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Description of infinite dimensional abelian regular Lie groups

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Abstract. It is shown that every abelian regular Lie group is a quotient of its Lie algebra via the exponential mapping.
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This paper is a sequel of [3] (see also [4], chapter VIII), where a regular Lie group is defined as a smooth Lie group modeled on convenient vector spaces such that the right logarithmic derivative has a smooth inverse $\mathrm{Evol} : C^\infty(\mathbb{R},\mathfrak{g}) \to C^\infty(\mathbb{R},G)$, the canonical evolution operator, where $\mathfrak{g}$ is the Lie algebra. We follow the notation and the concepts of this paper closely.

Lemma. *Let G be an abelian regular Lie group with Lie algebra $\mathfrak{g}$. Then the evolution operator is given by* $\mathrm{Evol}(X)(t) := \mathrm{Evol}^r(X)(t) = \exp\big(\int_0^t X(s)ds\big)$ *for* $X \in C^\infty(\mathbb{R},\mathfrak{g})$.

Proof. Since G is regular it has an exponential mapping $\exp : \mathfrak{g} \to G$ which is a smooth group homomorphism, because $s \mapsto \exp(sX)\exp(sY)$ is a smooth one-parameter group in G with generator $X+Y$, thus $\exp(X)\exp(Y) = \exp(X+Y)$ by uniqueness, [3], 3.6 or [4], 36.7. The Lie algebra $\mathfrak{g}$ is a convenient vector space with evolution mapping $\mathrm{Evol}_{\mathfrak{g}}(X)(t) = \int_0^t X(s)ds$, see [3], 5.4, or [4], 38.5. The mapping $\exp : \mathfrak{g} \to G$ is a homomorphism of Lie groups and thus intertwines the evolution operators by [3], 5.3 or [4], 38.4, hence the formula.

Another proof is by differentiating the right hand side, using [3], 5.10 or [4], 38.2. ∎

As a consequence we obtain that an abelian Lie group G is regular if and only if an exponential map exists. Furthermore, an exponential map is surjective

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on a connected abelian Lie group, because $\exp(\int_0^t \delta^r c(s)ds) = \mathrm{Evol}(\delta^r c)(t) = c(t)$ for any smooth curve $c: \mathbb{R} \to G$ with $c(0) = e$.

Theorem. *Let G be an abelian, connected and regular Lie group, then there is a c^∞-open neighborhood V of zero in $\mathfrak{g}$ so that $\exp(V)$ is open in G and $\exp : V \to \exp(V)$ is a diffeomorphism. Moreover, $\mathfrak{g}/\ker(\exp) \to G$ is an isomorphism of Lie groups.*

Proof. Given a connected, abelian and regular Lie group G, we look at the universal covering group $\tilde{G} \xrightarrow{\pi} G$, see [4], 27.14, which is also abelian and regular. Any tangent Lie algebra homomorphism from a simply connected Lie group to a regular Lie group can be uniquely integrated to a Lie group homomorphism by [5] or [3], 7.3 or [4], 40.3. Consequently, there exists a homomorphism $\Phi : \tilde{G} \to \mathfrak{g}$ with $\Phi' = id_{\mathfrak{g}}$. Since $\tilde{G}$ is regular there is a map from $\mathfrak{g}$ to $\tilde{G}$ extending id, which has to be the inverse of Φ and which is a fortiori the exponential map $\widetilde{\exp}$ of $\tilde{G}$, so Φ is an isomorphism of Lie groups. The universal covering projection π intertwines $\widetilde{\exp}$ and $\exp$, so the result follows. The quotient $\mathfrak{g}/\ker(\exp)$ is a Lie group since there are natural chart maps and the quotient space is a Hausdorff space by the Hausdorff property on G. ∎

Remarks. Given a convenient vector space E and a subgroup Z, it is not obvious how to determine simple conditions to ensure that E/Z is a Hausdorff space, because $c^\infty E$ is not a topological vector space in general (see [4], Chapter I): An additive subgroup Z of E is called "discrete" if there is a c^∞-open zero neighborhood V with $V \cap (Z + V) = \{0\}$ and for any $x \notin Z$ there is a c^∞-open zero neighborhood U so that $(x + Z + U) \cap (Z + U) = \emptyset$. The above kernel of exp naturally has this property, consequently any regular connected abelian Lie group is a convenient vector space modulo a "discrete" subgroup.

Let E be a Fréchet space, then a subgroup is "discrete" if and only if there is an open zero neighborhood V with $V \cap (Z + V) = \{0\}$, because $c^\infty E = E$. This leads immediately to a generalization of a result of Galanis ([2]), who proved that every abelian Fréchet-Lie group which admits an exponential map being a local diffeomorphism around zero is a projective limit of Banach Lie groups. With the above theorem one can easily write down this limit in general.

With the above methods it is necessary to assume regularity: Otherwise one obtains as image of Φ a dense arcwise connected subgroup of the convenient vector space $\mathfrak{g}$, which does not allow any conclusion in contradiction to the finite dimensional case. Note that the closed subgroup of integer-valued functions in $L^2([0,1], \mathbb{R})$ is arcwise connected but not a Lie subgroup (see [1]) so that Yamabe's theorem is already wrong on the level of infinite dimensional Hilbert spaces.

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