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The distribution of square-free numbers of the form $[n^c]$

par Xiaodong CAO et Wenguang ZHAI

Résumé. Nous montrons que pour $1 < c < \frac{61}{36} = 1.6944\cdots$, la suite $[n^c]$ $(n = 1, 2, \cdots)$ contient une infinité d'entiers sans facteur carré ; cela améliore un résultat antérieur dû à Rieger qui obtenait l'infinitude de ces entiers pour $1 < c < 1.5$.

Abstract. It is proved that the sequence $[n^c]$ $(n = 1, 2, \cdots)$ contains infinite squarefree integers whenever $1 < c < \frac{61}{36} = 1.6944\cdots$, which improves Rieger's earlier range $1 < c < 1.5$.

1. Introduction

A positive integer n is called squarefree if it is a product of different primes. Following a paper of Stux [15], Rieger showed in [11] that for all real c with $1 < c < 1.5$, the equation

$$S_c(x) = \sum_{\substack{n \le x \\ [n^c]\text{squarefree}}} 1 = \frac{6}{\pi^2}x + O(x^{\frac{2c+1}{4}+\varepsilon}) \tag{1.1}$$

holds, which is an immediate consequence of Deshouillers [4]. Here $[t]$ denotes the fractional part of t and ε is a positive constant small enough. It is an easy exercise to prove that

$$S_c(x) = \frac{6}{\pi^2}x + o(x) \tag{1.2}$$

for $0 < c \le 1$. When $1 < c < 2$, one still expects (1.2) to hold, but if $c = 2$, $[n^c]$ is always a square, so that $S_c(x) = 0$.

It is worth remarking that Stux [15] has shown that $S_c(x)$ tends to infinity for almost all positive real $1 < c < 2$ (in the sense of Lebegue measure), however this result provides no specific value of c.

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The aim of this paper is to further improve Rieger's range $1 < c < 1.5$ by the method of exponential sums.

Basic Proposition. *Let* $1 < c < 2, \gamma = \frac{1}{c}$*, and* $x > 1$*. Then we have*

$$S_c(x) = \frac{6}{\pi^2}x + \Delta_c(x), \tag{1.3}$$

with

$$\Delta_c(x) = \sum_{n \le x^c} |\mu(n)| \left(\psi(-(n+1)^\gamma) - \psi(-n^\gamma)\right) + O(x^{1-c/2}), \tag{1.4}$$

where $\psi(t) = t - [t] - 1/2$ *and* $\mu(n)$ *is the well-known Mőbius function.*

Using the simple one-dimensional exponent pair, we can prove immediately from the Basic Proposition that

Corollary. *Let* $1 < c < 1.625$*, then for* $\varepsilon > 0$

$$S_c(x) = \frac{6}{\pi^2}x + O(x^{\frac{8c+3}{16}+\varepsilon}). \tag{1.5}$$

Combining Fouvry and Iwaniec's new method in [5] and Heath-Brown's new idea in [6], we can prove the following better Theorem .

Theorem. *Let* c *be a real constant such that* $1 < c < 61/36$*, then*

$$S_c(x) = \frac{6}{\pi^2}x + O(x^{\frac{36(c+1)}{97}+\varepsilon}). \tag{1.6}$$

Notations. $f(x) \ll g(x)$ means that $f(x) = O(g(x))$, $m \sim M$ means $c_1 M < m \le c_2 M$ for some constants $c_1, c_2 > 0$. We also use notations $L = \log(x), e(x) = \exp(2\pi i x)$ and $\psi(\theta) = \theta - [\theta] - 1/2$. To simplify writing logarithms, we will assume that all parameters are bounded by a power of x. Throughout the paper we allow the constants implied by 'O' or '$\ll$' to depend on only arbitrarily small positive number ε and c when it occurs.

2. Proofs of Basic Proposition and Corollary

Proof of Basic Proposition. It is well-known that (see [9])

$$\sum_{n \le x} |\mu(n)| = \frac{6}{\pi^2}x + O(x^{\frac{1}{2}}) \tag{2.1}$$

and

$$|\mu(n)| = \sum_{d^2 | n} \mu(d). \tag{2.2}$$

Obviously, $[n^c]$ is square-free if and only if $m^\gamma \le n < (m+1)^\gamma, m$ square-free. Therefore

$$(2.3) \quad S_c(x) = \sum_{\substack{n\le x\\ [n^c]\text{squarefree}}} 1 = \sum_{\substack{m\le x^c\\ m \text{ squarefree}}} ([-m^\gamma] - [-(m+1)^\gamma]) + O(1)$$

$$= \sum_{m\le x^c} |\mu(m)| \left((m+1)^\gamma - m^\gamma\right) + E_{1c}(x)$$

$$= \gamma \sum_{m\le x^c} |\mu(m)| m^{\gamma-1} + E_{2c}(x),$$

where

$$E_{jc}(x) = \sum_{m\le x^c} |\mu(m)| \left(\psi(-(m+1)^\gamma) - \psi(-m^\gamma)\right) + O(1), \qquad j=1,2.$$

¿From (2.3), (2.1) and partial summation we can get the Basic Proposition at once.

The proof of Corollary will need the following two lemmas. Lemma 1 is well-known (see [1]), Lemma 2 is contained in Theorem 18 of Vaaler [16].

Lemma 1. *Let* $|g^{(m)}(x)| \sim YX^{1-m}$ *for* $1 < X < x \le 2X$ *and* $m = 1, 2, \cdots$. *Then*

$$(2.4) \qquad \sum_{X<n\le 2X} e(g(n)) \ll Y^\kappa X^\lambda + Y^{-1}$$

where (κ, λ) *is any exponent pair.*

Lemma 2. *Suppose* $J > 1$. *There is a function* $\psi^*(x)$ *such that*

$$(1) \qquad \psi^*(x) = \sum_{1\le|h|\le J} \gamma(h) e(hx),$$

$$(2) \qquad \gamma(h) \ll \frac{1}{|h|} \quad \textit{and} \quad (\gamma(h))' \ll \frac{1}{h^2},$$

$$(3) \qquad |\psi^*(x) - \psi(x)| \le \frac{1}{2(J+1)} \sum_{|h|\le J} (1 - \frac{|h|}{J}) e(hx).$$

By Lemma 1 and Lemma 2 we immediately obtain

Lemma 3. *Let* $y > 0, X > 1, 0 \le \sigma < 1$, $g(n) = (n+\sigma)^\gamma$. *Then*

$$(2.5) \qquad \sum_{n\sim X} \psi(yg(n)) \ll y^{\frac{\kappa}{1+\kappa}} X^{\frac{\lambda+\gamma\kappa}{1+\kappa}} + y^{-1}X^{1-\gamma}.$$

Proof of Corollary. Taking $M = x^{\frac{8c-5}{16}}$, by (2.2) we have

$$(2.6) \qquad \sum_{n\le x^c} |\mu(n)| \left(\psi(-(n+1)^\gamma) - \psi(-n^\gamma)\right)$$

$$
\begin{aligned}
&= \sum_{d\le x^{\frac{c}{2}}}\mu(d)\sum_{n\le x^c d^{-2}}\big(\psi(-(d^2n+1)^\gamma)-\psi(-(d^2n)^\gamma)\big)\\
&= \sum_{d\le M}\mu(d)\sum_{n\le x^c d^{-2}}\big(\psi(-(d^2n+1)^\gamma)-\psi(-(d^2n)^\gamma)\big)\\
&\quad+\sum_{M<d\le x^{\frac{c}{2}}}\mu(d)\sum_{n\le x^c d^{-2}}\big(\psi(-(d^2n+1)^\gamma)-\psi(-(d^2n)^\gamma)\big).
\end{aligned}
$$

□

By Lemma 3 with $(\kappa,\lambda)=(\frac{2}{7},\frac{4}{7})$ and simple splitting argument we have

$$
\begin{aligned}
(2.7)\qquad &\sum_{d\le M}\mu(d)\sum_{n\le x^c d^{-2}}\big(\psi(-(d^2n+1)^\gamma)-\psi(-(d^2n)^\gamma)\big)\\
&\ll L\sum_{d\le M}\left(\left(d^{2\gamma}\right)^{\frac{\kappa}{1+\kappa}}\left(x^c d^{-2}\right)^{\frac{\lambda+\gamma\kappa}{1+\kappa}}+(d^{2\gamma})^{-1}(x^c d^{-2})^{1-\gamma}\right)\\
&\ll L\sum_{d\le M}x^{\frac{4c+2}{9}}d^{\frac{-8}{9}}+x^{c-1}L\\
&\ll x^{\frac{4c+2}{9}}M^{\frac{1}{9}}L+x^{c-1}L\\
&\ll x^{\frac{8c+3}{16}+\varepsilon}.
\end{aligned}
$$

By Lemma 3 with $(\kappa,\lambda)=(\frac{1}{6},\frac{2}{3})$ we have

$$
\begin{aligned}
(2.8)\qquad &\sum_{M<d\le x^{\frac{c}{2}}}\mu(d)\sum_{n\le x^c d^{-2}}\big(\psi(-(d^2n+1)^\gamma)-\psi(-(d^2n)^\gamma)\big)\\
&\ll L\sum_{M<d\le x^{\frac{c}{2}}}\left(\left(d^{2\gamma}\right)^{\frac{\kappa}{1+\kappa}}\left(x^c d^{-2}\right)^{\frac{\lambda+\gamma\kappa}{1+\kappa}}+(d^{2\gamma})^{-1}(x^c d^{-2})^{1-\gamma}\right)\\
&\ll L\sum_{M<d\le x^{\frac{c}{2}}}x^{\frac{4c+1}{7}}d^{-\frac{8}{7}}+x^{c-1}L\\
&\ll x^{\frac{4c+1}{7}}M^{-\frac{1}{7}}L+x^{c-1}L\\
&\ll x^{\frac{8c+3}{16}+\varepsilon}.
\end{aligned}
$$

Now the Corollary follows from (2.6), (2.7), (2.8) and the Basic Proposition.

3. Some Lemmas

Lemma 4. *Let $0 < a < b \le 2a$. Let $f(\frac{z}{a})$ be holomorphic on an open convex set* **I** *containing the real line segement $[1, b/a]$. Assume also that (1) $\left|f''(\frac{z}{a})\right| \le M$ on* **I**, *(2) $f(x)$ is real when x is real, (3) $f''(x) \le -cM$ for some $c > 0$. Let $f'(b) = \alpha$, $f'(a) = \beta$. For each integer v in the range $\alpha \le v \le \beta$, define x_v by $f'(x_v) = v$. Then*

$$\sum_{a<n\le b} e(f(n)) = \sum_{\alpha<v\le\beta} \frac{e(f(x_v) - vx_v - \frac{1}{8})}{\sqrt{|f''(x_v)|}} + O\left(M^{-\frac{1}{2}} + \log(2 + M(b-a))\right).$$

For the proof of Lemma 4, see Heath-Brown [6], Lemma 6.

Lemma 5. *Suppose A_i, B_j, a_i and b_j are all positive numbers. If Q_1 and Q_2 are real with $0 < Q_1 \le Q_2$, then there exists some q such that $Q_1 \le q \le Q_2$ and*

$$\sum_{i=1}^{m} A_i q^{a_i} + \sum_{j=1}^{n} B_j q^{-b_j} \le$$
$$2^{m+n}\left(\sum_{i=1}^{m}\sum_{j=1}^{n}(A_i^{b_j}B_j^{a_i})^{\frac{1}{a_i+b_j}} + \sum_{i=1}^{m} A_i Q_1^{a_i} + \sum_{j=1}^{n} B_j Q_2^{-b_j}\right).$$

This is Lemma 3 of Srinivasan [14].

Lemma 6. *Let $0 < M \le N < \mu N \le \lambda M$, and let a_m be complex numbers with $|a_m| \le 1$. Then we have*

$$\sum_{N<m\le\mu N} a_m = \frac{1}{2\pi}\int_{-M}^{M}\left(\sum_{M<m\le\lambda M} a_m m^{-it}\right) N^{it}(\mu^{it} - 1)t^{-1}dt +$$
$$O(\log(2 + M)).$$

See Lemma 6 of Fouvry and Iwaniec [5].

Lemma 7. *Let α, β_1, β_2 be given real numbers with $\alpha(\beta_1 - 1)\beta_2 \ne 0$ and $\alpha \notin N$. Let $|a_m| \le 1$, $|b_{m_1m_2}| \le 1, y \ne 0$ and*

$$S = S(M,\ M_1,\ M_2) = \sum_{m\sim M}\sum_{m_1\sim M_1}\sum_{m_2\sim M_2} a_m b_{m_1m_2} e(ym^{\alpha}m_1^{\beta_1}m_2^{\beta_2}).$$

Let $F = |y|M^{\alpha}M_1^{\beta_1}M_2^{\beta_2}$. Then we have

$$SL^{-3} \ll F^{\frac{\kappa}{2(1+\kappa)}} M^{\frac{1+\kappa+\lambda}{2(1+\kappa)}} (M_1M_2)^{\frac{2+\kappa}{2(1+\kappa)}} +$$
$$M(M_1M_2)^{\frac{1}{2}} + M^{\frac{1}{2}}M_1M_2 + F^{-\frac{1}{2}}MM_1M_2$$

where (κ, λ) is any exponent pair.

Lemma 7 can be proved in the same way as the proof of Theorem 2 of Baker [1]. The idea of the proof is due to Heath-Brown [6].

Lemma 8. *Under the conditions of Lemma 7, if we further suppose that* $F \gg M$, *then*

$$SL^{-3} \ll \sqrt[22]{(M_1M_2)^{19}M^{13}F^3} + M_1M_2M^{\frac{5}{8}}\sqrt[16]{1+M^7F^{-4}} + \sqrt[4]{(M_1M_2)^3M^4(1+FM^{-2})}.$$

Proof. This is Theorem 3 of Liu [8], which is proved essentially by the large sieve inequality developed by Bombieri and Ivaniec [3]. But the term

$$\sqrt[32]{(M_1M_2)^{29}M^{28}F^{-2}\max(1, F^5M^{-10})}$$

in Liu's result is superfluous, since

$$\sqrt[32]{(M_1M_2)^{29}M^{28}F^{-2}} = \left(\sqrt[16]{F^{-4}M^{17}}M_1M_2\right)^{1/4}\left(M^{5/8}M_1M_2\right)^{3/8} \times \left(M\sqrt[4]{M_1^3M_2^3}\right)^{3/8},$$

$$\sqrt[32]{(M_1M_2)^{29}M^{18}F^3} = \left(\sqrt[22]{F^3M^{13}}M_1^{19}M_2^{19}\right)^{11/16} \times \left(M^{5/8}M_1M_2\right)^{5/16}M^{-5/128}.$$

□

Lemma 9. *Let* $\frac{1}{2} < \gamma < 1, M > 1, H > 1, |c(h)| \le 1, |b(d)| \le 1$. *Let* $N = N_j = x^cM^{-2}2^{-j}, (j = 1, 2, \cdots)$, $F = H(M^2N)^\gamma$ *and*

$$S(H, M, N) = \sum_{h\sim H}\sum_{d\sim M}\sum_{x^cd^{-2}2^{-j}<n\le x^cd^{-2}2^{-j+1}} c(h)b(d)e\left(h(d^2n)^\gamma\right),$$

then

$$S(H, M, N)L^{-4} \ll F^{\frac{1}{2}}\left(M^{1+\lambda+\kappa}N^\kappa H^{2+\kappa}\right)^{\frac{1}{2(1+\kappa)}} + (FM)^{\frac{1}{2}}H + M(NH)^{\frac{1}{2}} + MH + F^{-\frac{1}{2}}MNH. \tag{3.1}$$

Proof. By Lemma 4 and partial summation we get

$$S(H, M, N) \ll F^{-\frac{1}{2}}N\left|\sum_{h\sim H}\sum_{d\sim M}\sum_{\tilde{n}\in I} a(d)b(h,\tilde{n})e\left((hd^{2\gamma})^{\frac{1}{1-\gamma}}\tilde{n}^{\frac{\gamma}{\gamma-1}}\right)\right| + F^{-\frac{1}{2}}MNH + LMH, \tag{3.2}$$

where $|a(d)| \le 1, |b(h,\tilde{n})| \le 1$ and

$$I = (\gamma 2^{(j-1)(1-\gamma)}x^{1-c}hd^2, \gamma 2^{j(1-\gamma)}x^{1-c}hd^2] \subset [c_1\frac{F}{N}, c_2\frac{F}{N}]$$

for some $c_1, c_2 > 0$.

Applying Lemma 6 to the variable $\tilde{n}$, we obtain that for some $|b_1(h,\tilde{n})| \le 1$

(3.3)
$$S(H,M,N)L^{-1} \ll F^{-\frac{1}{2}}N\left|\sum_{h\sim H}\sum_{d\sim M}\sum_{\tilde{n}\sim F/N} a(d)b_1(h,\tilde{n})e\left((hd^{2\gamma})^{\frac{1}{1-\gamma}}\ \tilde{n}^{\frac{\gamma}{\gamma-1}}\right)\right| + F^{-\frac{1}{2}}MNH + LMH.$$

Now we use Lemma 7 to estimate the above sum, with M, H, F/N in place of M, M_1, M_2. This completes the proof of Lemma 9. □

Lemma 10. *Under the conditions of Lemma 9, we have*

$$S(H,M,N)L^{-4} \ll \sqrt[22]{F^{11}M^{13}N^3H^{19}} + \sqrt[8]{F^4M^5H^8} + \sqrt[16]{F^4M^{17}H^{16}} + \sqrt[4]{FM^4NH^3} + \sqrt[4]{F^2M^2NH^3} + F^{-\frac{1}{2}}MNH. \tag{3.4}$$

Proof. Applying Lemma 8 to estimate the sum in (3.3), with M, H, F/N in place of M, M_1, M_2, we can obtain the bound (3.4) if we notice

$$MH = \left(\sqrt[16]{F^4M^{17}}H^{16}\right)^{2/3}\left(F^{-1/2}MNH\right)^{1/3}\left(MN^8\right)^{-1/24}.$$

□

4. Proof of Theorem

Taking $Y = x^{\frac{c}{3}}$, we have

$$\begin{aligned}
&\sum_{n\le x^c}|\mu(n)|\left(\psi(-(n+1)^\gamma)-\psi(-n^\gamma)\right)\\
&= \sum_{d\le x^{\frac{c}{2}}}\mu(d)\sum_{n\le x^cd^{-2}}\left(\psi(-(d^2n+1)^\gamma)-\psi(-(d^2n)^\gamma)\right)\\
&= \sum_{d\le Y}\mu(d)\sum_{n\le x^cd^{-2}}\left(\psi(-(d^2n+1)^\gamma)-\psi(-(d^2n)^\gamma)\right)\\
&\quad+\sum_{n\le Y}\sum_{Y<d\le x^{\frac{c}{2}}n^{-\frac{1}{2}}}\mu(d)\left(\psi(-(d^2n+1)^\gamma)-\psi(-(d^2n)^\gamma)\right)\\
&= \sum\nolimits_1 + \sum\nolimits_2, \quad \text{say.}
\end{aligned} \tag{4.1}$$

Let $M > 1$ and $N = N_j = x^cM^{-2}2^{-j}$ for $j = 1, 2, \cdots$. Define

(4.2)
$$T(M,N) = \sum_{M<d\le 2M}\mu(d)\sum_{\frac{x^c}{d^22^j}<n\le\frac{x^c}{d^22^{j-1}}}\left(\psi(-(d^2n+1)^\gamma)-\psi(-(d^2n)^\gamma)\right).$$

Lemma 11. *We have*

$$T(M,N)L^{-5} \ll \left((M^2N)^{(1+\gamma)(1+\kappa)}M^{\lambda-\kappa}\right)^{\frac{1}{3+2\kappa}} + (M^2N)^{\frac{1+\gamma}{3}} \tag{4.3}$$

$$+MN^{\frac{1}{2}} + M^{1-\gamma}N^{1-\frac{\gamma}{2}} + \left((M^2N)^{9\gamma}M^2N^4\right)^{\frac{1}{16}},$$

where (κ,λ) *is any exponent pair.*

Proof. By Lemma 2 we get for any $J>0$

$$T(M,N) \ll MNJ^{-1} + \sum_H \frac{1}{H}|S(H,M,N)| + \sum_H \frac{1}{H}|S_1(H,M,N)|, \tag{4.4}$$

where $H = J, \frac{J}{2}, \frac{J}{2^2}, \cdots$ and

$$S_1(H,M,N) = \sum_{h\sim H} c(h) \sum_{M<d\leq 2M} b_1(d) \sum_{\frac{x^c}{d^2 2^j}<n\leq \frac{x^c}{d^2 2^{j-1}}} e(-h(d^2n+1)^\gamma),$$

$|c(h)| \ll 1, b_1(d) \ll 1, S(H,M,N)$ is defined in Lemma 9. $\square$

Write $\Phi_h(y) = e\left(h(y^\gamma - (y+1)^\gamma)\right) - 1$. By partial summation and Lemma 1, we have

$$\sum_{X<n\leq 2X} e\left(-h(d^2n+1)^\gamma\right) - e\left(-h(d^2n)^\gamma\right) \tag{4.5}$$

$$= \sum_{X<n\leq 2X} \Phi_h(d^2n)e\left(-h(d^2n)^\gamma\right)$$

$$\ll \max_{X\leq t\leq 2X} |\Phi_h(d^2t)| \left|\sum_{X<n\leq t} e\left(-h(d^2n)^\gamma\right)\right|$$

$$+ \int_X^{2X} \left|\frac{\partial \Phi_h(d^2t)}{\partial t}\right| \left|\sum_{X<n\leq t} e\left(-h(d^2n)^\gamma\right)\right| dt$$

$$\ll h(d^2X)^{\gamma-1} \max_{X\leq t\leq 2X} \left|\sum_{X<n\leq t} e\left(h(d^2n)^\gamma\right)\right|$$

$$\ll h(d^2X)^{\gamma-1}\{\left(hd^{2\gamma}X^{\gamma-1}\right)^\kappa X^\lambda + (hd^{2\gamma}X^{\gamma-1})^{-1}\}$$

$$\ll h^{1+\kappa}(d^2X)^{\gamma-1+\kappa\gamma}X^{\lambda-\kappa} + d^{-2}.$$

Here we used $\Phi_h(d^2t) \ll h(d^2X)^{\gamma-1}$ and $\frac{\partial \Phi_h(d^2t)}{\partial t} \ll hd^2(d^2X)^{\gamma-2}$ for $X\leq t\leq 2X$.

Now take $(\kappa,\lambda)=(2/7,4/7)$ in (4.5) we have

$$S_1(H,M,N)=\sum_{h\sim H}c(h)\sum_{M<d\le 2M}b_1(d)\sum_{\frac{x^c}{d^2 2^j}<n\le\frac{x^c}{d^2 2^{j-1}}}e(-hd^{2\gamma}n^\gamma) \tag{4.6}$$

$$+\sum_{h\sim H}c(h)\sum_{M<d\le 2M}b_1(d)\sum_{\frac{x^c}{d^2 2^j}<n\le\frac{x^c}{d^2 2^{j-1}}}e(-h(d^2n+1)^\gamma)-e(-hd^{2\gamma}n^\gamma)$$

$$=\sum_{h\sim H}c(h)\sum_{M<d\le 2M}b_1(d)\sum_{\frac{x^c}{d^2 2^j}<n\le\frac{x^c}{d^2 2^{j-1}}}e(-hd^{2\gamma}n^\gamma)$$

$$+\sum_h\sum_d O(h^{9/7}(d^2N)^{9\gamma/7-1}N^{2/7}+d^{-2})$$

$$=\sum_{h\sim H}c(h)\sum_{M<d\le 2M}b_1(d)\sum_{\frac{x^c}{d^2 2^j}<n\le\frac{x^c}{d^2 2^{j-1}}}e(-hd^{2\gamma}n^\gamma)$$

$$+O\left(H^{\frac{16}{7}}(M^2N)^{\frac{9\gamma}{7}}M^{-1}N^{-\frac{5}{7}}\right).$$

¿From (4.4)—(4.6) we get

$$T(M,N)\ll\frac{MN}{J}+J^{\frac{9}{7}}(M^2N)^{\frac{9\gamma}{7}}M^{-1}N^{-\frac{5}{7}}+\sum_H\frac{1}{H}|S(H,M,N)|, \tag{4.7}$$

where in $S(H,M,N)$ the coefficient of d is $b(d)$ or $b_1(d)$.

Now use Lemma 9 to estimate the above sum and then choose a best $J\in(0,+\infty)$ by Lemma 5, we obtain the bound (4.3).

Lemma 12. *We have*

$$T(M,N)L^{-5}\ll\sqrt[30]{(M^2N)^{11\gamma}M^{21}N^{11}}+\sqrt[12]{(M^2N)^{4\gamma}M^9N^4} \tag{4.8}$$
$$+\sqrt[20]{(M^2N)^{4\gamma}M^{21}N^4}+\sqrt[5]{(M^2N)^{2\gamma}M^3N^2}$$
$$+\sqrt[4]{(M^2N)^{\gamma}M^4N}+\sqrt[16]{(M^2N)^{9\gamma}M^2N^4}+M^{1-\gamma}N^{1-\frac{\gamma}{2}}.$$

Proof. In the proof of Lemma 11, using Lemma 10 in place of Lemma 9 to estimate the sum in (4.8), we can get Lemma 12. □

Lemma 13. *We have*

$$\sum\nolimits_1\ll x^{\frac{36(c+1)}{97}}L^8. \tag{4.9}$$

Proof. In the proof of Lemma 13, we will use $M^2N\ll x^c$.

To estimate $\sum_1$, we consider the following cases:

Case (i) $M \le x^{\frac{32c+323}{1164}}$.
Similar to (2.6), by Lemma 3 with $(\kappa,\lambda)=(\frac{13}{31},\frac{16}{31})$ we get

$$\begin{aligned}(4.10)\qquad &\sum_{M<d\le 2M}\mu(d)\sum_{n\le \frac{x^c}{d^2}}\psi(-(d^2n+1)^\gamma)-\psi(-(d^2n)^\gamma)\\ &\ll Lx^{\frac{16c+13}{44}}\sum_{M<d\le 2M}d^{-\frac{32}{44}}+x^{c-1}L\\ &\ll x^{\frac{16c+13}{44}}\left(x^{\frac{32c+323}{1164}}\right)^{\frac{12}{44}}L\\ &\ll x^{\frac{36(c+1)}{97}}L.\end{aligned}$$

Case (ii) $Z_1 = x^{\frac{32c+323}{1164}} < M \le x^{\frac{14(c+1)}{97}} = Z$.
By Lemma 11 with $(\kappa,\lambda)=(\frac{11}{53},\frac{33}{53})$(see [6, pp. 265]) we obtain

$$\begin{aligned}(4.11)\qquad &\sum_{M<d\le 2M}\mu(d)\sum_{n\le \frac{x^c}{d^2}}\big(\psi(-(d^2n+1)^\gamma)-\psi(-(d^2n)^\gamma)\big)\\ &\ll \{\Big((x^c)^{\frac{64(1+\gamma)}{53}}M^{\frac{22}{53}}\Big)^{\frac{53}{181}}+(x^c)^{\frac{1+\gamma}{3}}\\ &\quad +x^{\frac{c}{2}}+(x^c)^{1-\frac{\gamma}{2}}M^{-1}+\big((x^c)^{9\gamma+4}M^{-6}\big)^{\frac{1}{16}}\}L^7\\ &\ll \{\Big(x^{64(1+c)}Z^{22}\Big)^{\frac{1}{181}}+x^{\frac{1+c}{3}}+x^{\frac{c}{2}}+x^{c-\frac12}Z_1^{-1}+\big(x^{9+4c}Z_1^{-6}\big)^{\frac{1}{16}}\}L^7\\ &\ll x^{\frac{36(c+1)}{97}}L^7.\end{aligned}$$

Case (iii) $Z = x^{\frac{14(c+1)}{97}} < M \le Y$.
By Lemma 12 we have

$$\begin{aligned}(4.12)\qquad &\sum_{M<d\le 2M}\mu(d)\sum_{n\le \frac{x^c}{d^2}}\big(\psi(-(d^2n+1)^\gamma)-\psi(-(d^2n)^\gamma)\big)\\ &\ll \{\sqrt[30]{(x^c)^{11+11\gamma}M^{-1}}+\sqrt[12]{(x^c)^{4+4\gamma}M}+\sqrt[20]{(x^c)^{4+4\gamma}M^{13}}\\ &\quad+\sqrt[5]{(x^c)^{2+2\gamma}M^{-1}}+\sqrt[4]{(x^c)^{1+\gamma}M^2}+\sqrt[16]{(x^c)^{4+9\gamma}M^{-6}}\\ &\quad+(x^c)^{1-\frac{\gamma}{2}}M^{-1}\}L^7\\ &\ll \{\sqrt[30]{x^{11(1+c)}Z^{-1}}+\sqrt[12]{x^{4(1+c)}Y}+\sqrt[20]{x^{4(1+c)}Y^{13}}\\ &\quad+\sqrt[5]{x^{2(1+c)}Z^{-1}}+\sqrt[4]{x^{1+c}Y^2}+\sqrt[16]{x^{4c+9}Z^{-6}}+x^{c-\frac12}Z^{-1}\}L^7\\ &\ll x^{\frac{36(c+1)}{97}}L^7.\end{aligned}$$

Combining (4.10)—(4.12) completes the proof of Lemma 13. $\square$

Lemma 14. *We have*

$$\sum\nolimits_2 \ll x^{\max(\frac{7c}{12},\frac{32c+9}{66},\frac{5c+3}{12})}L^8. \tag{4.13}$$

Proof. Let $N \geq 1$, $M = M_j = \frac{x^{\frac{c}{2}}}{N^{\frac{1}{2}}2^j}$ for $j = 1, 2, \cdots$, and $F = H(M^2N)^\gamma$, take $J = (M^2N)^{\frac{1-\gamma}{2}}$. By lemma 2 we have

$$\begin{aligned} T_1(N,M) &= \sum_{n\sim N} \sum_{\frac{x^{\frac{c}{2}}}{n^{\frac{1}{2}}2^j}<d\leq\frac{x^{\frac{c}{2}}}{n^{\frac{1}{2}}2^{j-1}}} r(d)\left(\psi(-(d^2n+1)^\gamma)-\psi(-(d^2n)^\gamma)\right) \\ &\ll MNJ^{-1} + \sum_H \frac{1}{H}|T_1(H,N,M)| + \sum_H \frac{1}{H}|T_2(H,N,M)|, \end{aligned} \tag{4.14}$$

where H runs through $J, \frac{J}{2}, \frac{J}{2^2}, \cdots$, and

$$T_1(H,N,M) = \sum_{h\sim H}\sum_{n\sim N} \sum_{\frac{x^{\frac{c}{2}}}{n^{\frac{1}{2}}2^j}<d\leq\frac{x^{\frac{c}{2}}}{n^{\frac{1}{2}}2^{j-1}}} c(h)r(d)e\left(-h(d^2n)^\gamma\right), \tag{4.15}$$

$$T_2(H,N,M) = \sum_{h\sim H}\sum_{n\sim N} \sum_{\frac{x^{\frac{c}{2}}}{n^{\frac{1}{2}}2^j}<d\leq\frac{x^{\frac{c}{2}}}{n^{\frac{1}{2}}2^{j-1}}} c_1(h)r_1(d)e\left(-h(d^2n+1)^\gamma\right),$$

for some $c(h) \ll 1, c_1(h) \ll 1, r(d) \ll 1, r_1(d) \ll 1$.

Since $e\left(-h(d^2n+1)^\gamma\right) - e\left(-h(d^2n)^\gamma\right) \ll |h|(d^2n)^{\gamma-1}$, we have

$$\begin{aligned} T_2(H,N,M) &= \sum_{h\sim H}\sum_{n\sim N} \sum_{\frac{x^{\frac{c}{2}}}{n^{\frac{1}{2}}2^j}<d\leq\frac{x^{\frac{c}{2}}}{n^{\frac{1}{2}}2^{j-1}}} c_1(h)r_1(d)e\left(-h(d^2n)^\gamma\right) \\ &+ \sum_{h\sim H}\sum_{n\sim N} \sum_{\frac{x^{\frac{c}{2}}}{n^{\frac{1}{2}}2^j}<d\leq\frac{x^{\frac{c}{2}}}{n^{\frac{1}{2}}2^{j-1}}} c_1(h)r_1(d)\Phi_h(d^2n)e\left(-h(d^2n)^\gamma\right) \\ &\ll \sum_{h\sim H}\sum_{n\sim N} \sum_{\frac{x^{\frac{c}{2}}}{n^{\frac{1}{2}}2^j}<d\leq\frac{x^{\frac{c}{2}}}{n^{\frac{1}{2}}2^{j-1}}} c_1(h)r_1(d)e\left(-h(d^2n)^\gamma\right) \\ &\quad +MN(M^2N)^{\gamma-1}LH. \end{aligned}$$

So it suffices to bound $T_1(H,M,N)$.

Now first applying Lemma 6 to the variable d and then using Lemma 8 to estimate the sum directly with $(\alpha,\beta_1,\beta_2) = (2\gamma,\gamma,1)$, we get

$$L^{-4}T_1(N,M) \ll \sqrt{(MN)^2(M^2N)^{\gamma-1}} + \sum_H \frac{1}{H}\{ \sqrt[22]{(HN)^{19}M^{13}F^3}$$
$$+HNM^{\frac{5}{8}} \sqrt[16]{1+M^7F^{-4}} + \sqrt[4]{(HN)^3M^4(1+FM^{-2})}\}.$$

Using the bound $M^2N \ll x^c$, one has

$$L^{-6}T_1(N,M) \ll \sqrt{(x^c)^{\gamma}N} + \sqrt[44]{(x^c)^{13+6\gamma}N^{25}} + \sqrt[16]{x^{5c}N^{11}}$$
$$+ \sqrt[32]{x^{17c-8}N^{15}} + \sqrt[4]{x^{2c}N} + \sqrt[4]{x^{1+c}N^2}. \tag{4.16}$$

Now we note that $N \leq x^{\frac{c}{3}}$, by (4.16) and simple splitting argument, the bound (4.13) could be obtained at once. □

Finally, Combining Lemma 13, Lemma 14, (4.1) and the basic Proposition completes the proof of Theorem .

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References

[1] R.C. Baker, *The square-free divisor problem.* Quart. J. Math. Oxford Ser.(2) **45** (1994), no. 179, 269–277.
[2] R.C. Baker, G. Harman, J. Rivat, *Primes of the form [n^c].* J. Number Theory **50** (1995), 261–277.
[3] E. Bombieri, H. Iwaniec, *On the order of* $\zeta(\frac{1}{2}+it)$. Ann. Scula Norm. Sup. Pisa Cl. Sci.(4) **13** (1986), no. 3, 449–472.
[4] J.-M. Deshouillers, *Sur la répartition des nombres* $[n^c]$ *dans les progressions arithmétiques.* C. R. Acad. Sci. Paris Sér. A **277** (1973), 647–650.
[5] E. Fouvry, H. Iwaniec, *Exponential sums with monomials.* J. Number Theory **33** (1989), 311–333.
[6] D.R. Heath-Brown, *The Pjateckiĭ-Šapiro prime number theorem.* J. Number Theory **16** (1983), 242–266.
[7] M.N. Huxley, M. Nair, *Power free values of polynomials III.* Proc. London Math. Soc.(3) **41** (1980), 66–82.
[8] H.-Q. Liu, *On the number of abelian groups of given order.* Acta Arith. **64** (1993), 285–296.
[9] H.L. Montgomery, R.C. Vaughan, *The distribution of square-free numbers.* in Recent Progress in Number Theory **vol 1**, pp. 247–256, Academic Press, London-New York, 1981.
[10] M. Nair, *Power free values of polynomials.* Mathematika **23** (1976), 159–183.
[11] G.J. Rieger, *Remark on a paper of Stux concerning squarefree numbers in non-linear sequences.* Pacific J. Math. **78** (1978), 241–242.
[12] F. Roesler, *Über die Verteilung der Primzahlen in Folgen der Form* $[f(n+x)]$. Acta Arith. **35** (1979), 117–174.
[13] F. Roesler, *Squarefree integers in non-linear sequences.* Pacific J. math. **123** (1986), 223–225.
[14] B.R. Srinivasan, *Lattice problems of many-dimensional hyperboloids II.* Acta Arith. **37** (1963), 173–204.
[15] I.E Stux, *Distribution of squarefree integers in non-linear sequences.* Pacific J. Math. **59** (1975), 577–584.

[16] J.D. Vaaler, *Some extremal problems in Fourier analysis.* Bull. Amer. Math. Soc. **12** (1985), 183–216.

Xiaodong Cao
Beijing Institute of Petrochemical Technology,
Daxing, Beijing 102600
P. R. China

Wenguang Zhai
Department of Mathematics
Shandong Normal University
Jinan, Shandong 250014
P.R. China