Σ-ISOMORPHIC ALGEBRAIC STRUCTURES

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(Received September 24, 1993)

Summary. For an algebraic structure $\mathscr{A} = (A, F, R)$ or type τ and a set Σ of open formulas of the first order language $L(\tau)$ we introduce the concept of Σ-closed subsets of $\mathscr{A}$. The set $\mathscr{C}_\Sigma(\mathscr{A})$ of all Σ-closed subsets forms a complete lattice. Algebraic structures $\mathscr{A}$, $\mathscr{B}$ of type τ are called Σ-isomorphic if $\mathscr{C}_\Sigma(\mathscr{A}) \cong \mathscr{C}_\Sigma(\mathscr{B})$. Examples of such Σ-closed subsets are e.g. subalgebras of an algebra, ideals of a ring, ideals of a lattice, convex subsets of an ordered or quasiordered set etc. We study Σ-isomorphic algebraic structures in dependence on the properties of Σ.

Keywords: algebraic structure, closure system, subalgebra, ideal, Σ-closed subset, Σ-isomorphic structures

MSC 1991: 08A05, 04A05, 06B10

The concept of an algebraic structure was introduced in [6] and [8]. A type of a structure is a pair $\tau = \langle\{n_i\,;\ i \in I\}, \{m_j\,;\ j \in J\}\rangle$, where n_i and m_j are non-negative integers. A structure $\mathscr{A}$ of type τ is a triplet (A, F, R), where $A \neq \emptyset$ is a set and $F = \{f_i\,;\ i \in I\}$, $R = \{\varrho_j\,;\ j \in J\}$ are such that for each $i \in I$, $j \in J$, f_i is an n_i-ary operation on A and ϱ_j is an m_j-ary relation on A. Denote by $L(\tau)$ the first order language containing operational and relational symbols of type τ, see [6] for some details. If $R = \emptyset$, the structure $(A, F, \emptyset)$ is denoted by (A, F) and it is called an *algebra*. If $F = \emptyset$, the structure $(A, \emptyset, R)$ is denoted by (A, R) and it is called a *relational system*. A relational system (A, R) is called *binary* if each $\varrho_j \in R$ is binary; moreover, (A, R) is said to be *antisymmetrical* if each $\varrho_j \in R$ is an antisymmetrical relation.

Introduce the following concepts: for each $\gamma \in \Gamma$, where Γ is an index set, let $G_\gamma(x_1, \ldots, x_{k_\gamma}, y_1, \ldots, y_{s_\gamma}, z, f_i)$ be an open formula containing individual variables $x_1, \ldots, x_{k_\gamma}, y_1, \ldots, y_{s_\gamma}, z$ and a symbol f_i of an n_i-ary operation; for each $\lambda \in \Lambda$, where Λ is an index set, let $G_\lambda(x_1, \ldots, x_{k_\lambda}, y_1, \ldots, y_{s_\lambda}, z, \varrho_i)$ be an open formula containing individual variables $x_1, \ldots, x_{k_\lambda}, y_1, \ldots, y_{s_\lambda}, z$ and a symbol ϱ_j of an m_j-ary relation. Put $\Sigma = \{G_\gamma ; \gamma \in \Gamma\} \cup \{G_\lambda ; \lambda \in \Lambda\}$. The set $\Sigma = \{G_\gamma, \gamma \in \Gamma\} \cup \{G_\lambda, \lambda \in \Lambda\}$ of formulas of language $L(\tau)$ is called *limited* if there exist non-negative integers n, m such that $m = \max(\{k_\gamma, \gamma \in \Gamma\} \cup \{k_\lambda, \lambda \in \Lambda\})$ and $m = \max(\{s_\gamma, \gamma \in \Gamma\} \cup \{s_\lambda, \lambda \in \Lambda\})$. Let $\mathscr{A} = (A, F, R)$ be a structure of type τ and $B \subseteq A$.

Definition 1. A subset B of $\mathscr{A}$ is said to be Σ-*closed* if for each $\gamma \in \Gamma$, $\lambda \in \Lambda$ and every $a_1, \ldots, a_{k_\gamma}, a'_1, \ldots, a'_{k_\lambda} \in B$, $b_1, \ldots, b_{s_\gamma}, b'_1, \ldots, b'_{s_\lambda}, c, c' \in A$, if $G_\gamma(a_1, \ldots, a_{k_\gamma}, b_1, \ldots, b_{s_\gamma}, c, f_i)$ is satisfied in $\mathscr{A}$ then $c \in B$ and if $G_\lambda(a'_1, \ldots, a'_{k_\lambda}, b'_1, \ldots, b'_{s_\lambda}, c', \varrho_j)$ is satisfied in $\mathscr{A}$ then $c' \in B$. Denote by $\mathscr{C}_\Sigma(\mathscr{A})$ the set of all Σ-closed subsets of $\mathscr{A}$.

Since the concept of Σ-closed subsets is defined by the set of universal formulas, $B = \bigcap\{B_\delta ; \delta \in \Delta\}$ is also a Σ-closed subset of $\mathscr{A}$ provided $\mathscr{B}_\delta$ has this property for each $\delta \in \Delta$. We accept also the case $B = \emptyset$. Thus we have

Lemma 1. *Let $\mathscr{A} = (A, F, R)$ be a structure of type τ and Σ a set of open formulas of the language $L(\tau)$. Then the set $\mathscr{C}_\Sigma(\mathscr{A})$ of all Σ-closed subsets of $\mathscr{A}$ forms a complete lattice with respect to set inclusion with the greatest element A.*

Corollary 1. *For any $\mathscr{A}$, Σ and $M \subseteq A$ there exists the least Σ-closed subset $C_{\mathscr{A}}(M)$ containing M.*

If $M = \{a_1, \ldots, a_n\}$ then we will write briefly $C_{\mathscr{A}}(M) = C_{\mathscr{A}}(a_1, \ldots, a_n)$.

If the set Σ is implicitly known, we will use only the lattice $\mathscr{C}_\Sigma(\mathscr{A})$ to specify the closure system; we will use the more familiar notation for $\mathscr{C}_\Sigma(\mathscr{A})$ provided it was introduced in algebra, see the following examples.

Examples.

(1) Let $\mathscr{A} = (A, \leqslant)$ be an *ordered set*. Put $\Gamma = \emptyset$, $\Lambda = \{1\}$, $k_1 = 2$, $s_1 = 0$ and $\Sigma = \{G_1\}$, where $G_1(x_1, x_2, z, \leqslant)$ is the formula ($x_1 \leqslant z$ and $z \leqslant x_2$). Then the Σ-closed subsets of $\mathscr{A}$ are exactly the *convex subsets* of $(A, \leqslant)$.

(2) Let $\mathscr{A} = (A, F)$ be an *algebra*, $F = \{f_i ; i \in I\}$. Let $\Lambda = \emptyset$, $\Gamma = I$, $k_i = n_i$, $s_i = 0$ for $i \in I$. Put $\Sigma = \{G_i ; i \in I\}$, where $G_i(x_1, \ldots, x_{n_i}, z, f_i)$ is the formula $\big(f_i(x_i, \ldots, x_{n_i}) = z\big)$. Then the Σ-closed subsets of $\mathscr{A}$ are *subalgebras* of $\mathscr{A} = (A, F)$ and $\mathscr{C}_\Sigma(\mathscr{A}) = \mathrm{Sub}(\mathscr{A})$.

(3) Let $\mathscr{R} = (R, +, \cdot, 0)$ be a *ring*, $\Lambda = \emptyset$, $\Gamma = \{1, 2, 3\}$, $k_1 = 2$, $k_2 = k_3 = 1$, $s_1 = 0$, $s_2 = s_3 = 1$ and $\Sigma = \{G_1, G_2, G_3\}$, where G_1 is the formula $(x_1 + x_2 = z)$, G_2 is the formula $(x_1 \cdot y_1 = z)$ and G_3 is the formula $(y_1 \cdot x_1 = z)$. Then the Σ-closed subsets of $\mathscr{R}$ are the *ideals* of $\mathscr{R}$ and $\mathscr{C}_\Sigma(\mathscr{R}) = \mathrm{Id}\,\mathscr{R}$, the lattice of all ideals of $\mathscr{R}$.

Analogously we can introduce left or right ideals of $\mathscr{R}$.

(4) Similarly, if $\mathscr{L} = (L, \vee, \wedge)$ is a lattice, $\Lambda = \emptyset$, $\Gamma = \{1, 2\}$, $k_1 = 2$, $k_2 = 1$, $s_1 = 0$, $s_2 = 1$, $\Sigma = \{G_1, G_2\}$, where G_1 is the formula $(x_1 \vee x_2 = z)$ and G_2 is the formula $(x_1 \wedge y_2 = z)$, then the Σ-closed subsets are *lattice ideals*, i.e. $\mathscr{C}_\Sigma(\mathscr{L}) = \operatorname{Id} \mathscr{L}$.

(5) Let $\mathscr{L} = (L, \vee, \wedge)$ be a *lattice*, $\Gamma = \{1, 2\}$, $\Lambda = \{1'\}$, $k_1 = k_2 = k_{1'} = 2$, $s_1 = s_2 = s_{1'} = 0$, $\Sigma = \{G_1, G_2, G_{1'}\}$, where G_1 is the formula $(x_1 \vee x_2 = z)$, G_2 is the formula $(x_1 \wedge x_2 = z)$ and $G_{1'}$ is the formula $(x_1 \wedge z = x_1$ and $x_2 \vee z = x_2)$. Then the Σ-closed subsets are the *convex sublattices* of $\mathscr{L}$.

(6) Analogously, if $\mathscr{L} = (L, +, ., \leqslant)$ is a λ-*lattice* (see [10]), $\Gamma = \{1, 2\}$, $\Lambda = \{1'\}$, $k_1 = k_2 = k_{1'} = 2$, $s_1 = s_2 = s_{1'} = 0$, $\Sigma = \{G_1, G_2, G_{1'}\}$, where G_1 is the formula $(x_1 + x_2 = z)$, G_2 is the formula $(x_1 \cdot x_2 = z)$ and $G_{1'}$ is the formula $(x_1 \leqslant z$ and $z \leqslant x_2)$, then the Σ-closed subsets are just the *convex subλ-lattices* of $\mathscr{L}$.

(7) Analogously, if $\mathscr{A} = (A, \vee, \wedge, Q)$ is a *q-lattice* (see [3]), $\Sigma = \{G_1, G_2, G_{1'}\}$, where G_1 is the formula $(x_1 \vee x_2 = z)$, G_2 is the formula $(x_1 \wedge x_2 = z)$ and $G_{1'}$ is the formula $(x_1 Q z$ and $z Q x_2)$, then the Σ-closed subsets are the *convex sub-q-lattices* of $\mathscr{A}$.

(8) Let $\mathscr{A} = (A, f)$ be a *monounary algebra*, $\Lambda = \emptyset$, $\Gamma = \{1\}$, $k_1 = 2$, $s_1 = 0$, $\Sigma = \{G_1\}$, where G_1 is the formula $(x_1 \neq x_2$ and $x_2 \neq z$ and $z \neq x_1$ and $f(x_1) = z$ and $f^k(z) = x_2$ for some non-negative integer $k)$. Then the Σ-closed subsets are the *convex subsets of the monounary algebra* $\mathscr{A}$ defined in [7].

(9) Example (1) can be generalized as follows: For a *binary relational system* $\mathscr{A} = (A, R)$ with $R = \{\varrho_j\,;\, j \in J\}$ we call $\mathscr{C}_\Sigma(\mathscr{A})$ the *lattice of convex subsets* if $\Sigma = \{G_j\,;\, j \in J\}$ and every $G_j(x_1, x_2, z)$ is the formula $(x_1\ \varrho_j\ z$ and $z\ \varrho_j\ x_2)$; we denote $\mathscr{C}_\Sigma(\mathscr{A})$ by $\operatorname{Conv}(\mathscr{A})$.

(10) Examples (5), (6), (7) can be generalized as follows: An *algebraic structure* $\mathscr{A} = (A, F, R)$ is called a *binary algebraic structure* if a relational system (A, R) is binary. Let $\mathscr{A}$ be a binary algebraic structure, $\mathscr{A}_1 = (A, F)$, $\mathscr{A}_2 = (A, R)$, $\Sigma = \Sigma_1 \cup \Sigma_2$, where $\Sigma_1 = \{G_\gamma\,;\, \gamma \in \Gamma\}$ and $\Sigma_2 = \{G_\lambda\,;\, \lambda \in \Lambda\}$. The lattice $\mathscr{C}_\Sigma(\mathscr{A})$ is called the *lattice of convex subalgebras* of $\mathscr{A}$ if $\mathscr{C}_{\Sigma_1}(\mathscr{A}_1) = \operatorname{Sub} \mathscr{A}_1$ and $\mathscr{C}_{\Sigma_2}(\mathscr{A}_2) = \operatorname{Conv} \mathscr{A}_2$; $\mathscr{C}_\Sigma(\mathscr{A})$ is denoted by $C \operatorname{Sub} \mathscr{A}$.

We can also modify Definition 1 in the sense of the following remark.

R e m a r k 1. The concept of Σ-closed subsets can be generalized if we consider *term functions* instead of fundamental operations in formulas G_γ of Σ. Indeed, if $\mathscr{C} = (G, ., ^{-1}, e)$ is a *group*, $p(x, y)$ is the term function $p(x, y) = yxy^{-1}$ and $\Sigma = \{G_1, G_2, G_3, G_4\}$, where $G_1(x_1, x_2, z, .)$ is the formula $(x_1 \cdot x_2 = z)$, $G_2(x_1, z, ^{-1})$ is the formula $(x_1, ^{-1} = z)$, $G_3(z, e)$ is the formula $(e = z)$ and $G_4(x_1, y_1, z, p)$ is the formula $(p(x_1, y_1) = z)$ then $\mathscr{C}_\Sigma(\mathscr{C})$ is the *lattice of normal subgroups* of $\mathscr{C}$.

Similarly, we can also define *ideals of an ℓ-group* $\mathscr{G} = (G, \cdot, ^{-1}, e, \vee, \wedge)$, i.e. normal subgroups of the group $(G, \cdot, ^{-1}, e)$ which are convex sublattices of the lattice $(G, \vee, \wedge)$.

Definition 2. Let $\mathscr{A}$, $\mathscr{B}$ be structures of the same type τ and let Σ be a set of open formulas of the language $L(\tau)$. We say that $\mathscr{A}$, $\mathscr{B}$ are *Σ-isomorphic* if the lattices $\mathscr{C}_\Sigma(\mathscr{A})$ and $\mathscr{C}_\Sigma(\mathscr{B})$ are isomorphic.

Examples.

(10) Binary relational systems $\mathscr{A} = (A, R)$, $\mathscr{B} = (B, P)$ of the same type are called *convex isomorphic* if $\operatorname{Conv} \mathscr{A} \cong \operatorname{Conv} \mathscr{B}$. A special case of this concept is represented by convex isomorphic ordered sets. They were characterized in [1] and [4].

(11) Binary algebraic structures $\mathscr{A} = (A, F, R)$, $\mathscr{B} = (B, G, P)$ of the same type are called *convex isomorphic* if $C \operatorname{Sub} \mathscr{A} \cong C \operatorname{Sub} \mathscr{B}$. In particular, convex isomorphic lattices were characterized in [9] and convex isomorphic q-lattices were characterized in [5].

(12) Let $\mathscr{A} = (A, F)$, $\mathscr{B} = (B, F)$ be two algebras of the same type and $\mathscr{C}_\Sigma = \operatorname{Sub}$, i.e. Σ-closed subsets are *subalgebras*. Then $\mathscr{A}$, $\mathscr{B}$ are *Σ-isomorphic* if $\operatorname{Sub} \mathscr{A} \cong \operatorname{Sub} \mathscr{B}$.

(13) For rings or lattices, if $\mathscr{C}_\Sigma = \operatorname{Id}$, then $\mathscr{R}_1$, $\mathscr{R}_2$ are *Σ-isomorphic* if $\operatorname{Id} \mathscr{R}_1 \cong \operatorname{Id} \mathscr{R}_2$.

Definition 3. An algebraic structure $\mathscr{A} = (A, F, R)$ is called *Σ-separable* if $\{a\} \in \mathscr{C}_\Sigma(\mathscr{A})$ for each $a \in A$.

Definition 4. Let $\mathscr{A} = (A, F, R)$, $\mathscr{B} = (B, F, R)$ be Σ-separable structures of the same type τ which are Σ-isomorphic and let $h\colon \mathscr{C}_\Sigma(\mathscr{A}) \to \mathscr{C}_\Sigma(\mathscr{B})$ be the isomorphism. The mapping $\varphi_h : A \to B$ defined by the rule $\{\varphi_h(a)\} = h\{(a)\}$ is said to be *associated with the isomorphism h*.

For $M \subseteq A$ we put $\varphi_h(M) = \{\varphi_h(a), a \in M\}$.

Remark 2. If $\mathscr{A}$ is Σ-separable then $\mathscr{C}_\Sigma(\mathscr{A})$ is an *atomic lattice* whose atoms are exactly the sets $\{a\}$ for each $a \in A$. Moreover, every isomorphism of atomic lattices maps atoms onto atoms. Hence, Definition 4 is correct.

Lemma 2. *Let $\mathscr{A} = (A, F, R)$, $\mathscr{B} = (B, F, R)$ be Σ-separable and Σ-isomorphic structures of the same type. Let $h\colon \mathscr{C}_\Sigma(\mathscr{A}) \to \mathscr{C}_\Sigma(\mathscr{B})$ be the isomorphism. Then we have $\varphi_h\bigl(C_{\mathscr{A}}(M)\bigr) = C_{\mathscr{B}}\bigl(\varphi_h(M)\bigr)$ for any $M \subseteq A$.*

Proof. First, suppose $D \in \mathscr{C}_\Sigma(\mathscr{A})$. If $a \in D$, then $\{a\} \subseteq D$ and so $\{\varphi_h(a)\} = h(\{a\}) \subseteq h(D)$, thus $\varphi_h(D) \subseteq h(D)$. Conversely, if $b \in h(D)$, then $\{b\} \subseteq h(D)$ and

$\{\varphi_h^{-1}(b)\} = h^{-1}(\{b\}) \subseteq D$ because h is a bijection. Thus $\{\varphi_h^{-1}(b)\}$ is a singleton and $\varphi_h^{-1}(b) \in D$, i.e. $b \in \varphi_h(D)$, giving $h(D) \subseteq \varphi_h(D)$. So we have

$$\varphi_h(D) = h(D). \tag{1}$$

Now, let $M \subseteq A$. Since $M \subseteq C_{\mathscr{A}}(M)$, we obtain $\varphi_h(M) \subseteq \varphi_h\bigl(C_{\mathscr{A}}(M)\bigr)$. Furthermore, $\varphi_h\bigl(C_{\mathscr{A}}(M)\bigr) = h\bigl(C_{\mathscr{A}}(M)\bigr) \in \mathscr{C}_\Sigma(\mathscr{B})$ by (1) and so $C_{\mathscr{B}}\bigl(\varphi_h(M)\bigr) \subseteq \varphi_h\bigl(C_{\mathscr{A}}(M)\bigr)$. On the other hand, let $X \in \mathscr{C}_\Sigma(\mathscr{B})$ be such that $\varphi_h(M) \subseteq X$. Since h is surjective, there exists $Y \in \mathscr{C}_\Sigma(\mathscr{A})$ with $h(Y) = \varphi_h(Y) = X$. It follows that $M \subseteq Y$ and, therefore, $C_{\mathscr{A}}(M) \subseteq Y$. Consequently, $\varphi_h\bigl(C_{\mathscr{A}}(M)\bigr) \subseteq X$ and we can see that $\varphi_h\bigl(C_{\mathscr{A}}(M)\bigr) \subseteq C_{\mathscr{B}}\bigl(\varphi_h(M)\bigr)$. □

Theorem 1. *Let $\mathscr{A} = (A, F, R)$, $\mathscr{B} = (B, F, R)$ be Σ-separable structures of the same type for some limited $\Sigma = \{G_\gamma, \gamma \in \Gamma\} \cup \{G_\lambda, \lambda \in \Lambda\}$. Then the following conditions are equivalent:*

(i) *$\mathscr{A}$, $\mathscr{B}$ are Σ-isomorphic.*

(ii) *There exists a bijection $g\colon A \to B$ such that $g\bigl(C_{\mathscr{A}}(M)\bigr) = C_{\mathscr{B}}\bigl(g(M)\bigr)$ for any $M \subseteq A$.*

(iii) *There exists a bijection $g\colon A \to B$ such that*

$$g\bigl(C_{\mathscr{A}}(a_1, \ldots, a_n)\bigr) = C_{\mathscr{B}}\bigl(g(a_1), \ldots, g(a_n)\bigr)$$

for each $a_1, \ldots, a_n \in A$, where $n = \max(\{k_\gamma, \gamma \in \Gamma\} \cup \{k_\lambda, \lambda \in \Lambda\})$.

Proof. The condition (ii) follows from (i) by Lemma 2. The implication (ii) $\Rightarrow$ (iii) is trivial. Prove (iii) $\Rightarrow$ (i): Let g be a bijection satisfying (iii). Let $h\colon \operatorname{Exp} A \to \operatorname{Exp} B$ be a mapping defined as follows: $h(M) = \{g(a);\, a \in M\}$ for any $M \subseteq A$. Since g is a bijection, h is also a bijection. We are going to prove that for any Σ-closed subset D of $\mathscr{A}$ its image $h(D)$ is a Σ-closed subset of $\mathscr{B}$. Suppose $D \in \mathscr{C}_\Sigma(\mathscr{A})$. Let $\gamma \in \Gamma$, $G_\gamma(x_1, \ldots, x_{k_\gamma}, y_1, \ldots, y_{s_\gamma}, z, f_i) \in \Sigma$. Let $a'_1, \ldots, a'_n \in h(D)$, $b'_1, \ldots, b'_m, c' \in B$ and let the formula $G_\gamma(a'_n, \ldots, a'_n, b'_1, \ldots, b'_m, c', f_i)$ be satisfied in $\mathscr{B}$ for each $\gamma \in \Gamma$ (and, analogously, for each $\lambda \in \Lambda$, $G_\lambda \in \Sigma$). Then $c' \in C_{\mathscr{B}}(a'_1, \ldots, a'_n)$. Since g is a bijection there exist $a_1, \ldots, a_n, b_1, \ldots, b_m, c \in A$ such that $g(a_i) = a'_i$, $g(b_j) = b'_j$, $g(c) = c'$. We have $c \in C_{\mathscr{A}}(a_1, \ldots, a_n) \subseteq D$ according to (iii). Then $c' \in h(D)$, hence $h(D) \in \mathscr{C}_\Sigma(\mathscr{B})$. Analogously we can prove that if $h(D)$ is Σ-closed in $\mathscr{B}$ then also D is Σ-closed in $\mathscr{A}$. Thus the restriction of h onto $\mathscr{C}_\Sigma(\mathscr{A})$ is the isomorphism of $\mathscr{C}_\Sigma(\mathscr{A})$ onto $\mathscr{C}_\Sigma(\mathscr{B})$. □

Let $\mathscr{A} = (A, F, R)$ be a binary algebraic structure with $R = \{\varrho_j;\, j \in J\}$ and let $a, b \in A$. The set $\langle a, b\rangle = \{x \in A;\, a\, \varrho_j\, x \text{ and } x\, \varrho_j\, b \text{ for each } j \in J\}$ is called an *interval* of $\mathscr{A}$ determined by the elements a, b.

Corollary 2 (see Theorem 2.1 in [4]). *Two ordered sets* $\mathscr{A} = (A, \leqslant)$, $\mathscr{B} = (B, \leqslant)$ *are convex isomorphic if and only if there exists a bijection* $g\colon A \to B$ *such that for each* $a, b \in A$: $g(\langle a, b\rangle) = \langle g(a), g(b)\rangle$ *if* $a \leqslant b$ *and* $g(\{a, b\}) = \{g(a), g(b)\}$ *if* $a \parallel b$.

Corollary 3 (see Theorem 1 in [9]). *Two lattices* $\mathscr{L}_1 = (L_1, \vee, \wedge)$, $\mathscr{L}_2(L_2, \vee, \wedge)$ *are convex isomorphic if and only if there exists a bijection* $g\colon L_1 \to L_2$ *such that* $g(\langle a \wedge b, a \vee b\rangle) = \langle g(a) \wedge g(b), g(a) \vee g(b)\rangle$ *for each* $a, b \in A$.

The following assertion is evident:

Lemma 3. *Let* $\mathscr{A} = (A, F)$ *be an algebra and* $\mathscr{C}_\Sigma(\mathscr{A}) = \operatorname{Sub} \mathscr{A}$. *Then* $\mathscr{A}$ *is* Σ*-separable if and only if* $\mathscr{A}$ *is idempotent.*

Let $\mathscr{A} = (A, F)$ be an algebra, $a_1, \ldots, a_n \in A$. Denote by $[a_1, \ldots, a_n]$ the *subalgebra of* $\mathscr{A}$ *generated by the elements* $a_1, \ldots, a_n$.

Corollary 4. *Let* $\mathscr{A}$, $\mathscr{B}$ *be idempotent algebras of the same type* $\tau = \{n_i, i \in I\}$ *such that there exists* $n = \max\{n_i, i \in I\}$. *Then* $\operatorname{Sub} \mathscr{A} \simeq \operatorname{Sub} \mathscr{B}$ *if and only if there exists a bijection* $g\colon A \to B$ *such that* $g([a_1, \ldots, a_n]) = [g(a_1), \ldots, g(a_n)] \in \operatorname{Sub} \mathscr{B}$ *for any* $a_1, \ldots, a_n \in A$.

The concept of genomorphism was introduced in [2]: Let $\mathscr{A} = (A, F)$, $\mathscr{B} = (B, G)$ be algebras, not necessarily of the same type. A mapping $g\colon A \to B$ is called a *genomorphism*, if

a) g is *generative*, i.e. for each n-ary operation $f \in F$ and for each $a_1, \ldots, a_n \in A$ we have $g\big(f(a_1, \ldots, a_n)\big) \in [g(a_1), \ldots, g(a_n)]$,

b) g is *congruential*, i.e. for each n-ary operation $f \in F$ and for each $a_1, \ldots, a_n$, $a'_1, \ldots, a'_n \in A$ such that $g(a'_i) = g(a_i)$ $(i = 1, \ldots, n)$ we have $g\big(f(a_1, \ldots, a_n)\big) = g\big(f(a'_1, \ldots, a'_n)\big)$.

A bijective genomorphism is called an *isogenomorphism*. Evidently, every homomorphism is a genomorphism and every injective generative mapping is a genomorphism.

Corollary 5. *Let* $\mathscr{A}$, $\mathscr{B}$ *be idempotent algebras of the same type* $\tau = \{n_i\,;\, i \in I\}$ *such that there exists* $n = \max\{n_i\,;\, i \in I\}$. *Then* $\operatorname{Sub} \mathscr{A} \simeq \operatorname{Sub} \mathscr{B}$ *if and only if there exists an isogenomorphism of* $\mathscr{A}$ *onto* $\mathscr{B}$, *such that the inverse mapping is an isogenomorphism also.*

Clearly, any isomorphism and any antiisomorphism of lattices are isogenomorphism. Isogenomorphisms of lattices and semilattices were characterized in [11].

Definition 5. An algebraic structure $\mathscr{A} = (A, F, R)$ is called Σ-*semiseparable* if $C_{\mathscr{A}}(a) \neq C_{\mathscr{A}}(b)$ for each a, $b \in A$ with $a \neq b$. An element $a \in A$ is called a Σ-*idempotent* if $C_{\mathscr{A}}(a) = \{a\}$. An algebraic structure $\mathscr{A}$ is called Σ-*semiidempotent* if for each $\emptyset \neq X \in \mathscr{C}_{\Sigma}(\mathscr{A})$ there exists a Σ-idempotent a with $a \in X$.

Remark 3. If $\mathscr{A} = (A, F, R)$ is an algebraic structure and $a \in A$ is a Σ-idempotent then a is an idempotent element of the algebra $\mathscr{A} = (A, F)$. The converse assertion is not valid in general, e.g. if $\mathscr{A} = (A, \vee, \wedge, Q)$ is a q-lattice where Q is the induced quasiorder (i.e. aQb if and only if $a \vee b = b \vee b$). Let $\mathscr{A}$ be not a lattice and $\mathscr{C}_{\Sigma}(\mathscr{A}) = C\,\mathrm{Sub}\,\mathscr{A}$. Then there exist idempotent elements of $(A, \vee, \wedge)$ which are not Σ-idempotents.

If $\mathscr{A}$ is Σ-separable then it is also Σ-semiseparable, but not vice versa. For instance, if $\mathscr{A} = (A, \vee, \wedge)$ is a lattice and $\mathscr{C}_{\Sigma}(\mathscr{A}) = \mathrm{Id}\,\mathscr{A}$, then $C_{\mathscr{A}}(a)$ is the *principal ideal* of $\mathscr{A}$ generated by an element a and $\mathscr{A}$ is Σ-semiseparable but not Σ-separable.

Clearly, $\mathscr{A}$ is Σ-separable if and only if each element $a \in A$ is a Σ-idempotent. Denote by $T_{\Sigma}(\mathscr{A})$ the set of all Σ-idempotents of $\mathscr{A}$. If $\mathscr{A} = (A, F)$ is an algebra and $\mathscr{C}_{\Sigma}(\mathscr{A}) = \mathrm{Sub}\,\mathscr{A}$, then $T_{\Sigma}(\mathscr{A})$ is the set of all idempotent elements of $\mathscr{A}$. If $\mathscr{A} = (A, \leqslant)$ is an ordered set and $\mathscr{C}_{\Sigma}(\mathscr{A}) = \mathrm{Conv}\,\mathscr{A}$, then $T_{\Sigma}(\mathscr{A})$ is the set of all one-element intervals of A.

Theorem 2. *Let $\mathscr{A} = (A, F, R)$ be a Σ-semiseparable algebraic structure.*

(1) *If $T_{\Sigma}(\mathscr{A}) \neq \emptyset$ then $\mathscr{T} = \big\{\{x\};\, x \in T_{\Sigma}(\mathscr{A})\big\}$ is the set of all atoms of the lattice $\mathscr{C}_{\Sigma}(\mathscr{A})$.*

(2) *The lattice $\mathscr{C}_{\Sigma}(\mathscr{A})$ is atomic if and only if $\mathscr{A}$ is Σ-semiidempotent.*

Proof. (1) Denote by $\mathrm{At}(\mathscr{A})$ the set of all atoms of $\mathscr{C}_{\Sigma}(\mathscr{A})$. Clearly, $\mathscr{T} \subseteq \mathrm{At}(\mathscr{A})$. Suppose $P \in \mathrm{At}(\mathscr{A})$ and $P \notin \mathscr{T}$. Then there exist x_1, $x_2 \in P$, $x_1 \neq x_2$ such that $C_{\mathscr{A}}(x_1) \subseteq P$, $C_{\mathscr{A}}(x_2) \subseteq P$ and $C_{\mathscr{A}}(x_1) \neq C_{\mathscr{A}}(x_2)$, because $\mathscr{A}$ is Σ-semiseparable. This contradicts the assumption that P is an atom.

(2) If $\mathscr{C}_{\Sigma}(\mathscr{A})$ is atomic and $\emptyset \neq X \in \mathscr{C}_{\Sigma}(\mathscr{A})$ then there exists $P \in \mathrm{At}(\mathscr{A})$ such that $P \subseteq X$. According to (1) we have $P = \{x\}$ for some $x \in T_{\Sigma}(\mathscr{A})$, thus $x \in X$. Conversely, let $\mathscr{A}$ be Σ-semiidempotent. If $X \notin \mathrm{At}(\mathscr{A})$, then $\emptyset \neq P = \{x\} \subseteq X$. Hence, $\mathscr{C}_{\Sigma}(\mathscr{A})$ is an atomic lattice. $\square$

Remark 4. If $\mathscr{A}$ is not Σ-semiseparable then part (1) of Theorem 2 does not hold. E.g. if $\mathscr{A}$ is a q-lattice in Fig. 1, then $C\,\mathrm{Sub}\,\mathscr{A}$ (see Fig. 2) has two atoms but $T_{\Sigma}(\mathscr{A}) = \{1\}$, i.e. $\mathscr{T} = \big\{\{1\}\big\}$ is a one-element set.

Similarly, if $\mathscr{A}$ is not Σ-semiseparable then part (2) or Theorem 2 does not hold. E.g. if $\mathscr{A}$ is a q-lattice in Fig. 3 then $C\,\mathrm{Sub}\,\mathscr{A}$ in Fig. 4 is atomic but not Σ-semiidempotent.

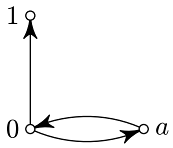

Fig. 1

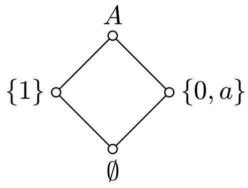

Fig. 2

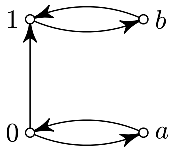

Fig. 3

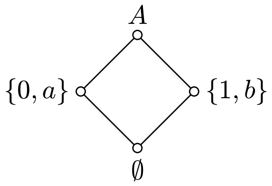

Fig. 4

Remark 5. If $\mathscr{A}$, $\mathscr{B}$ are Σ-semiseparable and Σ-semiidempotent algebraic structures which are Σ-isomorphic, then $\mathscr{C}_\Sigma(\mathscr{A})$, $\mathscr{C}_\Sigma(\mathscr{B})$ are atomic lattices according to Theorem 2 and $\mathrm{At}(\mathscr{A}) = \big\{\{x\};\, x \in T_\Sigma(\mathscr{A})\big\}$, $\mathrm{At}(\mathscr{B}) = \big\{\{y\};\, y \in T_\Sigma(\mathscr{B})\big\}$ are the sets of all atoms in $\mathscr{C}_\Sigma(\mathscr{A})$, $\mathscr{C}_\Sigma(\mathscr{B})$, respectively. Since every isomorphism of atomic lattices maps atoms onto atoms, there exists a bijection $t_h\colon T_\Sigma(\mathscr{A}) \to T_\Sigma(\mathscr{B})$ defined by the rule $\{t_h(x)\} = h(\{x\})$ for each $x \in T_\Sigma(\mathscr{A})$, where h is the isomorphism of $\mathscr{C}_\Sigma(\mathscr{A})$ onto $\mathscr{C}_\Sigma(\mathscr{B})$.

Clearly, if $\mathscr{A}$ is a Σ-separable algebraic structure then it is also Σ-semiidempotent, but not vice versa. Furthermore, if $\mathscr{A}$ is Σ-semiidempotent then $\mathscr{A}$ need not be Σ-semiseparable. For instance, let $\mathscr{A} = (Z, +, .)$ be the ring of all integers and $\mathscr{C}_\Sigma(\mathscr{A}) = \mathrm{Id}\,\mathscr{A}$. Then $\mathscr{A}$ is Σ-semiidempotent (any ideal of $\mathscr{A}$ contains zero, the only Σ-idempotent), but $\mathscr{A}$ is not Σ-semiseparable (e.g. $C_\mathscr{A}(2) = C_\mathscr{A}(-2)$), thus it is not Σ-separable.

Let $\mathscr{A} = (A, \vee, \wedge)$ be a lattice without the least element and $\mathscr{C}_\Sigma(\mathscr{A}) = \mathrm{Id}\,\mathscr{A}$. Then $\mathscr{A}$ is Σ-semiseparable but it is not Σ-semiidempotent.

Let $\mathscr{A} = (A, \vee, \wedge, Q)$ be a q-lattice which is not a lattice and $\mathscr{C}_\Sigma(\mathscr{A}) = C\,\mathrm{Sub}\,\mathscr{A}$. Then $\mathscr{A}$ is neither Σ-semiseparable nor Σ-semiidempotent.

Theorem 3. *Let $\mathscr{A} = (A, R)$ be a binary relational structure, $R = \{\varrho_j\,;\, j \in J\}$, $\Lambda = J$ and for each $j \in J$ let the formula G_j be of the form $(x_1\ \varrho_j\ z$ and $z\ \varrho_j\ x_2)$. The following conditions are equivalent:*

(i) *$\mathscr{A}$ is Σ-separable,*

(ii) *$\mathscr{A}$ is Σ-semiseparable,*

(iii) *$\mathscr{A}$ is antisymmetrical.*

P r o o f. The implication (i) $\Rightarrow$ (ii) is evident. Let $\mathscr{A}$ be Σ-semiseparable, $a\,\varrho_j\,b$ and $b\,\varrho_j\,a$ for some a, $b \in A$, $\varrho_j \in R$. Then $b \in C_{\mathscr{A}}(a)$, i.e. $C_{\mathscr{A}}(b) \subseteq C_{\mathscr{A}}(a)$ and $a \in C_{\mathscr{A}}(b)$, i.e. $C_{\mathscr{A}}(a) \subseteq C_{\mathscr{A}}(b)$, thus $C_{\mathscr{A}}(a) = C_{\mathscr{A}}(b)$ and so $a = b$. Hence we have (ii) $\Rightarrow$ (iii).

Prove (iii) $\Rightarrow$ (i): Let a be an arbitrary element of A and suppose $b \in C_{\mathscr{A}}(a)$. Then $a\,\varrho_j\,b$ and $b\,\varrho_j\,a$ for each $\varrho_j \in R$ and so $a = b$ because ϱ_j is antisymmetrical. Hence $C_{\mathscr{A}}(a) = \{a\}$ and $\mathscr{A}$ is Σ-separable. $\square$

Corollary 6. *Let $\mathscr{A} = (A, R)$, $\mathscr{B} = (B, P)$ be antisymmetrical binary relational systems of the same type. Then $\mathscr{A}$, $\mathscr{B}$ are convex isomorphic if and only if there exists a bijection $g\colon A \to B$ such that*

$$(*)\qquad g\big(C_{\mathscr{A}}(a,b)\big) = C_{\mathscr{B}}\big(g(a), g(b)\big) \quad \textit{for each } a, b \in A,$$

A binary algebraic structure $\mathscr{A} = (A, F, R)$ is called *antisymmetrical* if (A, R) is antisymmetrical, and it is called *idempotent* if (A, F) is an idempotent algebra.

Corollary 7. *Let $\mathscr{A} = (A, F, R)$, $\mathscr{B} = (B, G, P)$ be antisymmetrical idempotent algebraic structures of the same type. Then $\mathscr{A}$, $\mathscr{B}$ are convex isomorphic if and only if there exists a bijection $g\colon A \to B$ which satisfies the condition $(*)$.*

Definition 6. Let $\mathscr{A} = (A, F, R)$ be an algebraic structure of type τ, let Σ be a set of open formulas of the language $L(\tau)$. By a *graph* $\mathrm{Gr}_{\Sigma}(\mathscr{A})$ of $\mathscr{A}$ we mean a pair $\big(\mathscr{C}_{\Sigma}(\mathscr{A}), H\big)$, where the elements of $\mathscr{C}_{\Sigma}(\mathscr{A})$ form the *vertex set* and $\langle X, Y\rangle \in H$ for X, $Y \in \mathscr{C}_{\Sigma}(\mathscr{A})$ if and only if $X \cap Y \neq \emptyset$.

Theorem 4. *Let $\mathscr{A}, \mathscr{B}$ be algebraic structures of type τ and let Σ be a set of open formulas of the language $L(\tau)$. Then* (1) *implies* (2). *If, moreover, $\mathscr{A}$, $\mathscr{B}$ are Σ-separable then the conditions* (1), (2) *are equivalent, where:*

(1) $\mathscr{C}_{\Sigma}(\mathscr{A}) \simeq \mathscr{C}_{\Sigma}(\mathscr{B})$;

(2) $\mathrm{Gr}_{\Sigma}(\mathscr{A}) \simeq \mathrm{Gr}_{\Sigma}(\mathscr{B})$.

P r o o f. Let h be an isomorphism of $\mathscr{C}_{\Sigma}(\mathscr{A})$ onto $\mathscr{C}_{\Sigma}(\mathscr{B})$ and let X, $Y \in \mathscr{C}_{\Sigma}(\mathscr{A})$ be such that $X \cap Y \neq \emptyset$. Since $X \cap Y \in \mathscr{C}_{\Sigma}(\mathscr{A})$, we have $h(X \cap Y) \in \mathscr{C}_{\Sigma}(\mathscr{B})$ and, clearly, $h(X \cap Y) \neq \emptyset$. As h is an isomorphism, we have $h(X \cap Y) = h(X) \cap h(Y) \neq \emptyset$. On the other hand, if $h(X) \cap h(Y) \neq \emptyset$, then $h(X \cap Y) \neq \emptyset$, hence $X \cap Y \neq \emptyset$. Thus $X \cap Y \neq \emptyset$ if and only if $h(X) \cap h(Y) \neq \emptyset$ and so h is the isomorphism of graphs $\mathrm{Gr}_{\Sigma}(\mathscr{A})$ and $\mathrm{Gr}_{\Sigma}(\mathscr{B})$.

Now, let $\mathscr{A}, \mathscr{B}$ be Σ-separable and let g be an isomorphism of the graphs $\mathrm{Gr}_{\Sigma}(\mathscr{A})$, $\mathrm{Gr}_{\Sigma}(\mathscr{B})$. We will show that g is the isomorphism of the lattices $\mathscr{C}_{\Sigma}(\mathscr{A})$ and $\mathscr{C}_{\Sigma}(\mathscr{B})$

as well. Suppose X, $Y \in \mathscr{C}_\Sigma(\mathscr{A})$, $X \subseteq Y$ and $a \in g(X)$. Since $\mathscr{A}$ is Σ-separable, we have $\{a\} \in \mathscr{C}_\Sigma(\mathscr{A})$. Furthermore, $\{a\} \cap g(X) \neq \emptyset$, hence $g^{-1}(\{a\}) \cap X \neq \emptyset$. As $X \subseteq Y$, we get $g^{-1}(\{a\}) \cap Y \neq \emptyset$ and so $\{a\} \cap g(Y) \neq \emptyset$. Thus $a \in g(Y)$ and, consequently, $g(X) \subseteq g(Y)$. Similarly we can prove that the inclusion $g(X) \subseteq g(Y)$ implies $X \subseteq Y$. □

Let $\mathscr{A} = (A, F, R)$ be an algebraic structure and θ an equivalence on A. We call θ a *congruence of* $\mathscr{A}$ if it is a congruence of the algebra (A, F).

Definition 7. Let $\mathscr{A} = (A, F, R)$ be an algebraic structure of type τ, let Σ be a set of open formulas of the language $L(\tau)$ and $\theta \in \operatorname{Con} \mathscr{A}$. If $X \in \mathscr{C}_\Sigma(\mathscr{A})$, $a \in X$, $b \in [a]_\theta$ imply $b \in X$ for each a, $b \in A$ and every X of $\mathscr{C}_\Sigma(\mathscr{A})$, then $\mathscr{A}$ is called Σ-*coherent* with respect to θ.

Theorem 5. *Let $\mathscr{A} = (A, F, R)$ be an algebraic structure of type τ, let Σ be a limited set of open formulas of the language $L(\tau)$ and $\theta \in \operatorname{Con} \mathscr{A}$. Let $\mathscr{A}$ be Σ-coherent with respect to θ. Then $\mathscr{A}$ and $\mathscr{A}/\theta$ are Σ-isomorphic.*

P r o o f. Let us define a mapping $h\colon \mathscr{C}_\Sigma(\mathscr{A}) \to \operatorname{Exp}(\mathscr{A}/\theta)$ as follows: $h(\emptyset) = \emptyset$ and $h(X) = \{[a]_\theta\,;\, a \in X\}$ for $X \neq \emptyset$. Since $\mathscr{A}$ is Σ-coherent, h is clearly an injection. We will prove that h is an isomorphism of $\mathscr{C}_\Sigma(\mathscr{A})$ onto $\mathscr{C}_\Sigma(\mathscr{A}/\theta)$. Let $D \in \mathscr{C}_\Sigma(\mathscr{A})$, let $\gamma \in \Gamma$, where $G_\gamma(x_1, \ldots, x_{k_\gamma}, y_1, \ldots, y_{s_\gamma}, z, f_i)$ is the formula of Σ. Let $[a_1]_\theta, \ldots, [a_{k_\gamma}]_\theta \in h(D)$, $[b_1]_\theta, \ldots, [b_{s_\gamma}]_\theta$, $[c]_\theta \in \mathscr{A}/\theta$ and let $G_\gamma([a_1]_\theta, \ldots, [a_{k_\gamma}]_\theta, [b_1]_\theta, \ldots, [b_{s_\gamma}]_\theta, [c]_\theta.f_i)$ be satisfied in $\mathscr{A}/\theta$. Then $[c]_\theta \in C_{\mathscr{A}/\theta}([a_1]_\theta, \ldots, [a_n]_\theta) \in \mathscr{C}_\Sigma(\mathscr{A}/\theta)$ and $c \in C_{\mathscr{A}}(a_1, \ldots, a_n) \subseteq D$ where $n = \max(\{k_\gamma\,;\, \gamma \in \Gamma\} \cup \{k_\lambda\,;\, \lambda \in \Lambda\})$, because $\mathscr{A}$ is Σ-coherent. Hence $[c]_\theta \in h(D)$. Analogously it can be done for $\lambda \in \Lambda$ and the formula G_λ, i.e. $h(D)$ is Σ-closed in $\mathscr{A}/\theta$. Analogously we can prove that if $h(D)$ is Σ-closed in $\mathscr{A}/\theta$ then D is Σ-closed in $\mathscr{A}$, i.e. h is the isomorphism of $\mathscr{C}_\Sigma(\mathscr{A})$ onto $\mathscr{C}_\Sigma(\mathscr{A}/\theta)$. □

R e m a r k 6. Theorem 2 and 3 in [5] are consequences of Theorem 5 applied to q-lattices.

Let $\mathscr{A} = (A, F, R)$ be an algebraic structure f type τ, let Σ be a set of open formulas of the language $L(\tau)$. Let us define a binary relation θ_Σ on A as follows: $x\theta_\Sigma y$ if and only if $C_{\mathscr{A}}(x) = C_{\mathscr{A}}(y)$. This equivalence need not be a congruence of $\mathscr{A}$. For instance, if $\mathscr{Z} = (Z, +, ., 0)$ is the ring of integers and $\mathscr{C}_\Sigma(\mathscr{Z}) = \operatorname{Id} \mathscr{Z}$, then e.g. $2\theta_\Sigma 2$, $3\theta_\Sigma - 3$ but not $(2+3)\theta_\Sigma\bigl(2+(-3)\bigr)$. However, if $\mathscr{A} = (A, \vee, \wedge)$ is a q-lattice and $\mathscr{C}_\Sigma(\mathscr{A}) = C \operatorname{Sub} \mathscr{A}$, then $\theta_\Sigma \in \operatorname{Con} \mathscr{A}$; if this q-lattice $\mathscr{A}$ is not a lattice, then $\theta_\Sigma \neq \omega$ (the least congruence on $\mathscr{A}$). Evidently, $\theta_\Sigma = \omega$ for every Σ-semiseparable structure $\mathscr{A}$. Generally, we have

Theorem 6. *If θ_Σ is a congruence on $\mathscr{A}$, then $\mathscr{A}$ is Σ-coherent with respect to θ_Σ.*

Proof. Let $X \in \mathscr{C}_\Sigma(\mathscr{A})$, $a \in X$ and $b \in [a]_{\theta_\Sigma}$. Since $\theta_\Sigma \in \operatorname{Con}\mathscr{A}$, $b \in [a]_{\Sigma_\theta}$ implies $[b]_{\theta_\Sigma} = [a]_{\theta_\Sigma}$, i.e. $C_\mathscr{A}(a) = C_\mathscr{A}(b)$. However, $a \in X$ and $X \in \mathscr{C}_\Sigma(\mathscr{A})$ imply $C_\mathscr{A}(a) \subseteq X$, thus also $b \in X = [a]_{\theta_\Sigma}$, i.e. $\mathscr{A}$ is Σ-coherent with respect to θ_Σ. $\square$

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