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On partitions without small parts

par J.-L. NICOLAS et A. SÁRKÖZY

RÉSUMÉ. On désigne par $r(n,m)$ le nombre de partitions de l'entier n en parts supérieures ou égales à m. En appliquant la méthode du point selle à la série génératrice, nous donnons une estimation asymptotique de $r(n,m)$ valable pour $n \to \infty$, et $1 \le m \le c_1 \dfrac{n}{(\log n)^{c_2}}$.

ABSTRACT. Let $r(n,m)$ denote the number of partitions of n into parts, each of which is at least m. By applying the saddle point method to the generating series, an asymptotic estimate is given for $r(n,m)$, which holds for $n \to \infty$, and $1 \le m \le c_1 \dfrac{n}{(\log n)^{c_2}}$.

1. Introduction

Let $r(n,m)$ (resp. $p(n,m)$) denote the number of partitions of the positive integer n into parts, each of which is at least m (resp. at most m), that is, the number of partitions of n of the type

$$n = i_1 + i_2 + \dots + i_r, \quad m \le i_1 \le i_2 \le \dots \le i_r \le n,$$
$$(\text{resp. } 1 \le i_1 \le \dots \le i_r \le m),$$

and let $q(m,n)$ denote the number of partitions of n into *distinct* parts, each of which is at least m, i.e., the number of solutions of

$$n = j_1 + j_2 + \dots + j_r, \quad m \le j_1 < j_2 < \dots < j_r \le n.$$

The function $p(n,m)$ has been extensively studied by Szekeres in [6] and [7].

In the last several years, Dixmier and Nicolas [1], [2] have studied the function $r(n,m)$ while Erdős, Nicolas and Szalay [4], [3] have estimated the function $q(n,m)$; in all these papers m is relatively small in terms of n. In [5], Freiman and Pitman gave an asymptotic formula for $q(n,m)$ in terms of a certain parameter $\sigma = \sigma(n,m)$ in a quite wide range for m:

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Theorem A (G.A. Freiman and J. Pitman, [5]). *As $n \to +\infty$ we have*

$$q(n,m) = \frac{1}{(2\pi B^2)^{1/2}} e^{\sigma n} \prod_{j=m}^{n} (1 + e^{-j\sigma})(1+E), \tag{1.1}$$

where σ and B are given by

$$n = \sum_{j=m}^{n} \frac{j}{1+e^{\sigma j}} \tag{1.2}$$

and

$$B^2 = \sum_{j=m}^{n} \frac{j^2 e^{\sigma j}}{(1+e^{\sigma j})^2},$$

and

$$E = E(m,n) = O((\log\, n)^{9/2} \max\{n^{-1/4}, (m/n)^{1/2}\})$$

uniformly with respect to m such that

$$1 \le m \le \frac{K_0 n}{(\log\, n)^9}.$$

Here K_0 and the implied constants in the estimate of E are effective positive constants independent of m and n.

They used a probabilistic approach in proving this theorem, however, their proof could be presented as well in terms of the saddle point method, without any reference to probability theory. Moreover, in certain intervals for m, they succeeded in making (1.1) more explicit by eliminating the parameter σ, i.e., by giving an asymptotics for $q(n,m)$ in terms of m and n only.

In this paper our goal is to prove the $r(n,m)$ analog of Theorem A and, in fact, we will prove a sharper result with a series expansion instead of a factor of type $1+E$ as in (1.1). Before formulating our main theorem, first we have to introduce a parameter σ (playing the same role as the parameter σ in Theorem A).

We shall write $e^{2\pi i\alpha} = e(\alpha)$ and $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ will denote the set of the positive integers, real numbers and complex numbers, respectively. For all $\alpha \in \mathbb{R}$ and $0 < R < 1$, writing $z = R\, e(\alpha)$ clearly we have

$$T(z) \overset{\text{def}}{=} \prod_{j=m}^{n} \frac{1}{1-z^j} = \sum_{\ell=0}^{+\infty} r^*(\ell,m) z^\ell \tag{1.3}$$

where $r^*(\ell, m) = r(\ell, m)$ for $0 \le \ell \le n$ and $0 \le r^*(\ell, m) \le r(\ell, m)$ for $n < \ell$. Substituting $\alpha = 0$ and dividing by R^n, we obtain

$$\begin{aligned} U(R) &\stackrel{\text{def}}{=} R^{-n} \prod_{j=m}^{n} \frac{1}{1-R^j} \\ &= \cdots + \frac{r(n-1,m)}{R} + r(n,m) + r^*(n+1,m)R + \cdots \\ &= r(n,m) + F_{n,m}(R), \end{aligned}$$

say. Clearly,

$$\lim_{R \to 0+0} U(R) = +\infty, \qquad \lim_{R \to 1-0} U(R) = +\infty,$$

and $U(R)$ is continuous in $(0,1)$. Thus $U(R)$ has a minimum in $(0,1)$. Moreover, $U(R)$ is twice differentiable in $(0,1)$, thus its minimum is attained at a point R_0 satisfying the equation

$$\begin{aligned} U'(R) &= -nR^{-(n+1)}T(R) + R^{-n}\Big(\sum_{j=m}^{n} \Big(\frac{1}{1-R^j}\Big)'(1-R^j)\Big) \prod_{j=m}^{n} \frac{1}{1-R^j} \\ &= R^{-n}T(R)\Big(-\frac{n}{R} + \sum_{j=m}^{n} \frac{jR^{j-1}}{1-R^j}\Big) = 0 \end{aligned}$$

or, in equivalent form,

$$\sum_{j=m}^{n} \frac{jR^j}{1-R^j} = n. \tag{1.4}$$

Setting

$$\xi(x) = \sum_{j=m}^{n} \frac{j}{x^j - 1},$$

(1.4) can be rewritten in the form

$$\xi(1/R) = \sum_{j=m}^{n} \frac{j}{(1/R)^j - 1} = n. \tag{1.5}$$

Clearly we have

$$\lim_{x \to 1+0} \xi(x) = +\infty, \qquad \lim_{x \to +\infty} \xi(x) = 0,$$

and $\xi(x)$ is decreasing in $(1, +\infty)$. Thus (1.4) determines R (< 1) uniquely. Define σ by

$$R = e^{-\sigma}$$

(with the R defined by (1.4)) so that $\sigma > 0$, and (1.5) can be rewritten in the form

$$\sum_{j=m}^{n} \frac{j}{e^{\sigma j}-1} = n. \tag{1.6}$$

For $h \in \mathbb{N}$, $\ell \in \mathbb{N}$ write

$$A(h,\ell) = \sum_{j=m}^{n} \frac{j^h}{(e^{\sigma j}-1)^\ell}$$

so that by (1.6),

$$A(1,1) = n. \tag{1.7}$$

We shall also write

$$B^2 = 2A(2,2) + 2A(2,1) = 2\sum_{j=m}^{n} j^2\Big(\frac{1}{(e^{\sigma j}-1)^2} + \frac{1}{e^{\sigma j}-1}\Big). \tag{1.8}$$

Clearly, we have

$$\begin{aligned} r(n,m) &= \int_0^1 (R\, e(\alpha))^{-n}\, T(R\, e(\alpha))d\alpha \\ &= R^{-n}\int_0^1 e(-n\alpha)\prod_{j=m}^{n}\frac{1}{1-R^j e(j\alpha)}\, d\alpha \\ &= e^{\sigma n}\int_0^1 e(-n\alpha)\prod_{j=m}^{n}\frac{1}{1-e^{-\sigma j}e(j\alpha)}\, d\alpha = e^{\sigma n} J, \end{aligned} \tag{1.9}$$

say. By estimating the integral J we will prove:

Theorem 1. *There exist real numbers $d(x,y)$ with $x \in \mathbb{N}$, $y \in \{1, 2, ..., x\}$ such that writing*

$$L_x = d(x,1)A(x,1) + d(x,2)A(x,2) + ... + d(x,x)A(x,x), \tag{1.10}$$

for any fixed $k \in \mathbb{N}$, $k \geq 3$ as $n \to +\infty$ we have

$$r(n,m) = \frac{1}{\sqrt{\pi}\, B}\, e^{\sigma n}\prod_{j=m}^{n}\frac{1}{1-e^{-\sigma j}}\, Q \tag{1.11}$$

with

$$Q = 1 + \sum_{1\leq \ell \leq (3k-2)/2} (-1)^\ell Q_{2\ell} + E \tag{1.12}$$

where

$$Q_{2\ell} = 2^{-\ell}(2\ell-1)(2\ell-3)\cdot\ldots\cdot 1\cdot L_2^{-\ell} \sum_{\max\{1,\frac{2\ell-k+2}{2}\}\le t\le k} \frac{1}{t!} \times \sum_{\substack{h_1+h_2+\ldots+h_t=2\ell\\ 3\le h_1,\ldots,h_t\le k}} L_{h_1}L_{h_2}\ldots L_{h_t} \tag{1.13}$$

and, for a certain absolute constant $H\ge 2$,

$$E \ll \begin{cases} n^{-(k-1)/4}(\log n)^{k^2/2} & \text{for } m\le Hn^{1/2}\\ \left(\frac{m}{n}\right)^{(k-1)/2}(\log n)^{2k^2} & \text{for } m> Hn^{1/2}\end{cases} \tag{1.14}$$

uniformly with respect to m such that

$$1\le m\le c_1 n(\log n)^{-7k}. \tag{1.15}$$

In particular, we have

$$\begin{aligned} L_2 &= 2A(2,1)+2A(2,2) \quad (=B^2),\\ L_3 &= \frac{4}{3}A(3,1)+4A(3,2)+\frac{8}{3}A(3,3),\\ L_4 &= \frac{2}{3}A(4,1)+\frac{14}{3}A(4,2)+8A(4,3)+4A(4,4),\\ L_5 &= \frac{4}{5}A(5,1)+4A(5,2)+\frac{40}{3}A(5,3)+16A(5,4)\\ &\quad+\frac{32}{15}A(5,5),\\ L_6 &= \frac{4}{45}A(6,1)+\frac{124}{45}A(6,2)+16A(6,3)+\frac{104}{3}A(6,4)\\ &\quad+32A(6,5)+\frac{32}{3}A(6,1)\end{aligned} \tag{1.16}$$

and for $k=6$ (1.12) holds with

$$Q = 1+\frac{3}{4}L_2^{-2}L_4-\frac{15}{8}L_2^{-3}(L_6+\frac{1}{2}L_3^2)+\frac{105}{16}L_2^{-4}(L_3L_5+\frac{1}{2}L_4^2) -\frac{1155}{16}L_2^{-5}L_3^2L_4+\frac{5005}{256}L_2^{-6}L_3^4+E \tag{1.17}$$

where $L_2,\ldots,L_6$ are defined above and where

$$E \ll \begin{cases} n^{-5/4}(\log n)^{18} & \text{for } m\le Hn^{1/2}\\ \left(\frac{m}{n}\right)^{5/2}(\log n)^{72} & \text{for } m> Hn^{1/2}.\end{cases} \tag{1.18}$$

In Theorem 1 and throughout the rest of the paper, the constants $c_1, c_2, \ldots$ and the constants implied by the notations $O(\ldots)$, $\ll$, $\asymp$ as well are effectively computable, and they may depend only on the parameter k in Theorem 1 but are independent of any other parameters, in particular, of m and n. (We write $f \ll g$ if $f = O(g)$, and $f \asymp g$ means that both $f \ll g$ and $g \ll f$ hold.)

Note that an upper bound will be given for the numbers $A(h, \ell)$ with $1 \le \ell \le h$ and thus also for L_h in Lemma 2, and the order of magnitude of $B = L_2^{1/2}$ will be determined in Lemma 3.

The rest of the paper will be devoted to the proof of Theorem 1, and in Part II of this paper the result will be made more explicit by eliminating the parameter σ in a wide range for m and giving a sharp estimate for the numbers L_x in Theorem 1.

In [2], it has been proved that there exists a function g such that, for $\lambda > 0$, the following asymptotic estimation holds :

$$\log r(n, \lambda\sqrt{n}) \sim g(\lambda)\sqrt{n}, \quad n \to \infty.$$

Moreover, the function $\lambda \mapsto g(\lambda) - \lambda \log\, \lambda$ was proved to be analytic in a neighbourhood of 0, so that it can be written

$$g(\lambda) = \lambda \log\, \lambda + a_0 + a_1\lambda + a_2\lambda^2 + \cdots + a_j\lambda^j + \cdots$$

and, for $j \ge 2$, the values of the coefficients a_j are rational fractions of $C = \pi\sqrt{\frac{2}{3}}$:

$$a_0 = C, \quad a_1 = \log\, \frac{C}{2} - 1, \quad a_2 = -\frac{1}{4}\left(2C + \frac{1}{2C}\right), \quad a_3 = \frac{C^2 + 36}{288}, \quad \cdots$$

The following relation was also given in [2] :

$$r(n,m) \sim p(n)\left(\frac{C}{\sqrt{n}}\right)^{m-1}(m-1)!\exp\,\left[-\frac{1}{4}\left(2C + \frac{1}{2C}\right)\frac{m^2}{\sqrt{n}}\right], \qquad m \le n^{1/3-\varepsilon}$$

where, in the bracket, the coefficient a_2 shows up. In Part II, it will be proved that

$$r(n,m) \sim p(n)\left(\frac{C}{\sqrt{n}}\right)^{m-1}(m-1)!\exp\,\left[\sqrt{n}\,\widetilde{g}\left(\frac{m^2}{\sqrt{n}}\right)\right], \qquad m \le n^{1/2} \tag{1.19}$$

where $\widetilde{g}(\lambda) = g(\lambda) - \lambda \log\, \lambda - a_0 - a_1\lambda$. In the above relation, whenever $m < n^{1/2-\varepsilon}$, $\widetilde{g}$ can be replaced by the first terms of its power series expansion, for instance :

$$r(n,m) \sim p(n) \left(\frac{C}{\sqrt{n}}\right)^{m-1} (m-1)! \exp\left[a_2 \frac{m^2}{\sqrt{n}} + a_3 \frac{m^3}{n}\right], \quad m \le n^{3/8-\varepsilon}.$$

In Part II, the error term in the asymptotic estimation (1.19) of $r(n,m)$ will also be precised.

2. Estimate of σ

We shall prove

Lemma 1. *Let $k \ge 1$ be an integer. There is a positive number $H = H(k)$ such that $H \ge 2$ and*

$$\sigma \asymp n^{-1/2} \quad \text{for} \quad m \le H n^{1/2} \tag{2.1}$$

and

$$\begin{aligned} \sigma &= m^{-1}(\log(m^2/n) - \log\log(m^2/n) + O(1)) \\ &\quad (> \tfrac{1}{2} m^{-1} \log(m^2/n)) \ \text{for} \ Hn^{1/2} < m \ (\ll n(\log n)^{-7k}). \end{aligned} \tag{2.2}$$

Proof of Lemma 1. For large n we have

$$\sigma > \frac{5}{n} \tag{2.3}$$

since otherwise by (1.15) for large m we should have

$$\sum_{j=m}^{n} \frac{j}{e^{\sigma j}-1} \ \ge \sum_{j=n-300}^{n} \frac{j}{e^{\sigma j}-1} \ \ge \ 300\,\frac{n-300}{e^{\sigma n}-1} \ \ge \ 300 \frac{n-300}{e^5-1} \ > \ n$$

which contradicts (1.6).

On the other hand, we have

$$\sigma < 1 \tag{2.4}$$

since otherwise we had

$$\sum_{j=m}^{n} \frac{j}{e^{\sigma j}-1} \le \sum_{j=m}^{n} \frac{n}{e^{j}-1} < \sum_{j=m}^{n} \frac{2n}{e^j} \le 2n \sum_{j=1}^{+\infty} e^{-j} = n\,\frac{2}{e-1} < n$$

again in contradiction with (1.6).

We split the sum on the left hand side of (1.6) into two parts:

$$\begin{aligned} n = \sum_{j=m}^{n} \frac{j}{e^{\sigma j}-1} &= \sum_{m \le j \le 1/\sigma} \frac{j}{e^{\sigma j}-1} + \sum_{\max(m,1/\sigma) \le j \le n} \frac{j}{e^{\sigma j}-1} \\ &= \sum\nolimits_1 + \sum\nolimits_2 \end{aligned} \tag{2.5}$$

say. Clearly we have

$$\sum\nolimits_1 = 0 \quad \text{for} \quad 1/\sigma < m \tag{2.6}$$

and

$$\sum\nolimits_1 \asymp \sum_{m \le j \le 1/\sigma} \frac{j}{\sigma j} \asymp \frac{1}{\sigma}\Big(\frac{1}{\sigma} - m + 1\Big) \quad \text{for} \quad 1/\sigma \ge m. \tag{2.7}$$

Moreover, writing $M = \max(m, 1/\sigma)$ we have

$$\sum\nolimits_2 \asymp \sum_{M \le j \le n} \frac{j}{e^{\sigma j}} = \sum\nolimits_2', \tag{2.8}$$

say. The function xe^{-x} is decreasing for $x \ge 1$ and thus the terms in the sum $\sum_2'$ decrease as j increases. Thus we have

$$\int_{M+1}^{n} xe^{-\sigma x} dx \le \sum\nolimits_2' \le Me^{-M\sigma} + \int_{M}^{n} xe^{-\sigma x} dx. \tag{2.9}$$

Since

$$\int xe^{-\sigma x} dx = -\frac{x}{\sigma}\, e^{-\sigma x} - \frac{e^{-\sigma x}}{\sigma^2},$$

thus it follows from $M \ge 1/\sigma$, (2.3), (2.4) and (2.9) that

$$\sum\nolimits_2' \le Me^{-M\sigma} + \frac{M}{\sigma}\, e^{-\sigma M} + \frac{e^{-\sigma M}}{\sigma^2} \ll \frac{M}{\sigma}\, e^{-\sigma M} \tag{2.10}$$

and

$$\sum\nolimits_2' \ge \frac{M+1}{\sigma}\, e^{-\sigma(M+1)} - \frac{n}{\sigma}\, e^{-\sigma n} - \frac{e^{-\sigma n}}{\sigma^2} > \frac{M}{e\sigma}\, e^{-\sigma M} - \frac{2n}{\sigma}\, e^{-\sigma n}. \tag{2.11}$$

The function $f(x) = xe^{-\sigma x}$ is decreasing for $x \ge 1/\sigma$ and by (1.15) and (2.3) we have

$$M = \max(m, \frac{1}{\sigma}) \le \max(m, \frac{n}{5}) = \frac{n}{5}$$

and thus

$$ne^{-\sigma n} < (5M)e^{-\sigma(5M)} = 5Me^{-\sigma M - 4\sigma M} \le 5Me^{-\sigma M - 4}. \tag{2.12}$$

By (2.11) and (2.12) we have

$$\sum\nolimits_2' > \frac{M}{e\sigma}\, e^{-\sigma M} - (10e^{-4})\,\frac{M}{\sigma}\, e^{-\sigma M} > \frac{1}{10}\,\frac{M}{\sigma}\, e^{-\sigma M}. \tag{2.13}$$

It follows from (2.8), (2.10) and (2.13) that

$$\sum\nolimits_2 \asymp \frac{M}{\sigma}\, e^{-\sigma M}. \tag{2.14}$$

Now we have to distinguish two cases.

Case 1. Assume first that

$$\frac{1}{\sigma} \geq m. \tag{2.15}$$

Then by (2.5), (2.7), (2.14) and (2.15) we have

$$\begin{aligned} n = \sum\nolimits_1 + \sum\nolimits_2 \asymp \frac{1}{\sigma}\Big(\frac{1}{\sigma} - m + 1\Big) + \frac{M}{\sigma}\, e^{-\sigma M} \\ = \frac{1}{\sigma}\Big(\frac{1}{\sigma} - m + 1\Big) + \frac{1}{\sigma^2}\, e^{-1} \asymp \frac{1}{\sigma^2} \end{aligned} \tag{2.16}$$

whence

$$\sigma \asymp n^{-1/2} \quad (\text{for } \ 1/\sigma \geq m). \tag{2.17}$$

It follows from (2.15) and (2.17) that

$$m \ll n^{1/2} \quad (\text{for } \ 1/\sigma \geq m). \tag{2.18}$$

Case 2. Assume now that

$$\frac{1}{\sigma} < m. \tag{2.19}$$

Then by (2.5), (2.6), (2.14) and (2.19) we have

$$n = \sum\nolimits_1 + \sum\nolimits_2 = \sum\nolimits_2 \asymp \frac{m}{\sigma}\, e^{-\sigma m}. \tag{2.20}$$

Writing

$$f(x) = xe^x, \tag{2.21}$$

(2.20) can be rewritten in the form

$$f(m\sigma) \asymp \frac{m^2}{n} \quad \Big(\text{for } \ \frac{1}{\sigma} < m\Big). \tag{2.22}$$

By (2.19) here we have $m\sigma > 1$, and $f(x) > 1$ for $x > 1$. Thus (2.22) implies

$$\frac{m^2}{n} \gg 1$$

whence

$$m \gg n^{1/2} \quad (\text{for } \ 1/\sigma < m). \tag{2.23}$$

Moreover, for $x \to +\infty$ (2.21) implies

$$x = \log\, f(x) - \log\, \log\, f(x) + o(1),$$

and thus it follows from $y \to +\infty$, $f(x) \asymp y$ (note that this implies $x \to +\infty$) that

$$x = \log\, y - \log\, \log\, y + O(1) \quad (\text{for } \ y \to +\infty, \ \ f(x) \asymp y).$$

Thus it follows from (2.22) that

$$(2.24)\quad m\sigma = \log(m^2/n) - \log\log(m^2/n) + O(1) \qquad (\text{for } 1/\sigma < m,\ m^2/n \to +\infty).$$

Now we are ready to prove (2.1) and (2.2). By (2.23), there is a positive number K such that $m < Kn^{1/2}$ implies $1/\sigma \geq m$ and then by (2.17), σ satisfies (2.1) (for $m < Kn^{1/2}$). On the other hand, if H is large enough and $Hn^{1/2} < m$, then $1/\sigma < m$ by (2.18) and thus (2.2) holds by (2.24) (for $m > Hn^{1/2}$). Finally, if $Kn^{1/2} \leq m \leq Hn^{1/2}$, then for $1/\sigma \geq m$, (2.1) holds by (2.17) while for $1/\sigma < m$ it follows from (2.22) and this completes the proof of Lemma 1. $\square$

3. Estimate of the sums $A(h,\ell)$

Lemma 2. *For all $K \in \mathbb{N}$ there is a positive number $c_2 = c_2(K)$ such that*

$$(3.1)\qquad A(h,\ell) < \begin{cases} c_2 n^{(h+1)/2} & \text{for } m \leq Hn^{1/2} \\ c_2 m^{h+1}(\frac{n}{m^2})^{\ell}(\log(m^2/n))^{\ell-1} & \text{for } m > Hn^{1/2} \end{cases}$$

for $h = 1,2,\dots,K$ and $\ell = 1,2,\dots,h$ (with H defined in Lemma 1).

Proof of Lemma 2. We split the sum in the definition of $A(h,\ell)$ into two parts:

$$(3.2)\quad A(h,\ell) = \sum_{m \leq j \leq Hn^{1/2}} \frac{j^h}{(e^{\sigma j}-1)^{\ell}} + \sum_{\max(m,Hn^{1/2}) < j \leq n} \frac{j^h}{(e^{\sigma j}-1)^{\ell}} = \sum\nolimits_1 + \sum\nolimits_2,$$

say. Clearly,

$$(3.3)\qquad \sum\nolimits_1 = 0 \quad \text{for} \quad m > Hn^{1/2},$$

while for $m \leq Hn^{1/2}$ by (2.1) in Lemma 1 we have

$$(3.4)\quad \sum\nolimits_1 \leq \sum_{m \leq j \leq Hn^{1/2}} \frac{j^h}{(\sigma j)^{\ell}} = \sigma^{-\ell} \sum_{m \leq j \leq Hn^{1/2}} j^{h-\ell} \ll n^{\ell/2} \sum_{j \leq Hn^{1/2}} (Hn^{1/2})^{h-\ell} \ll n^{\ell/2}(n^{1/2})^{h-\ell+1} = n^{(h+1)/2} \qquad (\text{for } m \leq Hn^{1/2}).$$

Moreover, if $m \leq Hn^{1/2}$ then by (2.1) for all j in $\sum_2$ we have

$$\sigma j \gg n^{-1/2}(Hn^{1/2}) \gg 1$$

and thus

$$(e^{\sigma j}-1)^{-1} \ll e^{-\sigma j}. \tag{3.5}$$

For $m > Hn^{1/2}$ by (2.2) for all j in $\sum_2$ we have

$$\sigma j > \left(\frac{1}{2}\, m^{-1} \log(m^2/n)\right)m = \frac{1}{2}\log(m^2/n) \gg 1$$

so that (3.5) holds also in this case. By (3.5), writing $m_1 = \max(m, Hn^{1/2})$ we have

$$\sum\nolimits_2 \ll \sum_{m_1 < j \le n} j^h e^{-\sigma j \ell}.$$

An easy consideration shows that uniformly for $m_1 < j \le n$ and $j-1 \le x \le j$ we have

$$j^h\, e^{-\sigma j\ell} \ll x^h\, e^{-\sigma x \ell}$$

and thus

$$j^h\, e^{-\sigma j\ell} \ll \int_{j-1}^{j} x^h\, e^{-\sigma x\ell} dx$$

so that

$$\sum\nolimits_2 \ll \int_{m_1-1}^{n} x^h\, e^{-\sigma x \ell} dx. \tag{3.6}$$

Since we have

$$\int x^h\, e^{-\sigma x\ell} dx = -\Big(\frac{x^h}{\sigma\ell} + \sum_{t=2}^{h+1} \frac{h(h-1)\ldots(h-t+2)}{(\sigma\ell)^t}\, x^{h-t+1}\Big)\cdot e^{-\sigma x \ell}$$

(which can be shown easily by induction on h) thus it follows from (3.6) that

$$\sum\nolimits_2 \ll \Big(\frac{m_1^h}{\sigma\ell} + \sum_{t=2}^{h+1} \frac{h(h-1)\ldots(h-t+2)}{(\sigma\ell)^t}\, m_1^{h-t+1}\Big) e^{-\sigma(m_1-1)\ell}. \tag{3.7}$$

By Lemma 1, it follows from the definition of m_1 that $m_1\sigma \gg 1$. Moreover, by Lemma 1 we have $\sigma = o(1)$. Thus it follows from (3.7) that

$$\sum\nolimits_2 \ll \frac{m_1^h}{\sigma}\, e^{-\sigma m_1 \ell}$$

whence, again by Lemma 1 and the definition of m_1, we have

$$\sum\nolimits_2 \ll \frac{n^{h/2}}{n^{-1/2}} = n^{(h+1)/2} \quad \text{for } \ m \le Hn^{1/2} \tag{3.8}$$

and

$$\begin{aligned} \sum\nolimits_2 &\ll m^{h+1}(\log(m^2/n))^{-1} \exp((\log\log(m^2/n) - \log(m^2/n))\ell) \\ &= m^{h+1}\Big(\frac{n}{m^2}\Big)^{\ell} (\log(m^2/n))^{\ell-1} \quad \text{for } \ m > Hn^{1/2}. \end{aligned} \tag{3.9}$$

(3.1) follows from (3.2), (3.3), (3.4), (3.8) and (3.9) which completes the proof of Lemma 2. □

4. Estimate of B

Lemma 3. *We have*

$$B^2 = 2(A(2,2) + A(2,1)) \gg A(2,1) \gg \begin{cases} n^{3/2} & \textit{for } m \le Hn^{1/2} \\ mn & \textit{for } m > Hn^{1/2}. \end{cases}$$

Proof of Lemma 3. By Lemma 1 and (1.15) we have $m + 1/\sigma \le n$ and thus, again by Lemma 1,

$$\begin{aligned} A(2,1) &= \sum_{j=m}^{n} \frac{j^2}{e^{\sigma j} - 1} \ge \sum_{m+1/2\sigma \le j \le m+1/\sigma} \frac{j^2}{e^{\sigma j} - 1} \\ &\gg \frac{1}{\sigma} \frac{(m+1/2\sigma)^2}{e^{\sigma m + 1} - 1} \gg \frac{1}{\sigma} \frac{(m+1/2\sigma)^2}{e^{\sigma m}} \\ &\gg \begin{cases} \frac{1}{\sigma} \frac{(1/\sigma)^2}{e^{\sigma m}} \gg \sigma^{-3} \gg n^{3/2} & \text{for } m \le Hn^{1/2} \\ \frac{1}{\sigma} \frac{m^2}{e^{\sigma m}} \gg \frac{m}{\log(m^2/n)} \frac{m^2}{(m^2/n)(\log(m^2/n))^{-1}} = mn & \text{for } m > Hn^{1/2} \end{cases} \end{aligned}$$

which completes the proof of the lemma. □

5. Estimate of the integrand far from 0

We split the integral J defined in (1.9) into two parts. Set

$$\eta = \begin{cases} c^* n^{-3/4} (\log n)^{1/2} & \text{for } m \le Hn^{1/2} \\ c^* (nm)^{-1/2} (\log n)^{3/2} & \text{for } m > Hn^{1/2} \end{cases} \tag{5.1}$$

where c^* is a positive number, large enough in terms of k, which will be determined later. Let

$$\varphi_j(\alpha) = \frac{1}{1 - R^j\, e(j\alpha)} = \frac{1}{1 - e^{-\sigma j}\, e(j\alpha)}$$

and

$$\varphi(\alpha) = \prod_{j=m}^{n} \varphi_j(\alpha)$$

so that

$$J = \int_0^1 e(-n\alpha)\varphi(\alpha) d\alpha\,,$$

and write

$$
\begin{aligned}
(5.2) \qquad J &= \int_0^1 e(-n\alpha)\varphi(\alpha)d\alpha \\
&= \int_{-\eta}^{+\eta} e(-n\alpha)\varphi(\alpha)d\alpha + \int_{\eta\le|\alpha|\le 1/2} e(-n\alpha)\varphi(\alpha)d\alpha \\
&= J_1 + J_2,
\end{aligned}
$$

say. We will show that the integral J_1 gives the main contribution while J_2 gives only a small error term. Thus we shall need an asymptotic formula for J_1 and an upper bound for J_2. First we will estimate the integrand in J_2.

Lemma 4. *For $|\alpha| \le 1/2$, $m \in \mathbb{N}$, $k \in \mathbb{N}$, $k \ge 2$ we have*

$$\sum_{j=m}^{m+k-1} \sin^2 \pi j\alpha \ge \frac{k}{4} \min\{1, (\alpha k)^2\}.$$

Proof of Lemma 4. This is Lemma 8 in [5]. □

Lemma 5. *Let η be defined by (5.1). Uniformly for*

$$(5.3) \qquad \eta \le |\alpha| \le 1/2$$

we have

$$(5.4) \qquad |\varphi(\alpha)| \ll \varphi(0)n^{-k-2}.$$

Proof of Lemma 5. For all $0 < r < 1$ and $\beta \in \mathbb{R}$, writing $z = r\ e(\beta)$ we have

$$
\begin{aligned}
|1-z|^2 &= 1 + |z|^2 - 2\,Re z = 1 + r^2 - 2r \cos 2\pi\beta \\
&= (1-r)^2 + 2r(1-\cos 2\pi\beta) = (1-r)^2 + 4r \sin^2 \pi\beta \\
&= (1-r)^2(1 + \frac{4r}{(1-r)^2} \sin^2 \pi\beta)
\end{aligned}
$$

whence

$$\frac{(1-r)^2}{|1-z|^2} = (1 + \frac{4r}{(1-r)^2} \sin^2 \pi\beta)^{-1} \le (1 + 4r \sin^2 \ \pi\beta)^{-1}$$

since $(1-r)^2 < 1$. It follows that

$$\text{(5.5)} \qquad \begin{aligned} \frac{|\varphi(\alpha)|^2}{(\varphi(0))^2} &= \prod_{j=m}^{n} \frac{|\varphi_j(\alpha)|^2}{|\varphi_j(0)|^2} = \prod_{j=m}^{n} \frac{(1-e^{-\sigma j})^2}{|1-e^{-\sigma j}\, e(j\alpha)|^2} \\ &\le \prod_{j=m}^{n} \left(1+4e^{-\sigma j}\,\sin^2 \pi j\alpha\right)^{-1} \\ &\le \prod_{j=m}^{m+[1/\sigma]} \left(1+4e^{-\sigma j}\,\sin^2 \pi j\alpha\right)^{-1} \end{aligned}$$

since by (1.15) and Lemma 1 we have

$$m + [1/\sigma] < n.$$

For $m \le j \le m+[1/\sigma]$, the following inequality holds:

$$e^{-\sigma j} \ge e^{-\sigma(m+[1/\sigma])} \ge e^{-\sigma m-1}$$

and this implies

$$\text{(5.6)} \qquad \frac{|\varphi(\alpha)|^2}{(\varphi(0)^2} \le \prod_{j=m}^{m+[1/\sigma]} \left(1+4e^{-\sigma m-1}\,\sin^2 \pi j\alpha\right)^{-1}.$$

For $0 < x < 2$ a simple calculation shows that

$$\frac{1}{1+x} < e^{-x/2}$$

and, since $4e^{-\sigma m-1}\sin^2 \pi j\alpha < 4/e < 2$, it follows from (5.6) that

$$\text{(5.7)} \qquad \frac{|\varphi(\alpha)|^2}{(\varphi(0))^2} < \exp\left(-2e^{-\sigma m-1} \sum_{j=m}^{m+[1/\sigma]} \sin^2 \pi j\alpha\right).$$

By Lemma 1, Lemma 4, (5.1) and (5.3), it follows that

$$\text{(5.8)} \qquad \begin{aligned} \frac{|\varphi(\alpha)|^2}{(\varphi(0))^2} &< \exp\left(-e^{-\sigma m-1}\frac{1}{2\sigma}\min(1,(\alpha/\sigma)^2)\right) \\ &< \exp\left(-e^{-\sigma m-1}\frac{1}{2\sigma}\min(1,(\eta/\sigma)^2)\right) \\ &= \exp(-\frac{e^{-\sigma m-1}}{2}\eta^2\sigma^{-3}). \end{aligned}$$

Now, in the first case ($m \le H\sqrt{n}$), by (2.1), $\sigma m = O(1)$, and from (2.1) and (5.1), $\eta^2\sigma^{-3} \ge c_3 c^{*2} \log n$ so that, by choosing c^* large enough (in terms of k), (5.8) yields (5.4).

In the second case ($m > H\sqrt{n}$), by (2.2)

$$e^{-\sigma m} \gg \frac{n}{m^2} \log \frac{m^2}{n}$$

and from (2.2) and (5.1)

$$\frac{e^{-\sigma m-1}}{2}\eta^2\sigma^{-3} \ge c_4 c^{*2} \frac{(\log n)^3}{\left(\log \frac{m^2}{n}\right)^2} \ge c_4 c^{*2} \log n$$

since $m \le n$, and again, by choosing c^* large enough, (5.8) yields (5.4), which completes the proof of the lemma. $\square$

6. Estimate of J_2

Lemma 6. *We have*

$$|J_2| \ll B^{-1} n^{-k} \prod_{j=m}^{n} \frac{1}{1-e^{-\sigma j}} = B^{-1} n^{-k} \varphi(0). \tag{6.1}$$

Proof of Lemma 6. By Lemma 5 we have

$$\begin{aligned} |J_2| &= \left| \int_{\eta \le |\alpha| \le 1/2} e(-n\alpha)\varphi(\alpha) d\alpha \right| \\ &\le \int_{\eta \le |\alpha| \le 1/2} |\varphi(\alpha)| d\alpha \ll \int_{\eta \le |\alpha| \le 1/2} \varphi(0) n^{-k-2}\, d\alpha \\ &\le \varphi(0) n^{-k-2} = n^{-k-2} \prod_{j=m}^{n} \frac{1}{1-e^{-\sigma j}}. \end{aligned} \tag{6.2}$$

Moreover, by Lemma 2 and (1.8) we have

$$B^2 = 2A(2,2) + 2A(2,1) \ll n^3. \tag{6.3}$$

(6.1) follows from (6.2) and (6.3). $\square$

7. Estimate of the integrand near 0

For $m \le j \le n$, write

$$\begin{aligned} a_j &= \frac{1}{e^{\sigma j}-1}, \quad b_j = b_j(\alpha) = 1 - e(j\alpha), \\ b'_j &= b'_j(\alpha) = -\sum_{t=1}^{k} \frac{(2\pi i j\alpha)^t}{t!}, \\ u_j &= u_j(\alpha) = a_j b_j, \quad u'_j = u'_j(\alpha) = a_j b'_j \end{aligned}$$

so that

$$\frac{\varphi_j(\alpha)}{\varphi_j(0)} = \frac{1-e^{-\sigma j}}{1-e^{-\sigma j}\,e(j\alpha)} = \frac{e^{\sigma j}-1}{e^{\sigma j}-e(j\alpha)} = \frac{1}{1+a_j(\alpha)b_j(\alpha)} = \frac{1}{1+u_j(\alpha)}, \tag{7.1}$$

and the integrand in J can be rewritten as

$$\begin{aligned} e(-n\alpha)\varphi(\alpha) &= e(-n\alpha)\prod_{j=m}^{n}\varphi_j(\alpha) \\ &= e(-n\alpha)\varphi(0)\prod_{j=m}^{n}\frac{\varphi_j(\alpha)}{\varphi_j(0)} \\ &= e(-n\alpha)\varphi(0)\prod_{j=m}^{n}\frac{1}{1+u_j(\alpha)} \\ &= \prod_{j=m}^{n}\frac{1}{1-e^{-\sigma j}}\,e(-n\alpha)\prod_{j=m}^{n}\frac{1}{1+u_j(\alpha)}. \end{aligned} \tag{7.2}$$

Lemma 7. *Uniformly for*

$$|\alpha| \le \eta \tag{7.3}$$

and $m \le j \le n$ *we have*

$$|u_j| \ll \frac{j}{e^{\sigma j}-1}\,\eta \tag{7.4}$$

and

$$|u_j| \ll \frac{\eta}{\sigma} = o(1). \tag{7.5}$$

Proof of Lemma 7. For all α satisfying (7.3) we have

$$|b_j| = |1-e(j\alpha)| = |e(-j\alpha/2)-e(j\alpha/2)| = |\sin \pi j\alpha| \le \pi j|\alpha| \le \pi j\eta$$

so that

$$|u_j| = a_j|b_j| \le \frac{\pi j\eta}{e^{\sigma j}-1}$$

which proves (7.4).

Since $e^x - 1 > x$ for $x > 0$ thus it follows from (7.4) that

$$|u_j| \ll \frac{j}{e^{\sigma j}-1}\,\eta < \frac{j}{\sigma j}\,\eta = \frac{\eta}{\sigma}$$

which proves the first inequality in (7.5). Finally, $\eta/\sigma = o(1)$ follows from (1.15), (5.1) and Lemma 1, and this completes the proof of Lemma 7. □

Lemma 8. *Uniformly for α satisfying (7.3) and $m \le j < n$ we have*

$$|u_j - u'_j| \ll \frac{j^{k+1}}{e^{\sigma j}-1}\,\eta^{k+1} = o(1). \tag{7.6}$$

Proof of Lemma 8. For all j we have

$$|u_j - u'_j| = a_j|b_j - b'_j| = \frac{1}{e^{\sigma j}-1}\,|b_j - b'_j|. \tag{7.7}$$

If $j < \frac{1}{10\eta}$ then by (7.3),

$$2\pi j|\alpha| \le 2\pi j\eta < \frac{\pi}{5}$$

and thus

$$\begin{aligned} |b_j - b'_j| &= \left|\sum_{t=k+1}^{+\infty} \frac{(2\pi i j\alpha)^t}{t!}\right| \\ &\le \sum_{t=k+1}^{+\infty} \frac{(2\pi j|\alpha|)^t}{t!} \le \frac{1}{(k+1)!}\sum_{t=k+1}^{+\infty}(2\pi j\eta)^t \\ &< \frac{(2\pi j\eta)^{k+1}}{(k+1)!}\sum_{v=o}^{+\infty}\left(\frac{\pi}{5}\right)^v \ll (j\eta)^{k+1} \quad (\text{for } j < 1/10\eta). \end{aligned} \tag{7.8}$$

For $j \ge 1/10\eta$ we have

$$|b_j - b'_j| \le |b_j| + |b'_j| \le 2 + \sum_{t=1}^{k} \frac{(2\pi j\eta)^t}{t!} \ll (j\eta)^k \ll (j\eta)^{k+1}. \tag{7.9}$$

The first inequality in (7.6) follows from (7.7), (7.8) and (7.9).

For $j \le 1/\sigma$ clearly we have

$$\frac{j^{k+1}}{e^{\sigma j}-1}\,\eta^{k+1} \le \frac{j^{k+1}}{\sigma j}\,\eta^{k+1} = \frac{j^k}{\sigma}\,\eta^{k+1} \le (\frac{\eta}{\sigma})^{k+1} = o(1) \qquad (\text{for } j \le 1/\sigma) \tag{7.10}$$

since $\eta/\sigma = o(1)$ as we have seen in the proof of Lemma 7. Moreover, for $j > 1/\sigma$ we have

$$\frac{j^{k+1}}{e^{\sigma j}-1}\,\eta^{k+1} \ll \frac{j^{k+1}}{e^{\sigma j}}\,\eta^{k+1} = \frac{(\sigma j)^{k+1}}{e^{\sigma j}}\,(\frac{\eta}{\sigma})^{k+1}. \tag{7.11}$$

For $0 < x$ the function $f_k(x) = x^{k+1}e^{-x}$ is maximal at $x = k+1$, and the value of the maximum is $f_k(k+1) = (k+1)^{k+1}e^{-(k+1)}$. Thus it follows from (7.11) that

$$\frac{j^{k+1}}{e^{\sigma j}-1}\,\eta^{k+1} \ll (\frac{\eta}{\sigma})^{k+1} = o(1) \quad (\text{for } j > 1/\sigma). \tag{7.12}$$

(7.10) and (7.12) complete the proof of (7.6). □

Lemma 9. *If* $k \in \mathbb{N}$,

$$0 < \Delta < \frac{1}{4}, \tag{7.13}$$

$u \in \mathbb{C},\ u' \in \mathbb{C}$,

$$|u| < \frac{1}{4} \tag{7.14}$$

and

$$|u - u'| \le \Delta, \tag{7.15}$$

then we have

$$\frac{1}{1+u} = \exp\left(\sum_{\ell=1}^{k} (-1)^\ell \frac{(u')^\ell}{\ell} + R(u,u')\right) \tag{7.16}$$

with

$$|R(u,u')| < |u|^{k+1} + 2\Delta. \tag{7.17}$$

Proof of Lemma 9. By (7.14) we have

$$\begin{aligned} \frac{1}{1+u} &= \exp\left(-\log(1+u)\right) = \exp\left(\sum_{\ell=1}^{+\infty} (-1)^\ell \frac{u^\ell}{\ell}\right) \\ &= \exp\left(\sum_{\ell=1}^{k} (-1)^\ell \frac{(u')^\ell}{\ell} + R_1(u) + R_2(u,u')\right) \end{aligned} \tag{7.18}$$

where

$$R_1(u) = \sum_{\ell=k+1}^{+\infty} (-1)^\ell \frac{u^\ell}{\ell}$$

and

$$R_2(u,u') = \sum_{\ell=1}^{k} (-1)^\ell \frac{1}{\ell}(u^\ell - (u')^\ell).$$

By (7.14) clearly we have

$$\begin{aligned} |R_1(u)| &\le \sum_{\ell=k+1}^{+\infty} \frac{|u|^\ell}{\ell} \le \frac{|u|^{k+1}}{k+1} \sum_{j=0}^{+\infty} |u|^j = \frac{|u|^{k+1}}{k+1} \frac{1}{1-|u|} \\ &\le \frac{2}{k+1} |u|^{k+1} \le |u|^{k+1}. \end{aligned} \tag{7.19}$$

Moreover, by (7.13), (7.14) and (7.15) we have

$$
\begin{aligned}
(7.20)\quad |R_2(u,u')| &\le \sum_{\ell=1}^{k} \frac{1}{\ell}\,|u-u'| \sum_{j=0}^{\ell-1} |u|^j |u'|^{\ell-1-j} \\
&\le \Delta \sum_{\ell=1}^{k} \frac{1}{\ell} \sum_{j=0}^{\ell-1} |u|^j (|u| + |u'-u|)^{\ell-1-j} \\
&\le \Delta \sum_{\ell=1}^{k} \frac{1}{\ell} \sum_{j=0}^{\ell-1} |u|^j (|u| + \Delta)^{\ell-1-j} \\
&\le \Delta \sum_{\ell=1}^{k} \frac{1}{\ell} \sum_{j=0}^{\ell-1} (|u| + \Delta)^{\ell-1} = \Delta \sum_{\ell=1}^{k} (|u| + \Delta)^{\ell-1} \\
&< \Delta \sum_{\ell=1}^{+\infty} \left(\frac{1}{4} + \frac{1}{4}\right)^{\ell-1} = 2\Delta.
\end{aligned}
$$

(7.16) and (7.17) follow from (7.18), (7.19) and (7.20), and this completes the proof of the lemma. □

Lemma 10. *Uniformly for* $|\alpha| \le \eta$ *the integrand in* J_1 *is*

$$
(7.21)\qquad e(-n\alpha)\varphi(\alpha) = \varphi(0)\exp(-L_2\pi^2\alpha^2)\Big(1 + \sum_{v=1}^{3k-2} Z_v(\pi i\alpha)^v + E_0\Big)
$$

where

$$
(7.22)\qquad Z_v = \sum_{\max\{1, \frac{v-k+2}{2}\} \le t \le k} \frac{1}{t!} \sum_{\substack{3 \le h_1,\dots,h_t \le v \\ h_1+\dots+h_t = v}} L_{h_1} \dots L_{h_t}
$$

and

$$
(7.23)\qquad E_0 \ll \begin{cases} n^{-(k-1)/4}(\log\, n)^{k^2/2} & \textit{for } m \le Hn^{1/2} \\ \left(\frac{m}{n}\right)^{(k-1)/2} (\log\, n)^{2k^2} & \textit{for } m > Hn^{1/2}. \end{cases}
$$

Proof of Lemma 10. By (7.5) in Lemma 7 and by Lemma 8, (7.13) and (7.14) in Lemma 9 hold with u_j and u'_j in place of u and u', respectively. Thus by Lemma 9 we have

$$
(7.24)\qquad \frac{1}{1 + u_j(\alpha)} = \exp\left(\sum_{\ell=1}^{k} (-1)^\ell \frac{(u'_j)^\ell}{\ell} + R(u_j, u'_j)\right)
$$

where, by (7.4) in Lemma 7, Lemma 8 and (7.19) we have

$$
(7.25)\qquad |R(u_j, u'_j)| \ll \frac{j^{k+1}}{(e^{\sigma j} - 1)^{k+1}}\,\eta^{k+1} + \frac{j^{k+1}}{e^{\sigma j} - 1}\,\eta^{k+1}.
$$

It follows from (7.24) that

$$\prod_{j=m}^{n} \frac{1}{1+u_j(\alpha)} = \exp\left(\sum_{j=m}^{n} \sum_{\ell=1}^{k} (-1)^{\ell} \frac{(u_j')^{\ell}}{\ell} + \sum_{j=m}^{n} R(u_j, u_j') \right) \tag{7.26}$$

where by (7.25) the error term in the exponent is

$$\begin{aligned} \left| \sum_{j=m}^{n} R(u_j, u_j') \right| &\ll \left(\sum_{j=m}^{n} \left(\frac{j^{k+1}}{(e^{\sigma j}-1)^{k+1}} + \frac{j^{k+1}}{e^{\sigma j}-1} \right) \right) \eta^{k+1} \\ &= (A(k+1,k+1) + A(k+1,1))\eta^{k+1}. \end{aligned} \tag{7.27}$$

Moreover, the innermost term in the main term in the exponent in (7.26) can be rewritten as

$$\begin{aligned} \sum_{\ell=1}^{k} (-1)^{\ell} \frac{(u_j')^{\ell}}{\ell} &= \sum_{\ell=1}^{k} \frac{1}{\ell} a_j^{\ell} (-b_j')^{\ell} \\ &= \sum_{\ell=1}^{k} \frac{1}{\ell} \frac{1}{(e^{\sigma j}-1)^{\ell}} \left(\sum_{t=1}^{k} \frac{(2\pi i j \alpha)^t}{t!} \right)^{\ell}. \end{aligned} \tag{7.28}$$

Writing here

$$a = \frac{1}{e^{\sigma j}-1}, \quad x = \pi i j \alpha,$$

the last expression becomes a polynomial, in the variables a and x, of the form

$$\sum_{\ell=1}^{k} \frac{1}{\ell} a^{\ell} \left(\sum_{t=1}^{k} \frac{2^t}{t!} x^t \right)^{\ell} = \sum_{h=1}^{k^2} \left(\sum_{\ell=1}^{h} d(h,\ell,k) a^{\ell} \right) x^h. \tag{7.29}$$

If $k \geq h$ and $k' \geq h$, then clearly

$$d(h,\ell,k) = d(h,\ell,k'),$$

i.e., $d(h,\ell,k)$ is independent of k for $k \geq h$. Thus we may write

$$d(h,\ell,k) = d(h,\ell) \quad \text{for } k \geq h.$$

In particular, computing these numbers $d(h,\ell)$ for $h \le 6$ we obtain

(7.30)

$$\begin{aligned}
&d(1,1) = 2,\\
&d(2,1) = 2, \quad d(2,2) = 2,\\
&d(3,1) = \tfrac{4}{3}, \quad d(3,2) = 4, \quad d(3,3) = \tfrac{8}{3},\\
&d(4,1) = \tfrac{2}{3}, \quad d(4,2) = \tfrac{14}{3}, \quad d(4,3) = 8, \quad d(4,4) = 4,\\
&d(5,1) = \tfrac{4}{15}, \quad d(5,2) = 4, \quad d(5,3) = \tfrac{40}{3}, \quad d(5,4) = 16, \quad d(5,5) = \tfrac{32}{5}\\
&d(6,1) = \tfrac{4}{45}, \quad d(6,2) = \tfrac{124}{45}, \quad d(6,3) = 16, \quad d(6,4) = \tfrac{104}{3}, \quad d(6,5) = 32,\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad d(6,6) = \tfrac{32}{3}.
\end{aligned}$$

Using this notation, by (7.28) and (7.29) for $|\alpha| \le \eta$ the main term in the exponent in (7.26) becomes

$$\begin{aligned}
(7.31) \quad & \sum_{j=m}^{n} \sum_{\ell=1}^{k} (-1)^{\ell} \frac{(u_j')^{\ell}}{\ell}\\
&= \sum_{j=m}^{n} \sum_{h=1}^{k^2} \left(\sum_{\ell=1}^{h} d(h,\ell,k) \frac{1}{(e^{\sigma j}-1)^{\ell}} \right) (\pi i j \alpha)^h\\
&= \sum_{h=1}^{k^2} \left(\sum_{\ell=1}^{h} d(h,\ell,k) \sum_{j=m}^{n} \frac{j^h}{(e^{\sigma j}-1)^{\ell}} \right) (\pi i \alpha)^h\\
&= \sum_{h=1}^{k^2} \left(\sum_{\ell=1}^{h} d(h,\ell,k) A(h,\ell) \right) (\pi i \alpha)^h\\
&= \sum_{h=1}^{k} L_h (\pi i \alpha)^h + O\Big(\max_{\substack{k+1 \le h \le k^2\\ 1 \le \ell \le h}} A(h,\ell) \eta^h \Big)
\end{aligned}$$

where L_h and, in particular, L_2, L_3, L_4, L_5 and L_6 are defined as in the theorem, and by (1.7) and (7.30), the first term is

$$L_1(\pi i \alpha) = d(1,1)A(1,1)\pi i \alpha = 2n\pi i \alpha. \tag{7.32}$$

It follows from (7.2), (7.26), (7.27), (7.31) and (7.32) that for $|\alpha| \le \eta$ the integrand in J is

$$(7.33)\quad \begin{aligned} e(-n\alpha)\varphi(\alpha) &= \exp(-2\pi i n\alpha)\varphi(0) \\ &\times \exp\Big(2n\pi i\alpha + \sum_{h=2}^{k} L_h(\pi i\alpha)^h + O(\max_{\substack{k+1\le h\le k^2\\ 1\le \ell\le h}} A(h,\ell)\eta^h)\Big) \\ &= \varphi(0)\exp\left(\sum_{h=2}^{k} L_h(\pi i\alpha)^h + O(\max_{\substack{k+1\le h\le k^2\\ 1\le \ell\le h}} A(h,\ell)\eta^h)\right). \end{aligned}$$

By (5.1) and Lemma 2, for $m \le Hn^{1/2}$ in the error term we have

$$(7.34)\quad \begin{aligned} \max_{\substack{k+1\le h\le k^2\\ 1\le \ell\le h}} A(h,\ell)\eta^h &\ll \max_{k+1\le h\le k^2} n^{(h+1)/2}\,(n^{-3/4}(\log n)^{1/2})^h \\ &= \max_{k+1\le h\le k^2} n^{-(h-2)/4}\,(\log n)^{h/2} \\ &\ll n^{-(k-1)/4}\,(\log n)^{(k+1)/2} \quad (\text{for } m \le Hn^{1/2}) \end{aligned}$$

while for $m > Hn^{1/2}$, by $k \ge 3$ we have

$$(7.35)\quad \begin{aligned} &\max_{\substack{k+1\le h\le k^2\\ 1\le \ell\le h}} A(h,\ell)\eta^h \\ &\ll \max_{\substack{k+1\le h\le k^2\\ 1\le \ell\le h}} \Big(m^{h+1}(\frac{n}{m^2})^{\ell}(\log(m^2/n))^{\ell-1}\Big)\,((nm)^{-1/2}\,(\log n)^{3/2})^h \\ &\ll \max_{k+1\le h\le k^2} \Big(m^{h+1}\,\frac{n}{m^2}\Big)\,(nm)^{-h/2}\,(\log n)^{3h/2} \\ &= \max_{k+1\le h\le k^2} \Big(\frac{m}{n}\Big)^{(h-2)/2}\,(\log n)^{3h/2} \\ &= \Big(\frac{m}{n}\Big)^{(k-1)/2}\,(\log n)^{3(k-1)/2} \quad (\text{for } m > Hn^{1/2}). \end{aligned}$$

By (1.15), both upper bounds in (7.34) and (7.35) are $o(1)$ and thus in (7.33) we may write

$$(7.36)\qquad \exp(O(\max_{\substack{k+1\le h\le k^2\\ 1\le \ell\le h}} A(h,\ell)\eta^h)) = 1 + E_1$$

with

$$(7.37)\qquad E_1 = O(\max_{\substack{k+1\le h\le k^2\\ 1\le \ell\le h}} A(h,\ell)\eta^h) \quad (= o(1)).$$

Next in (7.33) we write

$$\exp\left(\sum_{h=2}^{k} L_h(\pi i\alpha)^h\right) = \exp(-L_2\pi^2\alpha^2)\exp(y_k(\alpha)) \tag{7.38}$$

where

$$y_k(\alpha) = \sum_{h=3}^{k} L_h(\pi i\alpha)^h. \tag{7.39}$$

Then for $|\alpha| \le \eta$ we have

$$\begin{aligned} y_k(\alpha) \ll \sum_{h=3}^{k} L_h\eta^h = \sum_{h=3}^{k}\left(\sum_{\ell=1}^{h} d(h,\ell)A(h,\ell)\right)\eta^h \\ \ll \max_{\substack{3\le h\le k\\ 1\le \ell\le h}} A(h,\ell)\eta^h. \end{aligned} \tag{7.40}$$

By using (5.1) and Lemma 2, it follows in the same way as in (7.34) and (7.35) that

$$y_k(\alpha) \ll \max_{3\le h\le k} n^{-(h-2)/4}(\log n)^{h/2} = n^{-1/4}(\log n)^{3/2} \quad \text{for } m \le Hn^{1/2} \tag{7.41}$$

and

$$y_k(\alpha) \ll \max_{3\le h\le k}\left(\frac{m}{n}\right)^{(h-2)/2}(\log n)^{3h/2} = \left(\frac{m}{n}\right)^{1/2}(\log n)^{9/2} \quad \text{for } m > Hn^{1/2}. \tag{7.42}$$

By (1.15), it follows from (7.41) and (7.42) that

$$y_k(\alpha) = o(1) \tag{7.43}$$

(uniformly for $|\alpha| \le \eta$). Thus the second factor in (7.38) can be written as

$$\exp(y_k(\alpha)) = 1 + \sum_{t=1}^{k}\frac{1}{t!}\,y_k^t(\alpha) + E_2 \tag{7.44}$$

with

$$E_2 \ll (y_k(\alpha))^{k+1}. \tag{7.45}$$

Moreover, the main term in (7.44) is

$$(7.46)\quad 1+\sum_{t=1}^{k}\frac{1}{t!}y_k^t(\alpha)=1+\sum_{t=1}^{k}\frac{1}{t!}\left(\sum_{h=3}^{k}L_h(\pi i\alpha)^h\right)^t$$
$$=1+\sum_{v=1}^{k^2}(\sum_{t=1}^{k}\frac{1}{t!}\sum_{\substack{3\le h_1,\dots,h_t\le k\\ h_1+\dots+h_t=v}}L_{h_1}\dots L_{h_t})(\pi i\alpha)^v.$$

Consider here the terms with $t<\frac{v-k+2}{2}$. Let $\max^*$ denote the maximum taken over the integers $v,t,h_1,...,h_t$ with $1\le v\le k^2$, $1\le t\le k$, $t<\frac{v-k+2}{2}$, $3\le h_1,...,h_t\le k$, $h_1+...+h_t=v$. Then uniformly for $|\alpha|\le\eta$, by (5.1) and Lemma 2 each of the terms with $t<\frac{v-k+2}{2}$ is

$$\begin{aligned}
&\ll\ W\overset{\text{def}}{=}\max{}^*\ L_{h_1}...L_{h_t}\eta^v\\
&=\ \max{}^*\left(\sum_{\ell_1=1}^{h_1}d(h_1,\ell_1)A(h_1,\ell_1)\right)\dots\left(\sum_{\ell_t=1}^{h_t}d(h_t,\ell_t)A(h_t,\ell_t)\right)\eta^v\\
&\ll\ \max{}^*(\max_{1\le\ell_1\le h_1}A(h_1,\ell_1))...(\max_{1\le\ell_t\le h_t}A(h_t,\ell_t))\eta^v\\
&\ll\ \max{}^*\ A(h_1,1)...A(h_t,1)\eta^v
\end{aligned}$$

so that for $m\le Hn^{1/2}$,

$$\begin{aligned}
(7.47)\qquad W\ &\ll\ \max{}^*\ n^{(h_1+1)/2}...n^{(h_t+1)/2}\,(n^{-3/4}\,(\log n)^{1/2})^v\\
&=\ \max{}^*\ n^{(v+t)/2-3v/4}\,(\log n)^{v/2}\\
&=\ \max{}^*\ n^{(2t-v)/4}\,(\log n)^{v/2}\\
&\le\ n^{-(k-1)/4}\,(\log n)^{k^2/2}\quad(\text{for }\ m\le Hn^{1/2}),
\end{aligned}$$

while for $m>Hn^{1/2}$,

$$\begin{aligned}
(7.48)\quad W\ &\ll\ \max{}^*\left(m^{h_1+1}\frac{n}{m^2}\right)\dots\left(m^{h_t+1}\frac{n}{m^2}\right)\\
&\qquad\times((nm)^{-1/2}\,(\log n)^{3/2})^v\\
&=\ \max{}^*\left(\frac{m}{n}\right)^{(v-2t)/2}\,(\log n)^{3v/2}\\
&\le\ \left(\frac{m}{n}\right)^{(k-1)/2}\,(\log n)^{3k^2/2}\le\left(\frac{m}{n}\right)^{(k-1)/2}\,(\log n)^{2k^2}.
\end{aligned}$$

Thus in (7.46) we may restrict ourselves to terms with $t \geq \frac{v-k+2}{2}$ at the expense of a small error term:

$$(7.49) \quad 1+\sum_{t=1}^{k} \frac{y_k^t(\alpha)}{t!}$$
$$= 1+\sum_{v=1}^{k^2}\left(\sum_{\max\{1,\frac{v-k+2}{2}\}\leq t\leq k} \frac{1}{t!} \sum_{\substack{3\leq h_1,\dots,h_t\leq k\\ h_1+\dots+h_t=v}} L_{h_1}\dots L_{h_t}\right)(\pi i\alpha)^v + E_3$$

with

$$(7.50) \qquad E_3 \ll \begin{cases} n^{-(k-1)/4}(\log n)^{k^2/2} & \text{for } m\leq Hn^{1/2}\\ \left(\frac{m}{n}\right)^{(k-1)/2}(\log n)^{2k^2} & \text{for } m> Hn^{1/2}.\end{cases}$$

Note that the summation over t is empty unless

$$\frac{v-k+2}{2}\leq k$$

whence

$$(7.51) \qquad v\leq 3k-2$$

so that the summation $\sum_{v=1}^{k^2}$ can be replaced by $\sum_{v=1}^{3k-2}$.

Defining Z_v by (7.22), it follows from (7.33), (7.36), (7.37), (7.38), (7.39), (7.43), (7.44), (7.45), (7.49), (7.50) and (7.51) that uniformly for $|\alpha|\leq\eta$ the integrand in J_1 is

$$e(-n\alpha)\varphi(\alpha) = \varphi(0)\exp(-L_2\pi^2\alpha^2)$$
$$\times\left(1+\sum_{v=1}^{3k-1} Z_v(\pi i\alpha)^v + E_2+E_3\right)(1+E_1)$$
$$= \varphi(0)\exp(-L_2\pi^2\alpha^2)\left(1+\sum_{v=1}^{3k-2} Z_v(\pi i\alpha)^v + E_4\right)$$

where

$$E_4 \ll E_1+E_2+E_3 \ll \begin{cases} n^{-(k-1)/4}(\log n)^{k^2/2} & \text{for } m\leq Hn^{1/2}\\ \left(\frac{m}{n}\right)^{(k-1)/2}(\log n)^{2k^2} & \text{for } m> Hn^{1/2}\end{cases}$$

and this completes the proof of Lemma 10. □

8. Estimate of J_1

Lemma 11. *We have*

$$J_1 = \pi^{-1/2} B^{-1} \varphi(0) \times \left(1 + \sum_{\ell=1}^{[(3k-2)/2]} (-1)^\ell 2^{-\ell} (2\ell-1)(2\ell-3)\cdot \ldots \cdot 1 \cdot L_2^{-\ell} Z_{2\ell} + E_5\right) \tag{8.1}$$

where

$$E_5 \ll \begin{cases} n^{-(k-1)/4} (\log n)^{k^2/2} & \text{for } m \leq Hn^{1/2} \\ \left(\frac{m}{n}\right)^{(k-1)/2} (\log n)^{2k^2} & \text{for } m > Hn^{1/2}. \end{cases} \tag{8.2}$$

Proof of Lemma 11. By Lemma 10 we have

$$\begin{aligned} J_1 &= \int_{-\eta}^{+\eta} e(-n\alpha)\varphi(\alpha)d\alpha \\ &= \varphi(0)\Big(\int_{-\eta}^{+\eta} \exp(-L_2\pi^2\alpha^2)(1+E_0)d\alpha \\ &\qquad + \sum_{v=1}^{3k-2} Z_v(\pi i)^v \int_{-\eta}^{+\eta} \alpha^v \exp(-L_2\pi^2\alpha^2)d\alpha\Big) \end{aligned} \tag{8.3}$$

with E_0 satisfying (7.23). It remains to estimate the integrals

$$I_v = \int_{-\eta}^{+\eta} \alpha^v \exp(-L_2\pi^2\alpha^2)d\alpha.$$

If v (≥ 0) is odd then the integrand is odd and thus

$$I_v = 0 \quad \text{for} \quad v \quad \text{odd}. \tag{8.4}$$

If $v = 2\ell$ is even, then substituting $x = (2L_2)^{1/2}\pi\alpha$ we obtain

$$I_{2\ell} = ((2L_2)^{1/2}\pi)^{-(2\ell+1)} \int_{-D}^{+D} x^{2\ell} \exp(-x^2/2)dx \tag{8.5}$$

with

$$D = (2L_2)^{1/2}\pi\eta.$$

It is easy to see by induction (using integration by parts) that we have

$$\begin{aligned} &\int_{-D}^{+D} x^{2\ell} \exp(-x^2/2)dx \\ &\qquad = (2\ell-1)(2\ell-3)\cdot \ldots \cdot 1 \int_{-D}^{+D} \exp(-x^2/2)dx. \end{aligned} \tag{8.6}$$

Moreover, we have

$$(8.7)\quad \int_{-D}^{+D} \exp(-x^2/2)dx = \int_{-\infty}^{+\infty} \exp(-x^2/2)dx -2\int_{D}^{+\infty} \exp(-x^2/2)dx$$
$$= (2\pi)^{1/2} - O\left(\int_{D}^{+\infty} \frac{x}{D} \exp(-x^2/2)dx\right)$$
$$= (2\pi)^{1/2} - O\left(\frac{1}{D} \exp(-D^2/2)\right)$$
$$= (2\pi)^{1/2} - O(L_2^{-1/2}\,\eta^{-1} \exp(-L_2\pi^2\eta^2))$$
$$= (2\pi)^{1/2} + E_6,$$

say. By (1.16), (5.1) and Lemma 3 we have

$$E_6 \ll L_2^{-1/2}\,\eta^{-1} \exp(-L_2\pi^2\eta^2) = (B\eta)^{-1} \exp(-B^2\pi^2\eta^2)$$
$$\ll \begin{cases} (\log n)^{-1/2} \exp(-c_5(c^*)^2 \log n) & \text{for } m \leq Hn^{1//2} \\ (\log n)^{-3/2} \exp(-c_5(c^*)^2 (\log n)^3) & \text{for } m > Hn^{1/2}. \end{cases}$$

If now we fix c^* to be large enough in terms of k (and so that (5.9) should also hold) then it follows that

$$(8.8)\qquad E_6 \ll n^{-k}.$$

By (8.3), (8.4), (8.5), (8.6), (8.7) and (8.8) (and since $Z_v = o(1)$ for $1 \leq v \leq k^2$ by the proof of Lemma 10) we have

$$J_1 = \varphi(0)\Bigg((1+E_0)(2L_2)^{-1/2}\,\pi^{-1} + \sum_{\ell=1}^{[(3k-2)/2]} (-\pi^2)^\ell((2L_2)^{1/2}\,\pi)^{-(2\ell+1)}\,Z_{2\ell}(2\ell-1)(2\ell-3)\cdot\ldots\cdot 1\Bigg) \times \int_{-D}^{+D} \exp(-x^2/2)\,dx$$
$$= \varphi(0)(2L_2)^{-1/2}\,\pi^{-1}((2\pi)^{1/2}+E_6) \times\Bigg(1+E_0+\sum_{\ell=1}^{[(3k-2)/2]} (-1)^\ell\,2^{-\ell}\,(2\ell-1)(2\ell-3)\cdot\ldots\cdot 1\cdot L_2^{-\ell}\,Z_{2\ell}\Bigg)$$

$$= \pi^{-1/2} L_2^{-1/2} \varphi(0) \times \Big(1 + \sum_{\ell=1}^{[(3k-2)/2]} (-1)^\ell 2^{-\ell} (2\ell-1)(2\ell-3)\cdot \ldots \cdot 1 \cdot L_2^{-\ell} Z_{2\ell} + O(E_0+E_6)\Big)$$

whence, by (1.16), (7.23) and (8.8), (8.1) and (8.2) follow. □

9. Completion of the proof of Theorem 1

The result follows from (1.9), (5.2), Lemma 6 and Lemma 11. In particular, the formulas for $L_2, \ldots, L_6$ are obtained from (1.10) by using the numbers $d(x,y)$ computed in the proof of Lemma 10, while (1.17) and (1.18) are the $k=6$ special cases of (1.12) and (1.14), respectively.

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J.-L. Nicolas
Institut Girard Desargues, UMR 5028
Mathématiques, Bâtiment 101
Université Claude Bernard (Lyon 1)
F-69622 Villeurbanne Cédex, FRANCE
E-mail : jlnicola@in2p3.fr

A. Sárközy
Department of Algebra and Number Theory
Eötvös Loránd University
Kecskeméti utca 10–12
H-1088 Budapest, Hungary
E-mail : sarkozy@cs.elte.hu