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APPROXIMATION NUMBERS OF COMPOSITION OPERATORS ON WEIGHTED HARDY SPACES

AJAY K. SHARMA[1*] AND AMBIKA BHAT[2]

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Abstract. In this paper we find upper and lower bounds for approximation numbers of compact composition operators on the weighted Hardy spaces $\mathcal{H}_\sigma$ under some conditions on the weight function σ.

1. Introduction and preliminaries

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$, $H(\mathbb{D})$ the class of all holomorphic functions on $\mathbb{D}$ and $H^\infty(\mathbb{D})$ the space of all bounded analytic function on $\mathbb{D}$ with the norm $||f||_\infty = \sup_{z\in\mathbb{D}} |f(z)|$. For $z \in \mathbb{D}$, let

$$\beta_z(w) = \frac{z-w}{1-\bar{z}w}, \quad z, w \in \mathbb{D},$$

that is, the involutive automorphism of $\mathbb{D}$ interchanging points z and 0. Let σ be a positive integrable function on $[0, 1)$. We extend σ on $\mathbb{D}$ defining $\sigma(z) = \sigma(|z|)$ for all $z \in \mathbb{D}$ and call it a weight or a weight function. By $\mathcal{H}_\sigma$ we denote the weighted Hardy space consisting of all $f \in H(\mathbb{D})$ such that

$$||f||^2_{\mathcal{H}_\sigma} = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 \sigma(z) dA(z) < \infty,$$

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* Corresponding author.

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where $dA(z) = \frac{1}{\pi} dxdy = \frac{1}{\pi} r dr d\theta$ is the normalized area measure on $\mathbb{D}$. A simple computation shows that a function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ belongs to $\mathcal{H}_\sigma$ if and only if

$$\sum_{n=0}^{\infty} |a_n|^2 \sigma_n < \infty,$$

where $\sigma_0 = 1$ and

$$\sigma_n = \sigma(n) = 2n^2 \int_0^1 r^{2n-1} w(r) dr, \quad n \in \mathbb{N}.$$

The sequence $(\sigma_n)_{n \in \mathbb{N}_0}$ is called the weight sequence of the weighted Hardy space $\mathcal{H}_\sigma$. The properties of the weighted Hardy space with the weight sequence $(\sigma_n)_{n \in \mathbb{N}_0}$, clearly depends upon σ_n.

Let $\mathcal{H}_\sigma$ be a weighted Hardy space with weight sequence $\{\sigma_n\}$. Then for each $\lambda \in \mathbb{D}$, the evaluation functional in $\mathcal{H}_\sigma$ at λ is a bounded linear functional and for $f \in \mathcal{H}_\sigma, \quad f(\lambda) = \langle f, K_\lambda \rangle$, where

$$K_\lambda(z) = \sum_{k=0}^{\infty} \frac{(\overline{\lambda} z)^k}{\sigma(k)} \quad \text{and} \quad ||K_\lambda||^2_{\mathcal{H}_\sigma} = \sum_{k=0}^{\infty} \frac{|\lambda|^{2k}}{\sigma(k)}.$$

Moreover,

$$|f(z)| \leq ||f||_{\mathcal{H}_\sigma} \Big(\sum_{k=0}^{\infty} r^{2k} (\sigma_k)^{-1} \Big)^{1/2} \tag{1.1}$$

$$|f'(z)| \leq ||f||_{\mathcal{H}_\sigma} \Big(\sum_{k=0}^{\infty} k^2 r^{2(k-1)} (\sigma_k)^{-1} \Big)^{1/2} \tag{1.2}$$

for $|z| \leq r$ where $\sigma(k) = ||z^k||^2_{\mathcal{H}_\sigma}$, see Theorem 2.10 in [2].

For more about weighted Hardy spaces and some related topics, see [2], [3] and [15].

Throughout the paper, a weight σ will satisfy the following properties:

- (W_1) σ is non-increasing;
- (W_2) $\frac{\sigma(r)}{(1-r)^{1+\delta}}$ is non-decreasing for some $\delta > 0$;
- (W_3) $\lim_{r \to 1} \sigma(r) = 0$.

We also assume that σ will satisfy one of the following properties:

- (W_4) σ is convex and $\lim_{r \to 1} \sigma(r) = 0$; or
- (W_5) σ is concave.

Such a weight function is called *admissible* (see [3]). If σ satisfies condition (W_1), (W_2), (W_3) and (W_4), then it is said that σ is I-*admissible*. If σ satisfies condition (W_1), (W_2), (W_3) and (W_5), then it is said that σ is II-*admissible*. I-admissibility corresponds to the case $\mathcal{H}^2 \subseteq \mathcal{H}_\sigma \subset \mathcal{A}^2_\alpha$ for some $\alpha > -1$, whereas II-admissibility corresponds to the case $\mathcal{D} \subseteq \mathcal{H}_\sigma \subset \mathcal{H}^2$. If we say that a weight is admissible it means that it is I-admissible or II-admissible.

Recall that for z and w in $\mathbb{D}$, the pseudohyperbolic distance d between z and w is defined by

$$d(z,w) = |\beta_z(w)|.$$

For $r \in (0,1)$ and $z \in \mathbb{D}$, denote by $D(z,r)$, the pseudohyperbolic disk whose pseudohyperbolic center is z and whose pseudohyperbolic radius is r, that is

$$D(z,r) = \big\{w \in \mathbb{D} : d(z,w) < r\big\}.$$

We need Carleson type Theorem for weighted Hardy spaces, see [11]

Theorem 1.1. *Let σ be an admissible weight, $r \in (0,1)$ fixed and μ be a positive Borel measure on $\mathbb{D}$. Then the following statements are equivalent:*

(1) *The following quantity is bounded*

$$C_1 := \sup_{z\in\mathbb{D}} \frac{\mu(D(z,r))}{\sigma(z)(1-|z|^2)^2};$$

(2) *There is a constant $C_2 > 0$ such that, for every $f \in H_\sigma$,*

$$\int_{\mathbb{D}} |f'(w)|^2 d\mu(w) \le C_2 \|f\|^2_{H_\sigma};$$

(3) *The following quantity is bounded*

$$C_3 := \sup_{z\in\mathbb{D}} \int_{\mathbb{D}} \frac{(1-|z|^2)^{2+2\gamma}}{\sigma(z)|1-\bar{z}w|^{4+2\gamma}} d\mu(w).$$

Moreover, the following asymptotic relationships hold

$$C_1 \asymp C_2 \asymp C_3.$$

The generalized Nevanlinna counting function shall play a key role in our work. The generalized Nevanlinna counting function associated to a weight function ω is defined for every $z \in \mathbb{D} \setminus \{\varphi(0)\}$ by

$$\mathfrak{N}_{\varphi,\sigma}(z) = \sum_{\varphi(\lambda)=z} \sigma(\lambda),$$

where $\mathfrak{N}_{\varphi,\sigma}(z) = 0$ when $z \notin \varphi(\mathbb{D})$. By convention, we define $\mathfrak{N}_{\varphi,\sigma}(z) = 0$ when $z = \varphi(0)$. When $\sigma(r) = \sigma_0(r) \asymp \log 1/r$, $\quad \mathfrak{N}_{\varphi,\sigma_0} = N_\varphi$, the usual Nevanlinna counting function associated to φ.
For more about generalized and classical Nevanlinna counting functions, see [2] and [3]. The generalized Nevanlinna counting function $\mathfrak{N}_{\varphi,\sigma}$ provides the following non-univalent change of variable formula (see [2], Theorem 2.32).

Lemma 1.2. *If g and σ are positive measurable function on $\mathbb{D}$ and φ a holomorphic self-map of $\mathbb{D}$, then*

$$\int_{\mathbb{D}} (g\circ\varphi)(z)|\varphi'(z)|^2\sigma(z)dA(z) = \int_{\mathbb{D}} g(z)\mathfrak{N}_{\varphi,\sigma}(z)dA(z).$$

Recall that the essential norm $||T||_e$ of a bounded linear operator on a Banach space X is given by

$$||T||_e = \inf\{||T-K|| : K \text{ is compact on } X\}.$$

It provides a measure of non-compactness of T. Clearly, T is compact if and only if $||T||_e = 0$.
Let φ be a non-constant analytic self-map (a so called Schur function) of $\mathbb{D}$ and let $C_\varphi : \mathcal{H}_\omega \to H(\mathbb{D})$ the associated composition operator:

$$C_\varphi f = f \circ \varphi.$$

For more about composition operators on weighted Hardy spaces, see [3], [11] and [15].
The next theorem can be found in [15].

Theorem 1.3. *Let σ_1 and σ_2 be two admissible weights ((I)-admissible or (II)-admissible) and φ be a holomorphic self-map of $\mathbb{D}$. Then $C_\varphi : \mathcal{H}_{\sigma_1} \to \mathcal{H}_{\sigma_2}$ is bounded if and only if*

$$\sup_{|z|<1} \frac{\mathfrak{N}_{\varphi,\sigma_2}(z)}{\sigma_1(z)} < \infty.$$

Moreover, if $C_\varphi : \mathcal{H}_{\sigma_1} \to \mathcal{H}_{\sigma_2}$ is bounded, then

$$||C_\varphi||^2_{\mathcal{H}_{\sigma_1} \to \mathcal{H}_{\sigma_2}} \asymp \sup_{|z|<1} \frac{\mathfrak{N}_{\varphi,\sigma_2}(z)}{\sigma_1(z)}.$$

As in [5], we first introduce the following notations. If

$$\varphi^\sharp(z) = \lim_{w \to z} \frac{\rho(\varphi(w), \varphi(z))}{\rho(w, z)} = \frac{|\varphi'(z)|(1 - |z|^2)}{1 - |\varphi(z)|^2}$$

is the pseudo-hyperbolic derivative of φ, we set:

$$[\varphi] = \sup_{z \in \mathbb{D}} \varphi^\sharp(z) = ||\varphi^\sharp||_\infty.$$

Also recall that the approximation (or singular) numbers $a_n(T)$ of an operator $T \in \mathcal{L}(H_1, H_2)$, between two Hilbert spaces H_1 and H_2 are defined by:

$$a_n(T) = \inf\{||T - R||; \;\; rank(R) < n\}, \;\; n = 1, 2, \cdots .$$

We have

$$a_n(T) = c_n(T) = d_n(T),$$

where the numbers c_n(resp. d_n) are the Gelfand (resp. Kolmogorov) numbers of T ([1], page 59 and page 51 respectively). In the sequel we shall need the following quantity:

$$\tau(T) = \liminf_{n \to \infty} [a_n(T)]^{1/n}.$$

These approximation numbers form a non-increasing sequence such that

$$a_1(T) = ||T||, \;\; a_n(T) = \sqrt{a_n(T^*T)}$$

are verify the so-called "ideal" and "subadditivity" properties ([4], see page 57 and page 68):

$$a_n(ATB) \leq ||A||a_n(T)||B||; \;\; a_{n+m-1}(S + T) \leq a_n(S) + a_m(T).$$

Moreover, the sequence $(a_n(T))$ tends to 0 if and only if T is compact. If for some p, $1 \leq p < \infty$, $(a_n(T)) \in l_p$, where

$$l_p = \Big\{a = \{a_n\}_{n=1}^{\infty} : ||a||_p = \Big(\sum_{n=1}^{\infty} |a_n|^p\Big)^{1/p} < \infty\Big\},$$

then we say that T belongs to the Schatten class S_p.
The upper and lower bounds for approximation numbers of composition operators on the Hardy space were computed by Li, Queffelec and Rodriguez-Piazza in [5]. In this paper, we generalized some of the results concerning upper and lower bounds for approximation numbers of composition operators to weighted Hardy spaces $\mathcal{H}_\sigma$ under some conditions on the weight function σ.
Throughout the paper constants are denoted by C, they are positive and not necessarily the same at each occurrence. The notation $A \lesssim B$ means that there is a positive constant C such that $\leq CB$. When $A \lesssim B$ and $B \lesssim A$, we write $A \asymp B$.

2. Lower Bound

We first show that, each Möbius transformations β_z always induce a bounded composition operator on $\mathcal{H}_\sigma$. This property ensures that, we may consider the operator C_φ under the assumption $\varphi(0) = 0$.

Proposition 2.1. *Let σ be an admissible weight. Then for each $z \in \mathbb{D}$, C_{β_z} is bounded on $\mathcal{H}_\sigma$.*

Proof. By the change of variable formula, we have

$$\begin{aligned}
\|C_{\beta_z} f\|^2_{\mathcal{H}_\sigma} &= |f(\beta_z(0))|^2 + \int_{\mathbb{D}} |f'(\beta_z(w))|^2 |\beta_z'(w)|^2 \sigma(w) dm(w) \\
&= |f(z)|^2 + \int_{\mathbb{D}} |f'(w)|^2 |\beta_z'(\xi_a(w))|^2 \sigma(\beta_z(w)) |\beta_z'(w)|^2 dm(w) \\
&= |f(z)|^2 + \int_{\mathbb{D}} |f'(w)|^2 |(\beta_z \circ \beta_z)'(w)|^2 \sigma(\beta_z(w)) dm(w) \\
&= |f(z)|^2 + \int_{\mathbb{D}} |f'(w)|^2 \sigma(\beta_z(w)) dm(z).(3) \qquad (2.1)
\end{aligned}$$

By Lemma 2.1 of [3], we have

$$\sigma(\beta_z(w)) \asymp \sigma(w). \qquad (4)$$

From (3) and (4), we have

$$\|C_{\beta_z} f\|^2_{\mathcal{H}_\sigma} \lesssim |f(z)|^2 + \|f\|^2_{\mathcal{H}_\sigma}$$

for each $f \in \mathcal{H}_\sigma$. This implies that $C_{\beta_z}(\mathcal{H}_\sigma) \subset \mathcal{H}_\sigma$. Thus by closed graph theorem, C_{β_z} is bounded on $\mathcal{H}_\sigma$. □

Proposition 2.2. *For each $z \in \mathbb{D}$, C_{β_z} is invertible.*

Proof. By Proposition 1, C_{β_z} is bounded. Now the proof is an easy consequence of Theorem 1.6 in [2].
In the following result, we show that if σ is II-admissible, or σ is I-admissible and C_φ is compact on $\mathcal{H}_\sigma$, then the approximation numbers of C_φ on $\mathcal{H}_\sigma$ cannot supersede a geometric speed. □

Theorem 2.3. *Let σ be an admissible weight and φ be a Schur function such that for $C_\varphi : \mathcal{H}_\sigma \to \mathcal{H}_\sigma$ is bounded. Suppose that C_φ is compact on $\mathcal{H}_\sigma$, whenever ω is I-admissible. Then there exist positive constant $C > 0$ and $0 < r < 1$ such that*

$$a_n(C_\varphi) \geq Cr^n, \qquad n = 1, 2, \cdots .$$

More precisely, one has $\beta(C_\varphi) \geq [\varphi]^2$ and hence for each $k < [\varphi]$ there exist a constant $C_k > 0$ such that

$$a_n(C_\varphi) \geq C_k k^{2n}.$$

For the proof we need the following lemma (see [5]).

Lemma 2.4. *Let $T : H \to H$ be a compact operator. Suppose that $(\lambda_n)_{n\geq 1}$ the sequence of eigenvalues of T rearranged in non-increasing order satisfies for some $\delta > 0$ and $r \in (0, 1)$*

$$|\lambda_n| \geq \delta r^n, \quad n = 1, 2, \cdots .$$

Then there exist $\delta_1 > 0$ such that

$$a_n(T) \geq \delta_1 r^{2n}, \quad n = 1, 2, \cdots .$$

In particular $\beta(T) \geq r^2$.

Proposition 2.5. *Let ω be an admissible weight and φ be a Schur function such that for $C_\varphi : \mathcal{H}_\omega \to \mathcal{H}_\omega$ is compact. Then $\tau(C_\varphi) \geq [\phi]^2$.*

Proof. The proof follows on same lines as the proof of Proposition 3.3 in [5]. We include it for completeness. For every $z \in \mathbb{D}$, let β_z be the involutive automorphism of $\mathbb{D}$. Then we have

$$\beta_z(z) = 0,\ \beta_z(0) = z,\ \beta_z'(z) = \frac{1}{|z|^2 - 1},\ \beta_z'(0) = |a|^2 - 1.$$

Let $\psi = \beta_{\varphi(z)} \circ \varphi \circ \beta_z$. Then 0 is a fixed point of ψ, whose derivative by the chain rule is

$$\psi'(0) = \beta'_{\varphi(z)}(\phi(z))\varphi'(z)\beta_z'(0) = \frac{\varphi'(z)(1 - |z|^2)}{1 - |\varphi(z)|^2} = \varphi^\sharp(z).$$

By Schwarz's lemma

$$\frac{(1 - |z|^2)}{1 - |\varphi(z)|^2}|\varphi'(z)| = |\psi'(0)| \leq 1.$$

Let us first assume that, the composition operator C_φ is compact on $\mathcal{H}_\sigma$. Then so is C_ψ, since we have

$$C_\psi = C_{\beta_z} \circ C_\varphi \circ C_{\beta_{\varphi(z)}}.$$

If $\psi'(0) \neq 0$, the sequence of eigenvalues of C_ψ the Hardy space H^2 is $([\psi'(0)]^n)_{n\geq 0}$ (see [2], page 96). Since II-admissibility corresponds to the case $\mathcal{H}_\sigma \subset H^2$, so the result given for H^2 holds for $\mathcal{H}_\sigma$ and would also holds for any space of analytic functions in $\mathbb{D}$ on which C_ψ is compact. By Lemma 2.4, we have

$$\tau(C_\psi) \geq |\psi'(0)| = |\varphi^\sharp(z)|^2 \geq 0.$$

This trivially still holds if $\psi'(0) = 0$. Now since C_{β_z} and $C_{\beta_{\varphi(z)}}$ are invertible operators, we have that $\tau(C_\varphi) = \tau(C_\psi)$ and therefore, we have

$$\tau(C_\varphi) = [\varphi]^2$$

for all $z \in \mathbb{D}$. By passing to the supremum on $z \in \mathbb{D}$, we end the proof of Proposition 2.5 and that of Theorem 2.3 in the compact case. If C_φ is not compact, the proposition trivially holds. Indeed, in this case, we have $\tau(C_\varphi) = 1 \geq [\varphi]^2$. □

3. Upper Bound

Theorem 3.1. *Let φ be a holomorphic self-map of $\mathbb{D}$ such that $\varphi(0) = 0$. Let σ be an admissible weight. Assume that $\sup \frac{\sigma(k)}{\sigma(k+n)} < \infty$ and $r \in (0,1)$ is fixed. Then the approximation number of $C_\varphi : \mathcal{H}_{\sigma_1} \to \mathcal{H}_{\sigma_2}$ has the upper bound*

$$a_n(C_\varphi) \lesssim \inf_{0<h<1} \left[(1-h)^{2n} \sum_{k=0}^{\infty} \frac{k^2(1-h)^{2(k-1)}}{\sigma_k} + (1-h)^{2n-2} \sum_{k=0}^{\infty} \frac{(1-h)^{2k}}{\sigma_k} \right] \left(\sup_{1 \leq j < \infty} \frac{\sigma_j}{\sigma_{j+n}} \right) + \sup_{z \in \mathbb{D}} \frac{\mu_{\sigma,\varphi,h}(D(z,r))}{\sigma(z)(1-|z|^2)^2}. \tag{3.1}$$

To prove the theorem, we need the following lemma.

Lemma 3.2. *Let $f(z) = \sum_{k=n}^{\infty} a_k z^k$ and $g(z) = z^n f(z)$. Then*

$$\|g\|^2_{\mathcal{H}_\sigma} \leq \sup_{1 \leq j < \infty} \frac{\sigma_j}{\sigma_{j+n}} \|f\|^2_{\mathcal{H}_\sigma}.$$

Proof. $\|g\|^2_{\mathcal{H}_\sigma} = \sum_{k=0}^{\infty} |a_{k+n}|\sigma_k = \sum_{k=0}^{\infty} |a_{k+n}|\sigma_{k+n} \frac{\sigma_k}{\sigma_{k+n}} \leq \sup_{1 \leq k < \infty} \frac{\sigma_k}{\sigma_{k+n}} \|f\|^2_{\mathcal{H}_\sigma}.$ □

Proof. We denote by P_n the projection operator defined by

$$P_n f = \sum_{k=0}^{n-1} \hat{f}(k) z^k$$

and we take $R = C_\varphi \circ P_n$, that is, if we have $f(z) = \sum_{k=0}^{\infty} \hat{f}(k) z^k \in \mathcal{H}_\sigma$ then

$$R(f) = \sum_{k=0}^{n-1} \hat{f}(k) \varphi^k$$

so that $(C_\varphi - R)f = C_\varphi(r)$. Then, we have

$$r(z) = \sum_{k=n}^{\infty} \hat{f}(k) z^k = z^n s(z),$$

where

$$||s||^2_{\mathcal{H}_\sigma} \leq C \sup \frac{\sigma(j)}{\sigma(j+k)} ||r||^2_{\mathcal{H}_\sigma}, \text{and} ||r||_{\mathcal{H}_\sigma} \leq ||f||_{\mathcal{H}_\sigma}. \tag{3.2}$$

Assume that $||f||_{\mathcal{H}_\sigma} \leq 1$ and $dm_{\varphi,\sigma} = \mathfrak{N}_{\varphi,\sigma}(z)dm(z)$. Fix $0 < h < 1$. Let

$$\mu_{\varphi,\sigma}(z) = (m_{\varphi,\sigma} \circ \varphi^{-1})(z)$$

and $\mu_{\varphi,\sigma,h}$ be the restriction of the measure $\mu_{\varphi,\sigma}(z)$ to the annulus $1-h < |z| \leq 1$. Then we have

$$\begin{aligned}
||(C_\varphi - R)f||^2_{\mathcal{H}_\sigma} &= ||C_\varphi(r)||^2_{\mathcal{H}_\sigma} \\
&= |r(\varphi(0))|^2 + \int_{\mathbb{D}} |r'(\varphi(z))|^2 |\varphi'(z)|^2 \sigma(z) dm(z) \\
&= \int_{\mathbb{D}} |r'(z)|^2 \mathfrak{N}_{\varphi,\sigma}(z) dm(z) \\
&\leq \int_{|z|\leq 1-h} |r'(z)|^2 \mathfrak{N}_{\varphi,\sigma}(z) dm(z) + \int_{1-h\leq|z|\leq 1} |r'(z)|^2 \mathfrak{N}_{\varphi,\sigma}(z) dm(z) \\
&= I_1 + I_2. \qquad (3.3)
\end{aligned}$$

Let $(z_n)_{n\in\mathbb{N}}$ be a sequence with a positive separation constant such that

$$\cup_{n=1}^\infty D(z_n, r) = \mathbb{D}$$

and every point in $\mathbb{D}$ belongs to at most M sets in the family $\{D(z_n, 2r)\}_{n\in\mathbb{N}}$. Since σ is an almost standard weight we have that for $0 < r_1 < r_2 < 1$

$$\left(\frac{1-r_2}{1-r_1}\right)^{t+1} w(r_1) \leq w(r_2) \leq w(r_1).$$

From this and since $1 - |z| \asymp 1 - |z_n|$, for $z \in D(z_n, 2r)$, we obtain

$$\sigma(z) \asymp \sigma(z_n), \quad z \in D(z_n, 2r).$$

Using these facts we obtain

$$\begin{aligned}
I_1 &= \int_{|z|\leq 1-h} |(z^n s'(z) + nz^{n-1}s)(z)|^2 \mathfrak{N}_{\varphi,\sigma}(z) dm(z) \\
&\leq \int_{|z|\leq 1-h} |z^n s'(z)|^2 \mathfrak{N}_{\varphi,\sigma}(z) dm(z) + n^2 \int_{|z|\leq 1-h} |z^{n-1} s(z)|^2 \mathfrak{N}_{\varphi,\sigma}(z) dm(z) \\
&\leq (1-h)^{2n} \int_{|z|\leq 1-h} |s'(z)|^2 \mathfrak{N}_{\varphi,\sigma}(z) dm(z) \\
&+ n^2 (1-h)^{2n-2} \int_{|z|\leq 1-h} |s(z)|^2 \mathfrak{N}_{\varphi,\sigma}(z) dm(z). \qquad (3.4)
\end{aligned}$$

Thus by Lemma 3.2, (2) and (6), we have

$$
\begin{aligned}
&(1-h)^{2n}\int_{|z|\le 1-h}|s'(z)|^2\mathfrak{N}_{\varphi,\sigma}(z)dm(z)\\
&\qquad\le (1-h)^{2n}\|s\|_{\mathcal{H}_\sigma}^2\left(\sum_{k=0}^{\infty}k^2(1-h)^{2(k-1)}\sigma_k^{-1}\right)\int\mathfrak{N}_{\varphi,\sigma}(z)dm(z)\\
&\qquad\lesssim (1-h)^{2n}\sup_{1\le j<\infty}\frac{\sigma_j}{\sigma_{j+n}}\|r\|_{\mathcal{H}_\sigma}^2\sum_{k=0}^{\infty}k^2(1-h)^{2(k-1)}\sigma_k^{-1}\|\varphi\|_{\mathcal{H}_\sigma}^2\\
&\qquad\lesssim (1-h)^{2n}\sup_{1\le j<\infty}\frac{\sigma_j}{\sigma_{j+n}}\|r\|_{\mathcal{H}_\sigma}^2\sum_{k=0}^{\infty}k^2(1-h)^{2(k-1)}\sigma_k^{-1}\|\varphi\|_{H^\infty}^2\\
&\qquad\lesssim (1-h)^{2n}\left(\sup_{1\le j<\infty}\frac{\sigma_j}{\sigma_{j+n}}\right)\sum_{k=0}^{\infty}k^2(1-h)^{2(k-1)}\sigma_k^{-1}. \qquad (3.5)
\end{aligned}
$$

Again by Lemma 3.2, (1) and (6), we have

$$
\begin{aligned}
&(1-h)^{2n-2}\int_{|z|\le 1-h}|s(z)|^2\mathfrak{N}_{\varphi,\sigma}(z)dm(z)\\
&\qquad\lesssim (1-h)^{2n-2}\|s\|_{\mathcal{H}_\sigma}^2\sum_{k=0}^{\infty}\frac{(1-h)^{2k}}{\sigma_k}\int\mathfrak{N}_{\varphi,\sigma}(z)dm(z)\\
&\qquad\lesssim (1-h)^{2n-2}\left(\sup_{1\le j<\infty}\frac{\sigma_j}{\sigma_{j+n}}\right)\|r\|_{\mathcal{H}_\sigma}^2\sum_{k=0}^{\infty}\frac{(1-h)^{2k}}{\sigma_k}\|\varphi\|_{\mathcal{H}_\sigma}^2\\
&\qquad\lesssim (1-h)^{2n-2}\left(\sup_{1\le j<\infty}\frac{\sigma_j}{\sigma_{j+n}}\right)\|r\|_{\mathcal{H}_\sigma}^2\sum_{k=0}^{\infty}\frac{(1-h)^{2k}}{\sigma_k}\|\varphi\|_{H^\infty}^2\\
&\qquad\lesssim (1-h)^{2n-2}\left(\sup_{1\le j<\infty}\frac{\sigma_j}{\sigma_{j+n}}\right)\sum_{k=0}^{\infty}\frac{(1-h)^{2k}}{\sigma_k}. \qquad (3.6)
\end{aligned}
$$

Combining (8), (9) and (10), we have

$$
\begin{aligned}
I_1 &\lesssim (1-h)^{2n}\left(\sup_{1\le j<\infty}\frac{\sigma_j}{\sigma_{j+n}}\right)\sum_{k=0}^{\infty}\frac{k^2(1-h)^{2(k-1)}}{\sigma_k}\\
&\qquad + (1-h)^{2n-2}\left(\sup_{1\le j<\infty}\frac{\sigma_j}{\sigma_{j+n}}\right)\sum_{k=0}^{\infty}\frac{(1-h)^{2k}}{\sigma_k}. \qquad (3.7)
\end{aligned}
$$

Again

$$
\begin{aligned}
I_2 &= \int_{1-h<|z|<1} |r'(z)|^2 \mathfrak{N}_{\varphi,\sigma}(z) dm(z) \\
&= \int_{\mathbb{D}} |r'(z)|^2 d\mu_{\sigma,\varphi,h}(z) \\
&\le \sum_{n=1}^{\infty} \int_{D(z_n,r)} |r'(z)|^2 d\mu_{\sigma,\varphi,h}(z) \\
&\le \sum_{n=1}^{\infty} \mu_{\sigma,\varphi,h}(D(z_n,r)) \sup_{\sigma\in D(z_n,r)} |r'(\sigma)|^2 \\
&\lesssim \sum_{n=1}^{\infty} \frac{\mu(D(z_n,r))}{\sigma(z_n)(1-|z_n|^2)^2} \int_{D(z_n,2r)} |r'(z)|^2 \sigma(z) dm(z) \\
&\lesssim \sup_{z\in\mathbb{D}} \frac{\mu_{\sigma,\varphi,h}(D(z,r))}{\sigma(z)(1-|z|^2)^2} \sum_{n=1}^{\infty} \int_{D(z_n,2r)} |r'(z)|^2 \sigma(z) dm(z) \\
&\lesssim \sup_{z\in\mathbb{D}} \frac{\mu_{\sigma,\varphi,h}(D(z,r))}{\sigma(z)(1-|z|^2)^2} \int_{\mathbb{D}} |r'(z)|^2 \sigma(z) dm(z) \\
&\lesssim \sup_{z\in\mathbb{D}} \frac{\mu_{\sigma,\varphi,h}(D(z,r))}{\sigma(z)(1-|z|^2)^2} \|r\|^2_{H_\sigma} \\
&\lesssim \sup_{z\in\mathbb{D}} \frac{\mu_{\sigma,\varphi,h}(D(z,r))}{\sigma(z)(1-|z|^2)^2}. \qquad (3.8)
\end{aligned}
$$

Combining (7), (11) and (12), we get the desired upper bound given in (5). □

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References

1. B. Carl, I. Stephani, Entropy, *Compactness and the Approximation of operators*, Cambridge Tracts in Mathematics 98, Cambridge University Press (1990).
2. C.C. Cowen and B.D. MacCluer, *Composition operators on spaces of analytic functions*, CRC Press Boca Raton, New York, 1995.
3. K. Kellay and P. Lefevre, *Compact composition operators on weighted Hilbert spaces of analytic functions,* J. Math. Anal. Appl. **386** (2012) 718–727.
4. H. Konig, *Eigenvalue Distribution of Compact operators,* Operator Theory, Advances and Application Vol. 16, Birkhauser, Basel-Boston-Stuttgart (1986).
5. D. Li, H. Queffelec and L. Rodriguez-Piazza, *On approximation numbers of composition operators,* J. Approx. Theory 164 (2014) 431–459.
6. P. Lefevre, D. Li , H. Queffelec, and L. Rodriguez-Piazza, *Some examples of compact composition operators on $\mathcal{H}^2$,* J. Funct. Anal. **255** (2008), 3098–3124.
7. P. Lefevre, D. Li , H. Queffelec, and L. Rodriguez-Piazza, *Compact composition operators on $\mathcal{H}^2$ and Hardy-Orlicz spaces,* J. Math. Anal. Appl. **354** (2009), 360–371.
8. P. Lefevre, D. Li , H. Queffelec, and L. Rodriguez-Piazza, *Nevanlinna counting function and Carleson function of analytic maps,* Math Ann. **351** (2011), 305–326.

9. P. Lefevre, D. Li , H. Queffelec, and L. Rodriguez-Piazza, *Some revisited results about composition operators on Hardy spaces,* Rev. Mat. Iberoam. **28** (2012), 57–76
10. J.H. Shapiro, *Composition operators and classical function theory,* Springer-Verlag, New York 1993.
11. A.K. Sharma, *Essential norm of generalized composition operators on weighted Hardy spaces,* Operators and Matrices **8** (2014), 399-409.
12. A.K. Sharma, *Weighted composition operators from Cauchy integral transforms to logarithmic weighted-type spaces,* Ann. Funct. Anal. **4** (2013), 163174.
13. A.K. Sharma and S. Ueki, *Composition operators from Nevanlinna type spaces to Bloch type spaces,* Banach J. Math. Anal. **6** (2012), 112–123.
14. A. Sharma and A.K. Sharma, *Carleson measures and a class of generalized integration operators on the Bergman space,* Rocky Mountain J. Math. **41** (2011), 1711–1724.
15. S. Stević and A.K. Sharma, *Essential norm of composition operators between weighted Hardy spaces,* Appl. Math. Comput. **217** (2011), 6192–6197.

[1] School of Mathematics, Shri Mata Vaishno Devi University, Kakryal, Katra-182320, J& K, India.

E-mail address: aksju_76@yahoo.com

[2] Ambika Bhat, School of Mathematics, Shri Mata Vaishno Devi University, Kakryal, Katra-182320, J& K, India.

E-mail address: ambikabhat.20@gmail.com