ON (j,k)-SYMMETRICAL FUNCTIONS

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Summary. In the present paper the authors study some families of functions from a complex linear space X into a complex linear space Y. They introduce the notion of (j,k)-symmetrical function ($k = 2,3,\ldots$; $j = 0,1,\ldots,k-1$) which is a generalization of the notions of even, odd and k-symmetrical functions. They generalize the well know result that each function defined on a symmetrical subset U of X can be uniquely represented as the sum of an even function and an odd function.

Keywords: (j,k)-symmetrical functions, holomorphic function, integral formulas, uniqueness theorem, mean value of a function, a variant of Schwarz lemma, fixed point, spectrum of an operator.

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Introduction. In the present paper the authors study some families of functions from a complex linear space X into a complex linear space Y. They introduce the notion of (j,k)-symmetrical function ($k = 2,3,\ldots$; $j = 0,1,\ldots,k-1$) which is a generalization of the notions of even, odd and k-symmetrical functions.

It has turned out that for every function x defined on a k-symmetrical subset U of X there exists exactly one sequence $(y^0, y^1, \ldots, y^{k-1})$ of (j,k)-symmetrical functions y^j such that $x = y^0 + y^1 + \ldots + y^{k-1}$ (Theorem 1). This result is a generalization of the well known fact that each function defined on a symmetrical subset U of X can be uniquely represented as the sum of an even function and an odd function.

Next the authors give an interpretation and some properties of the components of the above partition of x (Lemma 1, Theorems 2, 5, 9, 11) and present several methods of their determination (Theorem 7, Corollaries 3, 4). The authors also show that the theory of (j,k)-symmetrical functions which they have constructed has many interesting applications, for instance, for the investigation of the sets of fixed points of mappings (Theorem 15, Corollary 7), for the estimations of the absolute value of some integrals (Theorems 12, 13) and for obtaining some results of the type of Cartan uniqueness theorem for holomorphic mappings (Corollary 6).

Definitons and basic properties

Let k be an arbitrarily fixed integer not smaller than 2. It is clear that the set E_k of all roots of k-th degree of unity has the form

$$E_k = \{\varepsilon^0, \varepsilon^1, \ldots, \varepsilon^{k-1}\} \tag{1}$$

where $\varepsilon = \exp(2\pi \mathrm{i}/k)$.

Unless stated otherwise, the letters X and Y will represent vector spaces over the field $\mathbb{C}$ of complex number. By $\mathscr{F}(U,Y)$ we shall denote the vector space of all functions from the set $U \subset X$ into Y.

A set $U \subset X$ will be called k-fold symmetric if $\varepsilon U = U$. The family of all k-fold symmetric subsets U of X will be denoted by $\mathscr{S}_k(X)$.

For every integer j and a set $U \in \mathscr{S}_k(X)$ a function $x \in \mathscr{F}(U,Y)$ will be called (j,k)-symmetrical if for each $z \in U$

$$x(\varepsilon z) = \varepsilon^j x(z) \tag{2}$$

Of course, the set

$$\mathscr{F}_k^j(U,Y) := \{x \in \mathscr{F}(U,Y);\, x \text{ is } (j,k)-\text{symmetrical}\} \tag{3}$$

is a linear subspace of $\mathscr{F}(U,Y)$ and for $m,n = 0,1,\ldots,k-1$, $m \neq n$ we have

$$\mathscr{F}_k^m(U,Y) \cap \mathscr{F}_k^n(U,Y) = \{0\}.$$

Let us observe that the sets $\mathscr{F}_2^1(U,Y)$, $\mathscr{F}_2^0(U,Y)$, $\mathscr{F}_k^1(U,Y)$ are well-known families of odd functions,of even functions and of k-symmetrical functions, respectively.

Now we define an operator $L_k \colon \mathscr{F}(U,Y) \to \mathscr{F}(U,Y)$ such that for every $x \in \mathscr{F}(U,Y)$ and $z \in U$

$$L_k x(z) = x(\varepsilon z). \tag{4}$$

It is easy to see that L_k is a linear bijection of the space $\mathscr{F}(U,Y)$ into itself.

Now let us put for $j = 1,2,\ldots$

$$L_k^0 = I,\ L_k^j = L_k \circ \ldots \circ L_k,\ L_k^{-1} = (L_k)^{-1},\ L_k^{-j} = L_k^{-1} \circ L_k^j, \tag{5}$$

where I is the identity. In this way we define an operator L_k^j, which is a bijection of the space $\mathscr{F}(U,Y)$ into itself. Since

$$L_k^j x(z) = x(\varepsilon^j z) \tag{6}$$

for every integer j and for $x \in \mathscr{F}(U,Y)$, $z \in U$, so for every integers m, n we have

$$L_k^m \circ L_k^n = L_k^{m+n}, \quad L_k^{m+nk} = L_k^m \tag{7}$$

Now, for every integer l let

$$G_k^l := k^{-1} \sum_{j=0}^{k-1} \varepsilon^{-lj} L_k^j, \tag{8}$$

Of course G_k^l is a linear operator from the space $\mathscr{F}(U,Y)$ into $\mathscr{F}(U,Y)$ and for every integers m, n we have

$$G_k^{m+nk} = G_k^m, \tag{9}$$

Lemma 1. *For every integers n, m the following relations hold*

$$L_k^m \circ G_k^n = \varepsilon^{mn} G_k^n \tag{a}$$

$$G_k^m \circ G_k^n = \begin{cases} G_k^n & \text{if } k \mid m-n \\ 0 & \text{if } k \nmid n-m, \end{cases} \tag{b}$$

$$G_k^m(\mathscr{F}(U,Y)) \subset \mathscr{F}_k^m(U,Y). \tag{c}$$

P r o o f. (a) Using (8) and (9) we compute that

$$\begin{aligned} L_k^m \circ G_k^n x &= k^{-1} \sum_{j=0}^{k-1} \varepsilon^{-nj} L_k^m \circ L_k^j x = k^{-1} \sum_{j=0}^{k-1} \varepsilon^{-nj} L_k^{m+j} x \\ &= \varepsilon^{mn} k^{-1} \sum_{j=0}^{k-1} \varepsilon^{-n(m+j)} L_k^{m+j} x \\ &= \varepsilon^{nm} k^{-1} \sum_{j=m}^{m+k-1} \varepsilon^{-nj} L_k^j x = \varepsilon^{nm} G_k^n x \end{aligned}$$

(b) From (8) and from part (a) we have

$$G_k^m \circ G_k^n = k^{-1} \sum_{j=0}^{k-1} \varepsilon^{-mj} L_k^j \circ G_k^n x = k^{-1} \sum_{j=0}^{k-1} \varepsilon^{-mj} \varepsilon^{nj} G_k^n x = G_k^n x k^{-1} \sum_{j=0}^{k-1} \varepsilon^{j(n-m)}.$$

Now it is sufficient to apply the equality

$$\sum_{l=0}^{k-1} \varepsilon^{ls} = \begin{cases} k & \text{if } s \mid k \\ 0 & \text{if } s \nmid k. \end{cases} \tag{10}$$

The inclusion (c) follows directly from (3) and (8). □

Now we prove the main property of the operators G_k^l.

Theorem 1. *If $U \in \mathscr{S}_k(X)$, then each function $x \in \mathscr{F}(U,Y)$ can be written in the form*

$$x = \sum_{l=0}^{k-1} G_k^l x \tag{11}$$

and this partition is unique in the following sense: if $x = \sum_{l=0}^{k-1} y^l$, where $y^l \in \mathscr{F}_k^l(U,Y)$ for $l = 0, 1, \dots, k-1$, then $y^l = G_k^l x$.

P r o o f. Let $x \in \mathscr{F}(U,Y)$ and $z \in U$. Then, in view of (8), we have

$$\sum_{l=0}^{k-1} G_k^l x(z) = k^{-1} \sum_{l=0}^{k-1} \sum_{j=0}^{k-1} \varepsilon^{-lj} x(\varepsilon^j z).$$

To get the equality (11) it is sufficient to change the order of the addition and to use the formula (10).

To demonstrate the uniqueness, first we observe that if $y^l \in \mathscr{F}_k^l(U,Y)$, then

$$G_k^m y^l = \begin{cases} y^m & \text{for } l = m \\ 0 & \text{for } l = 0, 1, \dots, m-1, m+1, \dots, k-1. \end{cases} \tag{12}$$

Now supposing that $x = \sum_{l=0}^{k-1} y^l$, we have from (12) that

$$G_k^m x = \sum_{l=0}^{k-1} G_k^m y^l = y^m, m = 0, 1, \dots, k-1.$$

This completes the proof. □

Corollary 1.

$$\mathscr{F}(U,Y)=\bigoplus_{j=0}^{k-1}\mathscr{F}_k^j(U,Y) \text{ and } \mathscr{F}_k^j(U,Y)=G_k^j(\mathscr{F}(U,Y)),$$

where $\bigoplus$ denotes the simple sum.

Note. The functions G_2^0x, G_2^1x are sometimes called the odd and the even part of the function x, respectively (see e.g. [4]). This implies that Theorem 1 is a generalization of a well-known fact that every function $x\in\mathscr{F}(U,Y)$, $U\in\mathscr{S}_2(X)$, can be uniquely written as the sum of its odd and even parts. Analogously, the functions G_k^lx will be called (l,k)-symmetrical parts of the function x.

In the next theorem we give additional information about the partition (11).

Theorem 2. *If Y is a complex Hilbert space with a scalar product $\langle .\,,.\rangle$ and with the norm $\|y\|=\langle y,y\rangle^{1/2}$, then for every function $x\in\mathscr{F}(U,Y)$, where $U\in\mathscr{S}_k(X)$, we have*

$$\sum_{j=0}^{k-1}\|x(\varepsilon^jz)\|^2=k\sum_{l=0}^{k-1}\|G_k^lx(z)\|^2,\quad z\in U. \tag{13}$$

Proof. From the assumptions, in view of (8), (6) and (10) we obtain

$$\begin{aligned}k\sum_{l=0}^{k-1}\|G_k^lx(z)\|^2&=k\sum_{l=0}^{k-1}\left\langle G_k^lx(z),G_k^lx(z)\right\rangle\\&=k^{-1}\sum_{j,m=0}^{k-1}\left\langle x(\varepsilon^mz),x(\varepsilon^jz)\right\rangle\sum_{l=0}^{k-1}\varepsilon^{l(j-m)}\\&=\sum_{j,m=0,\ m=j}^{k-1}\left\langle x(\varepsilon^mz),x(\varepsilon^jz)\right\rangle=\sum_{j=0}^{k-1}\|x(\varepsilon^jz)\|^2.\end{aligned}$$

□

In the next two theorems we give some other properties of the operators L_k, G_k^j using the language of spectral theory.

For a linear operator A from a complex linear space X into a complex linear space Y by $\sigma(A)$ we will denote the point spectrum of the operator A, while by $V_A(\lambda)$, with $\lambda\in\sigma(A)$, we will denote the space of all eigenvectors of the operator A for the eigenvalue λ.

Theorem 3. *For every integer $k \geqslant 2$ and $j = 0, 1, \ldots, k-1$*

$$\sigma(L_k) = E_k \tag{14}$$

$$V_{L_k}(\varepsilon^j) = \mathscr{F}_k^j(U, Y) \tag{15}$$

P r o o f. If $\lambda \in E_k$, then there exists an integer $j = 0, 1, \ldots, k-1$ such that $\lambda = \varepsilon^j$ and in consequence every element x of the space $\mathscr{F}_k^j(U, Y)$ satisfies the equality

$$L_k x - \lambda x = 0 \tag{16}$$

This gives the inclusion $E_k \subset \sigma(L_k)$.

Now assume that $\lambda \in \sigma(L_k)$. Then there exists an element $x \neq 0$ in the space $\mathscr{F}(U, Y)$ which satisfies the equality (16). From this, according to Theorem 1 and Lemma 1, (a), we conclude that

$$\sum_{l=0}^{k-1} (\varepsilon^l - \lambda) G_k^l x = 0 \tag{17}$$

On the other hand, Theorem 1 gives for the function 0 the following unique partition $\sum\limits_{l=0}^{k-1} G_k^l 0 = 0$. This and (17) imply that $(\varepsilon^l - \lambda) G_k^l x = 0$, for $l = 0, 1, \ldots, k-1$, because $G_k^l 0 = 0$. Since $x \neq 0$, Theorem 1 shows that there exists an integer $j = 0, 1, \ldots, k-1$ such that $G_k^j x \neq 0$ and, in consequence, $\varepsilon^j = \lambda$. This gives $\lambda \in E_k$ and we have $\sigma(L_k) \subset E_k$. This completes the proof of the equality (14).

Now it is clear that the equality (15) holds, too.

It can be proved in a similar way. □

Theorem 4. *For every integer $k \geqslant 2$ and $l = 0, 1, \ldots, k-1$ we have*

$$\sigma(G_k^l) = \{0, 1\}; \quad V_{G_k^l}(0) = \bigoplus_{\substack{j=0 \\ j \neq 1}}^{k-1} \mathscr{F}_k^j(U, Y); \quad V_{G_k^l}(1) = \mathscr{F}_k^l(U, Y).$$

N o t e. Theorems 1–4 can also be proved on the basis of the results of the representation theory of finite groups. The adequate reference in this respect is for example [2].

Now we will give an interpretation of the functions G_k^lx.

Let $D := \{z_1, z_2, \ldots, z_m\} \subset \mathbb{C}$ and $y \in \mathscr{F}(D, Y)$. By $\Theta_D y$ we will denote the mean value of the function y in the set D, that is the element of the space Y given by the formula:

$$\Theta_D y = m^{-1} \sum_{j=0}^{m} y(z_j). \tag{18}$$

It is easy to check that the function $z\Theta_{zE_k} y$ is constant on the set ξE_k, if this function is defined on ξE_k.

Theorem 5. *If $U \in \mathscr{S}_k(\mathbb{C})$, then for each function $x \in \mathscr{F}(U, Y)$ and $z \in U$*

$$x(z) = \sum_{l=0}^{k-1} z^l \Theta_{zE_k} x_l \tag{19}$$

where the functions x_l are defined by the formula:

$$x_0 = x; \quad x_l(z) = \begin{cases} z^{-l} x(z) & \text{for } z \neq 0,\ l = 1, 2, \ldots, k-1 \\ 0 & \text{for } z = 0, \end{cases} \tag{20}$$

(For $z = 0$ in (19) we set $0^0 = 1$).

P r o o f. By Theorem 1 it suffices to show that

$$G_k^l x(z) = z^l \Theta_{zE_k} x_l, \quad l = 0, 1, \ldots, k-1. \tag{21}$$

If $z = 0$, then in view of (6) and (8) we have $G_k^0 x(0) = x(0)$, $G_k^l x(0) = x(0)$ for $l = 1, 2, \ldots, k-1$. On the other hand, according to (18) and (20) at the point $z = 0$, $z^l \Theta_{zE_k} x_l$ is equal to $\Theta_{\{0\}} x(0) = x(0)$ for $l = 0$, and 0 for $l = 1, 2, \ldots, k-1$. Therefore (21) for $z = 0$ holds.

Now let $z \neq 0$. Then

$$G_k^l x(z) = z^l k^{-1} \sum_{j=0}^{k-1} x(\varepsilon^j z) \varepsilon^{-lj} z^{-l} = z^l \Theta_{zE_k} x_l,$$

and (21) for $z \neq 0$ is proved. □

Theorems 1 and 5 yield

Corollary 2. *If $U \in \mathscr{S}_k(\mathbb{C})$, then every function $x \in \mathscr{F}(U,Y)$ can be uniquely represented in the form*

$$x(z) = \sum_{l=0}^{k-1} z^l f_l(z), z \in U \tag{22}$$

where f_l are constant functions on the sets zE_k for every $z \in U$.

N o t e. The announced interpretation of the functions $G_k^l x$ is presented in the equality (21).

Now we will give some connections of the notion of analyticity of functions with the functions $G_k^l x$ and the mean value $\Theta_{zE_k} x_l$.

Let X, Y be complex Banach spaces. For a fixed point $a \in X$ and a real number $r > 0$ we will denote the open ball $\{z \in X;\, \|z - a\| < r\}$ by $B_r(a)$. To simplify the notation we will write B_r and B for $B_r(0)$ and B_1, respectively. Of course $B_r \in \mathscr{S}_k(X)$ for every $k \geqslant 2$. Let U be an open subset of X. A mapping $x \in \mathscr{F}(U,Y)$ is said to be holomorphic or analytic, if for each $a \in U$ there exists a ball $B_r(a) \subset U$ and a sequence of homogeneous and continuous polynomials $P_m \in \mathscr{F}(X,Y)$ such that

$$x(z) = \sum_{m=0}^{\infty} P_m(z-a)$$

and the power series converges uniformly in $B_r(a)$.We shall put

$$\mathscr{H}(U,Y) = \{x \in \mathscr{F}(U,Y);\, x \text{ is holomorphic in } U\}.$$

The above definition and the properties of analytic functions can be found in Mujica, [5].

Theorem 6. *Let X, Y be complex Banach spaces, $U \in \mathscr{S}_k(X)$ an open connected set and $V \in \mathscr{S}_k(X)$ an open nonempty subset of U.*

If $x \in \mathscr{H}(U,Y)$ and $x\big|V \in \mathscr{F}_k^l(V,Y)$ for an integer $l = 0,1,\dots,k-1$, then $x \in \mathscr{F}_k^l(U,Y)$.

P r o o f. In view of (12), the condition $x\big|V \in \mathscr{F}_k^l(U,Y)$ gives that $G_k^m(x\big|V) = 0$ for $m \neq l$, so $(G_k^m x)\big|V = 0$, too. From this and from the fact that $G_k^m x \in \mathscr{H}(U,Y)$, in view of the identity principle (see [5, Pr. 5.7]) we deduce that $G_k^m x = 0$ for $m \neq l$. So $x = G_k^l x$ by Theorem 1. In order to complete the proof it is sufficient to apply Lemma 1(c). □

N o t e. Let us observe that if $X = \mathbb{C}$, then concerning V it is sufficient to assume that $V \in \mathscr{S}_k(X)$ and that it has a cluster point in U.

Theorem 7. *Let X, Y be complex Banach spaces and let $\sum_{m=0}^{\infty} P_m(z)$ be a power series of homogeneous and continuous polynomials $P_m \in \mathscr{F}(X, Y)$ which converges uniformly in a ball $B_r, r > 0$, and let x be its sum in B_r. Then*

$$G_k^l x(z) = \sum_{s=0}^{\infty} P_{l+sk}(z) \text{ for } z \in B_r.$$

P r o o f. From the assumptions,in view of (6), (8) and the homogeneity of the polynomials P_m, we have

$$G_k^l x(z) = k^{-1} \sum_{j=0}^{k-1} \varepsilon^{-jl} \sum_{m=0}^{\infty} \varepsilon^{jm} P_m(z)$$

and, changing the order of addition

$$G_k^l x(z) = k^{-1} \sum_{m=0}^{\infty} P_m(z) \sum_{j=0}^{k-1} \varepsilon^{(m-l)j}.$$

To complete the proof it is sufficient to use the formula (10). □

Corollary 3. *Let $X = \mathbb{C}$ and let Y be a complex Banach space. If $x \in \mathscr{H}(B_r, Y)$ has in the disc B_r the Taylor expansion $x(z) = \sum_{m=0}^{\infty} a_m z^m$, $a_m \in Y$ then*

$$G_k^l x(z) = z^l \sum_{s=0}^{\infty} a_{l+sk} z^s \text{ and } \Theta_{zE_k} x_l = \sum_{s=0}^{\infty} a_{l+sk} z^s \tag{23}$$

Theorem 8. *Let $X = \mathbb{C}$ and Y be a complex Banach space. If $x \in \mathscr{H}(\overline{B}_r, Y)$, then for $z \in B_r$*

$$x(z) = \sum_{l=0}^{k-1} z^l (2\pi \mathrm{i})^{-1} \int_{C_r} x(\zeta)\zeta^{k-l-1}(\zeta^k - z^k)^{-1} \, \mathrm{d}\zeta \tag{24}$$

where C_r denotes the boundary of the disc $B_r \subset \mathbb{C}$ with the positive orientation.

P r o o f. For the function x the Cauchy integral formula

$$x(z) = (2\pi \mathrm{i})^{-1} \int_{C_r} x(\zeta)(\zeta - z)^{-1}\,\mathrm{d}\zeta, \quad z \in B_r \tag{25}$$

holds (see e.g. [1, Th. 9.9.1.]). Since

$$\zeta^k - z^k = (\zeta - z) \sum_{l=0}^{k-1} z^l \zeta^{k-l-1}$$

we conclude from (25) that

$$x(z) = (2\pi \mathrm{i})^{-1} \int_{C_r} x(\zeta)(\zeta^k - z^k)^{-1} \sum_{l=1}^{k-1} z^l \zeta^{k-l-1}\,\mathrm{d}\zeta$$

The above, after the change of the order of the addition and integration, gives (24) and completes the proof. $\square$

Theorems 1 and 8 imply

Corollary 4. *Under the assumptions of Theorem* 8 *we have*

$$G_k^l x(z) = z^l (2\pi \mathrm{i})^{-1} \int_{C_r} x(\zeta)\zeta^{k-l-1}(\zeta^k - z^k)^{-1}\,\mathrm{d}\zeta, \quad z \in B_r.$$

Theorems 5 and 8 imply

Corollary 5. *Under the assumptions of Theorem* 8 *we have*

$$\Theta_{zE_k} x_l = (2\pi \mathrm{i})^{-1} \int_{C_r} x(\zeta)\zeta^{k-l-1}(\zeta^k - z^k)^{-1}\,\mathrm{d}\zeta, \quad z \in B_r.$$

Now we will prove the last theorem in this part of the paper.

Theorem 9. *Let $X = \mathbb{C}$, let Y be a complex Banach space and C_r—the boundary of the disc $B_r \subset \mathbb{C}$. If for the function $x \in \mathscr{F}(C_r, Y)$ the line integral $\int_{C_r} x(z)\,\mathrm{d}z$ exists, then*

$$\int_{C_r} x(z)\,\mathrm{d}z = \int_{C_r} G_k^{k-1} x(z)\,\mathrm{d}z.$$

P r o o f. First we observe that the definition of the line integral and the (l,k)-symmetry of G_k^l give

$$\varepsilon^{-1} \int_{C_r} G_k^l x(z)\,\mathrm{d}z = \int_{C_r} G_k^l x(\varepsilon z)\,\mathrm{d}z = \varepsilon^l \int_{C_r} G_k^l x(z)\,\mathrm{d}z.$$

Consequently, $\int_{C_r} G_k^l x(z)\,\mathrm{d}z = 0$ for $l = 0, 1, \ldots, k-2$. From this, according to Theorem 1, the assertion follows. $\square$

Suppose that X is a complex linear-topological space and $U \neq \emptyset$, X is a set of $\mathscr{S}_k(X)$. Let A_U denote an increasing family $\{U_r\}_{r\in(0,1)}$ consisting of those subsets U_r of U for which

i) U contains the closure $\overline{U}_r$

ii) $\bigcup\limits_{r\in(0,1)} U_r = U$.

We observe that if U, in particular, is the open unit ball B in a complex linear normed space X, then for example $A_B = (rB)_{r\in(0,1)}$.

Theorem 10. *Let X be a complex linear-topological space, Y a complex Hilbert space and $U \in \mathscr{S}_k(X)$. If for a real number $M > 0$, an integer $m = 0, 1, \dots, k-1$ and a family A_U, the function $x \in \mathscr{F}(U, Y)$ satisfies the assumptions*

$$\|x(z)\| \leqslant M \quad \text{for } z \in U, \tag{26}$$

$$\lim_{r\to 1^-} \Big(\inf_{\partial U_r} \|G_k^m x(z)\| \Big) = M, \tag{27}$$

$$\sup_{\overline{U}r} \|G_k^l x(z)\| = \sup_{\partial U_r} \|G_k^l x(z)\|, \quad r \in (0,1),\ 0 \leqslant 1 \leqslant k-1,\ l \neq m, \tag{28}$$

then $x = G_k^m x$.

Proof. By Theorem 1 it is sufficient to prove that $G_k^l x = 0$ for $l \neq m$, $0 \leqslant l \leqslant k-1$.

As $U \in \mathscr{S}_k(X)$, we obtain from the inequality (26) and Theorem 2 that

$$\sum_{l=0}^{k-1} \|G_k^l x(z)\|^2 \leqslant M^2 \quad \text{for } z \in U.$$

From this and from i) we obtain for $l = 0, 1, \dots, m-1, m+1, \dots, k-1$ the inequality

$$\sup_{\partial U_r} \|G_k^1 x(z)\|^2 \leqslant M^2 - \inf_{\partial U_r} \|G_k^m x(z)\|^2, \quad r \in (0,1).$$

Consequently,

$$\Big(\sum_{\partial U_r} \|G_k^l x(z)\|^2 \Big)^2 \leqslant M^2 - \big(\inf_{\partial U_r} \|G_k^m x(z)\| \big)^2, \quad r \in (0,1)$$

because $\sup\limits_J (.\,)^2 = (\sup\limits_J (.\,))^2$ and $\inf\limits_J (.\,)^2 = (\inf\limits_J (.\,))^2$, where $J = [0,1]$. This and (28) give

$$\big(\sup_{\overline{U}_r} \|G_k^l x(z)\|^2 \big)^2 \leqslant M^2 - \big(\inf_{\partial U_r} \|G_k^m x(z)\| \big)^2, \quad r \in (0,1),\ l \neq m. \tag{29}$$

Now we make the following observations:

As the family $\{U_r\}_{r\in(0,1)}$ is increasing, the left-hand side of (29) is a non-decreasing function of the variable $r \in (0,1)$. Of course, it is non-negative, too.

It follows form (27) that the right-hand side of (29) tends to zero, if $r \to 1^-$.

The above observations and (29) give

$$\sup_{\overline{U}_r} \|G_k^l x(z)\| = 0 \quad \text{for } r \in (0,1) \text{ and } l \neq m.$$

From this and ii) we get that $G_k^l = 0$ in U. This proves the theorem. □

Now, for an $n = 1, 2, \ldots$, let $X = Y = \mathbb{C}^n$ with the Euclidean inner product and the Euclidean norm. Put

$$B^k = \Big\{(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n;\ \sum_{j=1}^{n} |z_j|^{2k} < 1\Big\}. \tag{30}$$

Then the domain B^k belongs to the family $\mathscr{S}_k(\mathbb{C}^n)$ and B^1 is the open unit Euclidean ball in $\mathbb{C}^n$.

Theorem 10 yields the following interesting corollary, which is parallel to the result obtained by G. Janiec ([3, Th. 3 and 4]).

Corollary 6. *Let B^k be the domain defined in* (30). *If for $U = B^k$ and for a real number $M > 0$ the function $x \in \mathscr{H}(B^k, \mathbb{C}^n)$ satisfies the inequality* (26) *and $G_k^0 x = P_k \big| B^k$, where P_k denotes the k-homogeneous polynomial*

$$P_k(z) = M(z_1^k, z_2^k, \ldots, z_n^k), \quad z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n, \tag{31}$$

then $x = P_k \big| B^k$.

P r o o f. Let us put $A_{B^k} = (rB^k)_{r\in(0,1)}$. Then $\inf\limits_{\partial(rB^k)} \|G_k^0 x(z)\| = Mr^{1/2}$ for $r \in (0,1)$ and, consequently, the function $G_k^0 x$ satisfies the condition (27). On the other hand, the functions $G_k^l x$ for $l = 1, 2, \ldots, k-1$ are holomorphic in B^k, so, in view of the maximum norm principle (see e.g. [1; Th. 9.5.10 and Exercise 4b]) then all satisfy the equality (28). Theorem 10 gives also that $x = G_k^0 x$, which completeses the proof. □

The following generalization of the Schwarz lemma is an other interesting example of application of the above results.

Theorem 11. *Let $X = Y = \mathbb{C}$ and $x \in \mathscr{H}(B, B)$. Then*

i) *for every* $l = 1, 2, \ldots, k-1$ *and* $z \in B$,

$$|G_k^l x(z)| \leqslant |z|^l; \tag{32}$$

ii) *if* $x(0) = 0$, *then*

$$|G_k^k x(z)| \leqslant |z|^k \quad \textit{for } z \in B; \tag{33}$$

iii) *the equality in* (32) *holds for an* $l = 1, 2, \ldots, k$ *and an* $z^0 \in B$ *if and only if there exists* $\alpha \in \mathbb{C}$, $|\alpha| = 1$, *such that*

$$x(z) = \alpha z^l \tag{34}$$

for every $z \in B$.

Proof. i) Fix $l = 1, 2, \ldots, k$ and observe that in the disc B the function $G_k^l x \in \mathscr{H}(B, B)$ has the Taylor expansion of the form (23). From this, according to the Schwarz lemma (see e.g. [8; Th. 12.2]) we see that the inequality (32) is fulfilled, moreover the equality in (32) for $0 \neq z^0 \in B$ holds if and only if

$$G_k^l x(z) = \alpha z^l, \quad z \in B. \tag{35}$$

The proof of ii) is similar to that of part i).

iii) Fix $l = 1, 2, \ldots, k$ and suppose first that (34) holds. Then the (l, k)-symmetry of the function x from (34), in view of Theorem 1, implies (35) and, consequently, the equality in (32) and (33) at each point $z^0 \in B$.

Now suppose that there exist a point $0 \neq z^0 \in B$ such that $|G_k^l x(z^0)| = |z^0|^l$. It follows then from the proofs of i) and ii) that $G_k^l x$ is defined by the formula (35), so

$$\lim_{r \to 1^-} \Big(\inf_{|z|=r} |G_k^l x(z)| \Big) = 1.$$

As the function x satisfies also the remaining assumptions of Theorem 10, so from Theorem 10 we obtain that $x = G_k^l x$. This and (35) imply that x is represented by formula (34). This completes the proof. $\square$

Note. Theorem 11 generalize a result due to A. Pfluger ([6]).

Let $X = Y = \mathbb{C}$ and let μ be be the two dimensional Lebesgue measure in $\mathbb{C}$.

Theorem 12. *If a function* $x \in \mathscr{H}(B, B)$ *has the fixed point* $z = 0$ *and* $U \in \mathscr{S}_k(\mathbb{C})$ *is a measurable subset of* B, *then*

$$\left| \int_U x(z) \, \mathrm{d}\mu \right| < \frac{2\pi}{k+2}.$$

P r o o f. Using the theorem on change of variables in an integral for the function $G_k^l x$ we have for $l = 0, 1, \ldots, k-1$

$$\int_U G_k^l x(z)\,\mathrm{d}\mu = \int_U G_k^l x(\varepsilon z)\,\mathrm{d}u = \varepsilon^l \int_U G_k^l x(z)\,\mathrm{d}\mu.$$

Consequently, $\int_U G_k^l x(z)\,\mathrm{d}\mu = 0$ for $l = 1, 2, \ldots, k-1$. From Theorem 1 we get also $\int_U x(z)\,\mathrm{d}\mu = \int_U G_k^0 x(z)\,\mathrm{d}\mu$.

On the other hand, from Theorem 11 ii) and from the theorem on change of variables in an integral and Tonelli's theorem we have

$$\left|\int_U G_k^0 x(z)\,\mathrm{d}\mu\right| \leqslant \int_U |G_k^0 x(z)|\,\mathrm{d}\mu \leqslant \int_U |z|^k\,\mathrm{d}\mu \leqslant \int_B |z|^k\,\mathrm{d}\mu = \frac{2\pi}{k+2}.$$

Now we prove that the equality $|\int_U G_k^0 x(z)\,\mathrm{d}\mu| = 2\pi/(k+2)$ does not hold even if $U = B$.

Assume that $|\int_B G_k^0 x(z)\,\mathrm{d}\mu| = 2\pi/(k+2)$. Then

$$\left|\int_B G_k^0 x(z)\,\mathrm{d}\mu\right| = \int_B |G_k^0 x(z)|\,\mathrm{d}\mu = \int_B |z|^k\,\mathrm{d}\mu.$$

The first equality and the continuity of $G_k^0 x$, in view of [8; Th. 1.39, (c)] and $G_k^0 x \neq 0$, imply that there exist a complex constant α, $|\alpha| = 1$ such that $\alpha G_k^0 x(z) = |G_k^0 x(z)|$ in B. Therefore the function $\alpha G_k^0 x$ must be constant in B, because it is real and holomorphic. As $G_k^0 x(0) = 0$, we have $G_k^0 x = 0$, which contradicts our assumption. $\square$

N o t e. Since $B \in \mathscr{S}_k(\mathbb{C})$, for every $k \geqslant 2$ so $|\int_B x(z)\,\mathrm{d}\mu| < 2\pi/(k+2)$. Hence we obtain a new proof of the well known fact that $\int_B x(z)\,\mathrm{d}\mu = 0$ for every function $x \in \mathscr{H}(B, B)$ such that $x(0) = 0$.

Theorem 13. *Let for a $k = 2, 3, \ldots$ and $r \in (0, 1]$ $U := \bigcup\limits_{0 \leqslant s \leqslant r} sE_k$, and let μ be the one-dimensional Lebesgue measure on U. If $x \in \mathscr{H}(B, B)$, $x(0) = 0$, then*

$$\left|\int_U x(z)\,\mathrm{d}\mu\right| \leqslant \frac{r^{k+1}k}{k+1}.$$

Moreover, the equality holds if and only if there exists $\alpha \in \mathbb{C}$, $|\alpha| = 1$, such that $x(z) = \alpha z^k$ for $z \in B$.

P r o o f. We prove the above inequality in a way similar to the estimation in Theorem 12.

Now we will show the second part of the theorem.

If $|\int_U x(z)\,\mathrm{d}\mu| = r^{k+1}k/(k+1)$, then (see the proof of Theorem 12) we get $\int_U |G_k^0 x(z)|\,\mathrm{d}\mu = \int_U |z|^k\,\mathrm{d}\mu$. This and Theorem 11, part iii), in view of the formula (9), imply that there exists $\alpha \in \mathbb{C}$, $|\alpha| = 1$ such that $x(z) = \alpha z^k$ in B.

If we assume that $x(z) = \alpha z^k$ in B for an $\alpha \in \mathbb{C}$, $|\alpha| = 1$, then $x \in \mathscr{F}_k^0(U, \mathbb{C})$ and

$$\Big|\int_U x(z)\,\mathrm{d}\mu\Big| = k\Big|\int_{[0,r]} x(z)\,\mathrm{d}\mu\Big| = \frac{r^{k+1}k}{k+1}$$

. □

N o t e. If we put $k = 2$, in Theorem 13 then we obtain the result due to R. Mortini ([4]).

At the and we will give an application of the (l, k)-symmetrical functions in the theory of fixed points.

Let X, Y be complex linear spaces, $U \in \mathscr{S}_k(X)$ for a $k \geqslant 2$, and let $x \in \mathscr{F}(U, Y)$. Observe that if there exists a point $z^0 \in U$ which is a fixed point of the function $G_k^1 x$ and a zero of all functions $G_k^l x$, $l = 0, 2, 3, \ldots, k-1$, then according to Theorem 1, z^0 is a fixed point of x in U, too.

The inverse theorem is not true,nevertheless the following theorem holds.

Theorem 14. *Let the set of fixed points of a function $x \in \mathscr{F}(U, Y)$ be a k-symmetrical subset of the set $U \in \mathscr{S}_k(X)$. If z^0 is a fixed point of x, then z^0 is a fixed point of $G_k^1 x$ and a zero of $G_k^l x$ for $l = 0, 2, 3, \ldots, k-1$.*

P r o o f. From (6), (8) and from the assumptions we have

$$G_k^l x(z^0) = k^{-1}\sum_{j=0}^{k-1} \varepsilon^{-lj} x(\varepsilon^j z^0) = k^{-1} z^0 \sum_{j=0}^{k-1} \varepsilon^{j(1-l)}.$$

Hence, using the formula (10), we obtain that $G_k^l x(z^0) = 0$ for $l = 0, 2, 3, \ldots, k-1$ and $G_k^l x(z^0) = z^0$ for $l = 1$. This complete the proof. □

Corollary 7. *Let X be a complex strictly convex Banach space and $Y = X$. If the function $x \in \mathscr{H}(B, B)$ has a fixed points $z = 0$ and $z^0 \neq 0$, then z^0 is a fixed point of $G_k^1 x$ and zero of the functions $G_k^l x$ for $l = 0, 2, 3, \ldots, k-1$.*

P r o o f. It follows from the assumptions (see [7, Th. 2]) that the set of fixed points of x coincides with the set of fixed points of the mapping $Dx(0)\big|B$. This implies the k-symmetricality of the set of fixed points of x. Now it is sufficient to apply Theorem 14. □

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