma 5. *If N if finite then $N_{\mathbb{C}}$ is finite.*

Proof. Denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of G and H respectively. Since N is finite, the Lie algebra of $N_G(H)$ coincides with $\mathfrak{h}$. Let $\mathfrak{n}_{\mathbb{C}}$ be the Lie algebra of $N_{G^{\mathbb{C}}}(H^{\mathbb{C}})$; $\mathfrak{n}_{\mathbb{C}}$ coincides with the normalizer of $\mathfrak{h}^{\mathbb{C}}$ in $\mathfrak{g}^{\mathbb{C}}$ because $H^{\mathbb{C}}$ is connected. Since $\mathfrak{h}^{\mathbb{C}}$ is τ-invariant $\mathfrak{n}_{\mathbb{C}}$ is also τ-invariant. Hence $\mathfrak{n}_{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}}$. Therefore the identity component of $N_{G^{\mathbb{C}}}(H^{\mathbb{C}})$ coincides with $H^{\mathbb{C}}$. The group $N_{\mathbb{C}}$ is finite because $N_{G^{\mathbb{C}}}(H^{\mathbb{C}})$ is an algebraic group. ■

Lemma 6. *Let V be a finite-dimensional complex vector space, $G^{\mathbb{C}} \subseteq \mathrm{GL}(V)$ and the stabilizer of $v \in V$ contains H. If the group N is finite then $G^{\mathbb{C}}$-orbit of v is closed.*

Proof. By the Luna criterion (see [Theorem 6.17, 9], it is sufficient to prove that the orbit of normalizer of $H^{\mathbb{C}}$ in $G^{\mathbb{C}}$ is closed. This is an easy consequence of Lemma 5. ■

Lemma 7. *Let V be a finite-dimensional complex vector space, $G^{\mathbb{C}} \subseteq \mathrm{GL}(V)$ and the stabilizer of $v \in V$ contains H. If the group N is finite then the stabilizer of v in $G^{\mathbb{C}}$ is the complexification of the stabilizer of v in G.*

Proof. By Lemma 6 $G^{\mathbb{C}}$-orbit of v is closed. Then the stabilizer $\mathrm{Stab}_{\mathbb{C}}\, v$ of v in $G^{\mathbb{C}}$ is a complex reductive Lie group (see [Theorem 4.17, 9]). Hence $\mathrm{Stab}_{\mathbb{C}}\, v$ is the complexification of its maximal compact subgroup K_1. We may assume that $H \subseteq K_1$.

Let K be a maximal compact subgroup of $G^{\mathbb{C}}$ containing K_1. Denote by $\mathfrak{k}$ the Lie algebra of K. Then $\mathfrak{k}$ is a real compact form of $\mathfrak{g}^{\mathbb{C}}$. Since N is finite the center of $\mathfrak{g}$ is contained in $\mathfrak{h}$. Let $\mathfrak{g}_0^{\mathbb{C}}$, $\mathfrak{k}_0$ and $\mathfrak{g}_0$ be the semisimple parts of $\mathfrak{g}^{\mathbb{C}}$, $\mathfrak{k}$ and $\mathfrak{g}$ respectively. Denote by μ the antilinear involutive automorphism defining the compact real form $\mathfrak{k}_0$ of the Lie algebra $\mathfrak{g}_0^{\mathbb{C}}$.

Consider the automorphism $\theta = \mu\tau_0$ where τ_0 is the restriction of τ to $\mathfrak{g}_0^{\mathbb{C}}$. The standard proof of the fact that any two compact real forms of the complex semisimple Lie algebra are conjugated shows that there is an element $z \in \mathfrak{g}_0^{\mathbb{C}}$, such that $\theta^2 = \exp(\mathrm{ad}(4z))$, $\exp(\mathrm{ad}\, z)\mathfrak{g}_0 = \mathfrak{k}_0$, and adz commutes with all linear transformations of $\mathfrak{g}_0^{\mathbb{C}}$ commuting with θ^2 (for details see [8] or [3].

Since $\theta^2|_{\mathfrak{h}^{\mathbb{C}} \cap \mathfrak{g}_0^{\mathbb{C}}} = id$, z is contained in the centralizer of $\mathfrak{h}^{\mathbb{C}} \cap \mathfrak{g}_0^{\mathbb{C}}$ in $\mathfrak{g}_0^{\mathbb{C}}$. Hence vectors $\tau z + z$ and $i(\tau z - z)$ are elements of the centralizer of $\mathfrak{h} \cap \mathfrak{g}_0$ in $\mathfrak{g}_0$.

By Lemma 3 z is contained in the center of $\mathfrak{h}^{\mathbf{C}}$. Since $\mathfrak{h}$ includes the center of $\mathfrak{g}$, $\exp(\operatorname{ad} z)\mathfrak{g} = \mathfrak{k}$. Then $\exp(-\operatorname{ad} z)K \subseteq G$ because K is connected. Therefore

$$\exp(-\operatorname{ad} z)K_1 \subseteq \operatorname{Stab}_{\mathbf{C}}\, v \cap G.$$

The group $\exp(-\operatorname{ad} z)K_1$ is a maximal compact subgroup of $\operatorname{Stab}_{\mathbf{C}}\, v$. Thus $\operatorname{Stab}_{\mathbf{C}}\, v = (\operatorname{Stab}_{\mathbf{C}}\, v \cap G)^{\mathbf{C}}$. ∎

Proof of Theorem. The equivalence (2) $\Leftrightarrow$ (3) is proved in Lemma 3. The implication (1) $\Rightarrow$ (2) follows from Lemma 4.

Suppose that the group N is finite. Lemmas 1, 6 and 7 imply that every invariant finitely generated algebra on M is self-conjugated. By Lemma 2 every invariant algebra on M is self-conjugated. Therefore (2) implies (1). ∎

Corollary 1. *Let G be a compact connected Lie group, H is its closed connected subgroup. If the group $N_G(H)/H$ is finite then invariant function algebras on M are in one-to-one correspondence with the closed subgroups of G which include H.*

The class of homogeneous spaces satisfying the condition of Corollary 1 is rather wide. It contains, for example, all compact homogeneous symmetric spaces.

Corollary 2. *Let V be a finite-dimensional complex vector space, G be a compact connected Lie subgroup of $\mathrm{GL}(V)$. Suppose that H is a closed connected subgroup of G such that the group $N_G(H)/H$ is finite. If H is contained in the stabilizer of an element $v \in V$ then G-orbit Gv of v is polynomially convex and holomorphically convex.*

Proof. Denote by P the algebra of all polynomials on V. Let A be the uniform closure of the restriction of P to Gv. By the theorem A is self-conjugated. Clearly, P separates the points of Gv. By the Stone-Weierstrass theorem $A = C(Gv)$. Hence the set of all multiplicative linear functionals on $C(Gv)$ coincides with Gv. Since every point x of the polynomially convex hull of Gv defines the multiplicative linear functional on A (the evaluating functional at x) Gv is polynomially convex. Since the holomorphically convex hull of Gv is contained in the polynomially convex hull, Gv is holomorphically convex. ∎

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