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Natural divisors and the brownian motion

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Résumé. On propose un modèle du mouvement Brownien relatif aux diviseurs d'un entier, et on établit la convergence faible de la mesure associée dans l'espace $\mathbf{D}[0,1]$. On obtient un résultat analogue à celui obtenu par Erdös pour les diviseurs premiers [6] (cf. [14] pour une démonstration). Ces résultats et les recherches de l'auteur [15] étendent l'étude [9] de la distribution des diviseurs. Notre approche s'appuie sur les théorèmes limites fonctionnels en théorie des probabilités.

Abstract. A model of the Brownian motion defined in terms of the natural divisors is proposed and weak convergence of the related measures in the space $\mathbf{D}[0,1]$ is proved. An analogon of the Erdös arcsine law, known for the prime divisors [6] (see [14] for the proof), is obtained. These results together with the author's investigation [15] extend the systematic study [9] of the distribution of natural divisors. Our approach is based upon the functional limit theorems of probability theory.

1. Results.

The distribution of the prime factors of an integer determines that of the natural divisors. Therefore statements known for the primes often have their counterparts. That was perfectly demonstrated by R.R. Hall and G. Tenenbaum in monograph [9] for the normal orders of the k-th prime factor $p_k(m)$ and the k-th natural factor $d_k(m)$ of $m \in \mathbf{N}$. Let $\nu_x(\dots)$ denote the uniform probability measure on the set $\Omega_x = \{1, \dots, [x]\}$, $Lu = \log\max\{u, e\}$, and $L_k = L(L_{k-1})$. Then one has

$$\lim_{n\to\infty} \limsup_{x\to\infty} \nu_x\left(\max_{n\le k\le \omega(m)} |L_2 p_k(m) - k| \ge (1+\varepsilon)\sqrt{2kL_2k} \right) = 0$$

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and
(1)

$$\lim_{n\to\infty}\limsup_{x\to\infty}\nu_x\left(\max_{n\le k\le\tau(m)}|L_2 d_k(m)-\log_2 k|\ge(1+\varepsilon)\sqrt{2(\log_2 k)L_3 k}\right)=0.$$

Here and in what follows $\omega(m)$ and $\tau(m)$ denote the number of all different prime and natural factors respectively. Moreover, after the change of ε to $-\varepsilon$ these limits (even with $\liminf$ instead of $\limsup$) are equal to one. The same duality also holds for the sharpened estimates when the terms $\varepsilon\sqrt{2kL_2k}$ and $\varepsilon\sqrt{2(\log_2 k)L_3k}$ are substituted by some asymptotic expansions (c.f. [15]). In the paper [15] we extended the above mentioned results into the Strassen functional form, that is, both of them were included into more general relations for arithmetically defined stochastic processes. Proceeding along this way we can look for the duality in the functional limit theorems for arithmetic processes. The most of such the processes so far considered are defined in terms of additive functions. It means that the processes are related to the prime divisors (see [1], [3], [4], [8], [12], [13], [16], [19], [20] and other). We cannot give any reference concerning weak convergence of processes defined in terms of the natural divisors. But several results do indicate such a direction of possible investigations.

Let $\tau(m,u)=\mathrm{card}\{d\in\mathbf{N}: d|m, d\le u\}$. It is known [9], [18] that

$$\nu_x\big(\tau(m,x^t)-\tau(m,x^s)<u\tau(m)\big)\Rightarrow F_{st}(u),\qquad 0\le s<t\le 1,$$

where $\Rightarrow$ denotes weak convergence of distribution functions, $F_{st}(u)$ is some purely dicrete distribution function, and $x\to\infty$. This relation can be interpreted as referring to the increments of the arithmetic process $Y_x:=Y_x(m,t)=\tau(m,x^t)/\tau(m)$ having trajectories in the space $\mathbf{D}$:=$\mathbf{D}[0,1]$ of real-valued functions on $[0,1]$ which are right-continuous and have left-hand limits. Moreover, the remarkable asymptotic formula

$$\frac{1}{x}\sum_{m\le x}Y_x(m,t)=\frac{2}{\pi}\arcsin\sqrt{t}+o(1)=:As(t)+o(1),\qquad 0\le t\le 1,$$

which has been obtained in [5], shows the behaviour of the expectation of the process.

In this paper we look for a counterpart of the process

$$\psi_x:=\psi_x(m,t):=\frac{1}{\sqrt{L_2x}}\big(\omega(m,\exp\{(Lx)^t\})-tL_2x\big),\qquad 0\le t\le 1,$$

where $\omega(m,v) := \text{card}\{p - \text{prime} : p|m, p \leq v\}$, which "simulates" the Brownian motion $W = W(t)$ given on some probability space $\{\Omega, \mathcal{F}, P\}$. To be more precise, we will use some notations and concepts introduced in the book [2]. Let $\rho(\cdot,\cdot)$ denote the supremum metrics in the space $\mathbf{D}$, $\mathcal{D}$ be the Borel σ-algebra in $\mathbf{D}$ with respect to ρ, and $\nu_x \circ \psi_x^{-1}$ stand for the measure on $\mathcal{D}$ defined by $\nu_x(X_x \in A)$ where $A \in \mathcal{D}$. Put μ for the Wiener measure $P \circ W^{-1}$. In 1970 P. Billingsley [3] proved the following result.

Theorem B. *The sequence of measures $\nu_x \circ \psi_x^{-1}$ weakly converges to the Wiener measure μ as $x \to \infty$.*

More general results can be found in the above cited papers on the functional limit theorems.

Having in mind that statistically $\tau(m)$ behaves like the function $2^{\Omega(m)}$, where $\Omega(m)$ denotes the number of prime factors of $m \in \mathbf{N}$ counted according to their multiplicity, we will consider the limiting distribution of the process

$$X_x := X_x(m,t) := \frac{1}{\sqrt{L_2 x}}\big(\log_2 \tau(m, \exp\{(Lx)^t\}) - tL_2 x\big), \qquad 0 \leq t \leq 1,$$

with respect to the probability measure ν_x as $x \to \infty$. Denote $\mu_x = \nu_x \circ X_x^{-1}$. In what follows the limiting passage $x \to \infty$ is not explicitly indicated.

Theorem 1. *The sequence of measures μ_x weakly converges to the Wiener measure μ.*

The corollaries presented below follow from the relation

$$\nu_x(\varphi(X_x) < u) \Rightarrow P(\varphi(W) < u),$$

valid for each μ-almost everywhere continuous functional φ:$\mathbf{D} \to \mathbf{R}$. They describe new features of the sequence $d_k(m)$.

Corollary 1. *For each fixed $0 \leq s < t \leq 1$,*

$$\nu_x\big(X_x(m,t) - X_x(m,s)) < u\big) \Rightarrow \frac{1}{\sqrt{2\pi(t-s)}}\int_{-\infty}^{u} \exp\left\{-\frac{y^2}{2(t-s)}\right\}dy.$$

Hence if $s = 0$ and $t = 1$, we have the central limit theorem for the additive function $\log_2 \tau(m)$ belonging to the Kubilius class H (see [9]). But if $0 < t < 1$, then $\log_2 \tau(m,t)$ is only subadditive. That shows new direction of possible investigations, e.g. to extend the probabilistic number theory to the class of subadditive functions.

Corollary 2. *For each* $0 < t \leq 1$,

$$\nu_x\Big(\max_{0\leq s\leq t} X_x(m,s) < u\Big) \Rightarrow \sqrt{\frac{2}{\pi t}}\int_{-\infty}^{u} \exp\Big\{-\frac{y^2}{2t}\Big\}dy$$

and for $u \geq 0$,

$$\nu_x\Big(\max_{0\leq t\leq 1} |X_x(m,t)| < u\Big) \sim \nu_x\Big(\max_{1\leq k\leq \tau(m)} |\log_2 k - L_2 d_k(m)| < u\sqrt{L_2 x}\Big)$$
$$\Rightarrow \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\exp\Big\{-\frac{(2k+1)^2\pi^2}{8u^2}\Big\}.$$

The last assertion can be compared with the above mentioned estimates of the law of iterated logarithm ([9], [15]) illustrated by (1).

Let in what follows $As(u)$ be extended to $\mathbf{R}$ by equalities $As(u) = 0$ when $u < 0$ and $As(u) = 1$ when $u \geq 1$.

Corollary 3. *We have*

$$\nu_x\Big(\mathrm{meas}\big\{0 \leq t \leq 1 : \log_2 \tau\big(m, \exp\{(Lx)^t\}\big) > tL_2 x\big\} < u\Big) \Rightarrow As(u).$$

Now, since the points $t = t_j = L_2 d_j(m)/L_2 x$ are the jumps of the trajectories considered, calculating the Lebesgue measure in the last relation we obtain certain sums of the quantities

$$L\Big(\frac{Ld_{j+1}(m)}{Ld_j(m)}\Big)\Big/ L_2 x. \tag{2}$$

Even the estimate (1) is not sufficient to simplify the sums of the ratios (2). The following statement of P. Erdös [6] indicates that a more simple form of the arcsine law may be valid for the natural divisors.

Theorem E. *We have*

$$\nu_x\big(\mathrm{card}\{k \leq \omega(m) : L_2 p_k(m) < k\} < uL_2 x\big) \Rightarrow As(u).$$

Proof. see [14].

The counterpart of this Erdös theorem is our next result. Let I^+ denote the characteristic function of the set $\{y : y > 0\}$.

Theorem 2. *We have*

$$U_x(u) := \nu_x\bigg(\sum_{j\le\tau(m)}\frac{1}{j}I^+(\log_2 j - L_2 d_j(m)) < (L2)uL_2x\bigg) \Rightarrow As(u). \tag{3}$$

The proof is similar to that given in [14] for Theorem E.

2. Proof of Theorem 1.

The trajectories of $X_x(m,t)$ will be approximated "for almost all m" by $\psi_x(m,t)$. By the general theory [2] the assertion of Theorem 1 will follow from Theorem B and the estimate

$$\nu_x(\rho(\psi_x, X_x)\ge\varepsilon) = o(1) \tag{4}$$

for each $\varepsilon > 0$.

To prove (4), at first we observe that for each t, $\quad 0\le t\le 1$,

$$X_x(m,t)\le \Delta_x(m,t) := \frac{1}{\sqrt{L_2x}}\big(\log_2\delta(m,\exp\{(Lx)^t\}) - tL_2x\big), \tag{5}$$

where $\delta(m,v) = \mathrm{card}\{d\in\mathbf{N} : d|m,\ p(d)\le v\}$ and $p(d)$ denotes the maximal prime divisor of d. On prime numbers the additive functions $\omega(\cdot,v)$ and $\log_2\delta(\cdot,v)$ coincide, hence (see [12])

$$\nu_x(\rho(\psi_x,\Delta_x)\ge\varepsilon) = o(1) \tag{6}$$

for each $\varepsilon > 0$. Thus, the relations (5) and (6) yield the required upper estimate of $X_x(m,t)$ in terms of $\psi_x(m,t)$.

The lower estimation is based upon ideas suggested in Exercises to Chapter 1 of the book [9]. If $\tau(m,u,v) := \mathrm{card}\{d : d|m,\ d>u,\ p(d)\le v\}$, then we have [9]

$$2^{\omega(m,v)}\le\tau(m,u)+\tau(m,u,v), \qquad u,v,\ge 1, \tag{7}$$

and

$$\frac{1}{x}\sum_{m\le x}\tau(m,u,v)\ll (Lx)\exp\Big\{-c\frac{Lu}{Lv}\Big\} \tag{8}$$

uniformly in $1 \le u, v \le x$ with some positive $c > 0$. The last estimate follows from the following inequalities

$$\sum_{m \le x} \tau(m,u,v) \le x \sum_{\substack{d>u \\ p(d)\le v}} \frac{1}{d} \le x u^{-\lambda} \prod_{p \le v} \bigl(1 - p^{\lambda-1}\bigr)^{-1} \ll$$
$$\ll x \exp\Bigl\{ -\lambda Lx + \sum_{p \le v} p^{\lambda-1} \Bigr\},$$

where $\lambda = c/Lv$ with some $c > 0$.

Now we divide the interval $[0,1]$ into $N := \bigl[(L_2x)/(L_3x)^2\bigr]$ equal parts of the length $\gamma := N^{-1}$. For the values of $y := \exp\{(Lx)^t\}$ we choose the checkpoints

$$u_k := \exp\{(Lx)^{k\gamma}\}, \qquad 0 \le k \le N.$$

When $y \in [u_k, u_{k+1}]$, from (7) we obtain

$$\tau(m,y) \ge \tau(m,u_k) \ge 2^{\omega(m,u_{k-1})} - \tau(m,u_k,u_{k-1}) \tag{9}$$

provided that $k \ge 1$. Further, in virtue of (8) and $\tau(m,u_k,u_{k-1}) \in \mathbf{Z}^+$,

$$\nu_x\Bigl(\max_{1\le k\le N-1} \tau(m,u_k,u_{k-1}) > 0\Bigr) \ll \sum_{k=1}^{N} (Lx)^{(k-1)\gamma} \exp\{-c(LX)^{\gamma}\} \ll$$
$$\ll (Lx)\exp\bigl\{ -c\exp\{(L_3x)^2/2\}\bigr\} = o(1).$$

According to (9) this implies that for $y \in [u_k, u_{k+1}]$ uniformly in $1 \le k \le N-1$

$$\log_2 \tau(m,y) \ge \omega(m,u_{k-1}) \ge \omega(m,y) - \bigl(\omega(m,u_{k+1}) - \omega(m,u_{k-1})\bigr) \tag{10}$$

for almost all $m \le x$.

To estimate the second difference for almost all m, we will use Ruzsa's [17] following result.

Lemma 1. *Let $h_j(m)$, $j = 1, \ldots, s$, be real-valued additive functions. There exist a probability space $\{\Omega, \mathcal{F}, P\}$, independent random variables $\xi_p^{(j)}$, $p \le x$, defined on it by $P(\xi_p^{(j)} = h_j(p^r)) = p^{-r}(1-1/p)$, $r \ge 0$, such that*

$$\nu_x\Bigl(\max_{1\le j\le s} |h_j(m) - a_j| \ge v\Bigr) \ll P\Bigl(\max_{1\le j\le s} \Bigl|\sum_{p\le n} \xi_p^{(j)} - a_j\Bigr| \ge v/3\Bigr)$$

for arbitrary $a_j \in \mathbf{R}$ *and* $v \geq 0$, *with an absolute constant in the symbol* $\ll$.

Thus, applying Lemma 1 we obtain

$$\nu_x\left(\max_{1\leq k\leq N-1}\big(\omega(m,u_{k+1})-\omega(m,u_{k-1})\big)\geq \varepsilon\sqrt{L_2x}\right)\ll$$
$$\ll P\left(\max_{1\leq k\leq N-1}\sum_{u_{k-1}<p\leq u_{k+1}}\xi_p\geq \frac{\varepsilon}{3}\sqrt{L_2x}\right)=: P_x,$$

Now ξ_p, $p\leq x$, denote independent random variables defined by $P(\xi_p = 1) = 1-P(\xi_p=0)=1/p$. After elementary estimation of their expectations and variances from the exponential inequalities (see [11], p.254) we derive

$$P_x \ll \sum_{k=1}^{N}\exp\{-\varepsilon\sqrt{L_2x}/24\}=o(1)$$

for each $\varepsilon>0$. Hence and from (10) we conclude that

$$X_x(m,t)\geq \psi_x(m,t)+o(1) \tag{11}$$

uniformly in $\gamma\leq t\leq 1$ for almost all m.

The relation $\gamma=\gamma_x\sim (L_3x)^2/L_2x$ and Theorem B imply

$$\nu_x\left(\sup_{0\leq t\leq\gamma}|\psi_x(m,t)|\geq\varepsilon\right)=o(1).$$

Moreover, for sufficiently large x,

$$\nu_x\left(\sup_{0\leq t\leq\gamma}|X_x(m,t)|\geq\varepsilon\right)\leq \nu_x\left(\log_2\delta(m,\exp\{2(L_3x)^2\})\geq\frac{\varepsilon}{2}\sqrt{L_2x}\right)$$

which by the law of large numbers for additive functions [10] tends to zero. So, the estimate (11) remains valid uniformly in $0\leq t\leq 1$. That completes the lower estimation in (4). Theorem 1 is proved.

3. Proof of Theorem 2.

We split the proof of Theorem 2 into several lemmata. Denote

$$T(m,v):=\log_2\tau(m,v)-L_2v\,.$$

For arbitrary $k \geq 2$ we put $n_i = ik^{-1}L_2x$, $J_i = (n_{i-1}, n_i]$ where $1 \leq i \leq k$. For convenience in the space $\mathcal{L}$ of distribution functions we shall use the Lévy metrics defined by

$$\Lambda(F,G) = \inf\{\varepsilon > 0 : F(u-\varepsilon) - \varepsilon < G(u) < F(u+\varepsilon) + \varepsilon\}, \qquad F, G \in \mathcal{L}.$$

In what follows the symbol $o(1)$ may depend on some parameters which sometimes will be indicated, while the other symbol $\ll$ will contain absolute constants.

Lemma 2. *Let*

$$V_x(u) = \nu_x\biggl(\sum_{i=1}^{k} \sum_{\log_2 j \in J_i} \frac{1}{j} I^+(T(m, d_j(m))) < (L2)uL_2x\biggr). \tag{12}$$

Then the relation (3) is equivalent to $\Lambda(V_x, As) \to 0$.

Proof. In the double sum of (12) j runs over the natural numbers belonging to the interval $\bigl(1, 2^{L_2x}\bigr]$. To estimate the sum over $2^{L_2x} < j \leq \tau(m)$, we observe that by the law of large numbers for the additive function $\log_2 \tau(m)$ (see [10]) we have $|\log_2 \tau(m) - L_2x| < (L_2x)^{3/4}$ for all but $o(x)$ numbers $m \leq x$. Hence for these numbers

$$\sum_{L_2x < \log_2 j \leq \log_2 \tau(m)} \frac{1}{j} \ll \sum_{L_2x < \log_2 j \leq L_2x + (L_2x)^{3/4}} \frac{1}{j} \ll (L_2x)^{3/4}.$$

Now from the definition of the Lévy metrics we obtain $\Lambda(U_x, V_x) \to 0$. Lemma 2 is proved.

Observe, since the distribution function $As(u)$ is continuous, the convergence $\Lambda(V_x, As) \to 0$ implies uniform convergence in $u \in \mathbf{R}$.

Lemma 3. *Let* $N_i := \exp\{\exp\{n_i\}\}$. *We have*

$$\nu_x(\delta) := \nu_x\bigl(|T(m, d_j(m)) - T(m, N_i)| > \delta\sqrt{n_i}\bigr) \ll \delta^{-2}i^{-1} + o_{\delta,i}(1)$$

uniformly in j such that $\log_2 j \in J_i$, for each $i = 2, \ldots, k$ and $\delta > 0$.

Proof. Let

$$\mathcal{N}_x = \{m \leq x : |L_2 d_j(m) - \log_2 j| < 2\sqrt{n_i L_2 n_i}\}.$$

Then by (1) we obtain $\nu_x(m \notin \mathcal{N}_x\} = o(1)$, and further,

$$\begin{aligned}
\nu_x(\delta) &\le \nu_x\Big(m \in \mathcal{N}_x : |T(m, d_j(m)) - T(m, N_i)| > \delta\sqrt{n_i}\Big) + o(1) \\
&\le \nu_x\Big(\max\{|T(m,k) - T(m, N_i)| : N_{i-2} \le k \le N_{i+1}\} > \delta\sqrt{n_i}\Big) + o(1) \\
&=: \nu + o(1),
\end{aligned}$$

uniformly in j, $\log_2 j \in J_i$, provided x is sufficiently large. But according to Theorem 1 and the definition of the process $X_x(m,t)$,

$$\begin{aligned}
&\nu = \\
&= P\Big(\max\{|W(t) - W(i/k)| : (i-2)/k \le t \le (i+1)/k\} > \delta\sqrt{i/k}\Big) + o(1) \\
&\quad \ll P\big(|W((i-2)/k) - W(i/k)| > 2^{-1}\delta\sqrt{i/k}\big) + \\
&\quad + P\big(|W((i+1)/k) - W(i/k)| > 2^{-1}\delta\sqrt{i/k}\big) + o(1) \ll \delta^{-2} i^{-1} + o(1).
\end{aligned}$$

At the last stage we have used the well-known Lévy and Kolmogorov inequalities (see [11]). Lemma 3 is proved.

LEMMA 4. *We have*

$$\begin{aligned}
M_{xk} &:= \frac{1}{xL_2x}\sum_{i=1}^{k}\sum_{\log_2 j \in J_i}\frac{1}{j}\sum_{m \le x}|I^+(T(m, d_j(m))) - I^+(T(m, N_i))| \\
&\ll k^{-1/4} + o(1).
\end{aligned}$$

Proof. Suppose $\log_2 j \in J_i$ and

$$\begin{aligned}
D_{ij} : &= \frac{1}{[x]}\sum_{m \le x}|I^+(T(m, d_j(m)) - I^+(T(m, N_i))| = \\
&= \nu_x\big(T(m, d_j(m)) \le 0,\, T(m, N_i) > 0\big) + \\
&+ \nu_x\big(T(m, d_j(m)) > 0,\, T(m, N_i) \le 0\big) =: \\
&=: \mu^1 + \mu^2.
\end{aligned}$$

Let $\delta > 0$ be arbitrary, then

$$\mu^1 \le \nu_x\big(0 < T(m, N_i) < \delta\sqrt{n_i}\big) + \nu_x\big(T(m, N_i) - T(m, d_j(m)) \ge \delta\sqrt{n_i}\big)$$

and

$$\mu^2 \le \nu_x\big(-\delta\sqrt{n}_i < T(m,N_i) \le 0\big) + \nu_x\big(T(m,d_j(m)) - T(m,N_i) \ge \delta\sqrt{n}_i\big).$$

Hence and by Lemma 3

$$\begin{aligned} D_{ij} &\le \nu_x\big(|T(m,N_i)| < \delta\sqrt{n}_i\big) + \nu_x(\delta) \ll \\ &\ll \nu_x\big(|T(m,N_i)| < \delta\sqrt{n}_i\big) + \delta^{-2} i^{-1} + o(1) \end{aligned} \tag{13}$$

provided that $2 \le i \le k$. Theorem 1 implies the one-dimensional limit relation

$$\begin{aligned} &\nu_x\big(|T(m,N_i)| < \delta\sqrt{n}_i\big) = \nu_x\big(|X_x(m,i/k)| < \delta\sqrt{i/k}\big) = \\ &= P\big(|W(i/k)| < \delta\sqrt{i/k}\big) + o(1) \ll \delta + o(1). \end{aligned}$$

Thus, from (13) we obtain

$$D_{ij} \ll \delta + \delta^{-2} i^{-1} + o(1)$$

uniformly in j, $\log_2 j \in J_i$, for each $2 \le i \le k$ and $\delta > 0$. Now

$$M_{xk} \ll \frac{n_i}{L_2 x} + \frac{1}{L_2 x} \sum_{i=2}^{k} \big(\delta + \delta^{-2} i^{-1} + o(1)\big) \sum_{\log_2 j \in J_i} \frac{1}{j} \ll \delta + \frac{Lk}{\delta^2 k} + o(1).$$

Choosing $\delta = k^{-1/3}$ we complete the proof of Lemma 4.

Lemma 5. *Let $V_x(u)$ be defined in Lemma 1 and*

$$W_{xk}(u) := \nu_x\bigg(\frac{1}{k}\sum_{i=1}^{k} I^+(T(m,N_i)) < u\bigg).$$

Then $\Lambda(V_x, W_{xk}) \ll k^{-1/8} + o_k(1)$.

Proof. Lemma 4 implies

$$\begin{aligned} &\nu_x\bigg(\bigg|\frac{1}{(L2)L_2 x}\sum_{i=1}^{k}\sum_{\log_2 j \in J_i} \frac{1}{j}\big(I^+(T(m,d_j(m)) - I^+(T(m,N_i))\bigg| > \delta\bigg) \\ &\qquad \ll \delta^{-1} M_{xk} \ll \delta^{-1}(k^{-1/4} + o(1)) \end{aligned}$$

for each $\delta > 0$. Now if

$$\widehat{W}_{xk}(u) := \nu_x\bigg(\frac{1}{(L2)L_2 x}\sum_{i=1}^{k}\sum_{\log_2 j\in J_i}\frac{1}{j}I^+(T(m,N_i) < u\bigg),$$

then

$$\Lambda(V_x, \widehat{W}_{xk}) \ll \delta + \delta^{-1}(k^{-1/4} + o(1))$$

for each $\delta > 0$. The choice $\delta = k^{-1/8}$ yields

$$\Lambda(V_x, \widehat{W}_{xk}) \ll k^{-1/8} + o(1).$$

Since $\widehat{W}_{xk}(u) = W_{xk}(u + o(1))$, Lemma 5 is proved.

The main probabilistic ingreedient is the following result of P. Erdös and M. Kac.

Lemma 6. *Let Y_i, $i \geq 1$, be independent, normally distributed random variables such that $\mathbf{E}Y_i = 0$ and $\mathbf{E}Y_i^2 = i$. If*

$$P_k(u) := P\bigg(\frac{1}{k}\sum_{i=1}^{k} I^+(Y_i) < u\bigg),$$

then $\Lambda(P_k, As) \to 0$ as $k \to \infty$.

Proof see [7].

Proof of Theorem 2. Denote by $\chi_u(y_1,\dots,y_k)$ the characteristic function of the set

$$\{(y_1,\dots,y_k) \in \mathbf{R}^k : I^+(y_1) + \cdots + I^+(y_k) < uk\}.$$

For the distribution function W_{xk} defined in Lemma 5 we have

$$W_{xk}(u) = \int_{\mathbf{R}^k} \chi_u(y_1,\dots,y_k)d\nu_x\big(T(m,N_1) < y_1,\dots,T(m,N_k) < y_k\big).$$

Further, we substitute $y_i \mapsto y_i\sqrt{k^{-1}L_2 x}$ in the integral on the right–hand side. Since the functional limit result presented in Theorem 1 implies weak convergence of the k-dimensional distributions, we obtain

$$\begin{aligned}
W_{xk}(u) &= \\
&= \int_{\mathbf{R}^k} \chi_u(y_1,\dots,y_k)d\nu_x\big(X_x(m,1/k)\sqrt{k} < y_1,\dots,X_x(m,1)\sqrt{k} < y_k\big) \\
&= \int_{\mathbf{R}^k} \chi_u(y_1,\dots,y_k)dP\big(W(1/k)\sqrt{k} < y_1,\dots,W(1)\sqrt{k} < y_k\big) + o(1) \\
&= P_k(u) + o(1)
\end{aligned}$$

uniformly in $u \in \mathbf{R}$. Hence and by Lemma 6

$$\lim_{k\to\infty} \overline{\lim}\, \Lambda(W_{xk}, As) = 0.$$

According to Lemmas 5 and 2 the last equality yields the assertion of Theorem 2.

Finally, what about the process Y_x? We expect that the following conjecture is true.

PROPOSITION. *Let $\mathcal{D}_1$ be the σ–algebra in $\mathbf{D}$ of the Borel sets generated by the Skorokhod topology (see* [2] *for the definition). There exits a probability measure Q on $\mathcal{D}_1$ such that $\nu_x \circ Y_x^{-1}$ weakly converges to Q.*

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References

[1] G.J. Babu, *Probabilistic methods in the theory of arithmetic functions*, Ph.D. dissertation, The Indian Statistical Institute, Calcutta, (1973).

[2] P. Billingsley, *Convergence of Probability Measures*, Wiley & Sons, New York, (1968).

[3] P. Billingsley, *Additive functions and Brownian motion*, Notices Amer. Math. Soc. **17** (1970), 1050.

[4] P. Billingsley, *The probability theory of additive arithmetic functions*, The Annals of Probab. **2 No 5** (1974), 749–791.

[5] J.-M. Deshouillers, F. Dress & G. Tenenbaum, *Lois de répartition des diviseurs,1*, Acta Arithm **34** (1979), 273–285.

[6] P. Erdös, *On the distribution of prime divisors*, Aequationes Math. **2** (1969), 177–183.

[7] P. Erdös, M. Kac, *On the number of positive sums of independent random variables*, Bull. of the American Math. Soc. **53** (1947), 1011–1020.

[8] B. Grigelionis, R. Mikulevičius, *On the functional limit theorems of the probabilistic number theory*, Lithuanian Math.J. **24 No 2** (1984), 72–81, (Russian).

[9] R.R. Hall, G. Tenenbaum, *Divisors*, Cambridge University Press Cambridge (1988).

[10] J. Kubilius, *Probabilistic Methods in the Theory of Numbers Transl. Math. Monographs*, Amer.Math.Soc., Providence, R.I. **11** (1964).

[11] M. Loève, *Probability Theory*, D. van Nostrand Company, New York (1963), (3rd edition).

[12] E. Manstavičius, *Arithmetic simulation of stochastic processes*, Lithuanian Math. J. **24 No 3** (1984)), 276–285.

[13] E. Manstavičius, *An invariance principle for additive arithmetic functions*, Soviet Math. Dokl. **37 No 1** (1988), 259–263.

[14] E. Manstavičius, "Probability Theory and Mathematical Statistics. Proceedings of the Sixth Vilnius Conference" (1993) B.Grigelionis et al Eds) VSP/TEV, (1994), 533–539.

[15] E. Manstavičius, *Functional approach in the divisor distribution problems*, Acta Math. Hungarica **66 No 3** (1995), 343–359.

[16] W. Philipp, *Arithmetic functions and Brownian motion*, Proc. Sympos. Pure Math. **24** (1973), 233–246.

[17] I.Z. Ruzsa, *Effective results in probabilistic number theory, In*, Théorie élémentaire et analytique des nombres ed. J.Coquet, 107–130, Dépt. Math. Univ. Valenciennes,.

[18] G. Tenenbaum, *Lois de répartition des diviseurs, 4*, Ann. Inst. Fourier **29** (1979), 1–15.

[19] N.M. Timofeev, Ch.Ch. Usmanov, *On arithmetical modelling of the Brownian motion*, Dokl. Acad. Sci. Tadz.SSR **25 No 4** (1982), 207–211, (Russian).

[20] N.M. Timofeev, Ch.Ch. Usmanov, *On arithmetical modelling of random processes with independent increments*, Dokl. Acad. Sci. TadzSSR **27 No 10** (1984), 556–559, (Russian).

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