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Square-integrablity of tensor products

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Abstract. This paper is concerned with C_0-representations of locally compact groups. The focus is on the relationship between the C_0-property and square-integrability, the latter meaning that the representation is quasi-equivalent to a subrepresentation of the regular one. We show that for certain real algebraic groups every C_0-representation has a square-integrable tensor power and discuss some classes of groups enjoying this property. We point out to which extent this result supports a conjecture of Figà-Talamance and Piccardello concerning the radical of the Fourier algebra in the Fourier-Stietjes algebra. Finally, we give a simple criterion for a C_0-representation to be square-integrable.

1. Introduction

In this paper we are concerned with asymptotic properties of (strongly continuous) unitary representations $(\pi, \mathcal{H}_\pi)$ of a locally compact group G and with their relations to *square-integrability.* By the latter we mean that there is a dense subspace D of the representation space $\mathcal{H}_\pi$, such that for all $\xi, \eta \in D$ the *matrix coefficient*

$$\varphi_{\xi\eta} : G \to \mathbb{C}, \quad g \mapsto \langle \pi(g)\xi, \eta \rangle$$

is in $\mathrm{L}^2(G)$. Square-integrable representations have been extensively studied, and it follows from the results of Rieffel [40], Duflo–Moore [15] and others that a representation $(\pi, \mathcal{H}_\pi)$ is square-integrable if and only if it is quasi-equivalent in the sense of ([14],5.3.1) to a subrepresentation of the regular representation $(\lambda_G, \mathrm{L}^2(G))$. We write $\pi \overset{\mathrm{q}}{\leq} \lambda_G$. Beyond the classical theory, square-integrable representations recently gained interest, since they

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are the basic tool for the construction of continuous *wavelet transforms* from cyclic representations, see e.g., [29, 31, 25].

Thus it is highly desirable to be able to construct square-integrable representations or to decide when a given representation $(\pi, \mathcal{H}_\pi)$ is square-integrable and we present results concerning both questions (see Theorems 1.1 and 1.2).

Square-integrable representations are C_0*–representations*, which means that all matrix coefficients of π vanish at infinity, but not vice versa. The main theorem of this paper shows that, for certain real algebraic groups, every C_0–representation has a square integrable tensor power.

To be more precise, recall that a connected real algebraic group G has a unique largest unipotent radical N , and decomposes as

$$G = N \rtimes_\varphi H ,$$

where H is a reductive Levi-complement of N . N and H are Zariski-closed, N is simply connected with respect to the topology induced by $\mathrm{GL}(n, \mathbb{C})$ and H acts algebraically and reductively on the Lie algebra $\mathfrak{n}$ by the derived representation. The centralizer in G of a subset $S \in G$ is denoted by $\mathcal{Z}(S, G)$.

Theorem 1.1. *Let G be a connected real algebraic group and keep the above notation.*
Suppose that

(i) $\mathcal{Z}(H, G) \cap N = \{e\}$,

(ii) $\mathcal{Z}(\mathcal{Z}, G) \cap H$ *is compact, where* $\mathcal{Z}$ *is the center of* N .

Then there exists a finite $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ and for all representations $(\pi, \mathcal{H}_\pi)$ whose subrepresentations all have compact kernel

$$\pi^{\otimes k} \overset{\mathrm{q}}{\leq} \lambda_G .$$

Our second theorem gives a criterion for square-integrability of C_0–representations. Before formulating the theorem, let us recall that any irreducible representation $(\pi, \mathcal{H}_\pi)$ of a locally compact group G which contains a regularly embedded normal subgroup N is of the following form

$$(\pi, \mathcal{H}_\pi) = \mathrm{ind}_{G_{\tilde{\chi}}}^{G} \chi .$$

Here, $\tilde{\chi}$ is an irreducible representation of N , $G_{\tilde{\chi}}$ is the group of elements in G which preserve $\tilde{\chi}$ under conjugation and χ is an irreducible representation of $G_{\tilde{\chi}}$ such that $\chi|_N$ is a multiple of $\tilde{\chi}$.

Theorem 1.2. *Let N be a regularly embedded simply connected nilpotent normal subgroup of a Lie group G . Let $(\pi, \mathcal{H}_\pi)$ be a unitary representation of G and let $\pi = \int_{\hat{G}}^{\oplus} m(\varrho)\varrho d\mu(\varrho)$ be a decomposition in irreducibles. Then $\pi \overset{\mathrm{q}}{\leq} \lambda_G$ provided that*

(i) *for μ–almost all $\varrho = \mathrm{ind}_{G_{\tilde{\chi}}}^{G} \chi$, the fixed group $G_{\tilde{\chi}}$ of the character $\tilde{\chi}$, which $\chi|_N$ is a multiple of, is a compact extension of N;*

(ii) *for μ–almost all ϱ the G–orbit of the Kirillov-orbit associated to $\tilde{\chi}$ is open in $\mathfrak{n}^*$.*

Let us discuss these theorems. In Theorem 1.1, assumption (i) is a bit stronger than the requirement that N does not contain nontrivial subgroups which are central in G (such a group would be non-compact). As to the second requirement, identify G with the outer semidirect product $N \rtimes_\varphi H$ and observe that we may split off the kernel of φ from H , yielding

$$G = (N \rtimes H/\ker\varphi) \times \ker\varphi ,$$

(at least modulo finite subgroups). See ([27],p.87) for more details. Since the property, that any representation as in the theorem has a square-integrable tensor power, carries over to direct products and finite extensions, we may consider both factors separately. Then $\ker\varphi$ will enjoy this property if its center is compact and it is a Kazhdan group, by well-known results of Cowling [11] and Moore [38]. Our result 1.1 applies to the first factor if the quotient $(\ker(\varphi\,|\,_Z))/(\ker\varphi)$ is compact. If this requirement is not satisfied one may try to embed $H/\ker\varphi$ in an appropriate $\mathrm{Sp}(n,\mathbb{R})$ and to use a result of Howe and Moore [27]. We discuss some examples in Section 3.

In any case, we may apply Theorem 1.1 to the groups $G := V \rtimes H$, where V is a vector space and H the group of diagonal matrices or a semisimple Kazhdan group acting on V without nontrivial fixed points. We will give more elaborate examples in Section 2.

Several examples of groups are known such that, for every C_0–representation $(\pi, \mathcal{H}_\pi)$, a sufficiently large tensor power is square-integrable: This follows easily for semisimple Kazhdan groups from the results of Cowling [11] and Moore [38], for generalized motion groups from those of Liukonnen and Mislove [33, 34] and for $G := \mathbb{R}^2 \rtimes \mathrm{SL}(2,\mathbb{R})$ from [37]. By way of contrast, on groups with non-compact center and on non-compact nilpotent groups there exist C_0–representations whose tensor powers are far from being square-integrable. This was shown by Figá-Talamanca and Picardello generalizing methods of Varopoulos [23].

Observe that we do not require the representation π in Theorem 1.1 to be a C_0–representation, this is a part of the result. The fact that representations whose subrepresentations all have compact kernel are C_0 is of considerable interest in many areas, e.g., in ergodic theory ([7], [8]). This C_0–property is known for semisimple groups with finite center by the Howe–Moore theorem [27] and for connected so called *totally minimal groups*, treated below [35, 36].

The proof of Theorem 1.1 is based on the one given by Howe and Moore for the fact that irreducible representations of algebraic groups have a tensor power which is square-integrable modulo the projective kernel ([27],6.1). A somewhat different approach would use the methods in [33, 34], after getting rid of the compactness assumption made there.

Theorem 1.1 also bears significance to a conjecture by Figá-Talamanca and Picardello concerning *Fourier-Stieltjes algebras*. The Fourier-Stieltjes algebra $\mathrm{B}(G)$ is one of the most fruitful constructions of a dual object for a general locally compact group G , due to P. Eymard [16]. $\mathrm{B}(G)$ is the space of all matrix coefficients of strongly continuous unitary representations of G . Using sums, tensor products and contragredient representations, it is easily seen that $\mathrm{B}(G)$ carries the pointwise structure of an involutive commutative algebra. In addition, $\mathrm{B}(G)$ may be identified with the dual space of the group C^*–algebra $\mathrm{C}^*(G)$ via

$$\varphi_{\xi\eta} \mapsto (\mathrm{C}^*(G) \ni T \mapsto \langle \pi(T)\xi, \eta\rangle) ,$$

where $(\pi, \mathcal{H}_\pi)$ denotes also the lifting to $\mathrm{C}^*(G)$ and ξ, η are vectors in $\mathcal{H}_\pi$. The dual space norm makes $\mathrm{B}(G)$ an involutive (semisimple) Banach algebra which the group G acts upon by left and right translations. The matrix coefficients of the regular representation form a closed ideal, the *Fourier algebra* $\mathrm{A}(G)$. It coincides with the closure of all coefficient functions with compact support in $\mathrm{B}(G)$. For abelian G, Bochner's theorem identifies $\mathrm{B}(G)$ with the measure algebra $\mathrm{M}(\hat{G})$ on $\hat{G}$ and $\mathrm{A}(G)$ with $\mathrm{L}^1(\hat{G})$. For all these facts, see [16, 17].

More generally, denote by A_π the closure in $\mathrm{B}(G)$ of all coefficients of a unitary representation $(\pi, \mathcal{H}_\pi)$. By Arsac's theory ([1],3.1.II), A_π characterizes $(\pi, \mathcal{H}_\pi)$ up to quasi-equivalence. If

$$\mathrm{B}_0(G) := \{\psi \in \mathrm{B}(G) \mid \psi \text{ vanishes at infinity}\} ,$$

the above question on the square-integrability of tensor products is related to the question on the relationship between $\mathrm{B}_0(G)$ and the radical $\mathrm{A}_r(G)$ of the Fourier algebra, that is

$$\mathrm{A}_r(G) = \{\psi \in \mathrm{B}(G) \mid \exists k \in \mathbb{N} : \psi^k \in \mathrm{A}(G)\} .$$

Figà-Talamanca and Picardello [23] showed that $\mathrm{A}_r(G)$ is not norm dense in $\mathrm{B}_0(G)$ if the center of G is not compact or if G is a non-compact nilpotent group. Their conjecture reads as follows

Conjecture 1.3. Let G be an analytic group with compact center and without non-compact nilpotent direct factors. Then $\mathrm{A}_r(G)$ is dense in $\mathrm{B}_0(G)$.

Observe that $\mathrm{A}_r(G)$ might be a proper dense subspace of $\mathrm{B}_0(G)$. Consider for instance $G := \mathrm{SL}(2, \mathbb{R})$ and

$$\pi := \int_{[0,1]}^{\oplus} \kappa_s ds$$

where κ_s, $0 < s < 1$, denotes the complementary series representations (see e.g., [24],p. 246) and ds is a finite measure equivalent to the Lebesgue-measure. Then $\mathrm{A}_\pi \subseteq \overline{\mathrm{A}_r(G)}$, but $\mathrm{A}_\pi \not\subseteq \mathrm{A}_r(G)$.

Theorem 1.1 might be seen as a step towards proving Conjecture 1.3. Another class of groups satisfying this conjecture are linear reductive groups (with compact center), as follows immediately from the results of Cowling [11, 12]. Arbitrary compact central extensions of reductive Kazhdan groups are discussed in Chapter 3. Furthermore, in this chapter, we construct more groups supporting Conjecture 1.3, as $G := H_n \rtimes \mathrm{Sp}(n, \mathbb{R})$, where H_n is the $(2n+1)$–dimensional Heisenberg group with compact center, and related examples.

The proof of Theorem 1.2 is more or less a byproduct of the proof of Theorem 1.1. Let us discuss some appplications. Theorem 1.2 shows that all C_0-representations of the affine group of the line are square-integrable. Observing that the proof allows an immediate generalization to arbitrary local fields, we see that the same is true for the Fell group $\mathbb{Q}_p \rtimes \mathbb{Z}_p$. These two examples are well-known ([30],[41]). To give a new example, consider the following semidirect product: Let $\mathbb{R}^+$ act on the $(2n+1)$–dimensional Heisenberg group (identified with its Lie algebra)

$$\mathfrak{h}_n := \left\{ [(x,y),z] := \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \;\middle|\; x, y \in \mathbb{R}^n, z \in \mathbb{R} \right\}$$

via

$$t[(x,y),z]:=[(tx,ty),t^2z]\,.$$

The resulting group $G:=\mathfrak{h}_n\rtimes\mathbb{R}^+$ acts on $\mathfrak{h}_n^*$ via the coadjoint representation. Defining $(Z^*,X_1^*,\dots,X_n^*,Y_1^*,\dots,Y_n^*)$ to be the dual basis on $\mathfrak{h}_n^*$ w.r.t. the canonical basis, we see that the C_0–representations are induced by representations of $\mathfrak{h}_n$ corresponding to the linear forms $\pm Z^*$. Observe that $\mathbb{R}^+$ acts freely and without nontrivial fixed points on $\mathfrak{h}_n$. Thus Theorem (1.2) implies that on this group all C_0–representations are square-integrable.

The question when C_0-representations are square-integrable, has been widely discussed. One of the most important results is that the regular representation splits into irreducibles if every C_0–representation is square-integrable ([22], [3]), but this is only a necessary condition, at least in the nonunimodular case ([4],Ex.4). Baggett [2] showed by means of the Fell topology on the dual that the regular representation of a non-compact analytic unimodular group does not split.

From now on we will assume that G is a second countable and connected locally compact group and that all occuring Hilbert spaces are separable. A representation $(\pi,\mathcal{H}_\pi)$ is a strongly continuous unitary representation on a Hilbert space $\mathcal{H}_\pi$.

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2. Square-Integrability of Tensor Products on Linear Groups

a. The Proof of Theorem 1.1

The group $G=N\rtimes_\varphi H$ is by our assumptions a regular semidirect product and we may apply Mackey's theory to obtain the dual: Every $\varrho\in\hat{G}$ is of the form

$$\varrho\;=\;\mathrm{ind}_{NH_\chi}^G\chi'\otimes\sigma\;, \tag{1}$$

where

- $\chi\in\hat{N}$,
- $H_\chi:=\{h\in H|\,h\chi:=\chi\circ\varphi(h)^{-1}\simeq\chi\}$ is the fixed group of χ in H ,
- χ' and σ are multiplier representations in the following sense:

  There is a Borel function $\omega:H_\chi\times H_\chi\to\{z\in\mathbb{C}|\,|z|=1\}$, constant on the $N\times N$–cosets and satisfying

  $$\omega(e,x)=\omega(x,e)=1,\quad \omega(xy,z)\omega(x,y)=\omega(x,yz)\omega(y,z)$$

  (e denoting the group identity), such that χ' is an extension of χ to NH_χ with $\chi'(xy)=\overline{\omega(x,y)}\chi'(x)\chi'(y)$ and $\sigma(n)=1,\forall n\in N,\ \sigma(x,y)=\omega(x,y)\sigma(x)\sigma(y)$.

We denote the Mackey surjection $\hat{G}\to\hat{N}/G,\varrho\mapsto G.\chi$ by θ and recall that θ is continuous with respect to the Fell topology ([19],Lemma 3).

To fix notation, we shortly recall the Kirillov picture (for a complete discussion, see e.g., [10]): Let $\mathfrak{n}^*$ be the dual of the Lie algebra $\mathfrak{n}$ of N and denote by Ad^* the coadjoint action of N on $\mathfrak{n}^*$: $\mathrm{Ad}^*(n)f := f \circ \mathrm{Ad}(n)^{-1}$ for all $n \in N$, $f \in \mathfrak{n}^*$. There is a homeomorphism, the *Kirillov-map*,

$$\kappa : \mathfrak{n}^*/N \to \hat{N} \ ,$$

realized in the following way: Let $f \in \mathfrak{n}^*$ and $\mathfrak{p}$ be a *real polarization* of f . This means that $\mathfrak{p}$ is a maximal subalgebra which is *subordinate* to f , that is $f([\mathfrak{p},\mathfrak{p}]) = 0$. It should be noted that every subalgebra subordinate to f is contained in a real polarization. Now form the analytic subgroup $P := \exp \mathfrak{p}$ and obtain

$$\chi := \mathrm{ind}_P^N e^{if(\cdot)} \ . \tag{2}$$

Then χ depends only on the coadjoint orbit of f and (2) defines the Kirillov homeomorphism. Since the coadjoint orbits are closed, there exists a Borel section $F : \hat{N} \to \mathfrak{n}^*$ ([42],A.7).

Going back to our group $G = N \rtimes H$, we may identify $\mathfrak{n}$ with an ideal of the Lie algebra $\mathfrak{g} = \mathfrak{n} \rtimes \mathfrak{h}$. Hence G acts upon $\mathfrak{n}^*$ via

$$\mathrm{Ad}^*(n,h)f := f \circ \mathrm{Ad}(n,h)^{-1} \ .$$

Thus the fixed group H_χ consists of the elements $h \in H$ satisfying

$$\mathrm{Ad}^*(1,h)F(\chi) \in N \cdot F(\chi) \ .$$

Furthermore, since $\mathrm{Ad}^* g \mathrm{Ad}^* N f = \mathrm{Ad}^* N \mathrm{Ad}^* g f$, we have $\bar{\chi} \in G\chi \iff F(\bar{\chi}) \in GF(\chi)$. Now let $G(F(\chi)) := \{g \in G | \, \mathrm{Ad}^* g F(\chi) = F(\chi)\}$ be the stabilizer of $F(\chi)$ in G . This is an algebraic group and we may choose a Levi-complement H_2 of $G(F(\chi))$. As explained in ([27],p.87), we may conjugate $F(\chi)$ with a suitable $n \in N$ to obtain $H_2 = H_\chi$. Thus we find an $F' \in N \cdot F(\chi)$, which is stabilized by H_χ . In particular, $N \cdot H_\chi$ is algebraic.
Richardson's theorem ([32],p.132) together with the fact that the Zariski-topology is noetherian implies that there are closed subgroups $H_1, H_2, \ldots, H_s$ of H and a finite measurable invariant partition $\mathfrak{n}^* = D_1 \cup D_2 \cup \ldots \cup D_s$ defined by

$$F \in D_j \iff \{h \in H | hF \in N \cdot F\} \text{ is conjugate to } H_j, \quad 1 \le j \le s \ .$$

Since G acts smoothly upon $\hat{N}$, we find by the same argument as above a Borel section $L : \hat{N}/G \to \hat{N}$ and the composition map

$$R : \hat{G} \to \mathfrak{n}^*, \varrho \mapsto F \circ L \circ \theta(\varrho)$$

is Borel.

$$\begin{tikzcd} \hat{N}/G \arrow[r, "L"] & \hat{N} \arrow[d, "F"] \\ \hat{G} \arrow[u, "\theta"] \arrow[r, "R"'] & \mathfrak{n}^* \end{tikzcd}$$

Hence the mapping R induces a finite measurable partition of $\hat{G}$ in

$$\hat{G}_j := R^{-1}(D_j), \ 1 \le j \le s \ .$$

We will keep these notations all through this paper.

The proof of Theorem 1.1 runs roughly as follows:
Starting from a direct integral decomposition of a representation $(\pi, \mathcal{H}_\pi)$ as in the theorem, we obtain, by somewhat sophisticated calculations involving Mackey's tensor product theorem, the fact that π has no subrepresentations with non-compact kernel and requirement (ii), that

$$\pi^{\otimes k} \overset{\mathrm{q}}{\leq} \int_{\hat{G}^k}^{\oplus} \mathrm{ind}_N^G \int_{\mathfrak{n}^*}^{\oplus} \kappa(f) d\mu_{\varrho_1,\ldots,\varrho_k}(f) d\mu^{\otimes k}(\varrho_1,\ldots,\varrho_k) \; ,$$

where k is a natural number independent of π , μ is the spectral measure of π and $\mu_{\varrho_1,\ldots,\varrho_k}$ is the measure on the dual $\mathfrak{n}^*$ induced by the smooth mapping

$$G^k \to \mathfrak{n}^*, \; (g_1, g_2, \ldots, g_k) \mapsto \sum_{j=1}^{k} g_j R(\varrho_j) \; .$$

The reductiveness of the H–action, requirement (i) and again the fact that π has no subrepresentations with non-compact kernel yield that this mapping has full rank a.e. $\mu^{\otimes k}$, whence $\mu_{\varrho_1,\ldots,\varrho_k}$ is absolutely continuous to the Lebesgue-measure on $\mathfrak{n}^*$ for $\mu^{\otimes k}$– almost all $\varrho_1, \ldots, \varrho_k \in \hat{G}^k$. The theorem then follows from induction in stages.
Now let $(\pi, \mathcal{H}_\pi)$ be a representation such that all its subrepresentations have at most compact kernel. Since G is type I, π is quasi-equivalent to a multiplicity-free representation ([14],5.4.1). Hence we may assume that

$$\pi = \int_{\hat{G}}^{\oplus} \varrho d\mu(\varrho) \tag{3}$$

to be its decomposition in irreducibles. Thus $\pi = \bigoplus_{j=1}^{s} \pi_j$, where each π_j is supported on $\hat{G}_j, \; 1 \leq j \leq s$. If we find a $k \in \mathbb{N}$ such that $\pi_j^{\otimes k} \overset{\mathrm{q}}{\leq} \lambda_G$ for every $1 \leq j \leq s$ then $k_0 := ks$ will do the task for arbitrary representations as in the theorem (recall that the tensor product of a representation which is quasi-contained in the regular representation with any representation of G is quasi-contained in the regular representation, too). Thus we may restrict ourselves to the case, where μ is supported on a single $\hat{G}_{j_0}, \; 1 \leq j_0 \leq s$.

The first lemma repeats an argument of Howe and Moore [27] which we include for completeness:

Lemma 2.1. *There is an $r_0 \in \mathbb{N}, r_0 \leq \dim H_{j_0}$, such that there is a Zariski-open subset $\mathcal{D} \subseteq H_{j_0}^{r_0}$ with: For all $r \geq r_0$, $(h_1, h_2, \ldots, h_r) \in \mathcal{D} \times H^{r-r_0}$*

$$\left(h_1^{-1} H_{j_0} h_1 \cap h_2^{-1} H_{j_0} h_2 \cap \ldots \cap h_r^{-1} H_{j_0} h_r\right)_0 = \left(H_{j_0}^{\mathrm{nor}}\right)_0 \; .$$

Here, U_0 denotes the identity component of a subgroup U and $H_{j_0}^{\mathrm{nor}}$ the intersection of all conjugates of H_{j_0} , which is an algebraic normal subgroup.

Proof. Recall that the dimension function

$$H^r \ni (h_1, h_2, \ldots, h_r) \mapsto \dim(h_1^{-1} H_{j_0} h_1 \cap h_2^{-1} H_{j_0} h_2 \cap \ldots \cap h_r^{-1} H_{j_0} h_r)$$

is upper semicontinuous with respect to the Zariski-topology on H^r . Thus the minimum

$$d(r) := \min\{\dim(h_1^{-1} H_{j_0} h_1 \cap h_2^{-1} H_{j_0} h_2 \cap \ldots \cap h_r^{-1} H_{j_0} h_r) \,|\, h_1, h_2, \ldots, h_r \in H\}$$

is assumed on a Zariski-open subset $\mathcal{D}_r \subseteq H^r$. Clearly, after at most $\dim H_{j_0}$ steps, we have $d(r+1) = d(r)$. This implies that the identity component of $h_1^{-1} H_{j_0} h_1 \cap h_2^{-1} H_{j_0} h_2 \cap \ldots \cap h_r^{-1} H_{j_0} h_r$ is normal for all $h_1, h_2, \ldots, h_r \in \mathcal{D}_r$. Hence define $r_0 := r$, $\mathcal{D} := \mathcal{D}_r$. ∎

For the remainder of this section, recall the measurable mappings

$$\theta : \hat{G} \to \hat{N}/G, \quad L : \hat{N}/G \to \hat{N}, \quad R : \hat{G} \to \mathfrak{n}^*$$

from the beginning.

Lemma 2.2. *Keep the above notations. If $\tilde{S} \subseteq \hat{G}$ is μ-measurable and satisfies*

$$W := \operatorname{Span}(G \cdot R(\varrho) |\, \varrho \in \tilde{S}) \lneq \mathfrak{n}^*$$

then $\mu(\tilde{S}) = 0$.

Proof. Let $\tilde{S}$ be as in the lemma and define

$$M_W := \{x \in \mathfrak{n} |\, x \in \ker gR(\varrho) \ \forall g \in G, \varrho \in \tilde{S}\} .$$

By assumption, M_W is a closed non-compact normal subgroup of N and of G (after identifying N with $\mathfrak{n}$), subordinate to $gR(\varrho)$ for all $g \in G, \varrho \in \tilde{S}$. For each $\varrho \in \tilde{S}$, choose a real polarization $\mathfrak{p}$ containing M_W . Then M_W, being normal in N , is contained in the kernel of $L \circ \theta(\varrho) = \operatorname{ind}_{\mathfrak{p}}^N e^{iR(\varrho)}$. Since M_W is normal in G and since $\varrho = \operatorname{ind}_N^G ((L \circ \theta(\varrho))' \otimes \sigma(\varrho))$, the same argument shows that M_W is contained in $\ker \varrho$ for all $\varrho \in \tilde{S}$. Thus the kernel of $\int_{\tilde{S}}^{\oplus} \varrho d\mu$ were not compact if $\mu(\tilde{S}) \neq 0$. ∎

Corollary 2.3. *With the assumptions of Theorem* 1.1 , *we have for all $r \geq r_0$ and all $(h_1, h_2, \ldots, h_r) \in \mathcal{D} \times H^{r-r_0}$ (r_0 and $\mathcal{D}$ defined by* (2.1) *) that the subgroup*

$$h_1^{-1} H_{j_0} h_1 \cap h_2^{-1} H_{j_0} h_2 \cap \ldots \cap h_r^{-1} H_{j_0} h_r$$

is compact.

Proof. Since the groups under consideration are all algebraic, they have only finitely many connected components. By (2.1), it remains to show that $(H_{j_0}^{\mathrm{nor}})_0$ is compact. Indeed, every $h_0 \in (H_{j_0}^{\mathrm{nor}})_0$ satisfies

$$h_0 h R(\varrho) \in hNR(\varrho) = NhR(\varrho), \quad \forall h \in H,\ \varrho \in \operatorname{supp}\mu ,$$

in particular

$$h_0 \left(hR(\varrho)|_{\mathcal{Z}} \right) = hR(\varrho)|_{\mathcal{Z}}, \quad \forall h \in H ,$$

where $\mathcal{Z}$ denotes the center of $\mathfrak{n}$. By (2.2), the orbits $HNR(\varrho)$ span $\mathfrak{n}^*$. Thus every $f \in \mathcal{Z}^*$ is a linear combination of elements of $HNR(\varrho)|_{\mathcal{Z}}$ and $(H_{j_0}^{\mathrm{nor}})_0$ acts trivially on $\mathcal{Z}^*$, hence on $\mathcal{Z}$. By requirement (ii), $(H_{j_0}^{\mathrm{nor}})_0$ is compact. ∎

The aim of these lemmas is to apply Mackey's tensor product theorem.

Proposition 2.4. *With the above assumptions and notations, we have for* $k \geq r_0$:

$$\pi^{\otimes k} \overset{\mathrm{q}}{\leq} \int_{\hat{G}^k}^{\oplus} \mathrm{ind}_N^G \Big(\int_{\mathcal{D}\times H^{k-r_0}}^{\oplus} \bigotimes_{j=1}^{k} h_j \cdot (L\circ\theta)\,(\varrho_j)\, d\nu^{\otimes k}((h_1,\dots,h_k)) \Big) \times \times\, d\mu^{\otimes k}((\varrho_1,\dots,\varrho_k))\ ,$$

where ν^k *denotes a finite measure on* H^k *, equivalent to the Haar measure.*

Proof. Observe at first that the diagonal group $\Delta G \lhd G^k$ and $(NH_{j_0})^k$ are regularly related, since all these groups are algebraic. Thus, let $d\tilde{\nu}^{\otimes k}$ be the measure on the double coset space $\Delta G : (NH_{j_0})^k$ induced by a finite measure on G^k , equivalent to the Haar measure, and define for $g := (g_1,\dots,g_k) \in G^k$

$$G^{(g)} := g_1^{-1} NH_{j_0} g_1 \cap \ldots \cap g_k^{-1} NH_{j_0} g_k$$

and

$$\varrho^{(g)} : G^{(g)} \to \mathcal{U}\Big(\bigotimes_{j=1}^{k} \mathcal{H}_{(L\circ\theta(\varrho_j))'\otimes\sigma_j}\Big), \quad x \mapsto \bigotimes_{j=1}^{k} ((L\circ\theta(\varrho_j))'\otimes\sigma_j)(g_j x g_j^{-1})\ ,$$

where again $(L\circ\theta(\varrho_j))'$ denotes the extension of $(L\circ\theta)(\varrho_j)$ to NH_{j_0} and $\mathcal{U}(\mathcal{H}_\pi)$ denotes the group of unitary operators on $\mathcal{H}_\pi$. Then, by Mackey's tensor product theorem,

$$\bigotimes_{j=1}^{k} \varrho_j \;=\; \bigotimes_{j=1}^{k} \Big(\mathrm{ind}_{NH_{j_0}}^G (L\circ\theta(\varrho_j))'\otimes\sigma_j\Big) \;=\; \int_{\Delta G:(NH_{j_0})^k}^{\oplus} \mathrm{ind}_{G^{(g)}}^G \varrho^{(g)} d\tilde{\nu}^{\otimes k}\ .$$

But clearly $\Delta G : (NH_{j_0})^k$ is identified with $\Delta H : H_{j_0}^k$ and we may write

$$\bigotimes_{j=1}^{k} \Big(\mathrm{ind}_{NH_{j_0}}^G (L\circ\theta(\varrho_j))'\otimes\sigma_j\Big) \;=\; \int_{\Delta H:H_{j_0}^k}^{\oplus} \mathrm{ind}_{G^{(g)}}^G \varrho^{(g)} d\overline{\nu}^{\otimes k}\ ,$$

$\overline{\nu}$ being induced by a finite measure ν on H which is equivalent to the Haar measure. Now take $\mathcal{D} \subseteq H^{r_0}$ as in Lemma 2.1. Then $\mathcal{D}\times H^{k-r_0}$, being Zariski-open, is conull in H^k , hence we have certainly

$$\bigotimes_{j=1}^{k} \Big(\mathrm{ind}_{NH_{j_0}}^G (L\circ\theta(\varrho_j))'\otimes\sigma_j\Big) \;\leq\; \int_{\mathcal{D}\times H^{k-r_0}}^{\oplus} \mathrm{ind}_{G^{(g)}}^G \varrho^{(g)} d\nu^{\otimes k}\ .$$

Now, $G^{(h)}/N$ is compact for $h := (h_1,\dots,h_r) \in \mathcal{D}\times H^{k-r_0}$, by (2.3). Thus ([18],Lemma 4.2) implies $\mathrm{ind}_N^{G^{(h)}}(\varrho^{(h)}|_N) = \Big(\mathrm{ind}_N^{G^{(h)}} 1\Big)\otimes\varrho^{(h)} \geq \varrho^{(h)}$. On the other hand, $\varrho^{(h)}|_N \overset{\mathrm{q}}{\simeq} \bigotimes_{j=1}^k h_j(L\circ\theta)(\varrho_j)$. Putting all together and using induction in stages, we find (writing ϱ and h instead of $(vrh_1,\dots,\varrho_k)$ and $(h_1,\dots,h_k)$, respectively)

$$\begin{aligned}
\pi^{\otimes k} &= \int_{\hat{G}^k}^{\oplus} \bigotimes_{j=1}^{k} \varrho_j d\mu^{\otimes k}(\varrho) = \int_{\hat{G}^k}^{\oplus} \bigotimes_{j=1}^{k} \Big(\mathrm{ind}_{NH_{j_0}}^G (L\circ\theta(\varrho_j))'\otimes\sigma_j\Big)\, d\mu^{\otimes k}(\varrho) = \\
&\leq \int_{\hat{G}^k}^{\oplus} \int_{\mathcal{D}\times H^{k-r_0}}^{\oplus} \mathrm{ind}_N^G \Big(\varrho^{(h)}\,|_N\Big)\, d\nu^{\otimes k}(h) d\mu^{\otimes k}(\varrho) \\
&\overset{\mathrm{q}}{\simeq} \int_{\hat{G}^k}^{\oplus} \int_{\mathcal{D}\times H^{k-r_0}}^{\oplus} \mathrm{ind}_N^G \bigotimes_{j=1}^{k} h_j(L\circ\theta)(\varrho_j) d\nu^{\otimes k}(h)\, d\mu^{\otimes k}(\varrho) \\
&= \int_{\hat{G}^k}^{\oplus} \mathrm{ind}_N^G \Big(\int_{\mathcal{D}\times H^{k-r_0}}^{\oplus} \bigotimes_{j=1}^{k} h_j(L\circ\theta)(\varrho_j) d\nu^{\otimes k}(h) \Big) d\mu^{\otimes k}(\varrho).
\end{aligned}$$

This shows the formula. ∎

The strategy is now to prove that the inner integral

$$\int_{\mathcal{D}\times H^{k-r_0}}^{\oplus} \bigotimes_{j=1}^{k} h_j(L\circ\theta)(\varrho_j)\,d\nu^{\otimes k}(h_1,\ldots,h_k) \tag{4}$$

defines a square-integrable representation of N for large enough k. Induction in stages then yields the theorem. There are many results on the decomposition of tensor products on nilpotent Lie groups, e.g., in [10], [21], [5]. Here we need only the following qualitative version:

$$\bigotimes_{j=1}^{k} h_j(L\circ\theta)(\varrho_j) \overset{q}{\simeq} \int_{\mathfrak{n}^*}^{\oplus} \kappa(f)\,d\eta_1 * d\eta_2 * \ldots * d\eta_k\,,$$

where $\overset{q}{\simeq}$ denotes quasi-equivalence, κ the Kirillov map, $d\eta_j$ a finite quasi N–invariant measure on $\mathfrak{n}^*$, supported by the orbit of $h_jR(\varrho_j)$, and $*$ denotes the additive convolution on $\mathfrak{n}^*$. For each $1\le j\le k$, let $\bar{n_j}$ be the equivalence class of $n\in N$ in $\mathrm{Ad}^*(N)h_jR(\varrho_j)\simeq N/\mathfrak{s}(j)$, where $\mathfrak{s}(j)$ is the stabilizer of $h_jR(\varrho_j)$ and let $d\tilde{\eta}_j$ be a quasi N–invariant finite measure on $N/\mathfrak{s}(j)$. By uniqueness, $d\eta_j$ is quasi-equivalent to the measure induced by $d\tilde{\eta}_j$ and the canonical mapping $N/\mathfrak{s}(j)\ni\bar{n_j}\mapsto n_jh_jR(\varrho_j)$. Whence we may write the above intergral more explicitly as

$$\begin{aligned}\bigotimes_{j=1}^{k} h_j(L\circ\theta)(\varrho_j) \quad \overset{q}{\simeq} \quad & \int_{N/\mathfrak{s}(1)}^{\oplus}\int_{N/\mathfrak{s}(2)}^{\oplus}\cdots\int_{N/\mathfrak{s}(k)}^{\oplus} \kappa\left(\sum_{j=1}^{k} n_jh_js_jR(\varrho_j)\right)\times\\ & \times d\tilde{\eta}_1(\bar{n_1})\,d\tilde{\eta}_2(\bar{n_2})\ldots d\tilde{\eta}_k(\bar{n_k})\,.\end{aligned}$$

Now, let $d\sigma_j$ denote a finite measure on $\mathfrak{s}(j)$, equivalent to the Haar measure. We certainly don't change the quasi-equivalence class if we substitute

$$\begin{aligned}&\bigotimes_{j=1}^{k} h_j(L\circ\theta)(\varrho_j) \overset{q}{\simeq}\\ \overset{q}{\simeq}\quad & \int_{N/\mathfrak{s}(1)}^{\oplus}\int_{\mathfrak{s}(1)}^{\oplus}\cdots\int_{N/\mathfrak{s}(k)}^{\oplus}\int_{\mathfrak{s}(k)}^{\oplus} \kappa\left(\sum_{j=1}^{k} n_js_jh_jR(\varrho_j)\right)\times\\ & \times d\sigma_1(s_1)\,d\tilde{\eta}_1(\bar{n_1})\ldots d\sigma_k(s_k)\,d\tilde{\eta}_k(\bar{n_k})\,.\end{aligned}$$

All together, choosing finite measures dg, dn on G and N, equivalent to the respective Haar measures, and using Weil's formula, we find

$$\begin{aligned}&\int_{\mathcal{D}\times H^{k-r_0}}^{\oplus} \bigotimes_{j=1}^{k} h_j(L\circ\theta)(\varrho_j)\,d\nu^{\otimes k}(h) \overset{q}{\simeq}\\ \overset{q}{\simeq}\quad & \int_{\mathcal{D}\times H^{k-r_0}}^{\oplus}\int_{N^k}^{\oplus} \kappa\left(\sum_{j=1}^{k} n_jh_jR(\varrho_j)\right) dn^{\otimes k}(n)\,d\nu^{\otimes k}(h)\\ =\quad & \int_{N^k\times\mathcal{D}\times H^{k-r_0}}^{\oplus} \kappa\left(\sum_{j=1}^{k} g_jR(\varrho_j)\right) dg^{\otimes k}(g)\,.\end{aligned}$$

Now define for fixed $\varrho_1,\varrho_2,\ldots,\varrho_k$:

$$\psi_{\varrho_1,\ldots,\varrho_k}: G^k\to\mathfrak{n}^*,\quad (g_1,g_2,\ldots,g_k)\mapsto\sum_{j=1}^{k} g_jR(\varrho_j)\,. \tag{5}$$

The integral in Eq. (5) is just the integral over $\mathfrak{n}^*$ with respect to the measure $\mu_{\varrho_1,\ldots,\varrho_k}$ induced by $\psi_{\varrho_1,\ldots,\varrho_k}$ and a finite measure on G^k, equivalent to the Haar measure. Since the Plancherel measure of N corresponds to the Lebesgue measure $\lambda_{\mathfrak{n}^*}$ on $\mathfrak{n}^*$, it remains to prove the absolute continuity

$$\mu_{\varrho_1,\ldots,\varrho_k} \ll \lambda_{\mathfrak{n}^*} . \tag{6}$$

This will follow from an elementary argument from local differential geometry.

Lemma 2.5. *If the differential $d\psi_{\varrho_1,\ldots,\varrho_k}$ has full rank at one point, the induced measure $\mu_{\varrho_1,\ldots,\varrho_k}$ is absolutely continuous with respect to the Lebesgue measure.*

Proof. By assumption, there is a point in G^k, where $\psi_{\varrho_1,\ldots,\varrho_k}$ has maximal rank. Since this mapping is analytic, the points $g \in G^k$ satisfying $\operatorname{rank} d\psi_{\varrho_1,\ldots,\varrho_k}(g) \lneq \dim \mathfrak{n}^*$ form a submanifold of lower dimension, thus a nullset. The remainder of the proof is an immediate application of the implicit function theorem. ∎

In order to show that the condition of (2.5) is fullfilled for $\mu^{\otimes k}$–almost all $(\varrho_1, \varrho_2, \ldots, \varrho_n)$ and large enough k, we turn now to the more geometrical aspects of our special situation.

Let G be a linear Lie group, acting on a finite dimensional real vector space V and let $\mathfrak{g}$ be its Lie algebra. The derived representation of $\mathfrak{g}$ is defined by

$$Xy := \lim_{t\to 0} \frac{\exp(tX)y - y}{t}, \quad \forall y \in V, X \in \mathfrak{g} .$$

In particular, the tangent space of the orbit $G \cdot y$ at the point y is $\mathfrak{g} \cdot y$.

If $A \subseteq V$, we denote by $\operatorname{Aff}(A)$ the smallest affine subspace containing A. The following lemma is probably well-known. We include the proof for sake of completeness.

Lemma 2.6. *Keep the above notations. Let $x \in V$. Then $\operatorname{Aff}(Gx) = x + L_x$, where L_x is the vector space*

$$L_x := \operatorname{Span}(\{\mathfrak{g}gx, g \in G\}) .$$

Proof. Define $\mathcal{L} := \operatorname{Span}(g_2x - g_1x \,|\, g_1, g_2 \in G)$. Since $\mathcal{L}$ is closed, $\mathcal{L} \supset L_x$. To show the other direction, choose at first $g_1, g_2 \in G$, such that $g_2g_1^{-1}$ is contained in a Campbell–Hausdorff neighbourhood of the identity. Thus there are $t_0 \geq 0$ and $X \in \mathfrak{g}$ with $g_2 = \exp(t_0X)g_1$. Hence

$$\begin{aligned} g_2x - g_1x &= (\exp(t_0X) - 1)g_1x \\ &= \int_0^{t_0} \exp(sX)Xg_1x\,ds = \int_0^{t_0} X \underbrace{\exp(sX)g_1}_{=:g(s)} x\,ds \end{aligned}$$

Now, computing the integral as Riemannian sum yields

$$g_2x - g_1x = \lim_{N\to\infty} \frac{t_0}{N} \sum_{j=0}^{N-1} Xg\left(\frac{(j+1)}{N}t_0\right)x$$

Thus $g_2x - g_1x \in L_x$.

As to the general case, join g_1 and g_2 with a compact arc γ and cover γ with finitely many translates $U_1, U_2, \ldots, U_{n+1}$ of Campbell–Hausdorff neighbourhoods. Then pick

$$\tilde{g_1} := g_1,\ \tilde{g_j} \in U_{j-1} \cap U_j \cap \gamma\,(2 \leq j \leq n),\ \widetilde{g_{n+1}} := g_2\ .$$

By the first part, this yields

$$g_2 x - g_1 x \ = \ \sum_{j=1}^{n} \underbrace{(\widetilde{g_{j+1}} - \tilde{g_j})x}_{\in L_x} \in L_x\ . \qquad \blacksquare$$

Lemma 2.7. *Keep the above notations and let* $G = N \rtimes H$ *be as in Theorem* 1.1 , *acting upon* $\mathfrak{n}^*$ *via the coadjoint action.*
Then we have for all $f \in \mathfrak{n}^*$

$$\mathrm{Aff}(Gf) \ = \ L_f \ = \ \mathrm{Span}(Gf)\ .$$

Proof. We have $\mathrm{Aff}(Gf) = f_1 + L_f$.
Then $\mathrm{Span}(Gf) = \mathrm{Span}(f, L_f)$. Clearly, $L_f, f + L_f$ and $\mathrm{Span}(f, L_f)$ are H–invariant. Since H acts reductively, there is an at most one-dimensional H–invariant complement of L_f in $\mathrm{Span}(f, L_f)$, say $\mathbb{R} \cdot y$. We have to show that $y = 0$. Otherwise y is not fixed under H and, by connectedness, Hy is a ray through y. If necessary, we change the sign of y to obtain $Hy \cap (f + L_f) \neq \emptyset$. By invariance, $f + L_f$ contains the whole ray, thus it contains 0 and $f \in L_f$, as desired. $\blacksquare$

The following corollary finishes the proof of Theorem 1.1.

Corollary 2.8. *Keep the above notations. There is a* $k_0 \in \mathbb{N}$, $k_0 \leq \dim N$, *such that for all* $k \geq k_0$ *and for* $\mu^{\otimes k}$*–almost all* $(\varrho_1, \varrho_2, \ldots, \varrho_k) \in \hat{G}^k$ *there is a* $(g_1, g_2, \ldots, g_k) \in G^k$ *such that*

$$\mathrm{Span}(\mathfrak{g} g_1 R(\varrho_1), \ldots, \mathfrak{g} g_k R(\varrho_k)) \ = \ \mathfrak{n}^*\ .$$

Proof. Let k be the smallest natural number with:

(∗) There exists a measurable set $S \in \hat{G}^k$ with $\mu^{\otimes k}(S) > 0$ and for all $(\varrho_1, \ldots, \varrho_k) \in S,\ (g_1, \ldots, g_k) \in G^k$:

$$\dim\left(\mathrm{Span}(\mathfrak{g} g_1 R(\varrho_1), \ldots, \mathfrak{g} g_k R(\varrho_k))\right) \ = \ k - 1\ .$$

By (2.2) and (2.7), we have $k > 1$. Using Fubini's theorem we find

$$\mu^{\otimes k}(S) \ = \ \int_{\hat{G}} \mu^{\otimes (k-1)}(\{(\varrho_1, \ldots, \varrho_{k-1}) |\, (\varrho_1, \ldots \varrho_{k-1}, \varrho) \in S\}) d\mu(\varrho)\ .$$

Since $\mu^{\otimes k}(S) > 0$, there exists $(\varrho_1, \ldots, \varrho_{k-1}) \in \hat{G}^{k-1}$ such that

$$(\varrho_1, \ldots, \varrho_{k-1}, \varrho) \in S$$

for all ϱ in a subset $\bar{S} \subseteq \hat{G}$ of positive measure. In particular, for all $\varrho \in \bar{S}$ and for all $(g_1, \ldots, g) \in G^k$,

$$\mathfrak{g} g R(\varrho) \ \subseteq \ \mathrm{Span}(\mathfrak{g} g_1 R(\varrho_1), \ldots, \mathfrak{g} g_{k-1} R(\varrho_{k-1}))\ ,$$

whence, by (2.7),

$$L_{R(\varrho)} = \mathrm{Span}(G \cdot R(\varrho)) \subseteq \mathrm{Span}(\mathfrak{g}g_1R(\varrho_1), \ldots, \mathfrak{g}g_{k-1}R(\varrho_{k-1})) ,$$

Thus fix $(g_1, \ldots, g_{k-1}) \in G^{k-1}$ and observe that

$$W := \mathrm{Span}(\mathfrak{g}g_1R(\varrho_1), \ldots, \mathfrak{g}g_{k-1}R(\varrho_{k-1}))$$

is $(k-1)$–dimensional. But the above and (2.7) imply

$$\mathrm{Span}(\{G \cdot R(\varrho) | \varrho \in \bar{S}\}) \subseteq W .$$

But $\mu(\bar{S}) > 0$, whence by (2.2), $k > n+1$, as desired. ■

b. The Proof of Theorem 1.2

Let

$$\pi = \int_{\hat{G}}^{\oplus} m(\varrho)\varrho d\mu$$

be as in (1.2). By the compactness assumption and ([18],Lemma 4.2), we have for almost all ϱ :

$$\varrho = \mathrm{ind}_{G_{\tilde{\chi}}}^{G}\chi \leq \mathrm{ind}_N^G(\chi|_N) \simeq \mathrm{ind}_N^G g(\chi|_N)$$

for all $g \in G$. Thus, if dg denotes a finite measure on G , equivalent to the Haar measure,

$$\varrho \overset{\mathrm{q}}{\leq} \mathrm{ind}_N^G \int_G^{\oplus} g\chi|_N dg .$$

Since $G\chi|_N$ produces an open orbit in $\mathfrak{n}^*$, we see with (2.5)

$$\int_G^{\oplus} g\chi|_N dg \overset{\mathrm{q}}{\leq} \lambda_N .$$

Thus induction in stages shows $\varrho \overset{\mathrm{q}}{\leq} \lambda_G$, whence the theorem.

Examples 2.9. (i) Canonical examples for an application of Theorems 1.1 and 1.2 are the generalized affine group, where $N = \mathbb{R}^n$ and H is the group of diagonal matrices, or the action of $\mathbb{R}^+$ on the $(2n+1)$–dimensional Heisenberg group mentioned in the introduction.

(ii) Consider the space $\mathbb{R}^3$ as row vectors and the three-dimensional skew symmetric matrices Σ . Then $\mathfrak{n} := \mathbb{R}^3 \times \Sigma$ is a two step nilpotent algebra with the bracket

$$[(u, U), (v, V)] := (0, u^{\mathrm{t}}v - v^{\mathrm{t}}u) .$$

Take $H := \mathrm{SO}(3, \mathbb{R})$ and define

$$h(u, U) := (uh^{\mathrm{t}}, hUh^{\mathrm{t}}) .$$

Then $G := \mathfrak{n} \rtimes H$ satisfies the assumptions of 1.1.

(iii) Let G be a linear totally minimal group with abelian nilradical. By [35],2.5 $G = V \rtimes H$ for a vector group V . Then G has a finite extension $\tilde{G}$ satisfying

$$\tilde{G} = V \rtimes (K \times S_1 \ldots \times S_n) ,$$

where K is compact and S_j is non-compact simple with finite center for all $1 \leq j \leq n$. Let Σ_1 be the product of all the S_j , acting trivially on V and Σ_2 be the product of the remaining factors. Then

$$\tilde{G} = (V \rtimes \Sigma_2) \times \Sigma_1 \ .$$

Since each S_i is a finite extension of a linear algebraic group, Theorem 1.1 is applicable to the first factor and the second factor is handled below. This will yield $\overline{\mathrm{A}_r(G)} = \mathrm{B}_0(G)$.

(iv) The proofs of the Theorems 1.1 and 1.2 carry over without essential changes for connected algebraic groups over arbitrary local fields.

(v) The fact that each C_0–representation has a square-integrable tensor power is related to Haagerup's property (H), as discussed in [6]. A locally compact group is said to have property (H) if there is a C_0-representation which weakly contains the trivial representation. Since taking tensor products is continuous [20], the groups occuring in (1.1) have property (H) if and only if they are amenable.

c. Linear Reductive Groups with Compact Center

For these groups the solutions of the problems mentioned in the introduction are easily obtained by reformulation of the results of M. Cowling [11, 12, 13] and C.C. Moore [38]. Here, we need

Theorem 2.10. *Let G be a simple analytic group with finite center. Then*

(i) *For every $\pi \in \hat{G}$, there is $q = q(\pi) > 0$ and $C > 0$, such that all matrix coefficients $\varphi_{\xi\eta}$ of π are in $\mathrm{L}^{2q}(G)$ and satisfy*

$$\|\varphi_{\xi\eta}\|_{2q} \leq C\|\xi\|\|\eta\| \, .$$

Furthermore C is independent of π .

(ii) *If every matrix coefficient of a unitary representation $(\pi, \mathcal{H}_\pi)$ is in $\mathrm{L}^p(G)$, $(p \geq 1)$, the same is true for every representation weakly contained in π . In particular, the set*

$$\hat{G}_q := \{\pi \in \hat{G} | \text{ all matrix coefficients of } \pi \text{ are in } \mathrm{L}^{2q}(G)\}$$

is closed in $\hat{G}$. (This is true for an arbitrary locally compact group.)

(iii) $\mathrm{L}^2(G) * \mathrm{L}^2(G) = \mathrm{A}(G) \subseteq \mathrm{L}^{2+\varepsilon}(G)$ *for all $\varepsilon > 0$. This is called the Kunze-Stein phenomenon.*

The first two results carry over without difficulties to linear reductive groups with compact center. In order to prove (iii) one uses Herz' majorisation principle ([11],7.2). The question on the universality of q as in (i) is solved by results of Cowling and Moore:

Theorem 2.11. *A simple group with finite center has Kazhdan's property (T) if and only if there is a q such that the matrix coefficients of all nontrivial irreducible representations are in $\mathrm{L}^{2q}(G)$.*

This implies immediately

Corollary 2.12. *Let G be a linear reductive group with compact center. Then* $\mathrm{B}_0(G) = \overline{\mathrm{A}_r(G)}$, *and* $\mathrm{B}_0(G) = \mathrm{A}_r(G)$ *if and only if* G *is a Kazhdan group.*

G is Kazhdan if and only if there is $k \in \mathbb{N}$ such that $\pi^{\otimes k} \overset{\mathrm{q}}{\leq} \lambda_G$ for all C_0-representations $(\pi, \mathcal{H}_\pi)$.

Proof. There is a finite central extension $\tilde{G}$ of G satisfying

$$\tilde{G} = K \times S_1 \times \ldots \times S_n ,$$

where K is compact and $S_1, \ldots, S_n$ are non-compact simple groups with finite center. Clearly, it is enough to look at $\tilde{G}$. The dual of $\tilde{G}$ identifies topologically with $\hat{K} \times \widehat{S_1} \times \ldots \times \widehat{S_n}$, since any irreducible representation is an outer tensor product

$$\pi = \varrho \otimes \sigma_1 \otimes \ldots \otimes \sigma_n, \quad \varrho \in \hat{K}, \sigma_i \in \hat{S}_i, 1 \leq i \leq n .$$

Now let ψ be a C_0-representation of $\tilde{G}$ which is (without loss of generality) multiplicity free, and

$$\psi = \int_{\widehat{\tilde{G}}}^{\oplus} \pi_s d\mu(s)$$

its decomposition in irreducibles. Clearly, almost all π_s are nontrivial on every S_j, $1 \leq j \leq n$. By (2.10.ii), the set of these representations is the union of the closed sets $\bigcup_{q \geq 2} \widehat{\tilde{G}}_q$. By regularity of μ , one has for all $\xi, \eta \in \mathcal{H}_\psi$:

$$\langle \pi(\cdot)\xi, \eta \rangle = \mathrm{B}(G) - \lim_{q \to \infty} \int_{\widehat{\tilde{G}}_q} \langle \pi(\cdot)\xi_s, \eta_s \rangle d\mu(s).$$

Now if $\psi_q := \int_{\widehat{\tilde{G}}_q}^{\oplus} \pi d\mu$, all matrix coefficients of ψ_q are in $\mathrm{L}^{2q}(\tilde{G})$ and, by Hölder's inequality, $\psi_q^{\otimes q}$ has a dense set of square-integrable coefficients, hence is square-integrable. This shows one direction and the other follows from ([38],3.6). The second statement follows from the same arguments. ■

This proof is a very special case of the one given in [13].

3. Some Further Groups Satisfying $\mathrm{A}_r(G) = \mathrm{B}_0(G)$

In this section we discuss some groups not covered by Theorem 1.1. We start by continuing Example 2.9.i:

sl(2,R) $G := \mathfrak{h}_n \rtimes (\mathbb{R}^+ \times \mathrm{Sp}(n, \mathbb{R}))$:
Here $\mathfrak{h}_n \rtimes \mathbb{R}^+$ is the semidirect product of the Heisenberg group by $\mathbb{R}^+$ as in (2.9.i). The symplectic group $\mathrm{Sp}(n, \mathbb{R})$ acts upon $\mathfrak{h}_n$ via

$$A[(x, y), z] := [(x, y)A^{\mathrm{t}}, z] .$$

This action commutes with that of $\mathbb{R}^+$.
Recall the *metaplectic representations* $(\omega_r, \mathrm{L}^2(\mathbb{R}^n))$, $r \neq 0$, of the metaplectic group $\mathrm{Mp}(n, \mathbb{R})$, a two-fold covering of $\mathrm{Sp}(n, \mathbb{R})$ with covering homomorphism $p : \mathrm{Mp}(n, \mathbb{R}) \to \mathrm{Sp}(n, \mathbb{R})$. ω_r is defined in the following way: Let π_r be an infinite dimensional irreducible representation of $\mathfrak{h}_n$ whose restriction to the center $\mathcal{Z}$ is a multiple of $\chi_r : z \mapsto e^{irz}$. Since this requirement

determines π_r up to unitary equivalence, every $m \in \mathrm{Mp}(n,\mathbb{R})$ defines a unitary operator $\omega_r(m)$ satisfying:

$$\pi_r(p(m)n) = \omega_r(m)\pi_r(n)\omega_r(m)^{-1}, \quad \forall n \in \mathfrak{h}_n \ .$$

The mapping $m \mapsto \omega_r(m)$ is indeed a unitary representation of $\mathrm{Mp}(n,\mathbb{R})$ and thus uniquely induces a τ_r–representation $\tilde{\omega}_r$ on $\mathrm{Sp}(n,\mathbb{R})$, where τ_r is a multiplier of order 2, such that

$$\pi'_r(n,A) := \pi_r(n)\tilde{\omega}_r(A), \quad n \in \mathfrak{h}_n, A \in \mathrm{Sp}(n,\mathbb{R})\ , \tag{7}$$

extends π_r to an irreducible τ_r–representation of $\mathfrak{h}_n \rtimes \mathrm{Sp}(n,\mathbb{R})$. Furthermore we may take $\omega_r = \omega_s$ if $rs > 0$. Keeping these notations, we cite a result of Howe and Moore ([27],6.4):

Proposition 3.1. *There is a dense set D of vectors in $\mathrm{L}^2(\mathbb{R}^n)$, such that, for all $r \neq 0$, the absolute values of the matrix coefficients of π'_r to vectors in D belong to $\mathrm{L}^{4n+2+\varepsilon}((\mathfrak{h}_n \rtimes \mathrm{Sp}(n,\mathbb{R}))/\mathcal{Z})$ for all $\varepsilon > 0$, $\mathcal{Z}$ denoting the center of $\mathfrak{h}_n$.*

Now we return to our group $G = \mathfrak{h}_n \rtimes (\mathbb{R}^+ \times \mathrm{Sp}(n,\mathbb{R}))$. The G–orbits in $\mathfrak{h}_n^*$ are:

$$\{Z^* > 0\}, \{Z^* < 0\}, \{Z^* = 0\} \setminus \{0\}, \{0\}\ ,$$

where $(Z^*, X_1^*, \ldots X_n^*, Y_1^*, \ldots, Y_n^*)$ denotes the dual base.
Thus C_0–representations are supported on

$$\hat{G}_\pm := \{\mathrm{ind}_{h_n \rtimes \mathrm{Sp}(n,\mathbb{R})}^G \pi'_{\pm 1} \otimes \sigma |\ \sigma \text{ irreducible } \bar{\tau}_{\pm 1}\text{–representation of } \mathrm{Sp}(n,\mathbb{R})\}.$$

So, let $\pi := \int_{\hat{G}_+}^{\oplus} \mathrm{ind}_{h_n \rtimes \mathrm{Sp}(n,\mathbb{R})}^G \pi'_1 \otimes \sigma d\mu$ (we may assume again that π is multiplicity free). Then, for $k \in \mathbb{N}$, the same computation as in (2.4) yields

$$\pi^{\otimes k} \overset{\mathrm{q}}{\leq} \int_{\hat{G}_+^k}^{\oplus} \int_{\mathbb{R}^{+k}}^{\oplus} \mathrm{ind}_{h_n \rtimes \mathrm{Sp}(n,\mathbb{R})}^G \bigotimes_{i=1}^k \pi_1'^{t_i} \otimes \sigma_i^{t_i}\ dt^{\otimes k} d\mu^{\otimes k}\ ,$$

where dt is a finite measure on $\mathbb{R}^+$, equivalent to the Haar measure, and

$$\pi_1'^t \otimes \sigma^t(n,A) := \pi'_1(t \cdot n, A)\sigma(A) == \pi'_{t^2}(n,A)\sigma(A)\ .$$

If k is even, $\bigotimes_{i=1}^k \pi_1'^{t_i}$ and $\bigotimes_{i=1}^k \sigma_i$ are ordinary representations. Furthermore, by (3.1), the absolute value of the matrix coefficients of $\bigotimes_{i=1}^k \pi_1'^{t_i}$ associated to a dense set of vectors are square-integrable modulo the center $\mathcal{Z}$ for $k \geq 2n+2$, hence

$$\bigotimes_{i=1}^k \pi_1'^{t_i} \leq \mathrm{ind}_{\mathcal{Z}}^{h_n \rtimes \mathrm{Sp}(n,\mathbb{R})} \chi_{\sum_{i=1}^k t_i^2}\ .$$

This yields, by (2.5),

$$\begin{aligned}
\pi^{\otimes k} &\overset{\mathrm{q}}{\leq} \int_{\hat{G}_+^k}^{\oplus} \mathrm{ind}_{h_n \rtimes \mathrm{Sp}(n,\mathbb{R})}^G \left(\left(\mathrm{ind}_{\mathcal{Z}}^{h_n \rtimes \mathrm{Sp}(n,\mathbb{R})} \int_{\mathbb{R}^{+k}}^{\oplus} \chi_{\sum_{i=1}^k t_i^2} dt^{\otimes k} \right) \bigotimes_{i=1}^k \sigma_i \right) d\mu^{\otimes k} \\
&\overset{\mathrm{q}}{\leq} \int_{\hat{G}_+^k}^{\oplus} \mathrm{ind}_{h_n \rtimes \mathrm{Sp}(n,\mathbb{R})}^G \left(\left(\mathrm{ind}_{\mathcal{Z}}^{h_n \rtimes \mathrm{Sp}(n,\mathbb{R})} \lambda_{\mathcal{Z}} \right) \bigotimes_{i=1}^k \sigma_i \right)\ d\mu^{\otimes k} \\
&\overset{\mathrm{q}}{\leq} \int_{\hat{G}_+^k}^{\oplus} \mathrm{ind}_{h_n \rtimes \mathrm{Sp}(n,\mathbb{R})}^G \lambda_{h_n \rtimes \mathrm{Sp}(n,\mathbb{R})} d\mu^{\otimes k} \overset{\mathrm{q}}{\leq} \lambda_G,
\end{aligned}$$

since for every locally compact group U and for every representation ϱ of U : $\lambda_U \otimes \varrho \overset{\mathrm{q}}{\simeq} \lambda_U$. Decomposing an arbitrary C_0–representation $\pi = \pi_+ \oplus \pi_-$, where $\pi_\pm$ is supported on $\hat{G}_\pm$, we see:

Corollary 3.2. *Let π be a C_0-representation of $G = \mathfrak{h}_n \rtimes(\mathbb{R}^+ \times \mathrm{Sp}(n,\mathbb{R}))$, then for all $k > 4n+4$: $\pi^{\otimes k} \overset{q}{\leq} \lambda_G$.*

sl(2,R)Upper Triangular Matrices in $\mathrm{SL}(3,\mathbb{R})$:
Let

$$G := \left\{ \begin{pmatrix} \frac{1}{\alpha\beta} & x & z \\ 0 & \alpha & y \\ 0 & 0 & \beta \end{pmatrix} \middle| \, x, y, z \in \mathbb{R}, \ \alpha, \beta > 0 \right\} .$$

The representation theory of this group is a classical example due to Mackey, see ([32],III.B.Ex.6). Clearly, G is a semidirect product of the three-dimensional Heisenberg group $\mathfrak{h}_1$ with $\mathbb{R}^{+2}$, where the action is given by

$$(\alpha, \beta) \cdot (x, y, z) := (\frac{x}{\alpha^2\beta}, \frac{\alpha}{\beta} y, \frac{z}{\alpha\beta^2}) .$$

The C_0-representations are supported by

$$\mathrm{ind}_{h_1 \rtimes D'}^{G} \pi'_{\pm 1} \otimes \chi_\lambda ,$$

where $D' := \{(s^{-2}, s) | \, s \in \mathbb{R}^+\} \simeq \mathbb{R}^+$ is the fixed group of $\pi_{\pm 1}$, the irreducible representation of $\mathfrak{h}_1$ which, restricted to the center, is quasi-equivalent to $z \mapsto e^{\pm iz}$, $\pi'_{\pm 1}$ is an extension to $\mathfrak{h}_1 \rtimes D'$ and χ_λ denotes a character of $\mathbb{R}^+$. Here one uses the fact that $\mathbb{R}^+$ has only trivial cocycle representations. Observe that D' acts upon $\mathfrak{h}_1$ via $(s^{-2}, s)(x, y, z) = (s^3 x, \frac{1}{s^3} y, z)$. This defines an embedding $D' \hookrightarrow \mathrm{Sp}(1,\mathbb{R})$. By the uniqueness of the extension, the representation π'_+ of $\mathfrak{h}_1 \rtimes D'$ is the restriction of the cocycle representation $\pi_+ \tilde{\omega}_1$ of $\mathfrak{h}_1 \rtimes \mathrm{Sp}(1,\mathbb{R})$ to $\mathfrak{h}_1 \rtimes D'$, where $\tilde{\omega}_1$ denotes the metaplectic representation. By (3.1), there is a dense set of vectors in $\mathcal{H}_{\pi_+ \tilde{\omega}_1}$ producing coefficients of $\pi_+ \tilde{\omega}_1$ in $\mathrm{L}^k(\mathfrak{h}_1 \rtimes \mathrm{Sp}(n,\mathbb{R})/\mathcal{Z})$ for sufficiently large k $(k > 6)$. Denoting the Haar measures on $G_1 := (\mathfrak{h}_1 \rtimes \mathrm{Sp}(1,\mathbb{R}))/\mathcal{Z}$, $L_1 := (\mathfrak{h}_1 \rtimes D')/\mathcal{Z}$ and G_1/L_1 with dg, dh and $d\bar{g}$, respectively, we have for ξ, η contained in the above mentioned dense set:

$$\infty > \int_{G_1} |\varphi_{\xi\eta}|^k(g) dg = \int_{G_1/L_1} \Big(\int_{L_1} |\varphi_{\xi\eta}(gh)|^k dh \Big) d\bar{g} .$$

Thus the inner integral $\int_{L_1} |\varphi_{\xi\eta}(hg)|^k dh$ is finite for almost all $g \in G$ yielding enough k-integrable vectors. Whence π_+ and a fortiori $\pi_+ \otimes \chi_\lambda$ has a dense subset of L^k-vectors for all $\lambda \in \mathbb{R}$. Now, the same arguments used in the above example show:

Corollary 3.3. *Let π be a C_0-representation of G . Then for all $k \geq 8$:*

$$\pi^{\otimes k} \overset{q}{\leq} \lambda_G .$$

We now turn to nonlinear groups:

sl(2,R) $G := H_n \rtimes \mathrm{Sp}(n,\mathbb{R})$:
Here H_n denotes the $(2n+1)$-dimensional Heisenberg group with compact center, that is the image of an infinite dimensional irreducible representation of the simply connected group $\mathfrak{h}_n$. We parametrize H_n by $[(p,q),t]$, $p, q \in \mathbb{R}^n, t \in \mathbb{T}$. Then the action of $\mathrm{Sp}(n,\mathbb{R})$ on $\mathfrak{h}_n$ factors to an action on H_n . The center of the associated semidirect product $G := H_n \rtimes \mathrm{Sp}(n,\mathbb{R})$ is identified with the torus $\mathbb{T}$ and $G/\mathbb{T} \simeq \mathbb{R}^{2n} \rtimes \mathrm{Sp}(n,\mathbb{R})$ is a linear totally minimal group. The dual of G is determined by Mackey's theory and decomposes in essentially three measurable parts: $\hat{G}_{\mathrm{Sp}(n,\mathbb{R})} := \{\pi \in \hat{G} | \ker \pi \supset H_n\}, \hat{G}_{\mathbb{T}} := \{\pi \in \hat{G} | \ker \pi \supset \mathbb{T}\} \setminus \hat{G}_{\mathrm{Sp}(n,\mathbb{R})}$ and $\hat{G}_{\mathrm{faithful}} := \hat{G} \setminus (\hat{G}_{\mathrm{Sp}(n,\mathbb{R})} \cup \hat{G}_{\mathbb{T}})$. Since the center is compact here, the application of (3.1) is more easy than above:

Corollary 3.4. *Let π be a C_0-representation of $G = H_n \rtimes \mathrm{Sp}(n, \mathbb{R})$, then for sufficiently large k, $\pi^{\otimes k} \stackrel{q}{\leq} \lambda_G$.*

Proof. We write π as direct integral

$$\pi = \int_{\hat{G}_{\mathbb{T}}}^{\oplus} + \int_{\hat{G}_{\text{faithful}}}^{\oplus} \varrho d\mu(\varrho) =: \pi_{\mathbb{T}} \oplus \pi_{\text{faithful}} .$$

The first summand satisfies $\pi_{\mathbb{T}}^{\otimes m} \stackrel{q}{\leq} \lambda_G$ for large enough m, by Theorem 1.1, and the second $\pi_{\text{faithful}}^{\otimes 2n+2} \stackrel{q}{\leq} \lambda_G$, by (3.1). Thus, for k large enough, $\pi^{\otimes k} \stackrel{q}{\leq} \sum_{\ell=0}^{k} \pi_{\mathbb{T}}^{\otimes k-\ell} \otimes \pi_{\text{faithful}}^{\otimes \ell} \stackrel{q}{\leq} \lambda_G$. ■

sl(2,R)Compact Central Extensions of Linear Reductive Groups :
At first we consider an example. Let G be a simple analytic group with infinite center Z, e.g., the universal covering group of $\mathrm{Sp}(n, \mathbb{R})$. Then Z is a discrete, finitely generated abelian group, algebraically $Z \simeq \mathbb{Z}^n \times \mathrm{Tor}$, where Tor denotes the torsion part. Let $c_1, c_2, \ldots, c_n$ be generators of the free part of Z and $t \in \mathbb{T}$ be of infinite order. Then

$$L := \{((c_1^{k_1}, t^{k_1}), (c_2^{k_2}, t^{k_2}), \ldots (c_n^{k_n}, t^{k_n})), k_i \in \mathbb{Z}\}$$

is a discrete central subgroup of $G \times \mathbb{T}^n$. Define the canonical projection

$$p : G \times \mathbb{T}^n \to \overline{G} := (G \times \mathbb{T}^n)/L .$$

By the discreteness of L, we have $\mathcal{Z}(\overline{G}) = p(\mathcal{Z}(G \times \mathbb{T}^n)) = p(\mathrm{Tor} \times \mathbb{T}^n)$ ([26],III.3.2). Thus the center of $\overline{G}$ is compact and $\overline{G}/\mathcal{Z}(\overline{G}) \simeq G/Z \simeq \mathrm{Ad}G$ is a linear simple group. Hence $\overline{G}$ is totally minimal itself ([35],2.3). Observe that $\overline{G}$ contains a dense subgroup isomorphic to G, whence it is not linear. Using the results of Cowling and Moore presented in the last section, we can show the following

Theorem 3.5. *Let G be an analytic group with compact center Z, such that G/Z is a linear reductive Kazhdan group with compact center. Then* $\mathrm{B}_0(G) = \overline{\mathrm{A}_r(G)}$.

Proof. Let π be a C_0-representation of G. By the compactness of Z, π splits as $\pi = \sum_{\gamma \in \hat{Z}} \pi_\gamma$, where $\pi_\gamma|_Z \stackrel{q}{\leq} \gamma$ for all $\gamma \in \hat{Z}$. Thus it remains to consider π_γ. Then $\pi_\gamma \otimes \overline{\pi_\gamma}$ is a C_0-representation of the Kazhdan group G/Z, whence by (2.12), $(\pi_\gamma \otimes \overline{\pi_\gamma})^{\otimes k_0} \in \lambda_{G/Z}$ for a finite $k_0 \in \mathbb{N}$. By the Kunze-Stein phenomenon (2.10.iii), all coefficients of $\pi_\gamma \otimes \overline{\pi_\gamma}$ belong to $\mathrm{L}^{2k}(G/Z) \subseteq \mathrm{L}^{2k}(G)$ for all $k > k_0$. Thus, for all $\xi, \eta \in \mathcal{H}_\pi$: $|\varphi_{\xi\eta}|^{2k} \in \mathrm{L}^2(G)$. By Hölder's inequality, $\pi_\gamma^{\otimes 2k}$ is square-integrable, showing the theorem. ■

I suppose the result to be true in general.

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