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OPERATORS REVERSING ORTHOGONALITY AND CHARACTERIZATION OF INNER PRODUCT SPACES

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Abstract. In this short paper we answer a question posed by Chmieliński in [Adv. Oper. Theory **1** (2016), no. 1, 8–14]. Namely, we prove that among normed spaces of dimension greater than two, only inner product spaces admit nonzero linear operators which reverse the Birkhoff orthogonality.

1. Introduction and preliminaries

Let $(X, \|\cdot\|)$ be a normed space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. The *Birkhoff–orthogonality* of vectors x and y in X, is defined by:

$$x \perp_{\mathrm{B}} y \quad :\Leftrightarrow \quad \forall_{\lambda \in \mathbb{K}} \ \ \|x\| \le \|x + \lambda y\|.$$

Of course, in an inner product space we have $\perp_{\mathrm{B}} = \perp$. Clearly, the relation $\perp_{\mathrm{B}}$ is generally not symmetric. A (nonzero) linear mapping $T\colon X \to X$ which satisfies

$$\forall_{x,y \in X} \quad x \perp_{\mathrm{B}} y \ \Rightarrow \ Ty \perp_{\mathrm{B}} Tx \tag{1.1}$$

is called *reverses orthogonality*. This property is equivalent (see [3, p. 10]) to

$$\forall_{x,y \in X} \quad x \perp_{\mathrm{B}} y \ \Leftrightarrow \ Ty \perp_{\mathrm{B}} Tx. \tag{1.2}$$

If the orthogonality relation is symmetric, then (1.1) actually means the *orthogonality preserving property* and hence T is a linear similarity (see [3] and the references therein). Chmieliński [3] showed that on a two-dimensional normed space there may exist operators which reverse orthogonality *essentially* (i.e., they are not orthogonality preserving). Moreover, Chmieliński proved the following result.

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Theorem 1.1 ([3], Theorem 4.1)**.** *Let X be a smooth normed space such that* $\dim X \geq 3$*. Then there exists a nonzero linear operator $T\colon X \to X$ satisfying* (1.1) *if and only if X is an inner product space.*

Chmieliński posed the question [3, p. 13] whether the assumption of smoothness is necessary. In the next section we will give an answer. Namely, we will prove that the assumption of smoothness is redundant. Our proof will be based on the following characterization of inner product spaces.

Theorem 1.2 ([2, 4])**.** *Let X be a normed space such that* $\dim X \geq 3$*. Then X is an inner product space if and only if for each two-dimensional subspace M of X there exists a norm-one projection onto M, i.e., a bounded linear operator $P\colon X \to X$ such that $P(X) = M$, $P^2 = P$, and $\|P\| = 1$.*

It is worth mentioning that for real spaces it was proved by Kakutani [4]. Bohnenblust [2] extended it to complex spaces (cf. also [1, 12.4]).

2. Main result

Now we are able to strengthen Theorem 1.1.

Theorem 2.1. *Let $(X, \|\cdot\|)$ be a normed space such that* $\dim X \geq 3$*. Then there exists a nonzero linear operator $T\colon X \to X$ satisfying* (1.1) *if and only if X is an inner product space.*

Proof. If X is an inner product space, then the orthogonality relation is symmetric and the identity mapping satisfies (1.1). We prove the converse implication.

Assume now that X admits a nonzero linear operator $T\colon X \to X$ satisfying (1.1). We will prove that each three-dimensional linear subspace Y of X is an inner product space, which is sufficient to show that X is an inner product space.

Let Y be an arbitrary linear subspace of X with $\dim Y = 3$ and let M be a subspace of Y such that $\dim M = 2$. Since T is an injection (see [3, p. 10]), we have $\dim T(M) = 2$ and $\dim T(Y) = 3$. Moreover, $T(M) \subset T(Y)$. Since $\dim T(M) < \dim T(Y) < \infty$, it follows from Riesz Lemma (and from an easy compactness argument) that there exists a nonzero vector $w \in T(Y)$ such that $\|w\| \leq \|w - s\|$ for all $s \in T(M)$. So we get $w \perp_{\mathrm{B}} T(M)$. Moreover, we have $T(u) = w$ for some $u \in Y$. Thus $T(u) \perp_{\mathrm{B}} T(M)$. Combining it with (1.2), we obtain

$$M \perp_{\mathrm{B}} u. \tag{2.1}$$

It is easy to prove that $Y = M + \operatorname{span}\{u\}$ and $\{0\} = M \cap \operatorname{span}\{u\}$. Define a linear operator $P\colon Y \to Y$ by $P(m + \alpha u) := m$ for each $m + \alpha u \in Y = M + \operatorname{span}\{u\}$, where $m \in M$, $\alpha \in \mathbb{K}$. It is easy to check that $P(Y) = M$, $P^2 = P$ and $\|P\| = \|P^2\| \leq \|P\|^2$, so $\|P\| \geq 1$. On the other hand, for $m + \alpha u \in Y = M + \operatorname{span}\{u\}$ we get

$$\|P(m + \alpha u)\| = \|m\| \overset{(2.1)}{\leq} \|m + \alpha u\|$$

and hence $\|P\| \leq 1$. Finally, $\|P\| = 1$. Now, applying Theorem 1.2 we get that Y is an inner product space. □

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