Journal de Théorie des Nombres
de Bordeaux **18** (2006), 203–221

On some remarkable properties of the two-dimensional Hammersley point set in base 2

par Peter KRITZER

Résumé. Nous examinons une classe spéciale de $(0, m, 2)$-réseaux en base 2. Particulièrement, nous nous occupons du réseau de Hammersley en deux dimensions qui joue un rôle spécial parmi ce type de réseaux, puisque nous démontrons que c'est le plus mal distribué quant à la discrépance à l'origine. En le montrant, nous améliorons un majorant connu pour la discrépance à l'origine de $(0, m, 2)$-réseaux en base 2. De plus, nous démontrons qu'on peut obtenir des réseaux avec une discrépance à l'origine très basse en transformant le réseau de Hammersley d'une manière appropriée.

Abstract. We study a special class of $(0, m, 2)$-nets in base 2. In particular, we are concerned with the two-dimensional Hammersley net that plays a special role among these since we prove that it is the worst distributed with respect to the star discrepancy. By showing this, we also improve an existing upper bound for the star discrepancy of digital $(0, m, 2)$-nets over $\mathbb{Z}_2$. Moreover, we show that nets with very low star discrepancy can be obtained by transforming the Hammersley point set in a suitable way.

1. Introduction

For a point set $\mathbf{x}_0, \ldots, \mathbf{x}_{N-1}$ in $[0,1)^s$ the star discrepancy D_N^* is defined by

$$D_N^* := \sup_J \left| A_N(J) N^{-1} - \lambda(J) \right|,$$

where the supremum is extended over all intervals $J \subseteq [0,1)^s$ of the form $J = \prod_{j=1}^s [0, \alpha_j)$, $0 < \alpha_j \leq 1$, $A_N(J)$ denotes the number of i with $\mathbf{x}_i \in J$, and λ is the Lebesgue measure.

The concept of (digital) (t, m, s)-nets provides an efficient method to construct point sets with small star discrepancy. An extensive survey on this topic is given by Niederreiter in [8] (other related monographs are, for example, [3] and [6]). The general definition of a (t, m, s)-net can be found in [8].

Manuscrit reçu le 22 avril 2004.

Here, we study a special class of (t,m,s)-nets in base 2, so-called digital $(t,m,2)$-nets over $\mathbb{Z}_2$ (where $\mathbb{Z}_2$ is the finite field with two elements), which are point sets consisting of $N = 2^m$ points $\mathbf{x}_0,\dots,\mathbf{x}_{N-1}$ in $[0,1)^2$ generated as follows. Choose two $m\times m$-matrices C_1, C_2 over $\mathbb{Z}_2$ such that for all integers $d_1, d_2 \geq 0$ with $d_1 + d_2 = m - t$, the first d_1 rows of C_1 together with the first d_2 rows of C_2 form a linearly independent set over $\mathbb{Z}_2$. For $i \in \{0,\dots,2^m-1\}$ let i have base 2 representation $i = i_0 + i_1 2 + \cdots + i_{m-1}2^{m-1}$, with $i_k \in \mathbb{Z}_2$ for $0 \leq k \leq m-1$. For fixed i, multiply C_j, $1 \leq j \leq 2$, by the vector of digits of i, which gives

$$C_j \cdot (i_0,\dots,i_{m-1})^T =: (y_1^{(j)},\dots,y_m^{(j)})^T \in \mathbb{Z}_2^m.$$

Let

$$x_i^{(j)} := \sum_{k=1}^{m} \frac{y_k^{(j)}}{2^k}$$

and define $\mathbf{x}_i := (x_i^{(1)}, x_i^{(2)})$. Then the point set $\{\mathbf{x}_0,\dots,\mathbf{x}_{2^m-1}\}$ is a digital $(t,m,2)$-net over $\mathbb{Z}_2$ and the matrices C_1, C_2 are called the generating matrices of the digital net. However, we do not restrict ourselves to studying digital nets here, but we will make repeated use of the concept of digital nets that are digitally shifted by m-bit vectors. These are constructed by a slight variation in the net generating procedure outlined above: choose 2 vectors $\vec{\sigma}_1, \vec{\sigma}_2$ with

$$\vec{\sigma}_j = (\sigma_1^{(j)},\dots,\sigma_m^{(j)})^T \in \mathbb{Z}_2^m$$

and set, for each $i \in \{0,\dots,2^m-1\}$,

$$x_i^{(j)} := \sum_{k=1}^{m} \frac{y_k^{(j)} \oplus \sigma_k^{(j)}}{2^k}$$

for $1 \leq j \leq 2$, where $\oplus$ denotes addition modulo 2. Shifted digital $(t,m,2)$-nets are still $(t,m,2)$-nets, however, in general they are no digital nets any more, since they do not necessarily contain the origin. If we consider the set of all digital $(t,m,2)$-nets over $\mathbb{Z}_2$ that are digitally shifted by m-bit vectors (these include ordinary digital $(t,m,2)$-nets by choosing $\vec{\sigma}_1 = \vec{\sigma}_2 = \vec{0}$), this set corresponds to the set of all the $(t,m,2)$-nets in base 2 that can be constructed in a way proposed by Niederreiter in [8, p. 63] who defined digital nets in a more general manner than we did here.

It was shown by Niederreiter (see, e. g., [8]) that for the star discrepancy of any $(t,m,2)$-net in base 2 (not necessarily digital) we have

$$2^m D_{2^m}^* \leq 2^t \left\lfloor \frac{m-t}{2} + \frac{3}{2} \right\rfloor,$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. In [7], Larcher and Pillichshammer showed that, for any digital $(0,m,2)$-net over

$\mathbb{Z}_2$, the following upper bound on the star discrepancy holds:

$$2^m D^*_{2^m} \leq \frac{m}{3} + \frac{19}{9}.$$

It was also shown in [7] that the constant 1/3 in this bound is best possible, due to the following observation. The simplest example for a digital $(0, m, 2)$-net over $\mathbb{Z}_2$ is the well-known Hammersley net H which is generated by the matrices

$$C_1 = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix}, \; C_2 = \begin{pmatrix} 0 & \dots & 0 & 1 \\ \vdots & ⋰ & ⋰ & 0 \\ 0 & ⋰ & ⋰ & \vdots \\ 1 & 0 & \dots & 0 \end{pmatrix}.$$

The exact value of the star discrepancy of the Hammersley net over $\mathbb{Z}_2$ is given in several articles (see, e.g., [1], [5], [7]). For $m = 1$ it is 0.75, and for $m \geq 2$ we have

$$2^m D^*_{2^m}(H) = \frac{m}{3} + \frac{13}{9} - (-1)^m \cdot \frac{4}{9 \cdot 2^m}.$$

It is—due to the upper bound of Larcher and Pillichshammer—clear that the Hammersley net therefore is (in terms of the star discrepancy) the worst distributed digital $(0, m, 2)$-net over $\mathbb{Z}_2$ with respect to the leading term $m/3$. However, it still might be the case that there is a digital net with its star discrepancy lying in the gap between the upper bound and the star discrepancy of H. In this paper, we are going to show that this is impossible and that H (without any shift) is the worst distributed net of all the digital $(0, m, 2)$-nets over $\mathbb{Z}_2$ that are digitally shifted by m-bit vectors concerning the star discrepancy—not only with respect to the leading term but with respect to the precise value. By this result, the upper bound of Larcher and Pillichshammer is further improved (Section 2).

In [7, Theorem 6], Larcher and Pillichshammer mention special digital $(0, m, 2)$-nets over $\mathbb{Z}_2$ that have relatively low star discrepancy. Larcher and Pillichshammer even conjecture that these are the best digital $(0, m, 2)$-nets over $\mathbb{Z}_2$ with respect to the star discrepancy. In the same paper, the authors also give lower bounds on the star discrepancy of this special class of nets, namely

$$2^m D^*_{2^m} \geq m/5. \tag{1.1}$$

In spite of the bad distribution properties of the unshifted Hammersley net, certain shifts of the Hammersley net yield very well distributed $(0, m, 2)$-nets (with respect to the star discrepancy). Shifts of the Hammersley point set have already been examined by Halton and Zaremba [5], who showed that the star discrepancy of a special digital shift of H satisfies

$$2^m D^*_{2^m} = m/5 + O(1). \tag{1.2}$$

In [4], H. Faure improves (1.2) by giving a digital shift of the Hammersley net such that

$$2^m D^*_{2^m} \le m/6 + O(m^{\frac{1}{2}}). \tag{1.3}$$

In Section 3 we give special shift vectors by which the star discrepancy of the Hammersley net can be made very small, improving these results.

In [2], it is shown that for any digital $(t, m, 2)$-net over $\mathbb{Z}_2$ the following generalization, which is also best possible in the leading term, of the upper bound of Larcher and Pillichshammer holds.

$$2^m D^*_{2^m} \le 2^t\Big(\frac{m-t}{3} + \frac{19}{9}\Big). \tag{1.4}$$

A by-product of the results in Section 3 will be the construction of $(t, m, 2)$-nets with particularly low star discrepancy in comparison to this bound.

2. Upper Bounds for the Star Discrepancy of Shifted Digital $(0, m, 2)$-Nets

In the beginning of this section, let us introduce some notation. For given $k, l \ge 1$, we denote by $I^{k\times k}$ the $k \times k$-identity matrix and by $0^{k\times l}$ the $k \times l$-zero matrix. We denote by $\oplus$ addition modulo 2. If we use $\oplus$ with vectors, we mean componentwise addition modulo 2. For $\alpha \in [0, 1]$ we say that α is m-bit if α is of the form $\alpha = \alpha_1 2^{-1} + \cdots + \alpha_m 2^{-m}$ with $\alpha_i \in \mathbb{Z}_2$ for $1 \le i \le m$. We denote the vector of digits of α, $(\alpha_1, \ldots, \alpha_m)^T$, by $\vec{\alpha}$. Moreover, for $0 \le \alpha, \beta \le 1$, the discrepancy function $\Delta(\alpha, \beta)$ of a $(t, m, 2)$-net is defined by

$$\Delta(\alpha, \beta) := A_{2^m}\big([0, \alpha) \times [0, \beta)\big) - 2^m \alpha\beta.$$

If we want to stress that we are dealing with a special $(t, m, 2)$-net P, we might also write $\Delta(P, \alpha, \beta)$ and $A_{2^m}\big(P, [0, \alpha) \times [0, \beta)\big)$ respectively. It is well known (see for example [7]) that the supremum in the definition of the star discrepancy of a digital $(0, m, 2)$-net that is digitally shifted by m-bit vectors can be replaced by a maximum over a finite set with an error of at most $2/2^m - 1/2^{2m}$. We have

$$2^m D^*_{2^m} \le 2 + \max_{\substack{\alpha,\beta\\ m-\text{bit}}} |\Delta(\alpha, \beta)| - 2^{-m}.$$

We also have the following

Lemma 2.1. *The exact value of the star discrepancy of a (digitally shifted) digital* $(0, m, 2)$*-net* P *over* $\mathbb{Z}_2$ *is given by*

$$\max\left\{\max_{\substack{\alpha,\beta\\ m-\text{bit}}}\left|\frac{A_N\big([0,\alpha)\times[0,\beta)\big)}{N}-\alpha\beta\right|,\ \max_{\substack{\alpha,\beta\\ m-\text{bit}}}\left|\frac{A_N\big([0,\alpha]\times[0,\beta]\big)}{N}-\alpha\beta\right|\right\}.$$

Proof. The result is easily verified by employing the fact that all the coordinates of all the points of P are m-bit and that $\Delta(P, \alpha, \beta) = 0$ for $\alpha = 1$ and β m-bit or $\beta = 1$ and α m-bit. □

Multiplying the generating matrices, C_1, C_2, of a digital $(0, m, 2)$-net by the same regular $m \times m$-matrix from the right does not change the point set except for the order of the points. Thus, we can assume without loss of generality that $C_1 = I^{m\times m}$. Dealing with digital shifts of digital $(0, m, 2)$-nets over $\mathbb{Z}_2$, it is sufficient to consider only shifts in the second coordinate, since we have

Lemma 2.2. *Let* Y *be a net that is obtained by digitally shifting a digital* $(0, m, 2)$*-net with generating matrices* $C_1 = I^{m\times m}$ *and* C_2 *by vectors* $\vec{\sigma}_1, \vec{\sigma}_2 \in \mathbb{Z}_2^m$*. Then* Y *can also be obtained by replacing the vectors* $\vec{\sigma}_1, \vec{\sigma}_2$ *by vectors* $\vec{\tau}_1 = \vec{0}$ *and* $\vec{\tau}_2 = C_2 \cdot \vec{\sigma}_1 \oplus \vec{\sigma}_2 \in \mathbb{Z}_2^m$*.*

Proof. For $i \in \{0, \dots, b^m - 1\}$ denote the vector of digits of i by $\vec{i}$. Observe that

$$\begin{aligned}&\left\{(I^{m\times m}\cdot\vec{i}\oplus\vec{\sigma}_1, C_2\cdot\vec{i}\oplus\vec{\sigma}_2), 0\le i\le b^m-1\right\} =\\ &\qquad= \left\{(\vec{i}, C_2\cdot(\vec{i}\oplus\vec{\sigma}_1)\oplus\vec{\sigma}_2), 0\le i\le b^m-1\right\} =\\ &\qquad\quad= \left\{(I^{m\times m}\cdot\vec{i}, C_2\cdot\vec{i}\oplus C_2\cdot\vec{\sigma}_1\oplus\vec{\sigma}_2), 0\le i\le b^m-1\right\}.\end{aligned}$$

□

We are now ready to study digital shifts of a digital $(0, m, 2)$-net over $\mathbb{Z}_2$ and the star discrepancy of the nets obtained. By what has been outlined above, it is no loss of generality to assume that the generating matrices of the underlying digital net are $C_1 = I^{m\times m}$ and $C_2 = ((c_{i,j}))_{i,j=1}^m$, and it suffices to consider only shifts $\vec{\sigma} = (\sigma_1, \dots, \sigma_m)^T$ in the second coordinate. Corresponding to the notation introduced in [7, Section 2], let for $1 \le u \le m - 1$

$$C_2'(u) := \begin{pmatrix} c_{1,m-u+1} & \cdots & c_{u,m-u+1}\\ \vdots & \vdots & \vdots\\ c_{1,m} & \cdots & c_{u,m}\end{pmatrix}^{-1},$$

Moreover, for given m-bit α and β, let

$$\vec{\gamma} := C_2\cdot\vec{\alpha}\oplus\vec{\beta},\ \vec{\gamma}(u) := (\gamma_1, \dots, \gamma_u)^T,\ \vec{\sigma}(u) := (\sigma_1, \dots, \sigma_u)^T.$$

Further, let for $0 \le u \le m-1$

$$\mu(u) := \begin{cases} 0 & \text{if } u = 0 \\ 0 & \text{if } (\vec{\eta}(u)|C_2'(u)\cdot\vec{e}_1) = 1 \\ f(u) & \text{else,} \end{cases}$$

where $f(u) = \max\{1 \le j \le u : (\vec{\eta}(u)|C_2'(u)\cdot\vec{e}_i) = 0; i = 1,\ldots,j\}$, $\vec{\eta}(u) := \vec{\gamma}(u) \oplus \vec{\sigma}(u)$, and $\vec{e}_i$ is the i-th unit vector in $\mathbb{Z}_2^u$. We will make repeated use of the following lemma.

Lemma 2.3. *Let Y be a point set obtained by digitally shifting a digital $(0,m,2)$-net over $\mathbb{Z}_2$ (with generating matrices $C_1 = I^{m\times m}$, $C_2 = ((c_{i,j}))_{i,j=1}^m$) in the second coordinate by $\vec{\sigma} = (\sigma_1,\ldots,\sigma_m)^T \in \mathbb{Z}_2^m$. For any α and β m-bit, the discrepancy function satisfies*

$$\Delta(Y,\alpha,\beta) = \sum_{u=0}^{m-1} \|2^u\beta\| \, (-1)^{\sigma_{u+1}} g(u,\alpha,\beta,\vec{\eta},C_2), \tag{2.1}$$

where $\vec{\eta} := \vec{\gamma} \oplus \vec{\sigma}$, $\|.\|$ denotes the distance to the nearest integer function, and the function g satisfies $|g(u,\alpha,\beta,\vec{\eta},C_2)| \le 1$. In particular, if Y is a digitally shifted version of the Hammersley net H, we have

$$\Delta(Y,\alpha,\beta) = \sum_{u=0}^{m-1} \|2^u\beta\| \, (-1)^{\sigma_{u+1}} (\alpha_{m-u} \oplus \alpha_{m+1-(u-\mu(u))}), \tag{2.2}$$

where α_i is the i-th digit of α, and where we set $\alpha_{m+1} := 0$.

Proof. In the proof of Theorem 1 in [7] it is shown that for the discrepancy function of an unshifted digital $(0,m,2)$-net P over $\mathbb{Z}_2$ and m-bit α and β we have

$$\Delta(P,\alpha,\beta) = \sum_{u=0}^{m-1} \|2^u\beta\| \, g(u,\alpha,\beta,\vec{\gamma},C_2), \tag{2.3}$$

where the precise form of $g(u,\alpha,\beta,\vec{\gamma},C_2)$ is obtained by considering Walsh functions of the coordinates of the points of P. These Walsh functions are of the form $\mathrm{wal}_k(x) = (-1)^{(\vec{k}|\vec{x})}$, for an m-bit $x \in [0,1]$ and an integer $k \in \{0,\ldots,2^m-1\}$, where $\vec{k}$ and $\vec{x}$ are the digit vectors of k and x, respectively, and $(\cdot|\cdot)$ denotes the usual inner product. Since for m-bit $x, y \in [0,1]$ we have $\mathrm{wal}_k(x \oplus y) = \mathrm{wal}_k(x)\mathrm{wal}_k(y)$ (where $\oplus$ denotes digitwise addition modulo 2), it is, by following the proof of Theorem 1 in [7], no problem to show the generalization of (2.3) to (2.1) for a digitally shifted digital net Y.

If Y is the digitally shifted Hammersley point set, the special form of the function g is derived in complete analogy with Example 2 in [7]. □

Remark. Note that Lemma 2.3 implies

$$|\Delta(Y,\alpha,\beta)| \le \sum_{u=0}^{m-1} \|2^u\beta\| \tag{2.4}$$

for any m-bit α, β and for any $(0,m,2)$-net Y in base 2 that is obtained by shifting a digital $(0,m,2)$-net by m-bit vectors.

Before we state the main result of this section, we give some notation and auxiliary results that will be needed in the proof.

Let $m \ge 0$ be given. For $0 \le j \le m$ and $0 \le k \le 2^j - 1$ define

$$M_{k,j} := \{k2^{m-j}, \dots, (k+1)2^{m-j} - 1\}.$$

It is obvious that for $j \ge 1$ and even k

$$M_{k,j} \cup M_{k+1,j} = M_{\frac{k}{2},j-1}. \tag{2.5}$$

For short, we denote, for given k and j, the elements of $M_{k,j}$ in increasing order by

$$M_{k,j}^{(1)} < M_{k,j}^{(2)} < \dots < M_{k,j}^{(2^{m-j})},$$

i. e., $M_{k,j}^{(i)} = k2^{m-j} + i - 1$ for $1 \le i \le 2^{m-j}$.

Lemma 2.4. *Let a function $V_1 : \{0, \dots, 2^m - 1\} \to \{0, \dots, 2^m - 1\}$ be defined as follows. For $i \in \{0, \dots, 2^m - 1\}$, $i = i_0 + i_1 2 + \dots + i_{m-1}2^{m-1}$, let $\vec{i} = (i_0, \dots, i_{m-1})^T$ be the digit vector of i. We define, for given $\vec{\sigma} = (\sigma_1, \dots, \sigma_m)^T \in \mathbb{Z}_2^m$, $V_1(i)$ to be the number with the digit vector*

$$(i_0 \oplus \sigma_m, \dots, i_{m-1} \oplus \sigma_1)^T.$$

Then V_1 and $T_1 := V_1^{-1}$ are bijective and have the following additional property. For given $m \ge 0$, $0 \le j \le m$, and $0 \le k \le 2^j - 1$,

$$V_1(M_{k,j}) = M_{l_1,j}, \quad T_1(M_{k,j}) = M_{l_2,j}$$

with certain $l_1, l_2 \in \{0, \dots, 2^j - 1\}$.

Proof. Any set $M_{k,j}$ consists of all i with digit vectors $\vec{i} = (i_0, \dots, i_{m-1})^T$ where $i_0, \dots, i_{m-j-1}$ can be chosen arbitrarily and $i_{m-j}, \dots, i_{m-1}$ are fixed. From this the result follows easily. □

Lemma 2.5. *Let*

$$L = \left(\begin{array}{c|c} A & B \\ C & D \end{array}\right)$$

be a regular $m \times m$-matrix, where A is an $(m-j) \times (m-j)$-matrix and D is a $j \times j$-matrix. Let

$$L^{-1} = \left(\begin{array}{c|c} X & Y \\ U & V \end{array}\right),$$

where X is an $(m-j)\times(m-j)$-matrix and V is a $j\times j$-matrix. Then

$$\det L \det V = \det A \quad \text{and} \quad \det L \det X = \det D.$$

Proof. The first assertion is shown in [9, Theorem 2.3] for complex matrices. The generalization to matrices over arbitrary fields and the proof of the second assertion are straightforward. □

Lemma 2.6. *Let $D_1 = I^{m\times m}$ and $D_2 = ((d_{i,j}))_{i,j=1}^m$ be the generating matrices of an arbitrary digital $(0,m,2)$-net over $\mathbb{Z}_2$ and denote the row vectors of D_2 by $\vec{d}_1,\dots,\vec{d}_m$. Let $V_2 : \{0,\dots,2^m-1\} \to \{0,\dots,2^m-1\}$ be the following function. For $i \in \{0,\dots,2^m-1\}$, denote by $\vec{i} = (i_0,\dots,i_{m-1})^T$ the digit vector of i. We define $V_2(i)$ to be the number with the digit vector*

$$\big((\vec{d}_m|\vec{i}),\dots,(\vec{d}_1|\vec{i})\big)^T.$$

Then $T_2 := V_2^{-1}$ is bijective, linear and, for given $m \geq 0$, $0 \leq j \leq m$, and $0 \leq k \leq 2^j - 1$,

$$T_2(M_{k,j}) = \{M^{(1)}_{l_1,j}, M^{(2)}_{l_2,j}, \dots, M^{(2^{m-j})}_{l_{2^{m-j}},j}\}$$

for some $l_1,\dots,l_{2^{m-j}} \in \{0,\dots,2^j-1\}$.

Proof. Applying V_2 to an i, $\vec{i}$ is transformed into

$$\big((\vec{d}_m|\vec{i}),\dots,(\vec{d}_1|\vec{i})\big)^T = \begin{pmatrix} \vec{d}_m \\ \vdots \\ \vec{d}_1 \end{pmatrix} \cdot \begin{pmatrix} i_0 \\ \vdots \\ i_{m-1} \end{pmatrix}.$$

The matrix $(\vec{d}_m \dots \vec{d}_1)^T$ is a flipped version of D_2. By the regularity of D_2 it is obvious that V_2 and T_2 are isomorphisms. Moreover, the matrix can be written as

$$L := \left(\begin{array}{c|c} D_2^{(1)} & D_2^{(2)} \\ \hline D_2^{(3)} & D_2^{(4)} \end{array}\right), \quad \text{where } D_2^{(4)} = \begin{pmatrix} d_{j,m-j+1} & \dots & d_{j,m} \\ \vdots & \vdots & \vdots \\ d_{1,m-j+1} & \dots & d_{1,m} \end{pmatrix}.$$

Since D_1 and D_2 generate a digital $(0,m,2)$-net, and $D_1 = I^{m\times m}$, it follows that $D_2^{(4)}$ is regular. This, however, by Lemma 2.5, means that the matrix L^{-1} is such that

$$L^{-1} = \left(\begin{array}{c|c} L^{(1)} & L^{(2)} \\ \hline L^{(3)} & L^{(4)} \end{array}\right)$$

where $L^{(1)}$ is a regular $(m-j)\times(m-j)$-matrix.

An arbitrary $M_{k,j}$ consists of all i with digit vectors $\vec{i} = (i_0,\dots,i_{m-1})^T$ where $i_0,\dots,i_{m-j-1}$ can be chosen arbitrarily and $i_{m-j},\dots,i_{m-1}$ are fixed.

Applying T_2 to such an i means multiplying L^{-1} by $\vec{i}$. Since $L^{(1)}$ is regular, it follows that the matrix

$$\left(L^{(1)} \;\middle|\; L^{(2)}\right)$$

maps the i in $M_{k,j}$, where the first $m-j$ components of $\vec{i}$ run through $\mathbb{Z}_2^{m-j}$ and the rest is fixed, onto $\mathbb{Z}_2^{m-j}$. This yields the result. □

Lemma 2.7. *Let $V : \{0, \dots, 2^m - 1\} \to \{0, \dots, 2^m - 1\}$ be the function defined by $V := V_1 \circ V_2$, i.e. $V(i) = V_1(V_2(i))$, then V is bijective and for $T := V^{-1} = V_2^{-1} \circ V_1^{-1} = T_2 \circ T_1$ it is true that*

$$T(M_{k,j}) = \{M^{(1)}_{l_1,j}, M^{(2)}_{l_2,j}, \dots, M^{(2^{m-j})}_{l_{2^{m-j}},j}\} \tag{2.6}$$

for certain $l_i \in \{0, \dots, 2^j - 1\}$. Moreover, for $j \geq 1$ and for even k,

$$T(M_{k,j}) \cup T(M_{k+1,j}) = T(M_{\frac{k}{2},j-1}) = \{M^{(1)}_{l_1,j-1}, M^{(2)}_{l_2,j-1}, \dots, M^{(2^{m-j+1})}_{l_{2^{m-j+1}},j-1}\}$$

for certain $l_i \in \{0, \dots, 2^{j-1} - 1\}$.

Proof. The first assertion follows immediately from Lemmas 2.4 and 2.6. The second assertion follows from the first together with Equation (2.5). □

We are now ready to prove

Proposition 2.1. *Let $H = \{\mathbf{x}_0, \dots, \mathbf{x}_{2^m-1}\}$, $\mathbf{x}_i = (x_i^{(1)}, x_i^{(2)})$ for $0 \leq i \leq 2^m - 1$, be the $(0, m, 2)$-Hammersley net over $\mathbb{Z}_2$, generated by C_1 and C_2 as given above. Let $Y = \{\mathbf{y}_0, \dots, \mathbf{y}_{2^m-1}\}$, $\mathbf{y}_i = (y_i^{(1)}, y_i^{(2)})$ for $0 \leq i \leq 2^m - 1$, be the net that is obtained by shifting any fixed digital $(0, m, 2)$-net over $\mathbb{Z}_2$ in the second coordinate by an arbitrary vector $\vec{\sigma} = (\sigma_1, \dots, \sigma_m)^T \in \mathbb{Z}_2^m$. Then, for all α, β m-bit, we have*

$$A_N\big(Y, [0, \alpha] \times [0, \beta]\big) \leq A_N\big(H, [0, \alpha] \times [0, \beta]\big).$$

where $N = 2^m$.

Proof. Let α, β be m-bit. Define, for $j \in \{1, 2\}$,

$$H_j := \{i \in \{0, \dots, 2^m - 1\} : x_i^{(j)} \leq \alpha\},$$
$$Y_j := \{i \in \{0, \dots, 2^m - 1\} : y_i^{(j)} \leq \alpha\}$$

Then we have

$$A_N\big(H, [0, \alpha] \times [0, \beta]\big) = |H_1 \cap H_2|, \quad A_N\big(Y, [0, \alpha] \times [0, \beta]\big) = |Y_1 \cap Y_2|.$$

Let us now try to find out which $i \in \{0, \dots, 2^m - 1\}$ lie in H_1 and H_2. Concerning H_1, we first observe that, for $i = i_0 + i_1 2 + \dots + i_{m-1} 2^{m-1}$,

$$x_i^{(1)} = i_0 2^{-1} + i_1 2^{-2} + \dots + i_{m-1} 2^{-m},$$

Thus, i lies in H_1 if and only if

$$\sum_{j=0}^{m-1} i_{m-1-j} 2^j \leq a,$$

where $a = 2^m\alpha$. Note that $a \in \{0, \ldots, 2^m - 1\}$. We can order the $i \in \{0, \ldots, 2^m - 1\}$ according to the value of $x_i^{(1)}$, starting with $x_i^{(1)} = 0$ and then increasing. This gives a sequence

$$\begin{aligned} &i^{(0)} = 0, && i^{(1)} = 2^{m-1}, && i^{(2)} = 2^{m-2}, \\ &i^{(3)} = 2^{m-2} + 2^{m-1}, && i^{(4)} = 2^{m-3}, && i^{(5)} = 2^{m-3} + 2^{m-1}, \\ &i^{(6)} = 2^{m-3} + 2^{m-2}, \ldots \end{aligned}$$

Due to the special form of the sequence, it is not difficult to see that, for given j, the $i^{(n)}$ always hit the sets $M_{k,j}$ $(0 \leq j \leq m,\ 0 \leq k \leq 2^j - 1)$ defined above in a special order. The $i^{(n)}$ first hit $M_{0,j}^{(1)}$, then all $M_{k,j}^{(1)}$ with even k, and finally all $M_{k,j}^{(1)}$ with odd k. Then, the same pattern repeats itself for another index r, such that again $M_{0,j}^{(r)}$ is hit before all $M_{k,j}^{(r)}$ with even k which are again hit before all $M_{k,j}^{(r)}$ with odd k, etc. Thus,

$$|H_1 \cap M_{0,j}| \geq \left| H_1 \cap \{M_{k_1,j}^{(1)}, \ldots, M_{k_{2^{m-j}},j}^{(2^{m-j})}\} \right| \tag{2.7}$$

for $k_1, \ldots, k_{2^{m-j}} \in \{0, \ldots, 2^j - 1\}$, and

$$\left| \left| H_1 \cap \{M_{k_1,j}^{(1)}, \ldots, M_{k_{2^{m-j}},j}^{(2^{m-j})}\} \right| - \left| H_1 \cap \{M_{l_1,j}^{(1)}, \ldots, M_{l_{2^{m-j}},j}^{(2^{m-j})}\} \right| \right| \leq 1 \tag{2.8}$$

for any choice of $k_1, \ldots, k_{2^{m-j}}, l_1, \ldots, l_{2^{m-j}} \in \{0, \ldots, 2^j - 1\}$. To be more precise, there can be at most one index $r \in \{1, \ldots, 2^{m-j}\}$ such that

$$\left| H_1 \cap \{M_{k_r,j}^{(r)}\} \right| \neq \left| H_1 \cap \{M_{l_r,j}^{(r)}\} \right|.$$

Looking at H_2, it is clear by the form of C_2 that $i \in H_2$ if and only if

$$\sum_{j=0}^{m-1} i_j 2^j \leq b,$$

where $b = 2^m\beta \in \{0, \ldots, 2^m - 1\}$.

Let us now come to Y_1, Y_2. Y is a shifted digital net, where the digital net is generated by two matrices, D_1 and $D_2 = ((d_{i,j}))_{i,j=1}^m$. Again we can assume $D_1 = I^{m\times m}$, so $D_1 = C_1$ and $Y_1 = H_1$. On the other hand, by the way a digital net is constructed and by the way a digital net is shifted, it is easy to see that $i = i_0 + i_1 2 + \cdots + i_{m-1}2^{m-1}$ lies in Y_2 if and only if

$$\frac{(\vec{d}_1|\vec{i}) \oplus \sigma_1}{2} + \frac{(\vec{d}_2|\vec{i}) \oplus \sigma_2}{2^2} + \cdots + \frac{(\vec{d}_m|\vec{i}) \oplus \sigma_m}{2^m} \leq \beta.$$

Here, $\vec{d}_1, \ldots, \vec{d}_m$ denote the row-vectors of the matrix D_2. The latter condition is equivalent to

$$\sum_{j=0}^{m-1} \left((\vec{d}_{m-j} | \vec{i}) \oplus \sigma_{m-j} \right) \cdot 2^j \leq b.$$

Let now $V, T : \{0, \ldots, 2^m - 1\} \to \{0, \ldots, 2^m - 1\}$ be defined as in Lemma 2.7. The condition on the i in Y_2 means that $i \in Y_2$ if and only if $V(i) \in H_2$ which is of course equivalent to $i \in T(H_2)$. So, $T(H_2) = Y_2$ and, consequently, $Y_1 \cap Y_2 = H_1 \cap T(H_2)$. The crucial step is to show that

$$|Y_1 \cap Y_2| = |H_1 \cap T(H_2)| \leq |H_1 \cap H_2| \,.$$

Let p be maximal such that 2^p is a divisor of $b+1$, i.e., there is an integer l such that $l2^p = b+1$. Then we can write $H_2 = \{0, \ldots, b\}$ in the form

$$\{0, \ldots, 2^p - 1\} \cup \ldots \cup \{(l-1)2^p, \ldots, l2^p - 1\} = M_{0,m-p} \cup \ldots \cup M_{l-1,m-p}.$$

Note that l always satisfies $l \leq 2^{m-p}$. It is clear that

$$Y_2 = T(H_2) = T(M_{0,m-p} \cup \ldots \cup M_{l-1,m-p}) = T(M_{0,m-p}) \cup \ldots \cup T(M_{l-1,m-p}),$$

which results in

$$H_1 \cap T(H_2) = H_1 \cap \bigcup_{k=0}^{l-1} T(M_{k,m-p}) = \bigcup_{k=0}^{l-1} \left(H_1 \cap T(M_{k,m-p}) \right).$$

Since T is bijective and the $M_{k,m-p}$ are pairwise disjoint it follows that

$$|H_1 \cap T(H_2)| = \sum_{k=0}^{l-1} |H_1 \cap T(M_{k,m-p})| =: B(l-1, m-p).$$

On the other hand, however,

$$|H_1 \cap H_2| = \sum_{k=0}^{l-1} |H_1 \cap M_{k,m-p}| =: A(l-1, m-p).$$

We now show that, for any possible value of $m-p$,

$$B(l-1, m-p) \leq A(l-1, m-p)$$

by induction on the number l of sets $M_{k,m-p}$ involved. Note that $l \geq 2$ cannot be even. Indeed, if this would be the case, it would follow that p was not chosen maximal. For $l = 1$, $T(M_{0,m-p})$ is, by (2.6), of the form

$$\{M^{(1)}_{k_1,m-p}, \ldots, M^{(2^p)}_{k_{2^p},m-p}\}$$

for certain $k_1, \ldots, k_{2^p} \in \{0, \ldots, 2^{m-p} - 1\}$. Equation (2.7) then implies the result for any admissible value of $m-p$.

We have to do the induction step from l to $l+2 \leq 2^{m-p}$ where l is odd. Since $l+2 \geq 3$ is odd, it even follows that $l+2 \leq 2^{m-p} - 1$. This also implies $m-p \geq 2$ such that the sets $M_{k,m-p-1}$ and $M_{k,m-p-2}$ exist for suitable k. We have to show that $B(l+1,m-p) \leq A(l+1,m-p)$.

We first consider $A(l,m-p)$ and $B(l,m-p)$. As l is odd, by (2.5),

$$A(l,m-p) = A\big((l-1)/2, m-p-1\big), \quad B(l,m-p) = B\big((l-1)/2, m-p-1\big).$$

Suppose $(l-1)/2$ is even. Then $\bar{l} := (l-1)/2+1$ is odd and $\bar{l} \leq l$ since $l \geq 1$. Moreover, $\bar{l} \leq 2^{m-p-1}$. Since $(l+2)2^p = b+1$ and $\bar{l}$ is odd, it follows that $p+1$ is maximal such that $\bar{l}2^{p+1} = b+1-2^p$. By setting $\bar{p} := p+1$, the induction hypothesis yields

$$B\big((l-1)/2, m-\bar{p}\big) \leq A\big((l-1)/2, m-\bar{p}\big).$$

If $(l-1)/2$ is odd, we can again subtract 1 and divide by 2 and iterate the procedure until we finally sum over $k \in \{0,\dots,\bar{\bar{l}}\}$ with even $\bar{\bar{l}}$. Then again $\bar{\bar{l}}+1 \leq l$. In any case, it follows by the induction hypothesis that

$$\begin{aligned}(2.9)\quad B(l,m-p) = B\big((l-1)/2, m-p-1\big) \leq \\ \leq A\big((l-1)/2, m-p-1\big) = A(l,m-p).\end{aligned}$$

Suppose now that

$$B(l+1,m-p) > A(l+1,m-p).$$

Then, by Equation (2.9), and due to the fact that

$$\big|\,|H_1 \cap T(M_{l+1,m-p})| - |H_1 \cap M_{l+1,m-p}|\,\big| \leq 1,$$

we must have

$$(2.10)\quad B(l,m-p) = B\big((l-1)/2, m-p-1\big) = A\big((l-1)/2, m-p-1\big) = A(l,m-p)$$

and

$$|H_1 \cap T(M_{l+1,m-p})| > |H_1 \cap M_{l+1,m-p}|\,.$$

Suppose that

$$(2.11)\qquad \left|H_1 \cap T(M_{\frac{l+1}{2},m-p-1})\right| > \left|H_1 \cap M_{\frac{l+1}{2},m-p-1}\right|.$$

Consider now $A\big((l+1)/2, m-p-1\big)$ and $B\big((l+1)/2, m-p-1\big)$. For $l=1$, $(l+1)/2 = 1$. Then, by (2.5),

$$A(1,m-p-1) = |H_1 \cap M_{0,m-p-2}|\,, \quad B(1,m-p-1) = |H_1 \cap T(M_{0,m-p-2})|\,.$$

For $l \geq 3$, $(l+1)/2+1 \leq l$, and we can proceed as in the derivation of Equation (2.9). In both cases the induction hypothesis yields

$$B\big((l+1)/2, m-p-1\big) \leq A\big((l+1)/2, m-p-1\big).$$

If, however, Equation (2.11) holds, it would follow that

$$B\big((l-1)/2, m-p-1\big) < A\big((l-1)/2, m-p-1\big).$$

This would be a contradiction to Equation (2.10). So, we must have

$$|H_1 \cap T(M_{l+1,m-p})| > |H_1 \cap M_{l+1,m-p}|.$$

and

$$\left|H_1 \cap T(M_{\frac{l+1}{2},m-p-1})\right| \leq \left|H_1 \cap M_{\frac{l+1}{2},m-p-1}\right|, \tag{2.12}$$

Let

$$T(M_{l+1,m-p}) := \{M^{(1)}_{s_1,m-p}, \ldots, M^{(2^p)}_{s_{2^p},m-p}\},$$

where $s_1, \ldots, s_{2^p} \in \{0, \ldots, 2^{m-p}-1\}$. Let r be the index such that

$$\left|H_1 \cap \{M^{(r)}_{s_r,m-p}\}\right| > \left|H_1 \cap \{M^{(r)}_{l+1,m-p}\}\right|. \tag{2.13}$$

If s_r was odd, we would have a contradiction since $l+1$ is even and so $M^{(r)}_{l+1,m-p}$ must be hit by the sequence of the $i^{(n)}$ before $M^{(r)}_{s_r,m-p}$. So s_r must be even. By Equation (2.12) we must have

$$|H_1 \cap T(M_{l+2,m-p})| < |H_1 \cap M_{l+2,m-p}|.$$

Let

$$T(M_{l+2,m-p}) := \{M^{(1)}_{t_1,m-p}, \ldots, M^{(2^p)}_{t_{2^p},m-p}\},$$

where $t_1, \ldots, t_{2^p} \in \{0, \ldots, 2^{m-p}-1\}$. Due to the way the sequence of the $i^{(n)}$ hits the sets $M_{k,j}$ and due to Equation (2.13), it follows that

$$\left|H_1 \cap \{M^{(r)}_{t_r,m-p}\}\right| < \left|H_1 \cap \{M^{(r)}_{l+2,m-p}\}\right|.$$

Then, however, since $l+1$ is even and $l+2$ is odd, we would finally get that

$$\left|H_1 \cap \{M^{(r)}_{s_r,m-p}\}\right| > \left|H_1 \cap \{M^{(r)}_{l+1,m-p}\}\right| \geq$$
$$\geq \left|H_1 \cap \{M^{(r)}_{l+2,m-p}\}\right| > \left|H_1 \cap \{M^{(r)}_{t_r,m-p}\}\right|,$$

and so

$$\left| \left|H_1 \cap \{M^{(r)}_{s_r,m-p}\}\right| - \left|H_1 \cap \{M^{(r)}_{t_r,m-p}\}\right| \right| \geq 2.$$

This would be a contradiction since the sets involved have at most one element. This yields the result. □

From Proposition 2.1 we deduce

Theorem 2.1. *Let $H = \{\mathbf{x}_0, \ldots, \mathbf{x}_{2^m-1}\}$ be the $(0, m, 2)$-Hammersley net over $\mathbb{Z}_2$, generated by C_1 and C_2 as given above. Let $Y = \{\mathbf{y}_0, \ldots, \mathbf{y}_{2^m-1}\}$ be the net that is obtained by shifting a fixed digital $(0, m, 2)$-net over $\mathbb{Z}_2$ in the second coordinate by an arbitrary vector $\vec{\sigma} = (\sigma_1, \ldots, \sigma_m)^T \in \mathbb{Z}_2^m$. Then,*

$$D^*_N(Y) \leq D^*_N(H) = \left(\frac{m}{3} + \frac{13}{9} - (-1)^m \cdot \frac{4}{9 \cdot 2^m}\right) 2^{-m},$$

where $N = 2^m$. Moreover, this bound is sharp, and it is attained when Y is the Hammersley point set.

Proof. For calculating the star discrepancy of Y it is sufficient to consider only the intervals $[0,\alpha) \times [0,\beta)$ and $[0,\alpha] \times [0,\beta]$ with α, β m-bit (see Lemma 2.1). By Equation (2.4),

$$\left|A_N\big(Y,[0,\alpha)\times[0,\beta)\big)N^{-1}-\alpha\beta\right| = |\Delta(Y,\alpha,\beta)|\cdot\frac{1}{N} \le \frac{1}{N}\cdot\sum_{u=0}^{m-1}\|2^u\beta\|\,.$$

From [7] we get

$$\frac{1}{N}\cdot\sum_{u=0}^{m-1}\|2^u\beta\| \le \max_{\alpha,\beta\ m-\mathrm{bit}}|\Delta(H,\alpha,\beta)|\cdot\frac{1}{N} \le D_N^*(H),$$

so

$$\left|A_N\big(Y,[0,\alpha)\times[0,\beta)\big)N^{-1}-\alpha\beta\right| \le D_N^*(H). \tag{2.14}$$

Let us now consider the intervals $[0,\alpha]\times[0,\beta]$. On the one hand, we clearly have:

$$A_N\big(Y,[0,\alpha]\times[0,\beta]\big)N^{-1}-\alpha\beta \ge A_N\big(Y,[0,\alpha)\times[0,\beta)\big)N^{-1}-\alpha\beta.$$

On the other hand, by Proposition 2.1,

$$A_N\big(Y,[0,\alpha]\times[0,\beta]\big)N^{-1}-\alpha\beta \le A_N\big(H,[0,\alpha]\times[0,\beta]\big)N^{-1}-\alpha\beta. \tag{2.15}$$

The result now follows by Equation (2.14) and by observing that the absolute value of the right hand side in Equation (2.15) is bounded by $D_N^*(H)$. □

Remark. Theorem 2.1 improves the bound on $2^m D_{2^m}^*$ of (unshifted) digital $(0,m,2)$-nets over $\mathbb{Z}_2$ by Larcher and Pillichshammer. Moreover, we have found the worst among all digital $(0,m,2)$-nets over $\mathbb{Z}_2$ that are shifted by m-bit vectors with respect to the star discrepancy.

3. Digital Shifts of the Hammersley Net

Theorem 2.1 implies that any digital m-bit shift applied to H cannot have negative effects on the star discrepancy. In fact it can be shown that any digital shift different from $\vec{0}$ of the Hammersley net results in a real improvement of the star discrepancy.

Theorem 3.1. *Let H be the $(0,m,2)$-Hammersley net over $\mathbb{Z}_2$, and denote by S the Hammersley net that is shifted by a shift vector $\vec{\sigma}\in\mathbb{Z}_2^m\setminus\{\vec{0}\}$ in the second coordinate. Then, for $m\ge 3$, we have*

$$D_N^*(S) < D_N^*(H),$$

where $N = 2^m$.

Proof. For $m = 3$, the result is easily verified numerically. For $m \geq 4$ it is, by Lemma 2.1, sufficient to consider only the intervals $[0, \alpha) \times [0, \beta)$ and $[0, \alpha] \times [0, \beta]$ for m-bit α, β. Let us start with intervals of the form $[0, \alpha) \times [0, \beta)$. By using Equation (2.4) together with Theorem 2 in [7],

$$\left|A_N\big(S, [0, \alpha) \times [0, \beta)\big) - N\alpha\beta\right| \leq \sum_{u=0}^{m-1} \|2^u \beta\| \leq \frac{m}{3} + \frac{1}{9} - (-1)^m \frac{1}{9 \cdot 2^m} < ND_N^*(H).$$

Let us now turn to intervals of the form $[0, \alpha] \times [0, \beta]$ with α and β m-bit. By Proposition 2.1,

$$A_N\big(S, [0, \alpha] \times [0, \beta]\big) - N\alpha\beta \leq A_N\big(H, [0, \alpha] \times [0, \beta]\big) - N\alpha\beta \leq ND_N^*(H).$$

Suppose

$$A_N\big(S, [0, \alpha] \times [0, \beta]\big) - N\alpha\beta = ND_N^*(H).$$

This implies

$$A_N\big(H, [0, \alpha] \times [0, \beta]\big) - N\alpha\beta = ND_N^*(H).$$

In [7, Proof of Theorem 4b], the authors show that $D_N^*(H)$ is always attained for intervals of the form $[0, \alpha_0 - 2^{-m}] \times [0, \beta_0 - 2^{-m}]$, where α_0 and β_0 are m-bit, and they give the exact values of α_0 and β_0 for which ND_N^* is attained (for the exact values of α_0 and β_0, see [7, Theorem 4b]). Thus, $\alpha = \alpha_0 - 2^{-m}$, $\beta = \beta_0 - 2^{-m}$,

$$\begin{aligned} A_N\big(H, [0, \alpha] \times [0, \beta]\big) - N\alpha\beta &= A_N\big(H, [0, \alpha_0 - 2^{-m}] \times [0, \beta_0 - 2^{-m}]\big) \\ &\quad - N(\alpha_0 - 2^{-m})(\beta_0 - 2^{-m}) \\ &= \Delta(H, \alpha_0, \beta_0) + \alpha_0 + \beta_0 - 2^{-m}, \end{aligned}$$

and

$$A_N\big(S, [0, \alpha] \times [0, \beta]\big) - N\alpha\beta = \Delta\big(S, \alpha_0, \beta_0\big) + \alpha_0 + \beta_0 - 2^{-m}$$

for one of the pairs (α_0, β_0). This implies $\Delta(H, \alpha_0, \beta_0) = \Delta\big(S, \alpha_0, \beta_0\big)$. For all possible choices, it can easily be verified that (α_0, β_0) is such that

$$\Delta(H, \alpha_0, \beta_0) = \sum_{u=0}^{m-1} \|2^u \beta_0\|.$$

Moreover, β_0 is such that $\|2^u \beta_0\| > 0$ for all $u \in \{0, 1, \dots, m-1\}$. However, since $\vec{\sigma} \neq \vec{0}$, it then follows by Equation (2.2) in Lemma 2.3 that

$$\Delta\big(S, \alpha_0, \beta_0\big) < \Delta(H, \alpha_0, \beta_0)$$

and we have a contradiction. Therefore,

$$\begin{aligned} ND_N^*(H) &> A_N\bigl(S, [0,\alpha] \times [0,\beta]\bigr) - N\alpha\beta \\ &\geq A_N\bigl(S, [0,\alpha) \times [0,\beta)\bigr) - N\alpha\beta \\ &> -ND_N^*(H) \end{aligned}$$

and this yields the result. □

With Theorem 3.1, we know that non-trivial shifts of H yield an improvement in the star discrepancy. For some special shifts, this improvement is particularly remarkable, as the next theorem shows.

Theorem 3.2. *Let $m \geq 6$. Moreover, let*

$$\vec{\sigma} = (\underbrace{1, 1, \ldots, 1}_{\substack{k \text{ com-} \\ \text{ponents}}}, 0, 0, \ldots, 0)^T,$$

where $k = (m+6)/2$ if m is even, and $k = (m+5)/2$ if m is odd. Let S be the point set obtained by shifting the Hammersley point set by $\vec{\sigma}$ in the second coordinate. Then

$$ND_N^*(S) \leq \frac{m}{6} + c,$$

where $N = 2^m$, with $c = 7/6$ if m is even and $c = 4/3$ if m is odd.

Proof. We show the result for even m. The result for odd m is obtained similarly. Let m be even and let $k := (m+6)/2$. By Equation (2.2) and by Theorem 2 in [7], we easily find that, for α, β m-bit,

$$\Delta(S, \alpha, \beta) \geq -\sum_{u=0}^{k-1} \|2^u \beta\| \geq -\Bigl(\frac{k}{3} + \frac{1}{9} - \frac{(-1)^k}{9 \cdot 2^k}\Bigr).$$

Similarly, we have

$$\Delta(S, \alpha, \beta) \leq \sum_{u=k}^{m-1} \|2^u \beta\| = \sum_{u=0}^{m-k-1} \left\|2^{u+k} \beta\right\| \leq \frac{m-k}{3} + \frac{1}{9} - \frac{(-1)^{m-k}}{9 \cdot 2^{m-k}}.$$

Thus, it follows that

$$|\Delta(S, \alpha, \beta)| \leq \max\Bigl\{\frac{k}{3} + \frac{1}{9} - \frac{(-1)^k}{9 \cdot 2^k}, \frac{m-k}{3} + \frac{1}{9} - \frac{(-1)^{m-k}}{9 \cdot 2^{m-k}}\Bigr\}.$$

Since

$$\Delta(S, \alpha, \beta) \leq A_N\bigl(S, [0,\alpha] \times [0,\beta]\bigr) - N\alpha\beta \leq \Delta(S, \alpha, \beta) + 2,$$

we find that

$$ND_N^*(S) \leq \max\Bigl\{\frac{k}{3} + \frac{1}{9} - \frac{(-1)^k}{9 \cdot 2^k}, \frac{m-k}{3} + \frac{19}{9} - \frac{(-1)^{m-k}}{9 \cdot 2^{m-k}}\Bigr\}.$$

However, it can easily be seen that

$$\frac{k}{3} + \frac{1}{9} - \frac{(-1)^k}{9 \cdot 2^k} \leq \frac{m}{6} + \frac{10}{9} + \frac{1}{1152}.$$

Similarly,

$$\frac{m-k}{3} + \frac{19}{9} - \frac{(-1)^{m-k}}{9 \cdot 2^{m-k}} \leq \frac{m}{6} + \frac{10}{9} + \frac{1}{18}.$$

The result follows. □

A slight drawback of the result in Theorem 3.2 is that it only holds for $m \geq 6$. By a little modification we can extend Theorem 3.2 to the subsequent proposition which will then help to establish a result on $(t, m, 2)$-nets.

Proposition 3.1. *Let, for $m \geq 1$, S be the point set that is obtained as follows. First, shift H by the shift vector*

$$\vec{\sigma} = (\underbrace{1, 1, \ldots, 1}_{\substack{k \text{ com-} \\ \text{ponents}}}, 0, 0, \ldots, 0)^T,$$

where $k = m/2$ if m is even and $k = (m+1)/2$ if m is odd. Finally, add 2^{-m-1} to each coordinate of each point, i.e., the points are moved into the middle of the squares induced by the mesh with resolution 2^{-m} in $[0, 1)^2$. Then we have

$$ND_N^*(S) \leq \frac{m}{6} + c,$$

where $N = 2^m$ and c is a constant lower than $4/3$.

Proof. The proof is based on the same principle as the proof of Theorem 3.2. Since the points of S are $(m+1)$-bit, it is necessary to estimate the value of $\Delta(S, \alpha, \beta)$ for $(m+1)$-bit numbers α and β. This is achieved by bounding $\Delta(S, \alpha, \beta)$ in terms of $\Delta(S, \alpha', \beta')$ or $\Delta(S, \alpha'', \beta'')$, where α', β' are the largest m-bit numbers less than α and β, and α'', β'' are the smallest m-bit numbers greater than α and β, respectively. This part of the proof is a mere technicality which is dealt with by distinguishing different cases, according to whether the $(m+1)$-st digits of α and β are zero or not. The values of $\Delta(S, \alpha', \beta')$ and $\Delta(S, \alpha'', \beta'')$ can then conveniently be estimated by the same methods as in the proof of Theorem 3.2, since the number of points of a shifted Hammersley point set in a half-open m-bit interval does not change by moving the coordinates of the points by the quantity 2^{-m-1}.

In a similar way, one obtains the desired bounds on the expression $|A_N(S, [0, \alpha] \times [0, \beta]) - N\alpha\beta|$ for $(m+1)$-bit α and β. □

Remark. Note that the nets obtained by the constructions in Theorem 3.2 and Proposition 3.1 are essentially better than Larcher's and Pillichshammer's nets mentioned in the introduction (cf. Equation (1.1)), but they are no digital nets even though they are easily constructed from digital nets. Moreover, the bounds on the star discrepancy in Theorem 3.2 and Proposition 3.1 should be compared to the bounds in Equations (1.2) and (1.3).

From Proposition 3.1, we immediately get the following Theorem concerning the construction of $(t, m, 2)$-nets with particularly low star discrepancy (cf. Equation (1.4)).

Theorem 3.3. *For any $m \geq 1$, $0 \leq t \leq m$, there exists a $(t, m, 2)$-net P in base 2 that satisfies*

$$2^m D^*_{2^m}(P) \leq 2^t \Big(\frac{m-t}{6} + c \Big),$$

where $c < 4/3$.

Proof. The result follows by taking 2^t copies of the $(0, m-t, 2)$-Hammersley net, transforming them as outlined in Proposition 3.1, and applying a two-dimensional version of Theorem 2.6 in Chapter 2 in [6]. □

Acknowledgments

The research for this paper was supported by the Austrian Research Foundation (FWF) projects S8311-MAT and P17022-N12. Furthermore, the author would like to thank J. Dick (who gave the idea for Proposition 3.1), F. Pillichshammer, and W. Ch. Schmid for valuable discussions. Moreover, the author is grateful to the referee for his helpful suggestions for improving the paper.

References

[1] L. De Clerck, *A method for exact calculation of the stardiscrepancy of plane sets applied to the sequences of Hammersley.* Monatsh. Math. **101** (1986), 261–278.

[2] J. Dick, P. Kritzer, *Star-discrepancy estimates for digital $(t, m, 2)$-nets and $(t, 2)$-sequences over $\mathbb{Z}_2$.* Acta Math. Hungar. **109 (3)** (2005), 239–254.

[3] M. Drmota, R. F. Tichy, *Sequences, Discrepancies and Applications.* Lecture Notes in Mathematics **1651**, Springer, Berlin, 1997.

[4] H. Faure, *On the star-discrepancy of generalized Hammersley sequences in two dimensions.* Monatsh. Math. **101** (1986), 291–300.

[5] J. H. Halton, S. K. Zaremba, *The extreme and the L^2 discrepancies of some plane sets.* Monatsh. Math. **73** (1969), 316–328.

[6] L. Kuipers, H. Niederreiter, *Uniform Distribution of Sequences.* John Wiley, New York, 1974.

[7] G. Larcher, F. Pillichshammer, *Sums of distances to the nearest integer and the discrepancy of digital nets.* Acta Arith. **106** (2003), 379–408.

[8] H. Niederreiter, *Random Number Generation and Quasi-Monte Carlo Methods.* CBMS–NSF Series in Applied Mathematics **63**, SIAM, Philadelphia, 1992.

[9] F. Zhang, *Matrix Theory.* Springer, New York, 1999.

Peter Kritzer
Department of Mathematics
University of Salzburg
Hellbrunnerstr. 34
A-5020 Salzburg, Austria
E-mail : peter.kritzer@sbg.ac.at
URL: http://www.sbg.ac.at/mat/staff/kritzer