ON RADIALLY EXTREMAL DIGRAPHS

Ferdinand Gliviak, Martin Knor, Bratislava

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Summary. We define digraphs minimal, critical, and maximal by three types of radii. Some of these classes are completely characterized, while for the others it is shown that they are large in terms of induced subgraphs.

Keywords: Radius of a digraph, a digraph minimal (critical, maximal) by radius, induced subgraph

MSC 1991: 05C12, 05C35

1. Introduction

All digraphs considered in this paper are finite, without loops or multiple arcs.

Let D be a digraph. Then $V(D)$ denotes the node set of D and $E(D)$ the arc set of D. If $u, v \in V(D)$ then $d_D(u,v)$ denotes the length of a shortest path from u to v in D. If there is no path from u to v, we set $d_D(u,v) = \infty$. We note that $\infty > k$ for all natural k.

In what follows we recall definitions of probably the most usual radii in digraphs. Let D be a digraph and $u \in V(D)$. Then:

out-eccentricity of the node u is $\quad e_D^+(u) = \max\limits_{v \in V(D)} (d_D(u,v))$;

in-eccentricity of the node u is $\quad e_D^-(u) = \max\limits_{v \in V(D)} (d_D(v,u))$;

eccentricity of the node u is $\quad e_D(u) = \max\,(e_D^+(u), e_D^-(u))$.

The *out-radius* $r^+(D)$ (*in-radius* $r^-(D)$, *radius* $r(D)$) is the minimum value of $e_D^+(u)$ ($e_D^-(u)$, $e_D(u)$), $u \in V(D)$. We note that an upper bound for the number of arcs in digraphs with a prescribed number of nodes and a finite out-radius is given in [4].

Now we recall the definitions of minimality, criticity, and maximality:

Definition 1.1. Let D be a digraph (graph), and let f be an invariant of D. Then D is called:

minimal by f, if $f(D-e) \neq f(D)$ for every arc e of D;

critical by f, if $f(D-u) \neq f(D)$ for every node u of D;

maximal by f, if $f(D+e) \neq f(D)$ for every arc e of the complement of D.

For the sake of brevity, instead of writing out-radius, in-radius, and radius we use the symbols r^+, r^-, and r, respectively. We remark that the exact value of the out-radius, in-radius, and radius in the digraph D is always denoted by $r^+(D)$, $r^-(D)$, and $r(D)$, respectively.

The graphs minimal, critical, and maximal by connectivity, edge-connectivity, being block, arboricity, and chromatic number can be found in books [1] and [3]. A survey on graphs minimal, critical, and maximal by diameter and radius can be found in [2]. Here we recall the results on graphs minimal, critical, and maximal by radius.

Only for this paragraph, let r denote the usual radius in graphs. The classes of graphs minimal by r, critical by r, and maximal by r have been studied already. Trees are the only graphs minimal by r, see [5]. The graphs maximal by r with radius 2 are characterized in [7]. In [5] and [6] it is shown that each graph may be an induced subgraph of a graph that is critical by r, maximal by r, and has a prescribed radius t, $3 \leqslant t < \infty$. Here the research is continued for digraphs. We examine the digraphs minimal, critical, and maximal by out-radius, in-radius, and radius.

Deleting an arc e from a digraph D we cannot decrease distances between any two nodes. Hence, $r^+(D-e) \geqslant r^+(D)$, $r^-(D-e) \geqslant r^-(D)$, and $r(D-e) \geqslant r(D)$. Analogously, adding a new arc e to D we have $r^+(D+e) \leqslant r^+(D)$, $r^-(D+e) \leqslant r^-(D)$, and $r(D+e) \leqslant r(D)$. Thus, in the definition of digraphs minimal by r^+ (r^-, r), the symbol $\neq$ can be replaced by $>$, and in the definition of digraphs maximal by r^+ (r^-, r), the symbol $\neq$ can be replaced by $<$.

The following assertion enables us to restrict our considerations to radii r^+ and r only.

Proposition 1.2. *Let D be a digraph and let D' arise from D by reversing the orientation of all arcs. Then $r^+(D) = r^-(D')$.*

Proof. We have $e^+_D(u) = \max\limits_{v \in V(D)} (d_D(u,v)) = \max\limits_{v \in V(D')} (d_{D'}(v,u)) = e^-_{D'}(u)$. Thus, $r^+(D) = \min\limits_{u \in V(D)} e^+_D(u) = \min\limits_{u \in V(D')} e^-_{D'}(u) = r^-(D')$. □

In this paper we characterize some classes of minimal, critical, and maximal digraphs, while for the others we show that they are large in terms of induced subgraphs. Further results on minimal and critical digraphs will be presented in [8].

Since a single node is the only digraph with out-radius (radius) zero and it is minimal, critical, and maximal by r^+ and r, too, we restrict our considerations to digraphs with out-radius (radius) greater than zero.

The outline of this paper is as follows. In Section 2 we show that almost no digraph can be a subgraph of a digraph minimal by r. In Section 3 we show that each digraph may be an induced subgraph of a digraph that is critical by r with radius two. Moreover, in Sections 3 and 4 we describe all digraphs critical by r, critical by r^+, maximal by r, and maximal by r^+, for radii 1, 2, and ∞, except the digraphs critical by r with radius 2 on odd number of nodes. Finally, in Section 5 we show that each digraph may be an induced subgraph of a digraph that is critical by r^+, critical by r, and maximal by r^+ (maximal by r), and has a prescribed value t of out-radius or radius, $3 \leqslant t < \infty$.

Let D be a digraph. Then $\overline{D}$ denotes the complement of D. By $\mathrm{id}_D(u)$ we denote the input degree and by $\mathrm{od}_D(u)$ the output degree of a node $u \in V(D)$. If $u \in V(D)$, we denote

$$N_i^+(u) = \{v \in V(D)\colon d_D(u,v) = i\} \quad \text{for} \quad i = 0,1,2,\dots;$$
$$N_i^-(u) = \{v \in V(D)\colon d_D(v,u) = i\} \quad \text{for} \quad i = 0,1,2,\dots\,.$$

Definitions and notation not included here can be found in Buckley-Harary [2] or in any other elementary book on Graph Theory.

2. Minimal digraphs

This section is devoted to minimal digraphs. Note that there are no digraphs minimal by r^+ (r) with out-radius (radius) infinity, by Definition 1.1.

In [8] Kyš gives the following characterization of digraphs minimal by r^+:

Theorem 2.1. *A digraph D is minimal by r^+ if and only if D is a directed rooted out-tree (i.e. acyclic digraph with $\mathrm{id}_D(x) = 1$ for all $x \in V(D)$ except the root u for which $\mathrm{id}_D(u) = 0$).*

Hence, the digraphs minimal by r^+ are very simple. In what follows we consider only the digraphs minimal by r.

Proposition 2.2. *A digraph minimal by r with radius one consists of a collection of oriented two-cycles that share a node.*

Proof. Let D be a digraph minimal by r and $r(D) = 1$. Then there is a node $u \in V(D)$ such that $e_D(u) = 1$. Thus, $ux, xu \in E(D)$ for all $x \in V(D)$, $x \neq u$. Since D is minimal by r, D contains no more arcs. □

Let D be a digraph minimal by r. As mentioned above we have $r(D) < \infty$, and hence, D is strongly connected. On the other hand, if D is minimal by strong connectivity (i.e. D is strongly connected, but $D - e$ is not strongly connected for every $e \in E(D)$), D is minimal by r.

Let D consist of a collection of oriented cycles, having some common nodes, that have a tree structure. (See the digraphs in Proposition 2.2.) Then for each $u, v \in V(D)$ there is a unique u-v path in D, and hence, D is minimal by strong connectivity. However, there are other digraphs minimal by strong connectivity (see Fig. 2.1). Moreover, there are digraphs minimal by r that are not minimal by strong connectivity (see Fig. 2.2 and Fig. 2.3).

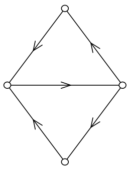

Fig. 2.1

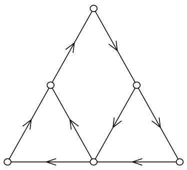

Fig. 2.2

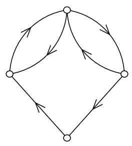

Fig. 2.3

Up to now we have not been able to characterize the digraphs minimal by r with radius greater than 1. However, we have the following proposition:

Proposition 2.3. *A digraph minimal by r does not contain the complete symmetric digraph on three nodes as a subgraph.*

Proof. Let D be a digraph minimal by r. As mentioned above, D is strongly connected. Let $u \in V(D)$ such that $e_D(u) = r(D)$. Then there are oriented paths from u to each node of D, and also paths from each node of D to u. Thus, there is a directed out-tree T^+, in-tree T^-, rooted at u, which is a spanning tree of D. We can assume that $r^+(T^+) = e^+(u)$ and $r^-(T^-) = e^-(u)$.

Suppose that there is an arc e in D such that $e \notin E(T^+)$ and $e \notin E(T^-)$. Then $e^+_{(D-e)}(u) = e^+_D(u)$ and $e^-_{(D-e)}(u) = e^-_D(u)$, which contradicts $r(D - e) > r(D)$. Hence, each arc of D belongs to either T^+ or T^-.

Let x, y, and z be three distinct nodes of D. Since each forest on three nodes contains at most two arcs, there are at most four arcs between the nodes x, y, and z in D. $\square$

Since almost all digraphs contain the complete symmetric digraph on three nodes as a subgraph (see e.g. [9]), we have the following corollary of Proposition 2.3:

Corollary 2.4. *Almost no digraph can be a subgraph of a digraph minimal by r.*

This section is devoted to digraphs critical by out-radius (radius).

Let G be an unoriented graph critical by radius. Let the digraph D arise from G by replacing the edges of G by pairs of opposite arcs. Then, clearly, D is critical by r, and also by r^+ and by r^-. However, there are digraphs critical by r^+ (r) which do not correspond to unoriented graphs.

In the first section of this paper we have shown that $r^+(D-e) \geqslant r^+(D)$ and $r(D-e) \geqslant r(D)$ for each $e \in E(D)$, and also that $r^+(D+e) \leqslant r^+(D)$ and $r(D+e) \leqslant r(D)$ for each $e \in E(\overline{D})$. Now we give analogous conditions for $r^+(D-u)$ and $r(D-u)$, where $u \in V(D)$.

Proposition 3.1. *Let D be a digraph and $u \in V(D)$. Then*
$r(D-u) \geqslant r(D) - 1$ *if* $r(D) < \infty$;
$r^+(D-u) \geqslant r^+(D) - 1$ *if* $r^+(D) < \infty$ *and* $\mathrm{id}_D(u) \geqslant 1$.

Proof. Let $r(D) < \infty$. Suppose that there is $u \in V(D)$ such that $r(D-u) \leqslant r(D)-2$. Then there is $z \in V(D-u)$ such that $e^+_{(D-u)}(z) \leqslant r(D)-2$ and $e^-_{(D-u)}(z) \leqslant r(D)-2$. Since $r(D) < \infty$, D is strongly connected. Thus, $\mathrm{id}_D(u) \geqslant 1$ and $\mathrm{od}_D(u) \geqslant 1$. Hence, $e^+_D(z) \leqslant r(D) - 1$ and $e^-_D(z) \leqslant r(D) - 1$. Thus, $r(D) \leqslant r(D) - 1$, a contradiction.

By an analogous argument the second part of the lemma can be proved using $\mathrm{id}_D(u) \geqslant 1$. □

The digraphs critical by r^+ (r) with out-radius (radius) infinity are characterized in Proposition 3.3 (Proposition 3.6). Hence, the following assertion characterizes the remaining digraphs that do not satisfy Proposition 3.1.

Proposition 3.2. *Let D be a digraph with $r^+(D) < \infty$, and let $v_0 \in V(D)$ be such that $\mathrm{id}_D(v_0) = 0$. Then D is critical by r^+ if and only if $V(D) = \{v_0, v_1, \ldots, v_{r^+(D)}\}$ and $N^+_i(v_0) = \{v_i\}$, $1 \leqslant i \leqslant r^+(D)$.*

Proof. Clearly, D is critical by r^+ if D satisfies the conditions in the assertion.

Now suppose that $r^+(D) < \infty$, $\mathrm{id}_D(v_0) = 0$, and D is critical by r^+. Since $d_D(x, v_0) = \infty$ for every $x \in V(D)$, $x \neq v_0$, we have $e^+_D(v_0) = r^+(D) < \infty$. Hence, there are nodes $v_1, v_2, \ldots, v_{r^+(D)}$ such that $v_i \in N^+_i$ and $v_{i-1}v_i \in E(D)$, $1 \leqslant i \leqslant r^+(D)$.

Suppose that $|V(D)| > r^+(D) + 1$. Let $j = \max\{i : |N^+_i(v_0)| \geqslant 2\}$, and let $z \in N^+_j(v_0)$, $z \neq v_j$. Since $|N^+_{j+1}(v_0)| \leqslant 1$, we have $e^+_{D-z}(v_0) = e^+_D(v_0)$. Hence, $N^+_i(v_0) = \{v_i\}$, $1 \leqslant i \leqslant r^+(D)$. □

The following class of digraphs critical by r^+ demonstrates Proposition 3.1 for the out-radius. Let D_n consist of two oriented cycles of length n that are joined by a pair of opposite arcs. Then D_n is critical by r^+ and $\{r^+(D_n - v) : v \in V(D_n)\} = \{n-1, n+1, n+2, \dots, 2n-2, \infty\}$.

From now on we consider only the digraphs with radii ∞, 1, and 2. We remark that the unique graph critical by radius with radius ∞ consists of two isolated nodes. Further, the unique graph critical by radius with radius 1 consists of two nodes joined by an edge. Finally, a graph G with radius 2 is critical by radius if and only if either G is a path on four nodes or G is a complete multipartite graph $K_{2,2,\dots,2}$ (except K_2), see [5].

First we characterize the digraphs critical by r^+.

Proposition 3.3. *Let D be a digraph critical by r^+. Then*
D consists of two isolated nodes, if $r^+(D) = \infty$;
D consists of two nodes joined by one or two arcs, if $r^+(D) = 1$.

Proof. Suppose that $r^+(D) = \infty$. Then obviously, D contains at least two nodes. Let $u \in V(D)$. Since D is critical by r^+, there is a node, say u', such that $e^+_{D-u}(u') < \infty$. Thus, for each $x \in V(D) - \{u, u'\}$ we have $d_D(u', x) < \infty$. Since $r^+(D) = \infty$, we have $\mathrm{id}_D(u) = 0$. Analogously, $\mathrm{id}_D(x) = 0$ for each $x \in V(D)$, and hence D is a discrete digraph. Since D is critical by r^+, it contains just two nodes.

Suppose that $r^+(D) = 1$. Then there is $u \in V(D)$ such that $ux \in E(D)$ for every $x \in V(D)$, $x \neq u$. Since D is critical by r^+, we have $|N^+_1(u)| = 1$. □

Now we describe the digraphs critical by r^+ with out-radius 2.

Theorem 3.4. *Let D be a digraph such that $r^+(D) = 2$ and $|V(D)| \geqslant 5$. Then D is critical by r^+ if and only if the complement of D consists of a collection of oriented cycles.*

Proof. Just for this proof we write that a node a is *friend-but*(b) if $ax \in E(D)$ for every $x \in V(D)$, $x \neq b$, and $ab \notin E(D)$.

Suppose that D is critical by r^+, $r^+(D) = 2$, and $|V(D)| \geqslant 5$. First we prove that $r^+(D - x) = 1$ for every $x \in V(D)$.

Let c be a node such that $e^+_D(c) = 2$ and $z \in N^+_2(c)$. Since D is critical by r^+ and $e^+_{(D-z)}(c) \leqslant 2$, we have $r^+(D - z) = 1$. Thus, there is a node u that is friend-but(z). Since $d_D(c, z) < \infty$, there is a node v such that $vz \in E(D)$. Then $v \in N^+_1(u)$ and $z \in N^+_2(u)$. Clearly, $e^+_{(D-x)}(u) \leqslant 2$ for every $x \in N^+_1(u)$ such that $x \neq v$. Thus, $r^+(D - x) = 1$ for every $x \in V(D)$ such that $x \neq u$ and $x \neq v$.

Suppose that $r^+(D - v) > 2$. Since $|V(D)| \geqslant 5$, we have $|N^+_1(u)| \geqslant 3$. Thus, there are two distinct nodes in $N^+_1(u)$, say w_1 and w_2, such that $w_1 \neq v$ and $w_2 \neq v$.

As shown above, we have $r^+(D-w_i)=1$, $1\leqslant i\leqslant 2$. Thus, there is a node w_i' that is friend-but(w_i), $1\leqslant i\leqslant 2$. Since $w_1\neq w_2$, we have $w_1'\neq w_2'$. Since $uw_i\in E(D)$, we have $w_i'\neq u$, $1\leqslant i\leqslant 2$. Since $e^+_{(D-v)}(z)\leqslant 2$ if $zu\in E(D)$, we have $w_i'\neq z$, $1\leqslant i\leqslant 2$. Thus, $w_i'\in N_1^+(u)$, $1\leqslant i\leqslant 2$. Since w_i' is friend-but(w_i), we have $w_i'z\in E(D)$, $1\leqslant i\leqslant 2$. Since $w_1'\neq w_2'$, we have $e^+_{(D-v)}(u)=2$, and hence $r^+(D-v)=1$, a contradiction. Thus, $r^+(D-x)=1$ for all $x\in V(D)$, $x\neq u$.

Since $r^+(D-v)=1$, there is a node, say y, that is friend-but(v). Since $z\neq v$, we have $u\neq y$. But using an analogous argument as above, we obtain that $e^+_{(D-u)}(y)\leqslant 2$, and hence $r^+(D-u)=1$. Thus, $r^+(D-x)=1$ for every $x\in V(D)$.

Now we describe the structure of $\overline{D}$.

Since $r^+(D)=2$, we have $\mathrm{od}_{\overline{D}}(x)\geqslant 1$ for every $x\in V(D)$. Since $r^+(D-z)=1$ for every $z\in V(D)$, for each $x\in V(D)$ there exists $x'\in V(D)$ that is friend-but(x). Since the xs are mutually distinct (the $V(D)$), also the x's are mutually distinct. Thus, the set of x's is just $V(D)$. Hence, $\mathrm{od}_{\overline{D}}(x)=1$ for every $x\in V(D)$.

Suppose that there is $y\in V(\overline{D})$ such that $\mathrm{id}_{\overline{D}}(y)=0$. Since $r^+(D-y)=1$, there is a node y' that is friend-but(y). Thus, $y'y\in E(\overline{D})$, a contradiction. Since $\mathrm{od}_{\overline{D}}(x)=1$ for every $x\in V(D)$, we have $|E(\overline{D})|=|V(D)|$, and hence $\mathrm{id}_{\overline{D}}(x)=1$ for all $x\in V(D)$. Thus, $\mathrm{id}_{\overline{D}}(x)=\mathrm{od}_{\overline{D}}(x)=1$ for every $x\in V(D)$, and $\overline{D}$ consists of a collection of oriented cycles.

Clearly, if $\mathrm{id}_{\overline{D}}(x)=\mathrm{od}_{\overline{D}}(x)=1$ for every $x\in V(D)$ and $|V(D)|\geqslant 3$, then D is critical by r^+ and $r^+(D)=2$. ☐

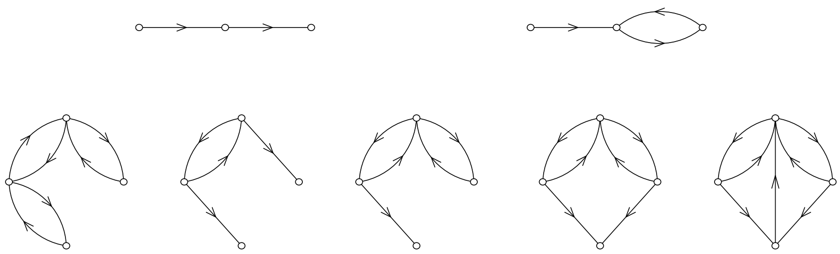

Fig. 3.1

There are just three digraphs D whose complement consists of a collection of oriented cycles and $3\leqslant|V(D)|<5$. One can verify that there are just seven digraphs critical by r^+ with out-radius 2 that do not satisfy the conditions in Theorem 3.4, namely the digraphs in Fig. 3.1. Thus, we have the following corollary:

Corollary 3.5. *Let D be a digraph critical by r^+ with out-radius two. Then either $|V(D)|\geqslant 3$ and the complement of D consists of a collection of oriented cycles, or D is one of the seven digraphs pictured in Fig.* 3.1.

Now we characterize the digraphs critical by r with radii ∞, 1 (Proposition 3.6), and the digraphs critical by r with radius 2 on an even number of nodes (Theorem 3.7).

Proposition 3.6. *Let D be a digraph critical by r. Then*
D consists of two nodes and at most one arc, if $r(D) = \infty$;
D consists of two opposite arcs, if $r(D) = 1$.

Proof. Suppose that $r(D) = \infty$. Then D is not strongly connected. Thus, D has at least two strongly connected components. Since $r(D - u) < \infty$ for every $u \in V(D)$, D has just two strongly connected components, each consisting of a single node.

Suppose that $r(D) = 1$. Then there is $u \in V(D)$ such that $ux, xu \in E(D)$ for all $x \in V(D)$, $x \neq u$. Since D is critical by r, we have $|V(D)| = 2$. □

The digraphs critical by r with radius 2 are rather complicated. However, the following theorem characterizes those of them that have an even number of nodes.

Theorem 3.7. *Let D be a digraph on an even number of nodes such that $r(D) = 2$ and $|V(D)| \geqslant 6$. Then D is critical by r if and only if the complement of D consists of a collection of independent arcs and oriented two-cycles.*

Proof. Clearly, D is critical by r if $\overline{D}$ consists of a couple of independent arcs and two-cycles.

Just for this proof we write that a node a is *friend-but*(b) if $ax, xa \in E(D)$ for every $x \in V(D)$, $x \neq b$, and ab or ba are not in $E(D)$. (We remark that the definition in the proof of Theorem 3.4 is slightly different.)

Let D be a digraph critical by r, such that $|V(D)| \geqslant 5$ and $r(D) = 2$. First we describe the structure of D if D contains two nodes, say u and u', such that u' is friend-but(u).

Since u' is friend-but(u) and D is strongly connected, we have $e_D(u') = 2$. Since $|V(D)| \geqslant 5$, we have $|V(D) - \{u, u'\}| \geqslant 3$. Thus, there is a node $v \in V(D)$ such that $d_{(D-v)}(u', u) \leqslant 2$ and $d_{(D-v)}(u, u') \leqslant 2$. Since D is critical by r, we have $r(D - v) = 1$ and there is $v' \in V(D)$ that is friend-but(v).

Clearly, u, u', and v are distinct nodes. Obviously, $v' \neq u'$ and $v' \neq v$. Since u is friend-but(v) implies that $uu', u'u \in E(D)$, we have $v' \neq u$. Hence, u, u', v and v' are distinct nodes.

Since $|V(D)| \geqslant 5$, there is one more node $z \in V(D)$ distinct from the u, u', v, v'. Since $v'u, uv' \in E(D)$, we have $e_{(D-z)}(u') = 2 > r(D-z)$. Hence, there is $z' \in V(D)$ distinct from all u, u', v, v', z that is friend-but(z).

Thus, u' is friend-but(u), v' is friend-but(v), and z' is friend-but(z). Continuing these considerations we obtain that $V(D) = \{u_i, u_i'; 1 \leqslant i \leqslant k\}$ for some $k \geqslant 3$, where u_i' is friend-but(u_i), $1 \leqslant i \leqslant k$.

However, $e_{(D-u_1')}(u_2') \leqslant 2$, since $u_3'u_2, u_2u_3' \in E(D)$. Thus, $r(D - u_1') = 1$, and there is a node x that is friend-but(u_1'). Since u_1' is friend-but(u_1), we have $x = u_1$. Analogously, u_i is friend-but(u_i'), $1 \leqslant i \leqslant k$. Thus, the complement of D consists of a collection of independent arcs and two-cycles as required.

In what follows we suppose that D contains no pair of nodes x, x' such that x' is friend-but(x). Then $r(D - x) \neq 1$ for each $x \in V(D)$.

Let u be a node such that $e_D(u) = 2$, and $y \in N_2^+(u)$. Since $2 \geqslant e_{(D-y)}(u) \geqslant r(D-y)$ if $d_D(y,u) = 2$, we have $yu \in E(D)$. Thus, for every $y \in N_2^+(u)$ we have $yu \in E(D)$.

Suppose that there is $x \in N_1^+(u)$ such that $xy \notin E(D)$ for every $y \in N_2^+(u)$, moreover, let $xu \notin E(D)$ whenever possible. Then $e^-_{(D-x)}(u) \leqslant 2$ and $e^+_{(D-x)}(u) \leqslant 2$. Hence $r(D-x) = 1$, a contradiction. Thus, for each $x \in N_1^+(u)$ there is $y \in N_2^+(u)$ such that $xy \in E(D)$.

Suppose that $|N_1^+(u)| > |N_2^+(u)|$. Since for each $x \in N_1^+(u)$ there is $y \in N_2^+(u)$ such that $xy \in E(D)$, there is $w \in N_1^+(u)$ such that $e_{(D-w)}(u) = 2$. Hence, $r(D - w) = 1$, a contradiction.

Suppose that $|N_1^+(u)| < |N_2^+(u)|$. Then there is $w \in N_2^+(u)$ such that $e_{(D-w)}(u) = 2$, a contradiction.

Thus $|N_1^+(u)| = |N_2^+(u)|$, and hence D has an odd number of nodes. □

Hence, the class of digraphs critical by r with radius 2 on an even number of nodes is pure in terms of induced subgraphs. However, for odd number of nodes we have the following theorem:

Theorem 3.8. *Let D be a digraph. Then there are infinitely many digraphs critical by r with radius two on an odd number of nodes, containing D as an induced subgraph.*

Proof. Let $V(D) = \{v_1, v_2, \dots, v_n\}$ and let $k \geqslant n+1$. Let $V(H_k) = \{u, v_1, v_2, \dots, v_k, z_1, z_2, \dots, z_k\}$ and let $E(H_k)$ consist of $E(D)$, uv_i, v_iz_i, z_iu, $1 \leqslant i \leqslant k$. In what follows we show that H_k satisfies the conditions in the theorem.

Clearly, $|V(H_k)| = 2k+1$, and H_k contains D as an induced subgraph. Since $k > n$, we have $e_H^+(v_i) = e_H^-(z_i) = 4$, $1 \leqslant i \leqslant k$, and $e_H(u) = 2$.

Since $\mathrm{id}_{(H-v_i)}(z_i) = 0$, we have $r(H-v_i) = \infty$, $1 \leqslant i \leqslant k$. Analogously, $r(H-x) = \infty$ if $x \in \{u, z_{n+1}, z_{n+2}, \dots, z_k\}$. Since $d_{(H-z_1)}(v_1, u) \geqslant 3$, $e^+_{(H-z_1)}(v_i) \geqslant 3$, and

$e^-_{(H-z_1)}(z_i) \geqslant 4$, $2 \leqslant i \leqslant k$, we have $r(H-z_1) \geqslant 3$. Analogously, $r(H-z_i) \geqslant 3$, $1 \leqslant i \leqslant n$. □

4. Maximal digraphs

In this section we characterize the digraphs maximal by r^+ (r) with out-radii (radii) ∞, 1, and 2.

We remark that a graph G with radius ∞ is maximal by radius if and only if G consists of two complete graphs. Further, a graph G with radius 1 is maximal by radius if and only if G is a complete graph on at least two nodes. Finally, a graph G with radius 2 is maximal by radius if and only if the complement of G consists of a collection of (at least two) stars (i.e. the complete bipartite graphs $K_{1,s}$, $s \geqslant 1$), see [7].

In what follows we characterize the digraphs maximal by r^+.

Proposition 4.1. *Let D be a digraph and $r^+(D) = \infty$. Then D is maximal by r^+ if and only if $V(D)$ can be partitioned into A_i, $1 \leqslant i \leqslant 3$, such that $A_1 \neq \emptyset$, $A_2 \neq \emptyset$, the digraph induced by A_i is complete, $1 \leqslant i \leqslant 3$, and for all $a_1 \in A_1$, $a_2 \in A_2$, $a_3 \in A_3$ we have $a_1a_3, a_2a_3 \in E(D)$ and there are no other arcs in D.*

Proof. Clearly, if D satisfies the conditions in Proposition 4.1, D is maximal by r^+ with out-radius ∞.

Now suppose that D is maximal by r^+ with out-radius ∞. Denote by $S_1, S_2, \dots, S_m$ the strongly connected components in D. Since $r^+(D) = \infty$, we have $m \geqslant 2$. Since D is maximal by r^+, each S_i, $1 \leqslant i \leqslant m$, is a complete symmetric digraph. Moreover, if $x, x' \in V(S_i)$, $y, y' \in V(S_j)$, and $xy \in E(D)$, we have $x'y' \in E(D)$.

Let D' be obtained from D by contracting every S_i to a single node s_i, and $s_is_j \in E(D')$ if and only if there are $x \in V(S_i)$ and $y \in V(S_j)$ such that $xy \in E(D)$, $i \neq j$.

Since $r^+(D) = \infty$, we have $r^+(D') = \infty$. Since D' contains no oriented cycle, there are at least two nodes in D', say s_1 and s_2, such that $\mathrm{id}_{D'}(s_1) = \mathrm{id}_{D'}(s_2) = 0$. Suppose that $|V(D')| \geqslant 3$. Since D is maximal by r^+, we have $|V(D')| = 3$ and $E(D') = \{s_1s_3, s_2s_3\}$. □

Proposition 4.2. *Let D be a digraph maximal by r^+. Then*
D is a complete symmetric digraph on at least two nodes, if $r^+(D) = 1$;
$|V(D)| \geqslant 3$ and $\mathrm{od}_{\overline{D}}(x) = 1$ for every node x of D, if $r^+(D) = 2$.

P r o o f. Suppose that $r^+(D) = 1$. Then D contains at least two nodes, and there is $u \in V(D)$ such that $ux \in E(D)$ for all $x \in V(D)$, $x \neq u$. Since D is maximal by r^+, D is a complete symmetric digraph.

Suppose that $r^+(D) = 2$. Clearly, $|V(D)| \geqslant 3$. Since D is maximal by r^+ and $r^+(D) = 2$, for each $x \in V(D)$ there is a unique node y of D such that $xy \in E(\overline{D})$. Thus, $\mathrm{od}_{\overline{D}}(x) = 1$ for every $x \in V(D)$. □

An acyclic digraph that arises from a complete bipartite graph $K_{1,s}$, $s \geqslant 1$, by replacing the edges by arcs (arbitrarily directed), we call an oriented star. The following assertion characterizes the digraphs maximal by r with radii ∞, 1, and 2.

Proposition 4.3. *Let D be a digraph maximal by r. Then*

D consists of two complete symmetric digraphs H_1 and H_2, and the arcs xy, $x \in V(H_1)$ and $y \in V(H_2)$, if $r(D) = \infty$;

D is a complete symmetric digraph on at least two nodes, if $r(D) = 1$;

the complement of D consists of a collection of oriented stars, if $r(D) = 2$.

P r o o f. Suppose that $r(D) = \infty$. Denote by $S_1, S_2, \ldots, S_m$ the strongly connected components in D. Since $r(D) = \infty$, we have $m \geqslant 2$. Since D is maximal by r, each S_i, $1 \leqslant i \leqslant m$, is a complete symmetric digraph. Moreover, if $x, x' \in V(S_i)$, $y, y' \in V(S_j)$, and $xy \in E(D)$, we have $x'y' \in E(D)$.

Let D' be obtained from D by contracting every S_i to a single node s_i, and $s_i s_j \in E(D')$ if and only if there are $x \in V(S_i)$ and $y \in V(S_j)$ such that $xy \in E(D)$, $i \neq j$. Clearly, D' contains a node, say s_1, such that $\mathrm{id}_{D'}(s_1) = 0$. Since D is maximal by r, D' consists of just two nodes s_1 and s_2 and the arc $s_1 s_2$.

Suppose that $r(D) = 1$. Then D contains at least two nodes, and there is $u \in V(D)$ such that $ux, xu \in E(D)$ for all $x \in V(D)$, $x \neq u$. Since D is maximal by r, D is a complete symmetric digraph.

Suppose that $r(D) = 2$. Then for each $x \in V(D)$ there is a node $y \in V(D)$ such that $xy \in E(\overline{D})$ or $yx \in E(\overline{D})$. Since D is maximal by r, $xy \in E(\overline{D})$ implies that $yx \notin E(\overline{D})$. Moreover, either $xz, zx \notin E(\overline{D})$ for all $z \in V(D) - \{x, y\}$, or $yz, zy \notin E(\overline{D})$ for all $z \in V(D) - \{x, y\}$. □

5. Existence theorems

In this section we show that the classes of digraphs critical by r^+, r^-, r, and also the classes of digraphs maximal by r^+, r^-, and r, are large in terms of induced subgraphs. Namely, we prove the following two theorems:

Theorem 5.1. *Let D be a digraph, and let t satisfy $3 \leqslant t < \infty$. Then there is an infinite number of digraphs H such that*

(1) *D is an induced subgraph of H;*
(2) *$r^+(H) = r^-(H) = r(H) = t$;*
(3) *H is critical by r^+, r^- and r;*
(4) *H is maximal by r^+ and r^-.*

Theorem 5.2. *Let D be a digraph, and let t satisfy $3 \leqslant t < \infty$. Then there is an infinite number of digraphs maximal by r with radius t, which contain D as an induced subgraph.*

In this section, K_n and C_n denote a digraph that arises from a complete graph or cycle on n nodes by replacing the edges by pairs of opposite arcs. By $\times$ we denote the Cartesian product. Moreover, we use the following conditions: Let S be a digraph and $u \in V(S)$. We say that u satisfies $(^*)$ if and only if

$$(^*) \qquad \forall x \in N_1^-(u) \;\; \exists y \in N_1^+(u), \; x \neq y, \text{ such that } xy \notin E(S),$$

and u satisfies $(^{*\prime})$ if and only if

$$(^{*\prime}) \qquad \forall x \in N_1^+(u) \;\; \exists y \in N_1^-(u), \; x \neq y, \text{ such that } yx \notin E(S).$$

For each t and m, $3 \leqslant t < \infty$ and $1 \leqslant m < \infty$, we construct digraphs $H_{t,m}$ and $F_{t,m}$ from D:

(1) Let a digraph D_1 arise from D by adding one new node u_1 for each $u \in V(D)$. (Hence, $|V(D_1)| = 2 \cdot |V(D)|$.) Moreover, if $\mathrm{id}_D(u) \geqslant 1$ we add the arc uu_1 to D_1, and if $\mathrm{od}_D(u) \geqslant 1$ we add the arc u_1u to D_1. Clearly, each node $u \in V(D)$ satisfies $(^*)$ and $(^{*\prime})$ in D_1.
(2) Let a digraph D_2 arise from D_1 by adding m isolated nodes, and let a digraph D_3 arise from D_2 by adding one new node w and the arcs xw and wx, $x \in V(D_2)$. Since $m \geqslant 1$, the node w satisfies $(^*)$ and $(^{*\prime})$ in D_3.
(3) Let D_3' be a copy of D_3. Denote by u' the node of D_3' corresponding to the node u of D_3. Let $V(D_4) = V(D_3) \cup V(D_3')$, and let the arc set of D_4 consist of $E(D_3)$, $E(D_3')$, and moreover for every $x, y \in V(D_3)$ let us have

$$(^{**}) \qquad yx',\, y'x \in E(D_4) \iff xy \notin E(D_3),$$

except the case $x = y$, where $x'x, xx' \notin E(D_4)$. It is easy to check that all nodes of D_4 satisfy $(^*)$ and $(^{*\prime})$. We note that a mapping φ such that $\varphi(u) = u'$ and $\varphi(u') = u$ for every $u \in V(D_3)$ is an automorphism of D_4.

(4) Finally, let $H_{3,m} = D_4$, $H_{4,m} = D_4 \times K_2$, and $H_{t,m} = D_4 \times C_{2(t-3)}$ if $t \geqslant 5$. Note that $2(t-3) \geqslant 4$ if $t \geqslant 5$. Clearly, all nodes of $H_{t,m}$ satisfy $(^*)$ and $(^{*\prime})$.

(5) Let a digraph D_5 arise from D_4 by adding the arcs xy', $x \in V(D_3)$ and $y' \in V(D_3')$.

(6) Finally, let $F_{3,m} = D_5$, $F_{4,m} = D_5 \times K_2$, and $F_{t,m} = D_5 \times C_{2(t-3)}$ if $t \geqslant 5$.

In what follows we prove two lemmas about $H_{t,m}$.

Lemma 5.3. *Let $u \in V(H_{t,m})$. Then there is $v \in V(H_{t,m})$ such that $d_{H_{t,m}}(u,v) = d_{H_{t,m}}(v,u) = t$. Moreover, $d_{H_{t,m}}(u,x) < t$ and $d_{H_{t,m}}(x,u) < t$ for every $x \in V(H_{t,m})$, $x \neq v$.*

Proof. First suppose that $t = 3$. We can assume that $u \in V(D_3)$.

If $x \in V(D_3)$, we have $d_{D_4}(u,x) \leqslant 2$, since uw, $wx \in E(D_3)$.

Let $x = z' \in V(D_3')$, $z' \neq u'$. Suppose that $uz' \notin E(D_4)$. Then $zu \in E(D_4)$ by $(^{**})$. Thus, there is $y \in V(D_4)$ such that $uy \in E(D_4)$ and $zy \notin E(D_4)$, by $(^*)$. Thus $yz' \in E(D_4)$ by $(^{**})$, and hence $d_{D_4}(u,z') \leqslant 2$.

Analogously, using $(^{*\prime})$ instead of $(^*)$ we obtain that $d_{D_4}(x,u) \leqslant 2$, if $x \neq u'$.

Clearly, $uu' \notin E(D_4)$. If $uy \in E(D_4)$, we have $yu' \notin E(D_4)$ by $(^{**})$. Thus, $d_{D_4}(u,u') = 3$. Analogously, $d_{D_4}(u',u) = 3$.

Now suppose that $t \geqslant 4$. Since $r^+(K_2) = r^-(K_2) = r(K_2) = 1$ and $r^+(C_{2l}) = r^-(C_{2l}) = r(C_{2l}) = l$, we have $r^+(H_{t,m}) = r^-(H_{t,m}) = r(H_{t,m}) = t$. Note that both K_2 and C_{2l} satisfy the lemma (each node has a unique node at the greatest distance). Thus, also the Cartesian products $D_4 \times K_2$ and $D_4 \times C_{2l}$ satisfy the lemma. □

Lemma 5.4. *Let $u \in V(H_{t,m})$, and let $v \in V(H_{t,m})$ be the unique node such that $d_{H_{t,m}}(u,v) = d_{H_{t,m}}(v,u) = t$. Then $d_{H_{t,m}}(u,x) + d_{H_{t,m}}(x,v) = d_{H_{t,m}}(v,x) + d_{H_{t,m}}(x,v) = t$ for every $x \in V(H_{t,m})$.*

Proof. First suppose that $t = 3$. We can assume that $u \in V(D_3)$. Then $v = u' \in V(D_3')$. If $ux \notin E(D_4)$ and $x \neq u'$, we have $xu' \in E(D_4)$ by $(^{**})$. Suppose that $ux \in E(D_4)$. If $d_{D_4}(x,u') \geqslant 3$, we have $x = u$ by Lemma 5.3. Thus, $d_{D_4}(u,x) + d_{D_4}(x,u') = 3$ for every $x \in V(D_4)$. Analogously, $d_{D_4}(u',x) + d_{D_4}(x,u) = 3$ for every $x \in V(D_4)$.

Now suppose that $t \geqslant 4$. Clearly, the Cartesian product of two digraphs, satisfying Lemma 5.4, satisfies the lemma, too. Since both K_2 and C_{2l} satisfy Lemma 5.4, $H_{t,m}$ satisfies the lemma as well. □

Now Theorem 5.1 can be proved.

Proof of Theorem 5.1. We show that $H_{t,m}$ satisfies the conditions in Theorem 5.1.

(1) Constructing the digraph $H_{t,m}$ we have never added an arc between two nodes of D. Hence, D is an induced subgraph of $H_{t,m}$.
(2) By Lemma 5.3, $r^+(H_{t,m}) = r^-(H_{t,m}) = r(H_{t,m}) = t$.
(3) Let $u \in H_{t,m}$. By Lemma 5.3 (the first part), there is a unique node $v \in V(H_{t,m})$ such that $d_{H_{t,m}}(u,v) = d_{H_{t,m}}(v,u) = t$. Hence, we have $e^+_{(H_{t,m}-u)}(v) = e^-_{(H_{t,m}-u)}(v) = t-1$ by Lemma 5.3 (the second part). Thus, $H_{t,m}$ is critical by r^+, r^-, and r.
(4) Let $uz \notin E(H_{t,m})$. By Lemma 5.3 there is a unique node $v \in V(H_{t,m})$ such that $d_{H_{t,m}}(u,v) = t$. Since $d_{H_{t,m}}(u,z) + d_{H_{t,m}}(z,v) = t$ by Lemma 5.4, we have $e^+_{(H_{t,m}+uz)}(u) \leqslant t-1$. Hence, $H_{t,m}$ is maximal by r^+. Analogously, $e^-_{(H_{t,m}+uz)}(z) \leqslant t-1$ if $uz \notin E(H_{t,m})$. Hence, $H_{t,m}$ is maximal by r^-.

Since $|V(H_{t,m_1})| \neq |V(H_{t,m_2})|$ if $m_1 \neq m_2$, the theorem is proved. $\square$

The digraph $F_{t,m}$ is not maximal by r in general, however, we have the following lemma:

Lemma 5.5. *The digraph D_5 is maximal by r with radius 3.*

Proof. Let $u \in V(D_3)$. Then $d_{D_4}(u',u) = 3$ by Lemma 5.3. Suppose that $d_{D_5}(u',u) \leqslant 2$. Then there is an arc $xy' \in E(D_5) - E(D_4)$ such that $d_{D_5}(u',x) + d_{D_5}(y',u) = 1$, a contradiction. Hence, $r(D_5) = 3$.

Let $xy \in E(\overline{D_5})$. Then $e^+_{(D_5+xy)}(x) \leqslant 2$ and $e^-_{(D_5+xy)}(y) \leqslant 2$ by Lemma 5.4. Suppose that $e^-_{(D_5+xy)}(x) \geqslant 3$. Then $x \in V(D_3)$, and hence $y \in V(D_3)$, too. Thus, $e^+_{(D_5+xy)}(y) \leqslant 2$ by Lemma 5.4. Thus $e_{(D_5+xy)}(y) \leqslant 2$, and hence D_5 is maximal by r. $\square$

Now Theorem 5.2 can be proved.

Proof of Theorem 5.2. Suppose that $t = 3$. Clearly, $F_{3,m}$ contains D as an induced subgraph. Moreover, $F_{3,m}$ is maximal by r with radius 3 by Lemma 5.5.

Now suppose that $t \geqslant 4$. Then $F_{t,m}$ is not necessarily maximal by r. However, from each digraph H with radius t' we can construct a digraph maximal by r with radius t', simply by adding arcs that do not decrease the radius. Let $F'_{t,m}$ be a digraph maximal by r that is constructed from $F_{t,m}$ by adding arcs that do not decrease the radius.

Since $r(D_5) = 3$ by Lemma 5.5, we have $r(F_{t,m}) = t$, and hence $r(F'_{t,m}) = t$.

Suppose that there is an arc $xy \in E(F'_{t,m}) - E(F_{t,m})$ such that $x, y \in V(D_5)$. Since $r(D_5+xy) \leqslant 2$ by Lemma 5.5, we have $r(F_{t,m}+xy) \leqslant t-1$ by Lemma 5.3. Hence, D is an induced subgraph of $F'_{t,m}$.

Since $|V(F'_{t,m_1})| \neq |V(F'_{t,m_2})|$ if $m_1 \neq m_2$, the theorem is proved. $\square$

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Author's address: *Ferdinand Gliviak*, Comenius University, Faculty of Mathematics and Physics, Mlynská dolina, 842 15 Bratislava, Slovakia, e-mail: `Gliviak@fmph.uniba.sk`; *Martin Knor*, Slovak Technical University, Faculty of Civil Engineering, Department of Mathematics, Radlinského 11, 813 68 Bratislava, Slovakia, e-mail: `Knor@cvt.stuba.sk`.