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ANISOTROPIC HERZ-MORREY SPACES WITH VARIABLE EXPONENTS

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Abstract. In this paper, the authors introduce the anisotropic Herz-Morrey spaces with two variable exponents and obtain some properties of these spaces. Subsequently as an application, the authors give some boundedness on the anisotropic Herz-Morrey spaces with two variable exponents for a class of sublinear operators, which extend some known results.

1. Introduction

In recent years, the theory of function spaces with variable exponents has developed since the paper [10] of Kováčik and Rákosník appeared in 1991. Lebesgue and Sobolev spaces with integrability exponent have been extensively investigated, see [6] and the references therein. In 2009, Izuki [8] defined the Herz-Morrey spaces with variable exponent. In 2012, Almeida and Drihem [1] introduced the Herz spaces with two variable exponents and proved the boundedness of some operators on these spaces. Meanwhile, extending classic function spaces arising in harmonic analysis of Euclidean spaces to other domains and non-isotropic settings is an important topic. In 2003, Bownik [2] introduced the anisotropic Hardy spaces associated with very general discrete groups of dilations. The above spaces include the classical isotropic Hardy space theory of Fefferman and Stein [7] and parabolic Hardy space theory of Calderón and Torchinsky [3, 4]. In 2015, Wang [12] defined the anisotropic Herz spaces with variable exponents and gave some applications.

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Inspired by the above results, we introduce the anisotropic Herz-Morrey spaces with two variable exponents which is a generalization of the anisotropic Herz-Morrey spaces and the Herz-Morrey spaces with variable exponent, and we obtain the boundedness of some sublinear operators on the anisotropic Herz-Morrey spaces with two variable exponents.

To be precise, we first briefly recall some standard notations in the remainder of this section. In Section 2, we will define the anisotropic Herz-Morrey spaces with two variable exponents $M\dot{K}^{\alpha(\cdot),\lambda}_{q(\cdot),p}(A;\mathbb{R}^n)$ and $MK^{\alpha(\cdot),\lambda}_{q(\cdot),p}(A;\mathbb{R}^n)$, and give some properties. In Section 3, we will give the boundedness of some sublinear operators on $M\dot{K}^{\alpha(\cdot),\lambda}_{q(\cdot),p}(A;\mathbb{R}^n)$ and $MK^{\alpha(\cdot),\lambda}_{q(\cdot),p}(A;\mathbb{R}^n)$.

Now, we first recall some notations in variable function spaces. Given an open set $\Omega \subset \mathbb{R}^n$, and a measurable function $p(\cdot) : \Omega \longrightarrow [1,\infty)$, $L^{p(\cdot)}(\Omega)$ denotes the set of measurable functions f on Ω such that for some $\lambda > 0$,

$$\int_\Omega \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx < \infty.$$

This set becomes a Banach function space when equipped with the Luxemburg-Nakano norm

$$\|f\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{\lambda > 0 : \int_\Omega \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx \le 1\right\}.$$

These spaces are referred to as variable Lebesgue spaces or, more simply, as variable L^p spaces, since they generalized the standard L^p spaces: if $p(x) = p$ is constant, then $L^{p(\cdot)}(\Omega)$ is isometrically isomorphic to $L^p(\Omega)$. The L^p spaces with variable exponent are a special case of Musielak-Orlicz spaces.

The space $L^{p(\cdot)}_{\mathrm{loc}}(\Omega)$ is defined by

$$L^{p(\cdot)}_{\mathrm{loc}}(\Omega) := \{f : f \in L^{p(\cdot)}(E) \text{ for all compact subsets } E \subset \Omega\}.$$

Define $\mathcal{P}(\Omega)$ to be set of $p(\cdot) : \Omega \longrightarrow [1,\infty)$ such that

$$p^- = \operatorname{ess\,inf}\{p(x) : x \in \Omega\} > 1, \quad p^+ = \operatorname{ess\,sup}\{p(x) : x \in \Omega\} < \infty.$$

Denote $p'(x) = p(x)/(p(x)-1)$. Let $\mathcal{B}(\Omega)$ be the set of $p(\cdot) \in \mathcal{P}(\Omega)$ such that the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\Omega)$.

In variable L^p spaces there are some important lemmas as follows.

Lemma 1.1 ([10])**.** *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. If $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $g \in L^{p'(\cdot)}(\mathbb{R}^n)$, then fg is integrable on $\mathbb{R}^n$ and*

$$\int_{\mathbb{R}^n} |f(x)g(x)|dx \le r_p \|f\|_{L^{p(\cdot)}} \|g\|_{L^{p'(\cdot)}},$$

where

$$r_p = 1 + 1/p^- - 1/p^+.$$

This inequality is named by generalized Hölder inequality with respect to the variable L^p spaces.

Lemma 1.2 ([5]). *Let $p(\cdot), q(\cdot), r(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ such that $1/p(x) = 1/q(x) + 1/r(x)$. If $f \in L^{q(\cdot)}(\mathbb{R}^n)$ and $g \in L^{r(\cdot)}(\mathbb{R}^n)$, then $fg \in L^{p(\cdot)}(\mathbb{R}^n)$ and*

$$\|fg\|_{L^{p(\cdot)}} \le C\|f\|_{L^{q(\cdot)}}\|g\|_{L^{r(\cdot)}},$$

where C is a constant independent of the functions f and g.

Lemma 1.3 ([9]). *Suppose $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Then there exists a constant $C > 0$ such that for all balls B in $\mathbb{R}^n$,*

$$\frac{1}{|B|}\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_B\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \le C.$$

Lemma 1.4 ([9]). *Let $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Then for all balls B in $\mathbb{R}^n$ and all measurable subsets $S \subset B$,*

$$\frac{\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_S\|_{L^{q(\cdot)}(\mathbb{R}^n)}} \lesssim \frac{|B|}{|S|},$$

$$\frac{\|\chi_S\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}} \lesssim \left(\frac{|S|}{|B|}\right)^{\delta_1}, \frac{\|\chi_S\|_{L^{q'(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{q'(\cdot)}(\mathbb{R}^n)}} \lesssim \left(\frac{|S|}{|B|}\right)^{\delta_2},$$

where $0 < \delta_1, \delta_2 < 1$ are constants.

Throughout this paper δ_2 is the same as in Lemma 1.4, and the notation $f \lesssim g$ means that there exists a constant $C > 0$ such that $f \le Cg$. If $f \lesssim g$ and $g \lesssim f$, then $f \approx g$.

We recall the following two definitions in [1].

Definition 1.5 ([1]). Let a function $g(\cdot) : \mathbb{R}^n \to \mathbb{R}$.

(1) $g(\cdot)$ is locally log-Hölder continuous, if there exists a constant $C > 0$ such that

$$|g(x) - g(y)| \le \frac{C}{\log(e + 1/|x-y|)}$$

for all $x, y \in \mathbb{R}^n$ and $|x - y| < 1/2$.

(2) $g(\cdot)$ is locally log-Hölder continuous at the origin (or has a log decay at the origin), if there exists a constant $C > 0$ such that

$$|g(x) - g(0)| \le \frac{C}{\log(e + 1/|x|)}$$

for all $x \in \mathbb{R}^n$.

(3) $g(\cdot)$ is locally log-Hölder continuous at infinity (or has a log decay at infinity), if there exist some $g_\infty \in \mathbb{R}$ and $C > 0$ such that

$$|g(x) - g_\infty| \le \frac{C}{\log(e + |x|)}$$

for all $x \in \mathbb{R}^n$.

By $\mathcal{P}_0(\mathbb{R}^n)$ and $\mathcal{P}_\infty(\mathbb{R}^n)$ we denote the class of all exponents $p \in \mathcal{P}(\mathbb{R}^n)$ which are locally log-Hölder continuous at the origin and at infinity, respectively.

Next we will introduce some basic definitions and properties of non-isotropic spaces associated with general expansive dilations. A $n \times n$ real matrix A is called an expansive matrix, sometimes called a dilation, if all eigenvalues λ of A

satisfy $|\lambda| > 1$. We suppose $\lambda_1, ..., \lambda_n$ are eigenvalues of A (taken according to the multiplicity) so that $1 < |\lambda_1| \leq ... \leq |\lambda_n|$. A set $\Delta \subset \mathbb{R}^n$ is said to be an ellipsoid if $\Delta = \{x \in \mathbb{R}^n : |Px| < 1\}$, for some nondegenerate $n \times n$ matrix P, where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^n$. For a dilation A, there exists an ellipsoid Δ and $r > 1$ such that $\Delta \subset r\Delta \subset A\Delta$, where $|\Delta|$, the Lebesgue measure of Δ, equals 1. Let $B_k = A^k\Delta$ for $k \in \mathbb{Z}$, then we have $B_k \subset rB_k \subset B_{k+1}$, and $|B_k| = b^k$, where $b = |\mathrm{det}A| > 1$. Let w be the smallest integer so that $2B_0 \subset A^w B_0 = B_w$. A quasi-norm associated with an expansive matrix A is a measurable mapping $\rho_A : \mathbb{R}^n \to [0, \infty)$ satisfying

$$\begin{aligned} \rho_A(x) &> 0 && \text{for } x \neq 0,\\ \rho_A(Ax) &= |\mathrm{det}A|\rho(x) && \text{for } x \in \mathbb{R}^n,\\ \rho_A(x+y) &\leq C(\rho_A(x) + \rho_A(y)) && \text{for } x, y \in \mathbb{R}^n, \end{aligned}$$

where $C \geq 1$ is a constant. One can show that all quasi-norms associated to a fixed dilation A are equivalent, see [2, Lemma 2.4]. Define the step homogeneous quasi-norm ρ on $\mathbb{R}^n$ induced by dilation A as

$$\rho(x) = \begin{cases} b^j & \text{if } x \in B_{j+1} \setminus B_j,\\ 0, & \text{if } x = 0. \end{cases}$$

For any $x, y \in \mathbb{R}^n$, we have

$$\rho(x+y) \leq b^w(\rho(x) + \rho(y)). \tag{1.1}$$

2. Some properties for the anisotropic Herz-Morrey spaces with two variable exponents

In this section, we first introduce the definition of anisotropic Herz-Morrey spaces with two variable exponents. Let $C_k = B_k \setminus B_{k-1}$ for $k \in \mathbb{Z}$. Denote $\mathbb{Z}_+$ and $\mathbb{N}$ as the sets of all positive and non-negative integers, $\chi_k = \chi_{C_k}$ for $k \in \mathbb{Z}$, $\tilde{\chi}_k = \chi_k$ if $k \in \mathbb{Z}_+$ and $\tilde{\chi}_0 = \chi_{B_0}$, where χ_{C_k} is the characteristic function of C_k.

Definition 2.1. Let $\alpha(\cdot) : \mathbb{R}^n \to \mathbb{R}$ with $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$, $0 < p < \infty$, $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $0 \leq \lambda < \infty$. The homogeneous anisotropic Herz-Morrey space $M\dot{K}^{\alpha(\cdot),\lambda}_{q(\cdot),p}(A;\mathbb{R}^n)$ associated with the dilation A is defined by

$$M\dot{K}^{\alpha(\cdot),\lambda}_{q(\cdot),p}(A;\mathbb{R}^n) = \{f \in L^{q(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q(\cdot),p}} < \infty\},$$

where

$$\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q(\cdot),p}} = \sup_{L \in \mathbb{Z}} b^{-L\lambda} \left\{ \sum_{k=-\infty}^{L} \|b^{k\alpha(\cdot)} f \chi_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \right\}^{1/p}.$$

The non-homogeneous anisotropic Herz-Morrey space $MK^{\alpha(\cdot),\lambda}_{q(\cdot),p}(A;\mathbb{R}^n)$ associated with the dilation A is defined by

$$MK^{\alpha(\cdot),\lambda}_{q(\cdot),p}(A;\mathbb{R}^n) = \{f \in L^{q(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{MK^{\alpha(\cdot),\lambda}_{q(\cdot),p}} < \infty\},$$

where

$$\|f\|_{MK^{\alpha(\cdot),\lambda}_{q(\cdot),p}} = \sup_{L\in\mathbb{Z}} b^{-L\lambda} \left\{ \sum_{k=0}^{L} \|b^{k\alpha(\cdot)} f\tilde{\chi}_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \right\}^{1/p}.$$

Remark 2.2. If $\lambda = 0$, then

$$M\dot{K}^{\alpha(\cdot),0}_{q(\cdot),p}(A;\mathbb{R}^n) = \dot{K}^{\alpha(\cdot),p}_{q(\cdot)}(A;\mathbb{R}^n)$$

and

$$MK^{\alpha(\cdot),0}_{q(\cdot),p}(A;\mathbb{R}^n) = K^{\alpha(\cdot),p}_{q(\cdot)}(A;\mathbb{R}^n),$$

where $\dot{K}^{\alpha(\cdot),p}_{q(\cdot)}(A;\mathbb{R}^n)$ and $K^{\alpha(\cdot),p}_{q(\cdot)}(A;\mathbb{R}^n)$ are the anisotropic Herz spaces associated with the dilation A.

Next we will consider some properties of $M\dot{K}^{\alpha(\cdot),\lambda}_{q(\cdot),p}(A;\mathbb{R}^n)$. There are similar properties for $MK^{\alpha(\cdot),\lambda}_{q(\cdot),p}(A;\mathbb{R}^n)$.

Theorem 2.3. *Suppose* $\alpha(\cdot),\alpha_1(\cdot),\alpha_2(\cdot) \in L^\infty(\mathbb{R}^n)\cap\mathcal{P}_0(\mathbb{R}^n)\cap\mathcal{P}_\infty(\mathbb{R}^n)$, $q(\cdot),q_1(\cdot)$, $q_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $1 < p,p_1,p_2 < \infty$, *and* $0 \le \lambda,\lambda_1,\lambda_2 < \infty$ *such that* $\alpha(\cdot) = \alpha_1(\cdot)+\alpha_2(\cdot)$, $1/q(\cdot) = 1/q_1(\cdot)+1/q_2(\cdot)$, $1/p = 1/p_1 + p_2$ *and* $\lambda = \lambda_1+\lambda_2$. *If* $f \in M\dot{K}^{\alpha_1(\cdot),\lambda_1}_{q_1(\cdot),p_1}(A;\mathbb{R}^n)$ *and* $g \in M\dot{K}^{\alpha_2(\cdot),\lambda_2}_{q_2(\cdot),p_2}(A;\mathbb{R}^n)$, *then* $fg \in M\dot{K}^{\alpha(\cdot),\lambda}_{q(\cdot),p}(A;\mathbb{R}^n)$ *and*

$$\|fg\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q(\cdot),p}} \le \|f\|_{M\dot{K}^{\alpha_1(\cdot),\lambda_1}_{q_1(\cdot),p_1}} \|g\|_{M\dot{K}^{\alpha_2(\cdot),\lambda_2}_{q_2(\cdot),p_2}}.$$

Proof. By Lemma 1.2 and the Hölder inequality we have

$$\begin{aligned}
&\|fg\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q(\cdot),p}} \\
&= \sup_{L\in\mathbb{Z}} b^{-L\lambda} \left\{ \sum_{k=-\infty}^{L} \|b^{k\alpha(\cdot)} fg\chi_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \right\}^{1/p} \\
&\le \sup_{L\in\mathbb{Z}} b^{-L\lambda_1} b^{-L\lambda_2} \left\{ \sum_{k=-\infty}^{L} \|b^{k\alpha_1(\cdot)} f\chi_k\|^p_{L^{q_1(\cdot)}(\mathbb{R}^n)} \|b^{k\alpha_2(\cdot)} g\chi_k\|^p_{L^{q_2(\cdot)}(\mathbb{R}^n)} \right\}^{1/p} \\
&\le \sup_{L\in\mathbb{Z}} b^{-L\lambda_1} \left\{ \sum_{k=-\infty}^{L} \|b^{k\alpha_1(\cdot)} f\chi_k\|^{p_1}_{L^{q_1(\cdot)}(\mathbb{R}^n)} \right\}^{1/p_1} \\
&\quad \times b^{-L\lambda_2} \left\{ \sum_{k=-\infty}^{L} \|b^{k\alpha_2(\cdot)} g\chi_k\|^{p_2}_{L^{q_2(\cdot)}(\mathbb{R}^n)} \right\}^{1/p_2} \\
&= \|f\|_{M\dot{K}^{\alpha_1(\cdot),\lambda_1}_{q_1(\cdot),p_1}} \|g\|_{M\dot{K}^{\alpha_2(\cdot),\lambda_2}_{q_2(\cdot),p_2}}.
\end{aligned}$$

So we complete the proof of Theorem 2.3. ◻

From Theorem 2.3 we can get the following Corollary.

Corollary 2.4. *Suppose* $\alpha(\cdot),\alpha_i(\cdot) \in L^\infty(\mathbb{R}^n)\cap\mathcal{P}_0(\mathbb{R}^n)\cap\mathcal{P}_\infty(\mathbb{R}^n)$, $q(\cdot),q_i(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $1 < p,p_i < \infty$, *and* $0 \le \lambda,\lambda_i < \infty (1 \le i \le m)$ *such that* $\alpha(\cdot) = \sum_{i=1}^{m} \alpha_i(\cdot)$,

$1/q(\cdot) = \sum_{i=1}^{m} 1/q_i(\cdot)$, $1/p = \sum_{i=1}^{m} 1/p_i$ *and* $\lambda = \sum_{i=1}^{m} \lambda_i$. *If* $f_i \in M\dot{K}_{q_i(\cdot),p_i}^{\alpha_i(\cdot),\lambda_i}(A;\mathbb{R}^n)$, *then* $\prod_{i=1}^{m} f_i \in M\dot{K}_{q(\cdot),p}^{\alpha(\cdot),\lambda}(A;\mathbb{R}^n)$ *and*

$$\|\prod_{i=1}^{m} f_i\|_{M\dot{K}_{q(\cdot),p}^{\alpha(\cdot),\lambda}} \leq \prod_{i=1}^{m} \|f_i\|_{M\dot{K}_{q_i(\cdot),p_i}^{\alpha_i(\cdot),\lambda_i}}.$$

Theorem 2.5. *Suppose* $\alpha(\cdot) \in L^\infty(\mathbb{R}^n) \cap \mathcal{P}_0(\mathbb{R}^n) \cap \mathcal{P}_\infty(\mathbb{R}^n)$, $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $1 < p < \infty$, *and* $0 \leq \lambda < \infty$. *If* $f, g \in M\dot{K}_{q(\cdot),p}^{\alpha(\cdot),\lambda}(A;\mathbb{R}^n)$, *then* $f + g \in M\dot{K}_{q(\cdot),p}^{\alpha(\cdot),\lambda}(A;\mathbb{R}^n)$ *and*

$$\|f+g\|_{M\dot{K}_{q(\cdot),p}^{\alpha(\cdot),\lambda}} \leq \|f\|_{M\dot{K}_{q(\cdot),p}^{\alpha(\cdot),\lambda}} + \|g\|_{M\dot{K}_{q(\cdot),p}^{\alpha(\cdot),\lambda}}.$$

Proof. By the Minkowski inequality we have

$$\begin{aligned}
&\|f+g\|_{M\dot{K}_{q(\cdot),p}^{\alpha(\cdot),\lambda}} \\
&= \sup_{L\in\mathbb{Z}} b^{-L\lambda} \left\{ \sum_{k=-\infty}^{L} \|b^{k\alpha(\cdot)}(f+g)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p \right\}^{1/p} \\
&\leq \sup_{L\in\mathbb{Z}} b^{-L\lambda} \left\{ \sum_{k=-\infty}^{L} \|b^{k\alpha(\cdot)} f\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p + \|b^{k\alpha(\cdot)} g\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p \right\}^{1/p} \\
&\leq \sup_{L\in\mathbb{Z}} b^{-L\lambda} \left\{ \left(\sum_{k=-\infty}^{L} \|b^{k\alpha(\cdot)} f\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p \right)^{1/p} + \left(\sum_{k=-\infty}^{L} \|b^{k\alpha(\cdot)} g\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p \right)^{1/p} \right\} \\
&\leq \|f\|_{M\dot{K}_{q(\cdot),p}^{\alpha(\cdot),\lambda}} + \|g\|_{M\dot{K}_{q(\cdot),p}^{\alpha(\cdot),\lambda}}.
\end{aligned}$$

So we obtain Theorem 2.5. □

From Theorem 2.5 we can get the following Corollary.

Corollary 2.6. *Suppose* $\alpha(\cdot) \in L^\infty(\mathbb{R}^n) \cap \mathcal{P}_0(\mathbb{R}^n) \cap \mathcal{P}_\infty(\mathbb{R}^n)$, $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $1 < p < \infty$, *and* $0 \leq \lambda < \infty$. *If* $f_i \in M\dot{K}_{q(\cdot),p}^{\alpha(\cdot),\lambda}(A;\mathbb{R}^n), 1 \leq i \leq m$, *then* $\sum_{i=1}^{m} f_i \in M\dot{K}_{q(\cdot),p}^{\alpha(\cdot),\lambda}(A;\mathbb{R}^n)$ *and*

$$\|\sum_{i=1}^{m} f_i\|_{M\dot{K}_{q(\cdot),p}^{\alpha(\cdot),\lambda}} \leq \sum_{i=1}^{m} \|f_i\|_{M\dot{K}_{q(\cdot),p}^{\alpha(\cdot),\lambda}}.$$

3. Boundedness of some sublinear operators

In this section, we will investigate the boundedness on the anisotropic Herz-Morrey spaces with two variable exponents for some sublinear operators.

Theorem 3.1. *Let $0 < p < \infty, q(\cdot) \in \mathcal{B}(\mathbb{R}^n), \alpha(\cdot) \in L^\infty(\mathbb{R}^n) \cap \mathcal{P}_0(\mathbb{R}^n) \cap \mathcal{P}_\infty(\mathbb{R}^n)$, $0 < 2\lambda < \alpha(0), \alpha_\infty < \delta_2$. If a sublinear operator T satisfies*

$$|Tf(x)| \lesssim \int_{\mathbb{R}^n} \frac{|f(y)|}{\rho(x-y)} dy, \quad x \notin \operatorname{supp} f, \tag{3.1}$$

for any $f \in L^{q(\cdot)}(\mathbb{R}^n)$ with a compact support and T is bounded on $L^{q(\cdot)}(\mathbb{R}^n)$, then T is bounded on $M\dot{K}^{\alpha(\cdot),\lambda}_{q(\cdot),p}(A;\mathbb{R}^n)$ and $MK^{\alpha(\cdot),\lambda}_{q(\cdot),p}(A;\mathbb{R}^n)$, respectively.

Proof. It suffices to prove that T is bounded on $M\dot{K}^{\alpha(\cdot),\lambda}_{q(\cdot),p}(A;\mathbb{R}^n)$. The non-homogeneous case can be proved in the similar way. Suppose $f \in M\dot{K}^{\alpha(\cdot),\lambda}_{q(\cdot),p}(A;\mathbb{R}^n)$. Similar to the method of Proposition 2.5 in [11] we get

$$\begin{aligned} \|Tf\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q(\cdot),p}} &\approx \max\Bigg\{ \sup_{L\le 0, L\in\mathbb{Z}} b^{-L\lambda}\Bigg(\sum_{k=-\infty}^{L} b^{k\alpha(0)p}\|(Tf)\chi_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\Bigg)^{1/p}, \\ &\quad \sup_{L>0, L\in\mathbb{Z}} \Bigg[b^{-L\lambda}\Bigg(\sum_{k=-\infty}^{-1} b^{k\alpha(0)p}\|(Tf)\chi_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\Bigg)^{1/p} \\ &\quad + b^{-L\lambda}\Bigg(\sum_{k=0}^{L} b^{k\alpha_\infty p}\|(Tf)\chi_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\Bigg)^{1/p}\Bigg]\Bigg\} \\ &= \max\{I, II\}. \end{aligned}$$

It suffices to prove that I is bounded on $M\dot{K}^{\alpha(\cdot),\lambda}_{q(\cdot),p}(A;\mathbb{R}^n)$, while the estimate of II is essentially similar to that of I. Denote $f_j = f\chi_j$ for each $j \in \mathbb{Z}$, then we have $f(x) = \sum_{j=-\infty}^{\infty} f_j(x)$. It is easy to see that

$$\begin{aligned} I^p &\lesssim \sup_{L\le 0, L\in\mathbb{Z}} b^{-L\lambda p} \sum_{k=-\infty}^{L} b^{k\alpha(0)p} \Bigg(\sum_{j=-\infty}^{k-w-1} \|(Tf_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Bigg)^p \\ &\quad + \sup_{L\le 0, L\in\mathbb{Z}} b^{-L\lambda p} \sum_{k=-\infty}^{L} b^{k\alpha(0)p} \Bigg(\sum_{j=k-w}^{L-1} \|(Tf_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Bigg)^p \\ &\quad + \sup_{L\le 0, L\in\mathbb{Z}} b^{-L\lambda p} \sum_{k=-\infty}^{L} b^{k\alpha(0)p} \Bigg(\sum_{j=L}^{\infty} \|(Tf_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Bigg)^p \\ &= I_1 + I_2 + I_3. \end{aligned}$$

Let us first estimate I_1. If $j \le k-w-1, x \in C_k$ and $y \in B_j$, by (1.1) we have

$$\rho(x-y) \ge b^{-w}\rho(x) - \rho(y) \ge b^{-w}\rho(x) - b^{-w-1}\rho(x) = b^{-w}(1-1/b)\rho(x).$$

Therefore by (3.1) and the generalized Hölder inequality, we get

$$\begin{aligned} |Tf_j(x)| &\le C\rho(x)^{-1}\int_{B_j} |f_j(y)|dy \\ &\le Cb^{-k}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}. \end{aligned}$$

So by Lemma 1.3 and Lemma 1.4, we have

$$
\begin{aligned}
\|(Tf_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} &\lesssim b^{-k}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\lesssim b^{-k}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\big(|B_k|\|\chi_{B_k}\|^{-1}_{L^{q'(\cdot)}(\mathbb{R}^n)}\big)\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \\
&\lesssim \|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_k}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}} \\
&\lesssim b^{\delta_2(j-k)}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
\tag{3.2}
$$

Therefore, when $0 < p \leq 1$, by $0 < \alpha(0) < \delta_2$, we get

$$
\begin{aligned}
I_1 &\lesssim \sup_{L\leq 0, L\in\mathbb{Z}} b^{-L\lambda p}\sum_{k=-\infty}^{L} b^{k\alpha(0)p}\sum_{j=-\infty}^{k-w-1} b^{\delta_2(j-k)p}\|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\lesssim \sup_{L\leq 0, L\in\mathbb{Z}} b^{-L\lambda p}\sum_{k=-\infty}^{L}\sum_{j=-\infty}^{k-w-1} b^{j\alpha(0)p}b^{(j-k)(\delta_2-\alpha(0))p}\|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\lesssim \sup_{L\leq 0, L\in\mathbb{Z}} b^{-L\lambda p}\sum_{j=-\infty}^{L-w-1} b^{j\alpha(0)p}\sum_{k=j+w+1}^{L} b^{(j-k)(\delta_2-\alpha(0))p}\|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\lesssim \sup_{L\leq 0, L\in\mathbb{Z}} b^{-L\lambda p}\sum_{j=-\infty}^{L-w-1} b^{j\alpha(0)p}\|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \lesssim \|f\|^p_{M\dot{K}^{\alpha(\cdot),\lambda}_{q(\cdot),p}}.
\end{aligned}
\tag{3.3}
$$

When $1 < p < \infty$, take $1/p + 1/p' = 1$. Since $0 < \alpha(0) < \delta_2$, by (3.2) and the Hölder inequality, we have

$$
\begin{aligned}
I_1 &\lesssim \sup_{L\leq 0, L\in\mathbb{Z}} b^{-L\lambda p}\sum_{k=-\infty}^{L} b^{k\alpha(0)p}\left(\sum_{j=-\infty}^{k-w-1} b^{\delta_2(j-k)}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p \\
&\lesssim \sup_{L\leq 0, L\in\mathbb{Z}} b^{-L\lambda p}\sum_{k=-\infty}^{L}\left(\sum_{j=-\infty}^{k-w-1} b^{j\alpha(0)p}b^{(j-k)[\delta_2-\alpha(0)]p/2}\|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\right) \\
&\quad\times\left(\sum_{j=-\infty}^{k-w-1} b^{(j-k)[\delta_2-\alpha(0)]p'/2}\right)^{p/p'} \\
&\lesssim \sup_{L\leq 0, L\in\mathbb{Z}} b^{-L\lambda p}\sum_{j=-\infty}^{L-w-1} b^{j\alpha(0)p}\sum_{k=j+w+1}^{L} b^{(j-k)(\delta_2-\alpha(0))p/2}\|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\lesssim \sup_{L\leq 0, L\in\mathbb{Z}} b^{-L\lambda p}\sum_{j=-\infty}^{L-w-1} b^{j\alpha(0)p}\|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \lesssim \|f\|^p_{M\dot{K}^{\alpha(\cdot),\lambda}_{q(\cdot),p}}.
\end{aligned}
\tag{3.4}
$$

Let us now estimate I_2. Similarly, we consider two cases for p. When $0<p\le 1$, by $0<\alpha(0)$ and $L^{q(\cdot)}(\mathbb{R}^n)$ boundedness of T, we have

$$\begin{aligned} I_2 &= \sup_{L\le 0, L\in\mathbb{Z}} b^{-L\lambda p} \sum_{k=-\infty}^{L} b^{k\alpha(0)p} \left(\sum_{j=k-w}^{L-1} \|(Tf_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^p \\ &\lesssim \sup_{L\le 0, L\in\mathbb{Z}} b^{-L\lambda p} \sum_{j=-\infty}^{L-1} b^{j\alpha(0)p} \|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \sum_{k=-\infty}^{j+w} b^{(k-j)\alpha(0)p} \\ &\lesssim \|f\|^p_{M\dot{K}^{\alpha(\cdot),\lambda}_{q(\cdot),p}}. \end{aligned} \tag{3.5}$$

When $1<p<\infty$, take $1/p+1/p'=1$. By $0<\alpha(0)$, $L^{q(\cdot)}(\mathbb{R}^n)$ boundedness of T and the Hölder inequality, we have

$$\begin{aligned} I_2 &\lesssim \sup_{L\le 0, L\in\mathbb{Z}} b^{-L\lambda p} \sum_{k=-\infty}^{L} b^{k\alpha(0)p} \left(\sum_{j=k-w}^{L-1} \|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^p \\ &\lesssim \sup_{L\le 0, L\in\mathbb{Z}} b^{-L\lambda p} \sum_{k=-\infty}^{L} \left(\sum_{j=k-w}^{L-1} b^{j\alpha(0)p} \|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} b^{(k-j)\alpha(0)p/2} \right) \\ &\quad \times \left(\sum_{j=k-w}^{L-1} b^{(k-j)\alpha(0)p'/2} \right)^{p/p'} \\ &\lesssim \sup_{L\le 0, L\in\mathbb{Z}} b^{-L\lambda p} \sum_{j=-\infty}^{L-1} b^{j\alpha(0)p} \|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \sum_{k=-\infty}^{j+w} b^{(k-j)\alpha(0)p/2} \\ &\lesssim \|f\|^p_{M\dot{K}^{\alpha(\cdot),\lambda}_{q(\cdot),p}}. \end{aligned} \tag{3.6}$$

For I_3, when $0<p\le 1$, by $0<2\lambda<\alpha(0)$ and $L^{q(\cdot)}(\mathbb{R}^n)$ boundedness of T, we get

$$\begin{aligned} I_3 &= \sup_{L\le 0, L\in\mathbb{Z}} b^{-L\lambda p} \sum_{k=-\infty}^{L} b^{k\alpha(0)p} \left(\sum_{j=L}^{\infty} \|(Tf_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^p \\ &\lesssim \sup_{L\le 0, L\in\mathbb{Z}} b^{-L\lambda p} \sum_{k=-\infty}^{L} \sum_{j=L}^{\infty} b^{(k-j)\alpha(0)p} b^{j\alpha(0)p} \|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &\lesssim \sup_{L\le 0, L\in\mathbb{Z}} b^{-L\lambda p} \sum_{k=-\infty}^{L} \sum_{j=L}^{\infty} b^{(k-j)\alpha(0)p} b^{j\lambda p} b^{-j\lambda p} \left(\sum_{m=-\infty}^{j} b^{m\alpha(0)p} \|f_m\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \right) \end{aligned} \tag{3.7}$$

$$
\begin{aligned}
&\lesssim \sup_{L\le 0, L\in\mathbb{Z}} b^{-L\lambda p}\sum_{k=-\infty}^{L}\sum_{j=L}^{\infty} b^{(k-j)\alpha(0)p} b^{j\lambda p}\|f\|^p_{M\dot{K}^{\alpha(\cdot),\lambda}_{q(\cdot),p}}\\
&\lesssim \sup_{L\le 0, L\in\mathbb{Z}} b^{-L\lambda p}\left(\sum_{k=-\infty}^{L} b^{k\alpha(0)p}\right)\left(\sum_{j=L}^{\infty} b^{(\lambda-\alpha(0))jp}\right)\|f\|^p_{M\dot{K}^{\alpha(\cdot),\lambda}_{q(\cdot),p}}\\
&\lesssim \sup_{L\le 0, L\in\mathbb{Z}} b^{-L\lambda p} b^{L\alpha(0)p} b^{(\lambda-\alpha(0))Lp}\|f\|^p_{M\dot{K}^{\alpha(\cdot),\lambda}_{q(\cdot),p}}\\
&\lesssim \|f\|^p_{M\dot{K}^{\alpha(\cdot),\lambda}_{q(\cdot),p}}. \qquad (3.8)
\end{aligned}
$$

When $1 < p < \infty$, take $1/p + 1/p' = 1$. Since $0 < 2\lambda < \alpha(0)$, by $L^{q(\cdot)}(\mathbb{R}^n)$ boundedness of T and the Hölder inequality, we have

$$
I_3 \lesssim \sup_{L\le 0, L\in\mathbb{Z}} b^{-L\lambda p}\sum_{k=-\infty}^{L} b^{k\alpha(0)p}\left(\sum_{j=L}^{\infty}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p
$$

$$
\begin{aligned}
&\lesssim \sup_{L\le 0, L\in\mathbb{Z}} b^{-L\lambda p}\sum_{k=-\infty}^{L}\left(\sum_{j=L}^{\infty} b^{j\alpha(0)p}\|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} b^{(k-j)\alpha(0)p/2}\right)\left(\sum_{j=L}^{\infty} b^{(k-j)\alpha(0)p'/2}\right)^{p/p'}\\
&\lesssim \sup_{L\le 0, L\in\mathbb{Z}} b^{-L\lambda p}\sum_{k=-\infty}^{L}\sum_{j=L}^{\infty} b^{(k-j)\alpha(0)p/2} b^{j\alpha(0)p}\|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&\lesssim \sup_{L\le 0, L\in\mathbb{Z}} b^{-L\lambda p}\sum_{k=-\infty}^{L}\sum_{j=L}^{\infty} b^{(k-j)\alpha(0)p/2} b^{j\lambda p}\|f\|^p_{M\dot{K}^{\alpha(\cdot),\lambda}_{q(\cdot),p}}\\
&\lesssim \sup_{L\le 0, L\in\mathbb{Z}} b^{-L\lambda p}\left(\sum_{k=-\infty}^{L} b^{k\alpha(0)p/2}\right)\left(\sum_{j=L}^{\infty} b^{(\lambda-\alpha(0)/2)jp}\right)\|f\|^p_{M\dot{K}^{\alpha(\cdot),\lambda}_{q(\cdot),p}}\\
&\lesssim \sup_{L\le 0, L\in\mathbb{Z}} b^{-L\lambda p} b^{L\alpha(0)p/2} b^{(\lambda-\alpha(0)/2)Lp}\|f\|^p_{M\dot{K}^{\alpha(\cdot),\lambda}_{q(\cdot),p}}\\
&\lesssim \|f\|^p_{M\dot{K}^{\alpha(\cdot),\lambda}_{q(\cdot),p}}. \qquad (3.9)
\end{aligned}
$$

Combining (3.3)-(3.9), we have

$$
I \lesssim \|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q(\cdot),p}}.
$$

Thus, the proof of Theorem 3.1 is completed. □

Remark 3.2. From the proof of Theorem 3.1, it is easy to see that the size condition (3.1) can be replaced by

$$
|Tf(x)| \le C\frac{\|f\|_{L^1}}{\rho(x)}, \quad \text{if } \inf_{y\in \mathrm{supp} f}\rho(x-y) \ge b^{-w}(1-1/b)\rho(x), \tag{3.10}
$$

for all $f \in L^{q(\cdot)}(\mathbb{R}^n)$ with compact support.

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References

1. A. Almeida and D. Drihem, *Maximal, potential and singular type operators on Herz spaces with variable exponents*, J. Math. Anal. Appl., **394** (2012), 781–795.
2. M. Bownik, *Anisotropic Hardy spaces and wavelets*, Mem. Amer. Math. Soc., **164** (2003), 122pp.
3. A.P. Calderón and A. Torchinsky, *Parabolic maximal functions associated with a distribution*, Adv. Math., **16** (1975), 1–64.
4. A.P. Calderón and A. Torchinsky, *Parabolic maximal functions associated with a distribution II*, Adv. Math., **24** (1977), 101–171.
5. D.V. Cruz-Uribe and A. Fiorenza, *Variable Lebesgue Spaces: Foundations and Harmonic Analysis, Applied and Numerical Harmonic Analysis*, Springer, Heidelberg, 2013.
6. L. Diening, P. Harjulehto, P. Hästö and M. Růžička, *Lebesgue and Sobolev spaces with variable exponents*, Lecture Notes in Math., vol. **2017**, Springer-Verlag, Berlin, 2011.
7. C. Fefferman and E.M. Stein, *H^p spaces of several variables*, Acta Math., **129** (1972), 137–193.
8. M. Izuki, *Boundedness of vector-valued sublinear operators on Herz-Morrey spaces with variable exponent*, Math. Sci. Res. J., **13** (2009), 243–253.
9. M. Izuki, *Boundedness of sublinear operators on Herz spaces with variable exponent and application to wavelet characterization*, Anal. Math., **36** (2010), 33–50.
10. O. Kováčik and J. Rákosník, *On spaces $L^{p(x)}$ and $W^{k,p(x)}$*, Czechoslovak Math. J., **41** (1991), 592–618.
11. Y. Lu and Y. Zhu, *Boundedness of multilinear Calderón-Zygmund singular operators on Morrey-Herz spaces with variable exponents*, Acta Math. Sin.(Engl. Ser.), **30** (2014), 1180–1194.
12. H. Wang, *Anisotropic Herz spaces with variable exponents*, Commun. Math. Anal., **18** (2015), 1–14.

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