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Maximal degenerate representations of SL(n+1,H)

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Abstract. The infinitesimal method is applied to study some maximal degenerate representations of $\mathrm{SL}(n+1,\mathbb{H})$. The irreducibility and unitarizability of these representations are determined and the composition series is given in the reducible case. The existence of intertwining operators is established for the differentiated Lie algebra representations. On the group level, the intertwining operators are given by an explicit integral formula.

Introduction

The intimate connection between maximal degenerate representations of $\mathrm{SL}(n+1,\mathbb{F})$ and canonical representations of $\mathrm{SU}(1,n;\mathbb{F})$ associated with a finite dimensional irreducible representation of $\mathrm{U}(1,\mathbb{F})$ has been recognized in [4], [3] and [5] respectively for $\mathbb{F}=\mathbb{R},\mathbb{C}$ and $\mathbb{H}$. The structure of these representations of $\mathrm{SL}(n+1,\mathbb{F})$ is therefore a key ingredient for the understanding of the corresponding canonical representations. The cases of $\mathrm{SL}(n+1,\mathbb{R})$ and $\mathrm{SL}(n+1,\mathbb{C})$ have been respectively treated in [4] and [7]. This paper deals with the case of $\mathrm{SL}(n+1,\mathbb{H})$.

Precisely, to each finite dimensional irreducible representation of $\mathrm{Sp}(1)$ we associate two families of degenerate representations of $\mathrm{SL}(n+1,\mathbb{H})$ induced by opposite maximal parabolic subgroups. We study the irreducibility and unitarizability of these representations, and, when reducible, we determine their composition series. Moreover, we determine the existence of intertwining operators among them. The explicit integral formula for these intertwining operators allows us to compute the so called η-function of [12] in a very direct manner.

Our analysis is based on the infinitesimal method, which easily produces very explicit results. After its first application by Bargmann for $\mathrm{SL}(2,\mathbb{R})$, this method has been successfully applied for the study of some degenerate principal series representations of several other classical groups. Among these we mention the following: $\mathrm{SO}_0(p,q)$, see [14], [15] and [8]; $\mathrm{SU}(p,q)$, see [8], and [9] for $p=q$; $\mathrm{Sp}(p,q)$, see [8] and [9] for $p=q$; $\mathrm{Sp}(2n,\mathbb{R})$, see [10] and [13]; $\mathrm{SL}(n+1,\mathbb{R})$, see [4]; $\mathrm{SL}(n+1,\mathbb{C})$, see [7].

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Our results are somehow related to those determined for $\mathrm{SL}(4(n+1),\mathbb{R})$, but for us the procedure to obtain them is more complicated because the restriction of our representations to the maximally compact subgroup $\mathrm{Sp}(n+1)$ of $\mathrm{SL}(n+1,\mathbb{H})$ is not multiplicity free. This forces us to modify the usual application of the infinitesimal method as employed in the case of either $\mathrm{SL}(n+1,\mathbb{R})$ or $\mathrm{SL}(n+1,\mathbb{C})$.

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1. Preliminaries

Let $\mathbb{H}$ denote the skew field of the quaternions, with units $1, i, j, k$ ($i^2 = j^2 = k^2 = -1$). If $q = a + ib + jc + kd \in \mathbb{H}$ (with $a, b, c, d \in \mathbb{R}$), then the quaternionic conjugate of q is $\bar{q} = a - ib - jc - kd$. The real and the imaginary parts of q are respectively $\Re q = a$ and $\Im q = ib + jc + kd$. If $q = \Im q$, then q is said to be purely quaternionic.

Let $\mathfrak{g} = \mathfrak{sl}(n+1,\mathbb{H})$ denote the real Lie algebra of the $(n+1)\times(n+1)$ matrices over $\mathbb{H}$ with purely quaternionic trace. The simply connected real Lie group $\mathrm{SL}(n+1,\mathbb{H})$ associated with $\mathfrak{sl}(n+1,\mathbb{H})$ will be also denoted by G. Using Dieudonné notion of determinant for square matrices with entries from $\mathbb{H}$ (described for example in [1], pp. 151–158), one can realize $\mathrm{SL}(n+1,\mathbb{H})$ as the group of the $(n+1)\times(n+1)$ matrices over $\mathbb{H}$ with determinant equal to 1. It turns out that $\mathrm{SL}(n+1,\mathbb{H})$ can also be identified with the group $SU^*(2(n+1))$ consisting of the $(2n+2)\times(2n+2)$ matrices over $\mathbb{C}$ with determinant 1 which are of the form

$$\begin{bmatrix} Z_1 & -\bar{Z}_2 \\ Z_2 & \bar{Z}_1 \end{bmatrix}$$

for $(n+1)\times(n+1)$ complex matrices Z_1 and Z_2.

Endow the right $\mathbb{H}$-vector space $\mathbb{H}^{n+1}$ with the inner product

$$(x, y) = \bar{y}_0 x_0 + \bar{y}_1 x_1 + \ldots + \bar{y}_n x_n.$$

The subgroup $U = \mathrm{Sp}(n+1)$ of G consisting of all matrices preserving $(\cdot,\cdot)$ is simply connected and maximally compact in G. Its Lie algebra $\mathfrak{u} = \mathfrak{sp}(n+1)$ consists of all the skew-Hermitian $(n+1)\times(n+1)$ matrices over $\mathbb{H}$: $X + \bar{X}^t = 0$ for $X \in \mathfrak{u}$, the symbol ^{t} denoting transposition. If $\mathfrak{p}$ indicates the vector space of the Hermitian elements of $\mathfrak{g}$, then $\mathfrak{g} = \mathfrak{u} + \mathfrak{p}$ is the Cartan decomposition of $\mathfrak{g}$ associated with the Cartan involution $\theta : X \mapsto -\bar{X}^t$. The corresponding involution on G is also denoted by θ (so $\theta(g) = (\bar{g}^t)^{-1}$).

Consider the maximal parabolic subgroups P^+ and $P^- = \theta(P^+)$ of G, respectively consisting of the matrices in G of the form

$$\begin{bmatrix} a & b \\ 0 & C \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a & 0 \\ b & C \end{bmatrix}$$

with $a \in \mathbb{H}\setminus\{0\}$ and C an $n\times n$ matrix.

For $l \in \mathbb{N}/2$, let τ_l denote the $(2l+1)$-dimensional irreducible representation of $\mathrm{Sp}(1) \cong \mathrm{SU}(2)$, realized as a unitary representation on the space $(V_l, (\cdot,\cdot)_l)$. The elements of $\mathrm{Sp}(1)$ will be often considered as quaternions λ with norm $(\bar\lambda\lambda)^{1/2} = 1$.

For $\mu \in \mathbb{C}$ define representations $\omega^\pm_{l,\mu}$ of $P^\pm$ by

$$\omega^\pm_{l,\mu}(p) = |a|^{\mu\pm\rho}\tau_l(a/|a|), \qquad p = \begin{bmatrix} a & \cdot \\ \cdot & \cdot \end{bmatrix} \in P^\pm, \tag{1}$$

where $\rho := 2(n+1)$.

Let $\pi^\pm_{l,\mu}$ denote the representation of G induced from $P^\pm$:

$$\pi^\pm_{l,\mu} = \mathrm{Ind}^G_{P^\pm}\omega^\pm_{l,\mu}. \tag{2}$$

The natural action of G on $\mathbb{H}^{n+1}_* := \mathbb{H}^{n+1} \setminus \{0\}$ is transitive. The subgroup P_1^+ of G stabilizing the point $e_0 = [1,0,\dots,0]^t$ consists of all elements of P^+ of the form $\begin{bmatrix} 1 & b \\ 0 & C \end{bmatrix}$. On it, $\omega^\pm_{l,\mu}$ is trivial. Let $\mathcal{D}_\mu(\mathbb{H}^{n+1}_*, \tau_l)$ be the space of compactly supported smooth V_l-valued functions on $\mathbb{H}^{n+1}_*$ satisfying

$$F(xa) = |a|^{-(\mu+\rho)}\tau_l(a/|a|)^{-1}F(x), \qquad x \in \mathbb{H}^{n+1}_*,\ a \in \mathbb{H} \setminus \{0\}. \tag{3}$$

A noncompact realization of $\pi^+_{l,\mu}$ can then be given on $\mathcal{D}_\mu(\mathbb{H}^{n+1}_*, \tau_l)$ by

$$[\pi^+_{l,\mu}(g)F](x) = F(g^{-1}x), \qquad g \in G,\ x \in \mathbb{H}^{n+1}_*. \tag{4}$$

(gx denoting multiplication of the matrix $g \in G$ by $x \in \mathbb{H}^{n+1}_*$ considered as a column vector).

Let $S := \{s \in \mathbb{H}^{n+1} : \|s\| = 1\}$ be the unit sphere in $\mathbb{H}^{n+1}$, where $\|x\| := (x,x)^{\frac{1}{2}}$ for $x \in \mathbb{H}^{n+1}$. G acts on S by

$$g \cdot s = \frac{gs}{\|gs\|}, \qquad g \in G,\ s \in S. \tag{5}$$

A compact realization of $\pi^+_{l,\mu}$ on the space $\mathcal{D}(S,\tau_l)$ of all smooth functions $\varphi : S \to V_l$ satisfying $\varphi(s\lambda) = \tau_l(\lambda)^{-1}\varphi(s)$ for all $s \in S$ and $\lambda \in \mathrm{Sp}(1)$ is given by

$$[\pi^+_{l,\mu}(g)\varphi](s) = \varphi(g^{-1}\cdot s)\|g^{-1}s\|^{-(\mu+\rho)}, \qquad g \in G,\ s \in S. \tag{6}$$

The realizations of $\pi^-_{l,\mu}$ are obtained by considering the transitive action of G on $\mathbb{H}^{n+1}_*$ given by $g : x \mapsto \theta(g)(x)$, for which the stabilizer of e_0 is $P_1^- = \theta(P_1^+)$. The noncompact and compact realizations of $\pi^-_{l,\mu}$ are again on the spaces $\mathcal{D}_{-\mu}(\mathbb{H}^{n+1}_*, \tau_l)$ and $\mathcal{D}(S,\tau_l)$, respectively. They are

$$[\pi^-_{l,\mu}(g)F](x) = F(\theta(g^{-1})x), \qquad g \in G,\ F \in \mathcal{D}_{-\mu}(\mathbb{H}^{n+1}_*, \tau_l),\ x \in \mathbb{H}^{n+1}_* \tag{7}$$

$$[\pi^-_{l,\mu}(g)\varphi](s) = \varphi(\theta(g^{-1})\cdot s)\|\theta(g^{-1})s\|^{\mu-\rho}, \qquad g \in G,\ \varphi \in \mathcal{D}(S,\tau_l),\ s \in S. \tag{8}$$

It follows, in particular, that $\pi^-_{l,\mu} = \pi^+_{-\mu,l} \circ \theta$.

In the remainder of the paper we will only work with the compact realizations. We endow every space of C^∞ functions with the Schwartz topology. The $\pi^\pm_{l,\mu}$ are therefore admissible differentiable representations of G. We refer to [2] Chapter III §7 for a discussion on the Schwartz topology and for the relationship between the notion of induced representation of this section and other notions commonly found in the literature.

2. Reduction of the problem

The usual application of the infinitesimal method (cf. f.i. [14] or [4]) cannot be directly employed for the representations $\pi^{\pm}_{l,\mu}$. In fact their restriction to the maximally compact subgroup $U = \mathrm{Sp}(n+1)$ is not multiplicity free. The problem is overcome by a trick similar to the one used in [8] for the study of the degenerate principal series representations of $\mathrm{Sp}(p,q)$. The main idea is to use the commutativity of the G-action (5) on S with the right multiplication by elements of $\mathrm{Sp}(1)$ in order to replace the representations $\pi^{\pm}_{l,\mu}$ of G with suitable "extended" representations of $G\times\mathrm{Sp}(1)$ having multiplicity free decomposition over $U\times\mathrm{Sp}(1)$.

We first substitute the representation $\pi^{\pm}_{l,\mu}$ of G with the representation $\pi^{\pm}_{l,\mu}\otimes\tau_l$ of $\tilde{G} := G\times\mathrm{Sp}(1)$. Since τ_l is irreducible and unitary, the irreducibility and unitarizability of (a submodule of) $\pi^{\pm}_{l,\mu}$ and of (the corresponding submodule in) $\pi^{\pm}_{l,\mu}\otimes\tau_l$ are equivalent.

The representation τ_l is self-dual (i.e. equivalent to its contragredient representation). Using [17] p. 99, it is possible to show the existence of an orthogonal basis $\{\mathbf{e}_j\}_{j=0}^{2l}$ for V_l with dual basis $\{\boldsymbol{\varepsilon}_j\}_{j=0}^{2l}$ for the dual space V_l^* so that the linear operator $A_l : V_l \to V_l^*$ of matrix $[A_{l;i,j}]_{i,j=0}^{2l}$ with

$$A_{l;i,j} = (-1)^i \delta_{j,2l-i} \tag{9}$$

intertwines τ_l and its contragredient representation. Here $\delta_{j,k}$ is the Kronecker delta: $\delta_{jk} = 1$ if $j = k$ and $= 0$ otherwise.

The bilinear form

$$\langle v, w\rangle_l := (A_l v)(w) \tag{10}$$

is nondegenerate, symmetric if $2l$ is even, skew-symmetric if $2l$ is odd, and it satisfies

$$\langle \tau_l(k)v, \tau_l(k)w\rangle_l = \langle v, w\rangle_l, \qquad k \in k,\ v, w \in V_l.$$

Let $\mathcal{D}(S)$ denote the space of C^∞ functions on S endowed with the C^∞-topology. Define $\boldsymbol{\beta}_l : \mathcal{D}(S,\tau_l)\otimes V_l \to \mathcal{D}(S)$ by

$$\boldsymbol{\beta}_l(\varphi\otimes v)(s) = \langle \varphi(s), v\rangle_l, \qquad \varphi \in \mathcal{D}(S,\tau_l), v \in V_l, s \in S. \tag{11}$$

Let $T^{\pm}_{l,\mu}$ be the representations of $\tilde{G}$ on $\mathcal{D}(S)$ given by

$$\begin{aligned} [T^{+}_{l,\mu}(g,\lambda)f](s) &= f(g^{-1}\cdot s\lambda)\|g^{-1}s\|^{-(\mu+\rho)} \\ [T^{-}_{l,\mu}(g,\lambda)f](s) &= f(\theta(g^{-1})\cdot s\lambda)\|\theta(g^{-1})s\|^{\mu-\rho} \end{aligned}$$

for $g \in G, \lambda \in \mathrm{Sp}(1), f \in \mathcal{D}(S), s \in S$. Then $\boldsymbol{\beta}_l$ is a continuous linear isomorphism of $\mathcal{D}(S,\tau_l)\otimes V_l$ onto its image $\mathcal{D}_l(S) := \boldsymbol{\beta}_l(\mathcal{D}(S,\tau_l)\otimes V_l)$ with the induced topology from $\mathcal{D}(S)$, and it intertwines the representations $\pi^{\pm}_{l,\mu}\otimes\tau_l$ and $T^{\pm}_{l,\mu}$ of $\tilde{G}$.

The study of the representation $\pi^{\pm}_{l,\mu}$ of G can be therefore equivalently replaced by the study of the representation $T^{\pm}_{l,\mu}$ of $\tilde{G}$.

3. Irreducibility of $\pi^{\pm}_{l,\mu}$

Let $\tilde{U} := \mathrm{Sp}(n+1)\times\mathrm{Sp}(1)$, which is maximally compact in $\tilde{G}$, and let $\mathcal{D}_l(S)_{\tilde{U}}$ denote the space of $\tilde{U}$-finite vectors in $\mathcal{D}_l(S)$. The representation $T^{\pm}_{l,\mu}$ induces a representation of the universal enveloping algebra $\mathfrak{U}(\tilde{\mathfrak{g}})$ of the complexification of the Lie algebra $\tilde{\mathfrak{g}}$ of $\tilde{G}$ on $\mathcal{D}_l(S)_{\tilde{U}}$. We also denote this representation by $T^{\pm}_{l,\mu}$. It is a well known property that the irreducibility of $\mathcal{D}_l(S)_{\tilde{U}}$ as a $\mathfrak{U}(\tilde{\mathfrak{g}})$-module is equivalent to the irreducibility of $\mathcal{D}_l(S)$ as a $\tilde{G}$-module. We can therefore recover the properties of the $\tilde{G}$-module out of those of the $\mathfrak{U}(\tilde{\mathfrak{g}})$-module.

Observe that for $(k,\lambda)\in\tilde{U}$ and $f\in\mathcal{D}_l(S)_{\tilde{U}}$

$$[T^{\pm}_{l,\mu}(k,\lambda)f](s) = f(k^{-1}s\lambda), \qquad s\in S. \tag{12}$$

$T^{\pm}_{l,\mu}$ is therefore the restriction to $\mathcal{D}_l(S)_{\tilde{U}}$ of the representation of $\tilde{U}$ on $\mathcal{D}(S)$ by the same formula (12). The $\tilde{U}$-type decomposition of this representation on $\mathcal{D}(S)$ has been studied by Johnson and Wallach [11] (see also [16]).

Let $\tilde{M}$ denote the group of all matrices in $\tilde{U}$ of the form

$$\begin{bmatrix} u & 0 & 0\\ 0 & V & 0\\ 0 & 0 & u\end{bmatrix}$$

with $u\in \mathrm{Sp}(1)$ and $V\in\mathrm{Sp}(n)$. Theorem 3.1(3) in [11] proves that the $\tilde{U}$-finite vectors in $\mathcal{D}(S)$ decompose according to

$$\mathcal{D}(S)_{\tilde{U}} = \sum\{V^{p,q} : p, q \text{ integers}, p\geq q\geq 0, p\equiv q \pmod 2\}\,. \tag{13}$$

Here $V^{p,q}$ is the $\tilde{U}$-cyclic subspace with $\tilde{M}$-fixed vector $e_{p,q}$ (unique up to a constant multiple) described as follows. Let $x_0, x_1, \dots, x_n$ be the usual coordinates in $\mathbb{H}^{n+1}$. Set

$$r^2 = \sum_{j=0}^{n} |x_j|^2\,, \qquad r\cos\xi = |x_0|\,, \qquad r\cos\xi\cos t = \Re x_0 \tag{14}$$

with $0\leq\xi\leq\frac{\pi}{2}$ and $0\leq t\leq\pi$. $e_{p,q}$ is the function on S (depending only on $|x_0|$ and on $\Re x_0$, i.e. only on ξ and t) given by

$$e_{p,q}(\xi,t) = \Xi_{q/2}(t)h_{p,q/2}(\xi) \tag{15}$$

where

$$\Xi_{q/2}(t) = \frac{\sin(q+1)t}{\sin t} \tag{16}$$

and

$$h_{p,q/2}(\xi) = \cos^p\xi\; F\left(\frac{-p+q}{2}, \frac{-p-q-2}{2}; 2n; -\tan^2\xi\right) \tag{17}$$

($F(a,b;c;z)$ is Gauss' hypergeometric function). If the point of coordinates (p,q) denotes $V^{p,q}$, then $\mathcal{D}(S)_{\tilde{U}}$ can be depicted as in Fig.1a.

Let $\chi_l(\lambda) := \operatorname{trace}\tau_l(\lambda)$ be the character of τ_l. Then $\Xi_l(t) = \chi_l(\lambda)$ if $\Re\lambda = \cos t$. The explicit expressions for $\boldsymbol{\beta}_{q/2}$ and $\langle\cdot,\cdot\rangle_{q/2}$ give

$$e_{p,q}(s) = \boldsymbol{\beta}_{q/2}\left(\sum_{j=0}^{q} F_{p,q;j}\otimes \mathbf{e}_j\right)(s) \tag{18}$$

where $\{\mathbf{e}_j\}_{j=0}^q$ is the fixed orthogonal basis for $V_{q/2}$ and

$$F_{p,q;j}(s) := (-1)^j\tau_{\frac{q}{2}}(s_0/|s_0|)^{-1}(\mathbf{e}_{2l-j})h_{p,q}(\mathrm{Arccos}\,|s_0|). \tag{19}$$

We obtain the following lemma.

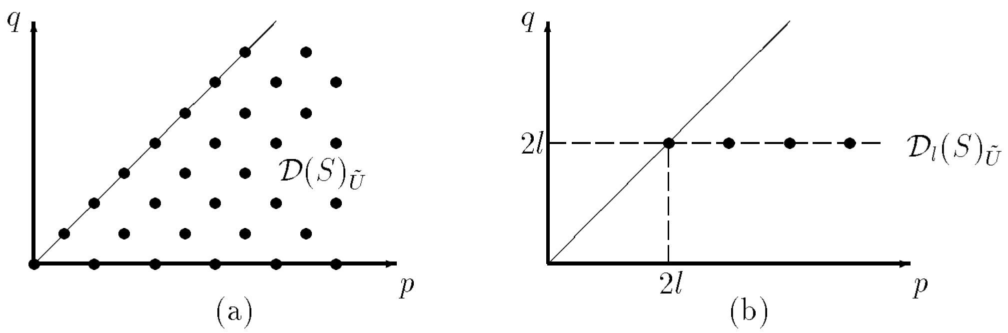

Figure 1: $\tilde{U}$-type decomposition for (a) $\mathcal{D}(s)$ and (b) $\mathcal{D}_l(S)$

Lemma 3.1. *$e_{p,q} \in \mathcal{D}_l(S)_{\tilde{U}}$ if and only if $q = 2l$.*

We therefore have the $\tilde{U}$-type decomposition

$$\mathcal{D}_l(S)_{\tilde{U}} = \sum \left\{ V^{p,2l} : p \,\text{integer} \geq 2l, p \equiv 2l \,(\text{mod } 2) \right\}, \tag{20}$$

which is represented in Fig.1b.

Let

$$H := \text{diag}\left(\frac{n}{n+1}, -\frac{1}{n+1}, \ldots, -\frac{1}{n+1} \right). \tag{21}$$

The group $\tilde{G}$ is generated by $\tilde{U}$ and by $\{\exp(tH) : t \in \mathbb{R}\} \equiv \{\exp(tH) : t \in \mathbb{R}\} \times \{1\}$. The action of $\mathfrak{U}(\tilde{\mathfrak{g}})$ on the $\tilde{U}$-types is therefore completely determined by the action of the differential operators $T^{\pm}_{l,\mu}(H) = \pm T^{+}_{l,\pm\mu}(H)$. In particular, it is enough to know the image of $e_{p,2l}$ under $T^{+}_{l,\mu}(H)$.

Let $\tau \in \mathbb{R}$. For the generic element $s \in S$, of coordinates ξ and t, let $\xi(\tau)$ and $t(\tau)$ respectively denote the values of the coordinates ξ and t of the element

$$s(\tau) := \frac{\exp(-\tau H)s}{\|\exp(-\tau H)s\|} \in S. \tag{22}$$

Then

$$\cos\xi(\tau) = \frac{e^{-\frac{n\tau}{n+1}} \cos\xi}{\left[e^{\frac{2\tau}{n+1}} \sin^2\xi + e^{-\frac{2n\tau}{n+1}} \cos^2\xi \right]^{1/2}} \tag{23}$$

$$\cos t(\tau) = \cos t. \tag{24}$$

On functions of (ξ, t), we have

$$T^{+}_{l,\mu}(H) = (\rho + \mu)\left(\frac{-1}{1+n} + \cos^2\xi \right) + \sin\xi\cos\xi \frac{\partial}{\partial\xi}. \tag{25}$$

The Gauss' relation (cf. [6] 2.8(33))

$$(\gamma - \alpha - \beta)F(\alpha, \beta; \gamma; z) + \alpha(1-z)F(\alpha+1, \beta; \gamma; z)$$
$$-(\gamma - \beta)F(\alpha, \beta - 1; \gamma; z) = 0$$

and the differentiation property (cf. [6] 2.8(21))

$$\frac{d}{dz}[z^{\alpha} F(\alpha, \beta; \gamma; z)] = \alpha z^{\alpha-1} F(\alpha+1, \beta; \gamma; z)$$

prove the following lemma.

Lemma 3.2.

(a) $\frac{b}{c} z F(a+1, b+1; c+1; z) = F(a+1, b; c; z) - F(a, b; c; z).$

(b) $$F(a+1, b; c; z) = -\frac{b}{c-a-b-1}(1-z) F(a+1, b+1; c; z) + \frac{c-a-1}{c-a-b-1} F(a, b; c; z).$$

(c) $$\begin{aligned} \frac{1}{1-z} F(a, b; c; z) &= \\ &= \frac{ab}{(c-a-b)(c-a-b+1)}(1-z) F(a+1, b+1; c; z) \\ &\quad - \frac{a(c-a)+(c-b-1)(b-1)}{(c-a-b-1)(c-a-b+1)} F(a, b; c; z) \\ &\quad + \frac{(c-b)(c-a)}{(c-a-b)(c-a-b+1)} \frac{1}{1-z} F(a-1, b-1; c; z). \end{aligned}$$

A long but straightforward computation gives

$$T^{+}_{l,\mu}(H) e_{p,2l} = \alpha_{+}(p, 2l, \mu) e_{p+2,2l} + \alpha_{0}(p, 2l, \mu) e_{p,2l} + \alpha_{-}(p, 2l, \mu) e_{p-2,2l} \tag{26}$$

where

$$\alpha_{+}(p, 2l, \mu) = \frac{(4n+p-2l)(4n+p+2l+2)}{4(2n+p+1)(2n+p)} \beta_{+}(p, \mu) \tag{27}$$

$$\alpha_{0}(p, 2l, \mu) = \frac{p(n-1)(4n+p+2)-4l(l+1)(n+1)}{2(n+1)(2n+p)(2n+p+2)} \beta_{0}(p, \mu) \tag{28}$$

$$\alpha_{-}(p, 2l, \mu) = \frac{(p-2l)(p+2l+2)}{4(2n+p+1)(2n+p)} \beta_{-}(p, \mu) \tag{29}$$

and

$$\beta_{+}(p, \mu) = \mu + \rho + p \tag{30}$$

$$\beta_{0}(p, \mu) = \mu \tag{31}$$

$$\beta_{-}(p, \mu) = \mu - \rho - p + 2. \tag{32}$$

Observe that the functions $\beta_{+}(p, \mu), \beta_{0}(p, \mu)$ and $\beta_{-}(p, \mu)$ are equal to the analogue functions for $\mathrm{SL}(4(n+1), \mathbb{R})$ [1]. $T^{\pm}_{l,\mu}$ is reducible if and only if

$$\beta_{-}(p, \mu) = 0 \quad \text{for some } p > 2l,\ p \equiv 2l \pmod 2$$

or

$$\beta_{+}(p, \mu) = 0 \quad \text{for some } p \geq 2l,\ p \equiv 2l \pmod 2.$$

We therefore have the following theorem.

[1]cf. [4], p.5. In [4] the groups $G = \mathrm{SL}(n, \mathbb{R})$ acts from the right, and no ρ-shift is considered in the inducing representation of $P^{\pm}$. Our $\beta_{+}(p, \mu)$ and $\beta_{-}(p, \mu)$ must therefore be compared with the functions $-\beta_1(-\mu-\rho, p)$ and $-\beta_2(-\mu-\rho, p)$ for $\mathrm{SL}(4(n+1), \mathbb{R})$, the functions β_1 and β_2 being as in [4].

Theorem 3.3. *The representations $T^{\pm}_{l,\mu}$ of $G\times\mathrm{Sp}(1)$ (hence the representations $\pi^{\pm}_{l,\mu}$ of G) are irreducible except for the following cases:*

(a) $\mu \in \mathbb{Z}$, $\mu \geq 2l+\rho$, $\mu \equiv 2l \pmod 2$.

(b) $\mu \in \mathbb{Z}$, $\mu \leq -(2l+\rho)$, $\mu \equiv 2l \pmod 2$.

In Case (a), *$T^{\pm}_{l,\pm\mu}$ has in $\mathcal{D}_l(S)$ one infinite dimensional irreducible submodule $E_{l,\mu}$ and one finite dimensional irreducible quotient $\tilde{E}_{l,\mu} = \mathcal{D}_l(S)/E_{l,\mu}$. In Case* (b), *$T^{\pm}_{l,\pm\mu}$ has in $\mathcal{D}_l(S)$ one finite dimensional irreducible submodule $E_{l,\mu}$ and one infinite dimensional irreducible quotient $\tilde{E}_{l,\mu} = \mathcal{D}_l(S)/E_{l,\mu}$.*

Consequently, in Case (a), *$\pi^{\pm}_{l\pm\mu}$ has in $\mathcal{D}_l(S,\tau_l)$ one infinite dimensional irreducible submodule $E_{l,\mu}$ and one finite dimensional irreducible quotient $\tilde{E}_{l,\mu} = \mathcal{D}_l(S,\tau_l)/E_{l,\mu}$. In Case* (b), *$T^{\pm}_{l,\pm\mu}$ has in $\mathcal{D}_l(S,\tau_l)$ one finite dimensional irreducible submodule $E_{l,\mu}$ and one infinite dimensional irreducible quotient $\tilde{E}_{l,\mu} = \mathcal{D}_l(S,\tau_l)/E_{l,\mu}$.*

For a fixed $l \in \mathbb{N}/2$, the situation for $T^{+}_{l,\mu}$ is described by the diagrams in Figure 2. The dot $\underset{m}{\bullet}$ denotes the $\tilde{U}$-type in $\mathcal{D}_l(S)_{\tilde{U}}$ corresponding to the parameter $(m, 2l)$. A barrier $\bullet\,]\,\bullet$ cannot be crossed from left to right using the $\mathfrak{U}(\tilde{\mathfrak{g}})$-action, but the crossing in the opposite direction is allowed. In this situation, the $\tilde{U}$-types on the left of the barrier form a $\mathfrak{U}(\tilde{\mathfrak{g}})$-invariant subspace; those at the right represent a $\mathfrak{U}(\tilde{\mathfrak{g}})$-invariant quotient. A similar interpretation is given to a barrier of the form $\bullet\,[\,\bullet$.

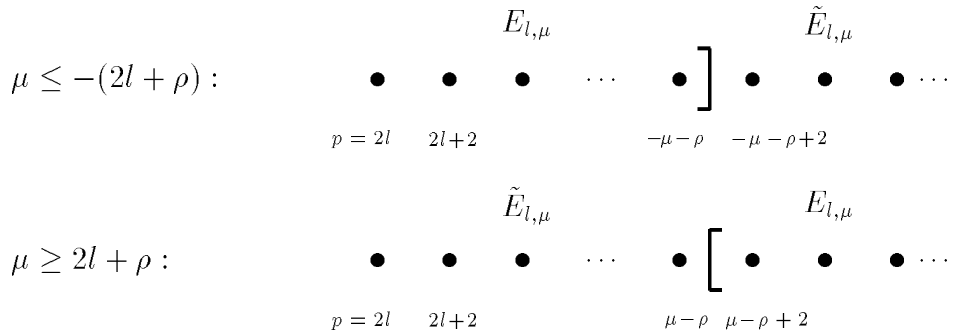

Figure 2: Composition series for the reducible $T^{+}_{l,\mu}$ ($\mu \in \mathbb{Z}, \mu \equiv 2l \pmod 2$)

4. Unitarizability

The representation $\pi^{\pm}_{l,\mu}$ is said unitarizable if the space of U-finite vectors in $\mathcal{D}(S,\tau)$ can be endowed with a positive definite Hermitian form $\langle\cdot,\cdot\rangle$ for which the $\mathfrak{g}$-action is given by skew-adjoint operators:

$$\langle \pi^{\pm}_{l,\mu}(X)\varphi, \psi\rangle = -\langle \varphi, \pi^{\pm}_{l,\mu}(X)\psi\rangle, \qquad X \in \mathfrak{g},\ \varphi, \psi \in \mathcal{D}(S,\tau_l)_U.$$

As for the irreducibility of the $\pi^{\pm}_{l,\mu}$, also the unitarizability of the $\pi^{\pm}_{l,\mu}$ can be deduced from the unitarizability of the $T^{\pm}_{l,\mu}$.

Let us first determine the existence of $\tilde{\mathfrak{g}}$-intertwining operators between the various $T^{\pm}_{l,\mu}$, that is the existence of (nonzero) linear operators $\tilde{A} : \mathcal{D}_l(S)_{\tilde{U}} \to \mathcal{D}_{l'}(S)_{\tilde{U}}$ satisfying

$$\tilde{A} \circ T^{\pm}_{l,\mu}(X) = T^{\pm}_{l',\mu'}(X) \circ \tilde{A}, \qquad X \in \tilde{\mathfrak{g}} \tag{33}$$

(in (33) any possible combination of $\pm$ signs is allowed). Let $\tilde{\mathfrak{u}} = \mathfrak{sp}(n+1) \oplus \mathfrak{sp}(1)$ be the Lie algebra of $\tilde{U}$. Since different $V^{p,q}$'s are inequivalent $\tilde{U}$-modules, such $\tilde{A}$ cannot exist unless $l = l'$. Moreover, the restriction of $\tilde{A}$ to each $V^{p,2l}$ must be a scalar multiple of the identity operator:

$$\tilde{A}|_{V^{p,2l}} = a_{p,2l} I$$

for some nonzero constants $a_{p,2l}$. These constants can be determined by applying both sides of the equation

$$\tilde{A} \circ T^{+}_{l,\mu}(H) = \pm T^{+}_{l,\pm\mu'}(H) \circ \tilde{A} \tag{34}$$

to the $\tilde{M}$-fixed vectors $e_{p,2l}$. In (34) the $+$ signs correspond to the case of interwining operators for $(T^{+}_{l,\mu}, T^{+}_{l,\mu'})$ or $(T^{-}_{l,-\mu}, T^{-}_{l,-\mu'})$, whereas the $-$ signs correspond to intertwining operators for $(T^{+}_{l,\mu}, T^{-}_{l,\mu'})$ or $(T^{-}_{l,-\mu}, T^{+}_{l,-\mu'})$.

Using (27)–(29), we obtain

$$\begin{aligned} a_{p+2,2l}\,(\mu + \rho + p) &= a_{p,2l}\,(\mu' \pm \rho \pm p) \\ a_{p,2l}\,\mu &= a_{p,2l}\,\mu' \\ a_{p-2,2l}\,(\mu - \rho - p + 2) &= a_{p,2l}\,(\mu' \pm (2 - \rho - p)), \end{aligned}$$

i.e.

$$a_{p+2,2l}\,(\mu + \rho + p) = a_{p,2l}(\mu \pm \rho \pm p) \tag{35}$$

Suppose first $\mu \in \mathbb{C}$ is so that $T^{\pm}_{l,\mu}$ is irreducible. Then we can uniquely determine $a_{p,2l}$ in terms of $a_{2l,2l}$. When the $+$ signs are considered, (35) gives $a_{p,2l} = a_{2l,2l}$ for all $p \geq 2l, p \equiv 2l \pmod 2$. So the unique $\tilde{\mathfrak{g}}$-intertwining operators between $T^{\pm}_{l,\mu}$ and itself are the scalar multiples of the identity operator. When the $-$ signs are considered, (35) shows that the unique $\tilde{\mathfrak{g}}$-intertwining operators for $(T^{+}_{l,\mu}, T^{-}_{l,\mu})$ and for $(T^{-}_{l,-\mu}, T^{+}_{l,-\mu})$ are given by $\tilde{A}|_{V^{p,2l}} = a_{p,2l} I$ with

$$\begin{aligned} a_{p,2l} &= \frac{\Gamma\left(\frac{\beta_-(2l,\mu)}{2}\right)\Gamma\left(\frac{\beta_+(2l,\mu)}{2}\right)}{\Gamma\left(\frac{\beta_-(p,\mu)}{2}\right)\Gamma\left(\frac{\beta_+(p,\mu)}{2}\right)}\, a_{2l,2l} && (36) \\ &= \frac{\Gamma\left(-\frac{\beta_-(p,\mu)}{2}+1\right)\Gamma\left(-\frac{\beta_+(p,\mu)}{2}+1\right)}{\Gamma\left(-\frac{\beta_-(2l,\mu)}{2}+1\right)\Gamma\left(-\frac{\beta_+(2l,\mu)}{2}+1\right)}\, a_{2l,2l}. && (37) \end{aligned}$$

Suppose then $\mu \in \mathbb{C}$ corresponds to a reducible $T^{\pm}_{l,\mu}$.

Let $T^{\pm,0}_{l,\mu}$ and $\tilde{T}^{\pm}_{l,\mu}$ respectively denote the restriction of $T^{\pm}_{l,\mu}$ to $E_{l,\mu}$ and the factor representation of $T^{\pm}_{l,\mu}$ on $\tilde{E}_{l,\mu}$. We indicate with the same symbols also the corresponding differentiated representations of $\tilde{\mathfrak{g}}$. The choice of the $+$ signs in (35) gives the scalar multiples of the identity as the unique $\tilde{\mathfrak{g}}$-intertwining operators between each of these representations and itself. When the $-$ signs are considered in (35), we have to distinguish two further cases, $\mu \le -(\rho+2l)$ and $\mu \ge \rho+2l$. In the first case, (35) is solved for $2l \le p \le -\mu-\rho$ by (37), which defines intertwining operators for $(T^{+,0}_{l,\mu}, \tilde{T}^{-}_{l,\mu})$ and for $(T^{-,0}_{l,-\mu}, \tilde{T}^{+}_{l,-\mu})$. The solution for $p \ge -\mu-\rho+2$ is

$$a_{p,2l} = (-1)^{\frac{\beta_+(p,\mu)}{2}+1} \frac{\Gamma\left(-\frac{\beta_-(p,\mu)}{2}+1\right)}{\Gamma(-\mu+1)\,\Gamma\left(\frac{\beta_+(p,\mu)}{2}\right)}\, a_{-\mu-\rho+2,2l}. \tag{38}$$

Observe that the right-hand side of (38) vanishes for $2l \le p \le -\mu-\rho$, so to define intertwining operators for $(\tilde{T}^{+}_{l,\mu}, T^{-,0}_{l,\mu})$ and for $(\tilde{T}^{-}_{l,-\mu}, T^{+,0}_{l,-\mu})$. In the second case $\mu \ge \rho+2l$, (35) is solved for $2l \le p \le \mu-\rho$ by (36), which vanishes for $p \ge \mu-\rho+2$. Interwining operators for $(\tilde{T}^{+}_{l,\mu}, T^{-,0}_{l,\mu})$ and for $(\tilde{T}^{-}_{l,-\mu}, T^{+,0}_{l,-\mu})$ are therefore defined. For $p \ge \mu-\rho+2$, (35) is solved by

$$a_{p,2l} = (-1)^{\frac{\beta_-(p,\mu)}{2}} \frac{\Gamma\left(-\frac{\beta_-(p,\mu)}{2}+1\right)\Gamma(\mu+1)}{\Gamma\left(\frac{\beta_+(p,\mu)}{2}\right)}\, a_{\mu-\rho+2,2l}. \tag{39}$$

This solution gives intertwining operators for $(T^{+,0}_{l,\mu}, \tilde{T}^{-}_{l,\mu})$ and for $(T^{-,0}_{l,-\mu}, \tilde{T}^{+}_{l,-\mu})$.

We collect the results in the following proposition.

Proposition 4.1. *Besides the trivial case of scalar multiples of the identity which intertwine a representation with itself, there exist $\tilde{\mathfrak{g}}$-intertwining operators only for the following pairs.*

Irreducible case: $(T^{+}_{l,\mu}, T^{-}_{l,\mu})$ and $(T^{-}_{l,\mu}, T^{+}_{l,\mu})$.

Reducible case: $(T^{+,0}_{l,\mu}, \tilde{T}^{-}_{l,\mu})$, $(\tilde{T}^{-}_{l,\mu}, T^{+,0}_{l,\mu})$, $(T^{-,0}_{l,\mu}, \tilde{T}^{+}_{l,\mu})$ and $(\tilde{T}^{+}_{l,\mu}, T^{-,0}_{l,\mu})$.

In particular, the following $\tilde{\mathfrak{g}}$-representations are equivalent: $T^{+}_{l,\mu}$ and $T^{-}_{l,\mu}$ (when irreducible); $T^{+,0}_{l,\mu}$ and $\tilde{T}^{-}_{l,\mu}$; $T^{-,0}_{l,\mu}$ and $\tilde{T}^{+}_{l,\mu}$.

Observe that a $\tilde{\mathfrak{g}}$-intertwining operator $\tilde{A}$ for the $T^{\pm}_{l,\mu}$ corresponds to a $\tilde{\mathfrak{g}}$-intertwining operator $\tilde{\tilde{A}}$ for the differentiated representations $\mathrm{d}(\pi^{\pm}_{l,\mu} \otimes \tau_l) \equiv \pi^{\pm}_{l,\mu} \otimes I + I \otimes \tau_l$ of $\tilde{\mathfrak{g}}$ on $(\mathcal{D}_l(S,\tau_l) \otimes V_l))_{\tilde{U}} = \mathcal{D}_l(S,\tau_l)_U \otimes V_l$. In turn, because of the irreducibility of τ_l, the $\tilde{\tilde{A}}$'s are exactly the linear operators of the form $A \otimes I$ for a $\mathfrak{g}$-intertwining operator $A : \mathcal{D}_l(S,\tau_l)_U \to \mathcal{D}_l(S,\tau_l)_U$ of the $\pi^{\pm}_{l,\mu}$'s.

Let $\pi^{\pm,0}_{l,\mu}$ and $\tilde{\pi}^{\pm}_{l,\mu}$ respectively denote the restriction of $\pi^{\pm}_{l,\mu}$ to $E_{l,\mu}$ and the factor representation of $\pi^{\pm}_{l,\mu}$ on $\tilde{E}_{l,\mu}$. We indicate with the same symbols also the corresponding differentiated representations of $\mathfrak{g}$. Then Proposition 4.1 immediately implies

Corollary 4.2. *Besides the trivial case of scalar multiples of the identity which intertwine a representation with itself, there exist $\mathfrak{g}$-intertwining operators only for the following pairs.*

Irreducible case: $(\pi^{+}_{l,\mu}, \pi^{-}_{l,\mu})$ and $(\pi^{-}_{l,\mu}, \pi^{+}_{l,\mu})$.

Reducible case: $(\pi^{+,0}_{l,\mu}, \tilde{\pi}^{-}_{l,\mu})$, $(\tilde{\pi}^{-}_{l,\mu}, \pi^{+,0}_{l,\mu})$, $(\pi^{-,0}_{l,\mu}, \tilde{\pi}^{+}_{l,\mu})$ and $(\tilde{\pi}^{+}_{l,\mu}, \pi^{-,0}_{l,\mu})$.

In particular, the following $\mathfrak{g}$-representations are equivalent: $\pi^{+}_{l,\mu}$ and $\pi^{-}_{l,\mu}$ (when irreducible); $\pi^{+,0}_{l,\mu}$ and $\tilde{\pi}^{-}_{l,\mu}$; $\pi^{-,0}_{l,\mu}$ and $\tilde{\pi}^{+}_{l,\mu}$.

We now determine which among the $T^{\pm}_{l,\mu}$ and their irreducible subrepresentations and subquotients are unitarizable.

Note first that the standard inner product in $L^2(S)$

$$(\varphi|\psi) = \int_S \varphi(s)\overline{\psi(s)}\,ds \tag{40}$$

is invariant with respect to the pairs $(T^{+}_{l,\mu}, T^{+}_{l,-\bar\mu})$ and $(T^{-}_{l,\mu}, T^{-}_{l,-\bar\mu})$. Therefore $T^{\pm}_{l,\mu}$ is always unitarizable when $\bar\mu = -\mu$.

In the general case, if a $\tilde{\mathfrak{g}}$-invariant Hermitian form $(\cdot,\cdot)$ exists for $T^{\pm}_{l,\mu}$ (or for some of its subrepresentations or quotients when reducible), then its restriction to each $\tilde U$-type must be a constant multiple of the standard inner product from $L^2(S)$, say

$$(\varphi,\psi) = a_{p,2l}(\varphi|\psi) \quad \text{on } V^{p,2l}.$$

Moreover, different $\tilde U$-types must be mutually orthogonal, so that $(\cdot,\cdot)$ is completely determined by the $a_{p,2l}$'s. The operator $\tilde A : \mathcal{D}_l(S)_{\tilde U} \to \mathcal{D}_l(S)_{\tilde U}$ given by $\tilde A|_{V^{p,2l}} = a_{p,2l} I$ then intertwines $(T^{\pm}_{l,\mu}, T^{\pm}_{l,-\bar\mu})$. Proposition 4.1 implies $-\bar\mu = \mu$. Therefore a $\tilde{\mathfrak{g}}$-invariant Hermitian form $(\cdot,\cdot)$ exists on $\mathcal{D}_l(S)_{\tilde U}$ if and only if $\mu \in i\mathbb{R}$, and in this case it is a scalar multiple of the standard inner product of $L^2(S)$. A similar analysis can be done for the reducible $T^{\pm}_{l,\mu}$.

Theorem 4.3. *An irreducible $T^{\pm}_{l,\mu}$ (hence $\pi^{\pm}_{l,\mu}$) is unitarizable if and only if $\mu \in i\mathbb{R}$. When $T^{\pm}_{l,\mu}$ (or $\pi^{\pm}_{l,\mu}$) is reducible, neither its irreducible submodule nor its irreducible quotients are unitarizable.*

We remark that the inner product making $\pi^{\pm}_{l,\mu}$ unitary is the standard inner product in $L^2(S, V_l)$, that is

$$(\varphi|\psi)_l := \int_S (\varphi(s), \psi(s))_l\,ds. \tag{41}$$

5. Intertwining operators and the η-function

An intertwining operator for $(\pi^{\pm}_{l,\mu}, \pi^{\pm}_{l',\mu'})$, with arbitrary choice of signs, is a (nonzero) continuous linear map $A : \mathcal{D}(S,\tau_l) \to \mathcal{D}(S,\tau_{l'})$ satisfying

$$A \circ \pi^{\pm}_{l,\mu}(g) = \pi^{\pm}_{l',\mu'}(g) \circ A, \qquad g \in G.$$

In particular, A maps $\mathcal{D}(S,\tau_l)_U$ into itself and, by continuity, it intertwines the differentiated representations. Corollary 4.2 implies that in the irreducible case a nontrivial intertwining operator can exist only for $(\pi^{+}_{l,\mu}, \pi^{-}_{l,\mu})$ and $(\pi^{-}_{l,\mu}, \pi^{+}_{l,\mu})$. In this case, it is uniquely determined by its restriction to the U-finite vectors, so it is unique up to constant multiples. With the same argument of Lemma 5.3 in [14], it is possible to show that the $\mathfrak{g}$-intertwining operators $\tilde A$ determined by (36) indeed extend to intertwining operators for $(\pi^{+}_{l,\mu}, \pi^{-}_{l,\mu})$ and $(\pi^{-}_{l,-\mu}, \pi^{+}_{l,-\mu})$. We will not pursue this argument further. Rather, we explicitely present them in integral form.

Recall that an operator $A_\mu : \mathcal{D}(S,\tau_l) \to \mathcal{D}(S,\tau_l)$ is said to be holomorphic (resp. meromorphic) in μ if $\omega(A_\mu\varphi)$ is a holomorphic (resp. meromorphic) function of μ for all $\varphi \in \mathcal{D}(S,\tau_l)$ and all $\omega \in \mathcal{D}(S,\tau_l)'$, the dual space of $\mathcal{D}(S,\tau_l)$.

For every $\varphi \in \mathcal{D}(S,\tau_l)$

$$A_{l,\mu}\varphi(s) = \int_S |(s,t)|^{\mu-\rho}\tau_l(\,\overline{(s,t)}/|(s,t)|\,)\,\varphi(t)\,dt, \qquad s \in S \tag{42}$$

defines a norm convergent V_l-valued integral provided $\Re\mu > 2(n+1)$. On this region, $A_{l,\mu}$ is holomorphic in μ and the regularization of (42) gives a meromorphic extension of $A_{l,\mu}$ to $\mathbb{C}$.

It can be easily checked that $A_{l,\mu}$ intertwines $(\pi^+_{l,\mu},\pi^-_{l,\mu})$ and $(\pi^-_{l,-\mu},\pi^+_{l,-\mu})$. It follows that $A_{l,-\mu}\circ A_{l,\mu}$ intertwines $\pi^\pm_{l,\mu}$ with itself, so it is a scalar multiple of the identity operator:

$$A_{l,-\mu}\circ A_{l,\mu} = \eta_{l,\mu} I \tag{43}$$

for some even meromorphic function $\eta_{l,\mu}$.

We now want to determine the eigenvalues of $A_{l,\mu}$ on the U-types and the function $\eta_{l,\mu}$. Since $A_{l,\mu}$ intertwines $\pi^+_{l,\mu}$ and $\pi^-_{l,\mu}$, the operator $\tilde{A}_{l,\mu} := \boldsymbol{\beta}_l\circ(A_{l,\mu}\otimes I)\circ\boldsymbol{\beta}_l^{-1}$ intertwines $T^+_{l,\mu}$ and $T^-_{l,\mu}$.

Observe that if $f = \boldsymbol{\beta}_l(\varphi\otimes v) \in \mathcal{D}_l(S)$, then

$$(\tilde{A}_{l,\mu}f)(s) = \langle A_{l,\mu}\varphi(s), v\rangle_l = \int_S |(s,t)|^{\mu-\rho}\langle\tau_l\left(\frac{\overline{(s,t)}}{|(s,t)|}\right)\varphi(t), v\rangle_l\,dt.$$

Setting $s = e_0$, we obtain

$$(\tilde{A}_{l,\mu}f)(e_0) = \int_S |t_0|^{\mu-\rho}\langle\tau_l(\,\overline{t_0}/|t_0|)\varphi(t), v\rangle_l\,dt.$$

For every $p \equiv 2l \pmod 2$, $p \geq 2l$, $e_{p,2l}$ is an eigenfunction of $\tilde{A}_{l,\mu}$. Let $a_{p,l;\mu}$ be the corresponding eigenvalue. $a_{p,l;\mu}$ can be determined by evaluating at e_0 both sides of the equation

$$\tilde{A}_{l,\mu}e_{p,2l} = a_{p,l;\mu}e_{p,2l}.$$

Because of (36) it is enough to know $\gamma_{l,\mu} := a_{2l,l;\mu}$. From (18) we have

$$e_{2l,2l}(s) = \boldsymbol{\beta}_l\left(\sum_{j=0}^{2l} F_{2l,2l;j}\otimes\mathbf{e}_j\right)(s)$$

with

$$F_{2l,2l;j}(s) = (-1)^j\tau_l(\,s_0/|s_0|\,)^{-1}(\mathbf{e}_{2l-j})|s_0|^{2l}.$$

Hence $e_{2l,2l}(e_0) = 2l+1$ and

$$\begin{aligned}
\tilde{A}_{l,\mu}e_{2l,2l}(e_0) &= \int_S |t_0|^{\mu-\rho+2l}\sum_{j=0}^{2l}(-1)^j\langle\tau_l(t_0/|t_0|)\tau_l(t_0/|t_0|)^{-1}\mathbf{e}_{2l-j},\mathbf{e}_j\rangle_l\,dt \\
&= \int_S |t_0|^{\mu-\rho+2l}\left[\sum_{j=0}^{2l}(-1)^j\langle\mathbf{e}_{2l-j},\mathbf{e}_j\rangle_l\right]dt \\
&= (2l+1)\int_S |t_0|^{\mu-\rho+2l}\,dt
\end{aligned}$$

Lemma 5.1. *Let $F : S \to \mathbb{C}$ be of the form $F(t) = f(|t_0|)$ for some function $f \in L^1(0,1)$ and let dt denote the U-invariant measure on S normalized by the condition $dt(S) = 1$ then*

$$\int_S F(t)\,dt = 4n(2n+1)\int_0^1 f(x)(1-x^2)^{2n-1}x^3\,dx.$$

Lemma 5.1 and the integral defining the Beta function therefore give

$$\begin{aligned}
\gamma_{l,\mu} &= \Gamma(\rho)\frac{\Gamma\left(\frac{\mu-\rho+2l+4}{2}\right)}{\Gamma\left(\frac{\mu+\rho+2l}{2}\right)}\\
a_{p,2l;\mu} &= \Gamma(\rho)\frac{\Gamma\left(\frac{\mu-\rho-2l+2}{2}\right)\Gamma\left(\frac{\mu-\rho+2l+4}{2}\right)}{\Gamma\left(\frac{\mu-\rho-p+2}{2}\right)\Gamma\left(\frac{\mu+\rho+p}{2}\right)},\quad p\equiv 2l \ (\mathrm{mod}\ 2),\ p\geq 2l\\
\eta_{l,\mu} &= \gamma_{l,\mu}\gamma_{l,-\mu} = \Gamma(\rho)^2\frac{\Gamma\left(\frac{\mu-\rho+2l+4}{2}\right)}{\Gamma\left(\frac{\mu+\rho+2l}{2}\right)}\frac{\Gamma\left(\frac{-\mu-\rho+2l+4}{2}\right)}{\Gamma\left(\frac{-\mu+\rho+2l}{2}\right)}
\end{aligned}$$

Observe that, as in Proposition 7.4 of [12], we have

$$\begin{aligned}
&\gamma_{l,\overline{\mu}} = \overline{\gamma_{l,\mu}},\\
&\eta_{l,\mu} = \eta_{l,-\mu},\\
&\eta_{l,i\mu} \geq 0 \text{ for } \mu\in\mathbb{R}.
\end{aligned}$$

Moreover the normalized intertwining operators $\mathcal{A}_{l,\mu} := \gamma_{l,\mu}^{-1}A_{l,\mu}$ satisfy

$$\begin{aligned}
&\mathcal{A}_{l,\mu}\mathcal{A}_{l,-\mu} = I,\\
&\mathcal{A}^*_{l,\mu} = \mathcal{A}_{l,-\bar{\mu}} \text{ the adjoint being defined } U\text{−type by } U\text{−type},\\
&\mathcal{A}_{l,\mu} \text{ is a unitarily equivalence between } \pi^+_{l,\mu} \text{ and } \pi^-_{l,\mu} \text{ for } \mu\in i\mathbb{R}.
\end{aligned}$$

References

[1] Artin, E., "Geometric Algebra," Interscience, New York, 1957.

[2] Borel, A., and N. Wallach, "Continuous cohomology, discrete subgroups and representations of reductive groups, " Princeton University Press, Princeton, 1980.

[3] van Dijk, G., and S. C. Hille, *Maximal degenerate representations, Berezin kernels and canonical representations*, in: B. Komrakov, J. Krasil'shchik, G. Litvinov and A. Sossinky, Editors, "Lie groups and Lie algebras, their representations, generalizations, and applications," 1–15, Kluwer Academic, Dordrecht, 1997.

[4] van Dijk, G., and V.F. Molchanov, *Tensor product of maximal degenerate series representations of the group* $\mathrm{SL}(n,\mathbb{R})$, J. Math. Pures et Appl. (1997), to appear.

[5] van Dijk, G., and A. Pasquale, *Canonical representations of* $\mathrm{Sp}(1,n)$ *associated with representations of* $\mathrm{Sp}(1)$, Preprint, Leiden University, 1998.

[6] Erdélyi, A., et al, "Higher transcendental functions," McGraw-Hill, New York, 1953.

[7] Hille, S. C., *Maximal degenerate representations of* $\mathrm{SL}(n+1,\mathbb{C})$, Preprint, Leiden University, 1997.

[8] Howe, R. E., and E. Tan, *Homogeneous functions on light cones: the infinitesimal structure of some degenerate principal series representations*, Bull. Amer. Math. Soc. **28** (1993), 1–74.

[9] Johnson, K. D., *Degenerate principal series and compact groups*, Math. Ann. **287** (1990), 703–718.

[10] —, *Degenerate principal series on tube domains*, Contemporary Math. **138** (1992), 175–187.

[11] Johnson, K. D., and N. R. Wallach, *Composition and intertwining operators for the spherical principal series*, Trans. Amer. Math. Soc. **229** (1977), 137–173.

[12] Knapp, A. W., and E. M. Stein, *Intertwining operators for semisimple Lie groups, II*, Inventiones Math. **60** (1980), 9–84.

[13] Lee, S. T., *Degenerate principal series representations of* $\mathrm{Sp}(2n,\mathbb{R})$, Compositio Math. **103** (1996), 123–151.

[14] Molchanov, V. F., *Representations of pseudo-orthogonal groups associated with a cone*, Math USSR Sbornik **10** (1970), 333–347.

[15] Rossmann, W., *Analysis on real hyperbolic spaces*, J. Funct. Anal **30** (1978), 448–477.

[16] Smith, R. T., *The spherical representations of groups transitive on* S^n, Indiana Univ. Math. Journal **24** (1974), 307–325.

[17] Zelobenko, D. P., "Compact Lie groups and their representations," Translations of Mathematical Monographs 40, Amer. Math. Soc., Providence, 1973.

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